Symbolic Determinant Evaluation

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The HOLONOMIC ANSATZ II.
Automatic DISCOVERY(!) and PROOF(!!)
of Holonomic Determinant Evaluations

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- $A_n = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ is an $n \times n$ matrix,
- $a_{i,j}$ is a bivariate holonomic sequence, not depending on n,
- $b_n \neq 0$ for all $n \geqslant 1$.

Some Examples

$$\det_{1\leqslant i,j\leqslant n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}}$$

$$\det_{0\leqslant i,j\leqslant n-1} \binom{2i+2a}{j+b} = 2^{n(n-1)/2} \prod_{k=0}^{n-1} \frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!}$$

$$\det_{0\leqslant i,j\leqslant n-1} \sum_k \binom{i}{k}\binom{j}{k} 2^k = 2^{n(n-1)/2}$$

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$$\boxed{\det_{1\leqslant i,j\leqslant n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!}\prod_{k=1}^{n-1}\frac{(k!)^2}{(k+1)_{n-1}}}$$

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Toy Example

$$A_n := (a_{i,j})_{1 \leqslant i,j \leqslant n} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

with $a_{i,j} := \dfrac{1}{i+j-1}$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

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- $A_n^{(i,j)}$: matrix A_n with row i and column j deleted

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- Define $c_{n,j} := (-1)^{n+j} M_{n,j} / M_{n,n}$
- We obtain $\displaystyle\sum_{j=1}^{n} a_{i,j} c_{n,j} = \delta_{i,n} \frac{\det A_n}{\det A_{n-1}}$

Toy Example

We can explicitly compute the numbers $c_{n,j}$:

$$\begin{array}{ccccccc}
n=1 & n=2 & n=3 & n=4 & n=5 & n=6 & n=7 \\
\begin{pmatrix} 1 \end{pmatrix} &
\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} &
\begin{pmatrix} \frac{1}{6} \\ -1 \\ 1 \end{pmatrix} &
\begin{pmatrix} -\frac{1}{20} \\ \frac{3}{5} \\ -\frac{3}{2} \\ 1 \end{pmatrix} &
\begin{pmatrix} \frac{1}{70} \\ -\frac{2}{7} \\ \frac{9}{7} \\ -2 \\ 1 \end{pmatrix} &
\begin{pmatrix} -\frac{1}{252} \\ \frac{5}{42} \\ -\frac{5}{6} \\ \frac{20}{9} \\ -\frac{5}{2} \\ 1 \end{pmatrix} &
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From this we guess that

$$c_{n,j} = (-1)^{j+n}\binom{n-1}{j-1}\binom{j+n-2}{j-1}\binom{2n-2}{n-1}^{-1}.$$

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Then we evaluate the sum (e.g., using Zeilberger's algorithm)

$$\begin{aligned}\sum_{j=1}^{n} a_{n,j} c_{n,j} &= \sum_{j=1}^{n} \frac{(-1)^{j+n}}{n+j-1} \binom{n-1}{j-1} \binom{j+n-2}{j-1} \binom{2n-2}{n-1}^{-1} \\ &= \frac{1}{2n-1} \binom{2n-2}{n-1}^{-2} = \frac{\det A_n}{\det A_{n-1}}.\end{aligned}$$

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$$= \frac{1}{2n-1} \binom{2n-2}{n-1}^{-2} = \frac{\det A_n}{\det A_{n-1}}.$$

Therefore:

$$\det A_n = \prod_{k=1}^{n} \frac{1}{2k-1} \binom{2k-2}{k-1}^{-2} = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}}.$$

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$\longrightarrow$ Use holonomic functions!

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From this we guess that (e.g., by Kauers' Guess package)

$$c_{n,j} = \frac{(j-n-1)(j+n-2)}{(j-1)^2}\, c_{n,j-1},$$

$$c_{n,j} = \frac{(n-1)(j+n-2)}{2(2n-3)(j-n)}\, c_{n-1,j}.$$

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Implementations are available in F. Chyzak's Maple package `Mgfun` and our Mathematica package `HolonomicFunctions`; here we will use the latter one.

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has a unique solution, namely $c_{n,j} = (-1)^{n+j} M_{n,j}/M_{n,n}$.

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Now use $c_{n,j}$ to do Laplace expansion of A_n w.r.t. the last row:

$$\det A_n = \sum_{j=1}^{n} M_{n,n} c_{n,j} a_{n,j}.$$

Showing that the sum evaluates to b_n completes the induction step.

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- Work with an implicit (recursive) definition of $c_{n,j}$.

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Instead:

- Hope that $c_{n,j}$ is holonomic (may be the case or not).
- Work with an implicit (recursive) definition of $c_{n,j}$.
- The values of $c_{n,j}$ can be computed for concrete $n, j \in \mathbb{N}$.

How to get $c_{n,j}$

We cannot expect to be able to compute $c_{n,j}$ explicitly!
(at least not for symbolic n)

Instead:

- Hope that $c_{n,j}$ is holonomic (may be the case or not).
- Work with an implicit (recursive) definition of $c_{n,j}$.
- The values of $c_{n,j}$ can be computed for concrete $n, j \in \mathbb{N}$.
- If recurrences exist they can be guessed automatically (e.g. with M. Kauers's Mathematica package `Guess`)

Zeilberger's Holonomic Ansatz

1. Compute many values of $c_{n,j}$ (e.g. for $1 \leqslant j \leqslant n \leqslant 100$).
2. Guess linear recurrences for $c_{n,j}$ from that data.
3. Prove the following identities using holonomic closure properties and creative telescoping:

$$c_{n,n} = 1 \qquad (n \geqslant 1), \tag{1}$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0 \qquad (1 \leqslant i < n), \tag{2}$$

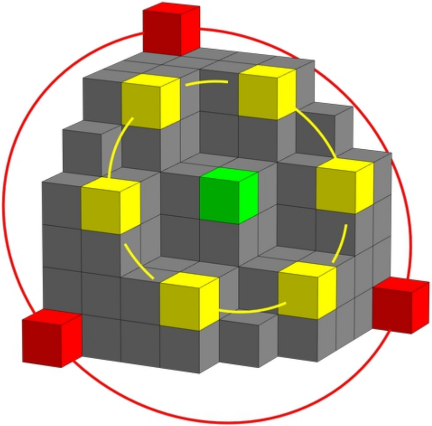
$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \qquad (n \geqslant 1). \tag{3}$$

Note: all these steps can be executed automatically!

Part I

Proof of the q-TSPP Conjecture

(joint work with M. Kauers and D. Zeilberger)



A Success Story of the Holonomic Ansatz

C. Koutschan, M. Kauers, D. Zeilberger:

Proof of George Andrews's and David Robbins's
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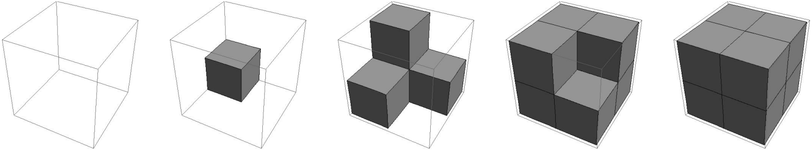
By evaluating the q-holonomic determinant (Okada 1989)

$$\det_{1\leqslant i,j\leqslant n}\left(q^{i+j-1}\begin{bmatrix}i+j-2\\ i-1\end{bmatrix}_q + q^{i+j}\begin{bmatrix}i+j-1\\ i\end{bmatrix}_q + (1+q^i)\delta_{i,j} - \delta_{i,j+1}\right)$$

$$= \prod_{1\leqslant i\leqslant j\leqslant k\leqslant n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2$$

a long-standing combinatorial problem (first stated in 1983) was solved, the q-enumeration of totally symmetric plane partitions.

Let $T(n)$ denote set of TSPPs with largest part at most n.



Stembridge's Theorem:
(Stembridge 1995)
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q-TSPP conjecture:
(KKZ theorem, 2011)

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leqslant i \leqslant j \leqslant k \leqslant n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

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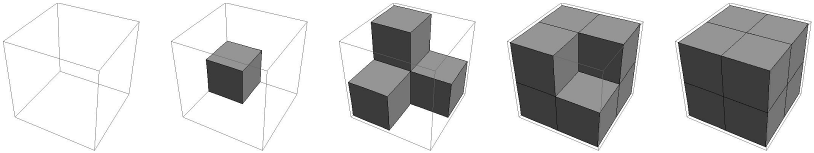

q^0

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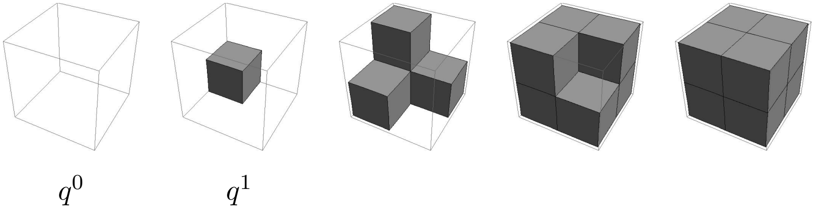

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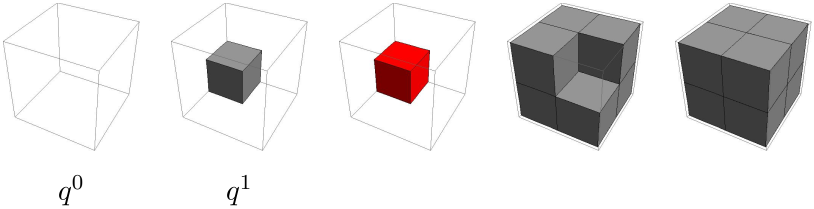

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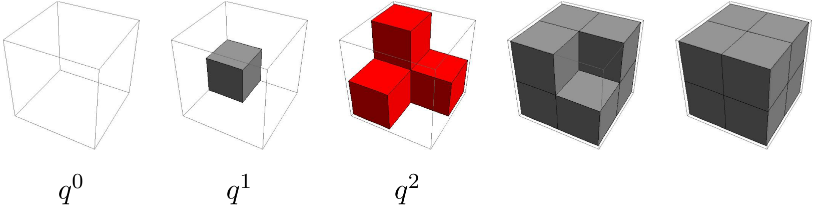

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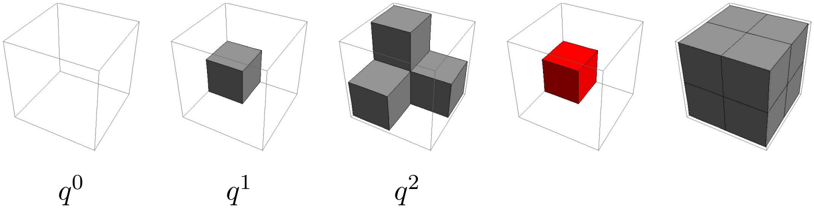

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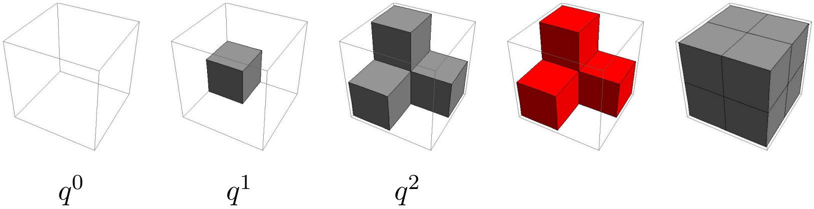

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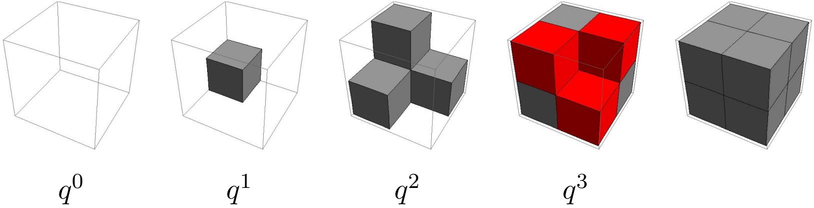

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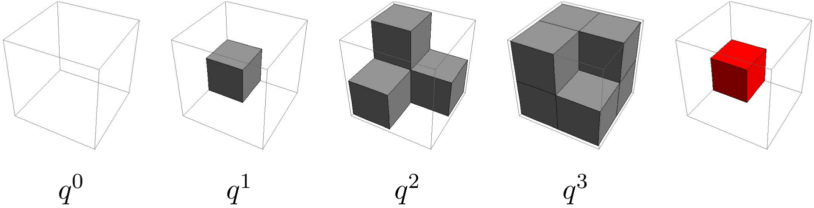

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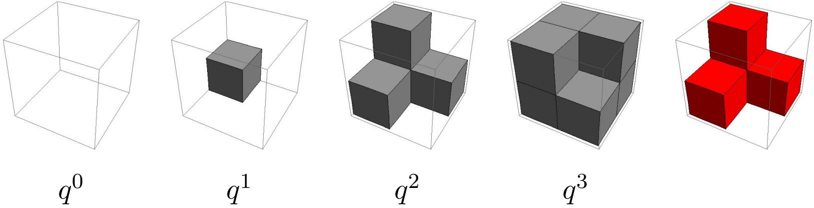

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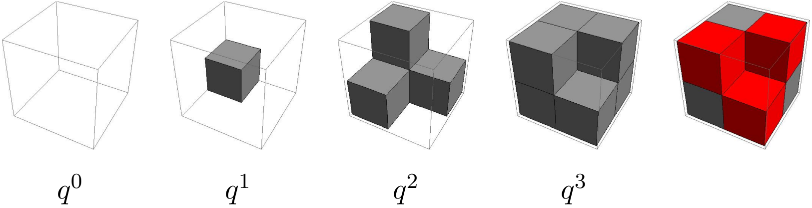

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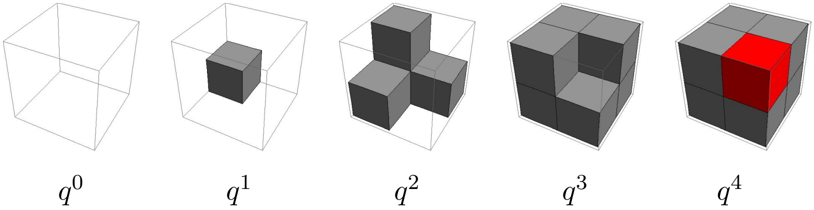

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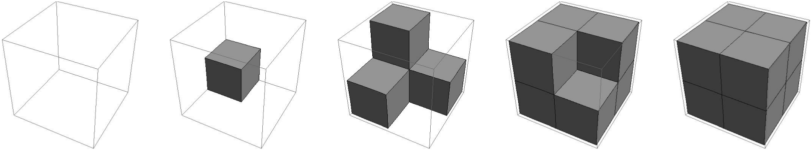

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Part II

Solving Krattenthaler's Conjectures
by
Variations of the Holonomic Ansatz

(joint work with T. Thanatipanonda)

Krattenthaler's Work on Determinants

- In 1999, C. Krattenthaler published the classic

  Advanced Determinant Calculus

  in *Séminaire Lotharingien de Combinatoire* **42**:1–67.

  For example, the q-TSPP conjecture was mentioned therein.

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- In 2005, C. Krattenthaler published

  Advanced Determinant Calculus: A Complement

  in *Linear Algebra and its Applications* **411**:68–166.

  We solved Conjectures 34, 35, and 36 from this paper
  which are related to combinatorial problems (rhombus tilings).

Conjecture 34 $\longrightarrow$ Theorem 1

Let the determinant $D_1(n)$ be defined by

$$D_1(n) := \det_{1\leqslant i,j\leqslant n}\left(\delta_{i,j} + \binom{\mu+i+j-2}{j}\right)$$

where μ is an indeterminate and $\delta_{i,j}$ is the Kronecker delta.
Then the following relation holds:

$$\frac{D_1(2n)}{D_1(2n-1)} = (-1)^{(n-1)(n-2)/2} 2^n \frac{\left(\frac{\mu}{2}+n\right)_{\lceil n/2\rceil}\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}}{(n)_n\left(-\frac{\mu}{2}-2n+\frac{3}{2}\right)_{\lceil (n-2)/2\rceil}}.$$

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This conjecture was posed by G. Andrews in 1980; it appeared in the context of enumerating certain classes of plane partitions.

Result of the Holonomic Ansatz

Some Data

$$
\begin{aligned}
D_1(1) &= \mu + 1 \\
D_1(2) &= (\mu+1)(\mu+2) \\
D_1(3) &= \tfrac{1}{12}(\mu+1)(\mu+2)(\mu+3)(\mu+14) \\
D_1(4) &= \tfrac{1}{72}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+9)(\mu+14) \\
D_1(5) &= \tfrac{1}{8640}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+9) \\
&\quad \times(\mu^3+45\mu^2+722\mu+3432) \\
D_1(6) &= \tfrac{1}{518400}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6) \\
&\quad \times(\mu+8)(\mu+13)(\mu+15)(\mu^3+45\mu^2+722\mu+3432) \\
D_1(7) &= \tfrac{1}{870912000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_1(8) &= \tfrac{1}{731566080000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_1(9) &= \tfrac{1}{221225582592000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+21)^2 \\
&\quad \times(\mu^6+142\mu^5+8505\mu^4+277100\mu^3+5253404\mu^2+52937808 \\
D_1(10) &= \tfrac{1}{334493080879104000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+25)(\mu+27) \\
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Modified Holonomic Ansatz

Idea: use the subsequence $\tilde{c}_{n,j} := c_{2n,j}$.

Modified Holonomic Ansatz

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Then we have to prove the following the identities:

$$\tilde{c}_{n,2n} = 1 \qquad (n \geqslant 1), \tag{1a}$$

$$\sum_{j=1}^{2n} \tilde{c}_{n,j} a_{i,j} = 0 \qquad (1 \leqslant i < 2n), \tag{2a}$$

$$\sum_{j=1}^{2n} \tilde{c}_{n,j} a_{2n,j} = \frac{b_{2n}}{b_{2n-1}} \qquad (n \geqslant 1). \tag{3a}$$

The Meta-Certificate $\tilde{c}_{n,j}$

With the modified ansatz we obtain very small recurrences for $\tilde{c}_{n,j}$:

$$\begin{aligned}
&2n(j+1)(2n-1)(2j+\mu)(j-2n)(j-2n+1)\\
&\qquad\times(\mu+4n-5)(\mu+4n-3)(j+\mu+2n-1)\tilde{c}_{n,j} =\\
&j(j+\mu-1)(2j+\mu-1)(j-2n+3)(\mu+4n-3)\\
&\qquad\times(j^4+2j^3\mu+\ldots\langle 24 \text{ terms}\rangle+12)\tilde{c}_{n-1,j+1} -\\
&(j+1)(j+\mu+2n-3)(2j^6\mu+8j^6n+\ldots\langle 92 \text{ terms}\rangle-210\mu n)\tilde{c}_{n-1,j}
\end{aligned}$$

$$\begin{aligned}
&(j-1)(j+\mu-3)(j+\mu-2)(2j+\mu-4)(j-2n)\\
&\qquad\times(j+\mu+2n-1)\tilde{c}_{n,j} =\\
&j(j+\mu-3)(4j^4+8j^3\mu+\ldots\langle 26 \text{ terms}\rangle+16)\tilde{c}_{n,j-1} -\\
&j(j-1)(j+\mu-2)(2j+\mu-2)(j-2n-2)(j+\mu+2n-3)\tilde{c}_{n,j-2}
\end{aligned}$$

Conjecture 35 $\longrightarrow$ Theorem 2

Let μ be an indeterminate and n be a nonnegative integer.
If n is even, then the following determinant evaluation holds:

$$\det_{1\leqslant i,j\leqslant n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j}\right)=$$

$$(-1)^{n/2}2^{n(n+2)/4}\frac{\left(\frac{\mu}{2}\right)_{n/2}}{\left(\frac{n}{2}\right)!}\left(\prod_{i=0}^{(n-2)/2}\frac{i!^2}{(2i)!^2}\right)\times$$

$$\left(\prod_{i=0}^{\lfloor (n-4)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{5}{2}\right)_{(n-4i-2)/2}^{2}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+3\right)_{(n-4i-4)/2}^{2}\right)$$

Conjecture 35 ⟶ Theorem 2

Let μ be an indeterminate and n be a nonnegative integer.
If n is odd, then the following determinant evaluation holds:

$$\det_{1\leqslant i,j\leqslant n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j}\right)=$$

$$(-1)^{(n-1)/2}2^{(n+3)(n+1)/4}\left(\frac{\mu-1}{2}\right)_{(n+1)/2}\left(\prod_{i=0}^{(n-1)/2}\frac{i!(i+1)!}{(2i)!(2i+2)!}\right)\times$$

$$\left(\prod_{i=0}^{\lfloor (n-3)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{5}{2}\right)_{(n-4i-3)/2}^{2}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+\frac{3}{2}\right)_{(n-4i-1)/2}^{2}\right)$$

Result of the Holonomic Ansatz

Result of the Holonomic Ansatz

Instead of b_n/b_{n-1} we would like to study the quotients

$$\frac{b_{2n}}{b_{2n-2}} \quad \text{and} \quad \frac{b_{2n+1}}{b_{2n-1}}.$$

The Double Step Method

Based on the formula for the determinant of a block matrix

$$\det(M) = \det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2)$$

we obtain the following proof scheme:

$$c'_{n,n-1} = c''_{n,n} = 1, \qquad c'_{n,n} = c''_{n,n-1} = 0, \tag{1b}$$

$$\sum_{j=1}^{n} a_{i,j} c'_{n,j} = \sum_{j=1}^{n} a_{i,j} c''_{n,j} = 0, \qquad (1 \leqslant i \leqslant n-2) \tag{2b}$$

$$\begin{aligned} \frac{b_n}{b_{n-2}} = & \left(\sum_{j=1}^{n} a_{n-1,j}\, c'_{n,j}\right)\left(\sum_{j=1}^{n} a_{n,j}\, c''_{n,j}\right) \\ & - \left(\sum_{j=1}^{n} a_{n-1,j}\, c''_{n,j}\right)\left(\sum_{j=1}^{n} a_{n,j}\, c'_{n,j}\right). \end{aligned} \tag{3b}$$

Alternative: the Desnanot-Jacobi Approach

Denote $b_n(I,J) := \det_{\substack{I \leqslant i \leqslant n-1+I \\ J \leqslant j \leqslant n-1+J}} a_{i,j}$ (our determinant is $b_n(1,1)$).

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With this notation, the Desnanot-Jacobi identity reads

$$b_n(0,0)b_{n-2}(1,1) = b_{n-1}(0,0)b_{n-1}(1,1) - b_{n-1}(0,1)b_{n-1}(1,0)$$

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Substitute $n \to 2n+1$ and $n \to 2n$, and use $b_{2n-1}(0,0) = 0$:

$$b_{2n}(0,0)b_{2n}(1,1) = b_{2n}(0,1)b_{2n}(1,0)$$
$$b_{2n}(0,0)b_{2n-2}(1,1) = -b_{2n-1}(0,1)b_{2n-1}(1,0)$$

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$$b_{2n}(0,0)b_{2n}(1,1) = b_{2n}(0,1)b_{2n}(1,0)$$
$$b_{2n}(0,0)b_{2n-2}(1,1) = -b_{2n-1}(0,1)b_{2n-1}(1,0)$$

From these two equations we obtain the desired quotient:

$$\frac{b_{2n}(1,1)}{b_{2n-2}(1,1)} = -\frac{b_{2n}(1,0)}{b_{2n-1}(1,0)} \cdot \frac{b_{2n}(0,1)}{b_{2n-1}(0,1)}$$

Alternative: the Desnanot-Jacobi Approach

Similarly, using $b_{2n}(0,0) = -b_{2n-1}(1,1)$, we get

$$\frac{b_{2n+1}(1,1)}{b_{2n-1}(1,1)} = -\frac{b_{2n+1}(0,1)}{b_{2n}(0,1)} \cdot \frac{b_{2n+1}(1,0)}{b_{2n}(1,0)}.$$

Alternative: the Desnanot-Jacobi Approach

Similarly, using $b_{2n}(0,0) = -b_{2n-1}(1,1)$, we get

$$\frac{b_{2n+1}(1,1)}{b_{2n-1}(1,1)} = -\frac{b_{2n+1}(0,1)}{b_{2n}(0,1)} \cdot \frac{b_{2n+1}(1,0)}{b_{2n}(1,0)}.$$

Remarks:

- $b_{2n-1}(0,0) = 0$: show that the rows of this matrix are dependent, by guessing the coefficients of the corresponding linear combination.
- $b_{2n}(0,0) = -b_{2n-1}(1,1)$ can be shown by yet another variation of the holonomic ansatz.
- $b_{2n}(1,0)/b_{2n-1}(1,0)$ etc. can be treated with our first variation of the original holonomic ansatz.

Conjecture 36 $\longrightarrow$ Theorem 3

Let μ be an indeterminate. For any odd nonnegative integer n there holds

$$\det_{1\leqslant i,j\leqslant n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j+1}\right)=$$

$$(-1)^{(n-1)/2}2^{(n-1)(n+5)/4}(\mu+1)\frac{\left(\frac{\mu}{2}-1\right)_{(n+1)/2}}{\left(\frac{n+1}{2}\right)!}\left(\prod_{i=0}^{(n-1)/2}\frac{i!^2}{(2i)!^2}\right)$$

$$\times\left(\prod_{i=0}^{\lfloor(n-1)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{3}{2}\right)^2_{(n-4i-1)/2}\right)$$

$$\times\left(\prod_{i=0}^{\lfloor(n-3)/4\rfloor}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+\frac{5}{2}\right)^2_{(n-4i-3)/2}\right)$$

Proof

Relate this determinant to the previous problem:

$$\det_{1\leqslant i,j\leqslant 2n-1}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j+1}\right)=$$

$$\det_{2\leqslant i,j\leqslant 2n}\left(-\delta_{i,j}+\binom{(\mu-2)+i+j-2}{j}\right)=b_{2n-1}(2,2,\mu-2)$$

Proof

Relate this determinant to the previous problem:

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$$\det_{2\leqslant i,j\leqslant 2n}\left(-\delta_{i,j}+\binom{(\mu-2)+i+j-2}{j}\right)=b_{2n-1}(2,2,\mu-2)$$

A variation of the holonomic ansatz yields a recurrence for

$$\frac{b_{2n}(1,1,\mu-2)}{b_{2n-1}(2,2,\mu-2)}.$$

But $b_{2n}(1,1,\mu-2)$ is already known (previous result).

Our Conjecture

We found a beautiful formula for Andrews' determinant $D_1(n)$.

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We found a beautiful formula for Andrews' determinant $D_1(n)$. Let

$$C(n) = \frac{(-1)^n + 3}{2} \prod_{i=1}^{n} \frac{\left\lfloor \frac{i}{2} \right\rfloor !}{i!},$$

$$E(n) = (\mu + 1)_n \left(\prod_{i=1}^{\left\lfloor \frac{3}{2} \left\lfloor \frac{1}{2}(n-1) \right\rfloor - 2 \right\rfloor} \Big(\mu + 2i + 6 \Big)^{2 \left\lfloor \frac{1}{3}(i+2) \right\rfloor} \right)$$

$$\times \left(\prod_{i=1}^{\left\lfloor \frac{3}{2} \left\lfloor \frac{n}{2} \right\rfloor - 2 \right\rfloor} \Big(\mu + 2i + 2 \left\lfloor \frac{3}{2} \left\lfloor \frac{n}{2} + 1 \right\rfloor \right\rfloor - 1 \Big)^{2 \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{3}(i-1) \right\rfloor - 1} \right),$$

$$F_m(n) = \left(\prod_{i=1}^{\left\lfloor \frac{1}{4}(n-1) \right\rfloor} (\mu + 2i + n + m)^{1-2i-m} \right)$$

$$\times \left(\prod_{i=1}^{\left\lfloor \frac{n}{4} - 1 \right\rfloor} (\mu - 2i + 2n - 2m + 1)^{1-2i-m} \right),$$

Our Conjecture

…further let…

$$F(n) = \begin{cases} E(n)F_0(n), & \text{if } n \text{ is even,} \\ E(n)F_1(n) \displaystyle\prod_{i=1}^{\frac{1}{2}(n-5)} (\mu + 2i + 2n - 1), & \text{if } n \text{ is odd,} \end{cases}$$

$$\begin{aligned} T(k) = {} & 55296k^6 + 41472(\mu-1)k^5 + 384(30\mu^2 - 66\mu + 53)k^4 \\ & + 96(\mu-1)(15\mu^2 - 42\mu + 61)k^3 \\ & + 4(19\mu^4 - 122\mu^3 + 419\mu^2 - 544\mu + 72)k^2 \\ & + (\mu-1)(\mu^4 - 14\mu^3 + 101\mu^2 - 160\mu - 84)k \\ & + 2(\mu-3)(\mu-2)(\mu-1)(\mu+1), \end{aligned}$$

Our Conjecture

...and let...

$$
\begin{aligned}
S_1(n) = \sum_{k=1}^{n-1} &\Big(2^{6k} (\mu + 8k - 1) \left(\tfrac{1}{2}\right)_{2k-1}^2 \left(\tfrac{1}{2}(\mu+5)\right)_{2k-3} \\
&\quad \times \left(\tfrac{1}{2}(\mu+4k+2)\right)_{k-2} \left(\tfrac{1}{2}(\mu+4k+2)\right)_{2n-2k-2} T(k) \Big) \\
&\quad \Big/ \Big((2k)! \left(\tfrac{1}{2}(\mu+6k-3)\right)_{3k+4} \Big), \\
S_2(n) = \sum_{k=1}^{n-1} &\Big(2^{6k} (\mu + 8k + 3) \left(\tfrac{1}{2}\right)_{2k}^2 \left(\tfrac{1}{2}(\mu+5)\right)_{2k-2} \\
&\quad \times \left(\tfrac{1}{2}(\mu+4k+4)\right)_{k-2} \left(\tfrac{1}{2}(\mu+4k+4)\right)_{2n-2k-2} T\big(k+\tfrac{1}{2}\big) \Big) \\
&\quad \Big/ \Big((2k+1)! \left(\tfrac{1}{2}(\mu+6k+1)\right)_{3k+5} \Big),
\end{aligned}
$$

Our Conjecture

$$P_1(n) = 2^{3n-1} \frac{\left(\frac{1}{2}(\mu+6n-3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-3}} \times \left(\frac{\left(\frac{1}{2}(\mu+2)\right)_{2n-2}}{(\mu+3)^2} + \frac{\mu(\mu-1)S_1(n)}{2^{13}} \right),$$

$$P_2(n) = 2^{3n-1} \frac{\left(\frac{1}{2}(\mu+6n+1)\right)_{3n-1}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-2}} \times \left(\frac{(\mu+14)\left(\frac{1}{2}(\mu+4)\right)_{2n-2}}{(\mu+7)(\mu+9)} + \frac{\mu(\mu-1)S_2(n)}{2^{9}} \right),$$

$$G(n) = \begin{cases} P_1\left(\frac{1}{2}(n+1)\right), & \text{if } n \text{ is odd,} \\ P_2\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Then for every positive integer n we have

$$\det_{1\leqslant i,j\leqslant n} \left(\delta_{i,j} + \binom{\mu+i+j-2}{j} \right) = C(n)F(n)G\left(\left\lfloor \tfrac{1}{2}(n+1) \right\rfloor\right).$$

Towards a Proof of our Conjecture

Recent news: together with T. Thanatipanonda, we applied the Desnanot-Jacobi theorem successively to this determinant, in order to split it into simpler subproblems:

$D_{1,0}(2n) = 0$

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$D_{0,0}(2n)/D_{0,0}(2n-1)$

$D_{2,0}(2n)/D_{2,0}(2n-1)$

$D_{-1,1}(2n+1)/D_{-1,1}(2n)$

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	$\mu = 5$
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$D_{0,1}(2n) = 0$	✓
$D_{0,0}(2n)/D_{0,0}(2n-1)$	✓
$D_{2,0}(2n)/D_{2,0}(2n-1)$	✓
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$D_{2,0}(2n)/D_{2,0}(2n-1)$	✓	✓
$D_{-1,1}(2n+1)/D_{-1,1}(2n)$	✓	✗

Part III

The Holonomic Ansatz adapted to the Evaluation of Pfaffians

(joint work with M. Ishikawa)

Pfaffians

Consider a skew-symmetric matrix A, i.e., $A = -A^T$:

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots \\ -a_{1,2} & 0 & a_{2,3} & \cdots \\ -a_{1,3} & -a_{2,3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(it is easy to see that $\det A = 0$ if A has odd dimensions).

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(it is easy to see that $\det A = 0$ if A has odd dimensions).

Now let A be a skew-symmetric matrix of size $2n \times 2n$.
Then the Pfaffian of A is defined as

$$\operatorname{Pf} A := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

Note that $(\operatorname{Pf} A)^2 = \det A$.

Try to Apply Determinant Techniques

Zeilberger's holonomic ansatz doesn't work, since it requires

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We could apply our variant that considers the quotient

$$\frac{\det A_n}{\det A_{n-2}}.$$

- This double step method works in theory,
- but is complicated in practice and
- leads to very large computations.

Laplace Expansion for Pfaffians

- Let $A = (a_{i,j})_{1 \leqslant i,j \leqslant 2n}$ be a skew-symmetric matrix.
- Denote by $A(i,j)$ the $(2n-2) \times (2n-2)$ matrix which is obtained by deleting the rows and columns i and j from A.
- Define the cofactors $\Gamma_{i,j} := \begin{cases} (-1)^{j-i-1} \operatorname{Pf} A(i,j) & \text{if } i < j, \\ (-1)^{i-j} \operatorname{Pf} A(j,i) & \text{if } j < i, \\ 0 & \text{if } i = j. \end{cases}$

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Then there exists a Laplace-type expansion for the Pfaffian of A:

$$\delta_{j,k} \operatorname{Pf} A = \sum_{i=1}^{2n} a_{j,i} \Gamma_{k,i} = \sum_{i=1}^{2n} a_{i,j} \Gamma_{i,k}.$$

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Setting $j = k = 2n$ leads to

$$\operatorname{Pf} A = \sum_{i=1}^{2n} a_{2n,i} \Gamma_{2n,i} = \sum_{i=1}^{2n} a_{i,2n} \Gamma_{i,2n}.$$

Pfaffian Evaluation: Proof by Induction

Problem: Prove that $\operatorname{Pf} A_{2n} = \operatorname{Pf}\left(a_{i,j}\right)_{1 \leqslant i,j \leqslant 2n} = b_n$ for all $n \in \mathbb{N}$.

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Induction step: the assumption implies that the linear system

$$\begin{pmatrix} a_{1,1} & \cdots & a_{2n-2,1} & a_{2n-1,1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{1,2n-2} & \cdots & a_{2n-2,2n-2} & a_{2n-1,2n-2} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{2n,1} \\ \vdots \\ c_{2n,2n-2} \\ c_{2n,2n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

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has a unique solution, namely $c_{2n,j} = \Gamma_{j,2n}/\Gamma_{2n-1,2n} = \Gamma_{j,2n}/b_{n-1}$.

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has a unique solution, namely $c_{2n,j} = \Gamma_{j,2n}/\Gamma_{2n-1,2n} = \Gamma_{j,2n}/b_{n-1}$.

Now use $c_{2n,j}$ in the expansion formula for the Pfaffian of A_{2n}:

$$\operatorname{Pf} A_{2n} = \sum_{j=1}^{2n-1} b_{n-1} c_{2n,j} a_{j,2n}.$$

Showing that the sum evaluates to b_n completes the induction step.

The Holonomic Ansatz for Pfaffians

Now the holonomic ansatz can be formulated for Pfaffians:

1. Compute many values of $c_{2n,j}$ (e.g. for $1 \leqslant j \leqslant 2n \leqslant 100$).
2. Guess linear recurrences for $c_{2n,j}$ from that data.
3. Prove the following identities using holonomic closure properties and creative telescoping:

$$c_{2n,2n-1} = 1 \qquad (n \geqslant 1), \tag{P1}$$

$$\sum_{j=1}^{2n-1} c_{2n,j} a_{j,i} = 0 \qquad (1 \leqslant i < 2n), \tag{P2}$$

$$\sum_{j=1}^{2n-1} c_{2n,j} a_{j,2n} = \frac{b_n}{b_{n-1}} \qquad (n \geqslant 1). \tag{P3}$$

Turn Conjectures into Theorems

Theorem. Let $M_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k}$ denote the n-th Motzkin number. Then for $n \in \mathbb{N}$ we have

$$\operatorname{Pf}\left((j-i)M_{i+j-3}\right)_{1\leqslant i,j\leqslant 2n} = \prod_{k=0}^{n-1}(4k+1).$$

Theorem. Let $D_n = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}$ denote the n-th central Delannoy number. Then for $n \in \mathbb{N}$ we have

$$\operatorname{Pf}\left((j-i)D_{i+j-3}\right)_{1\leqslant i,j\leqslant 2n} = 2^{(n+1)(n-1)}(2n-1)\prod_{k=1}^{n-1}(4k-1).$$

Theorem. Let $N_n(x)$ denote the n-th Narayana polynomial defined by $N_0(x) = 1$ and $N_n(x) = \sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^k$, $n \geqslant 1$. Then for $n \in \mathbb{N}$ we have

$$\operatorname{Pf}\left((j-i)N_{i+j-2}(x)\right)_{1\leqslant i,j\leqslant 2n} = x^{n^2}\prod_{k=0}^{n-1}(4k+1).$$

A Considerably Shorter Proof of q-TSPP?

S. Okada:

On the generating functions for
certain classes of plane partitions

Journal of Combinatorial Theory, Series A, **53**:1–23, 1989.

In this paper, the q-TSPP conjecture was translated into a determinant evaluation (as we saw before). On the way, there was a similar statement involving a certain Pfaffian.

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Idea: Apply the holonomic ansatz for Pfaffians to this one and hope that this yields a simpler proof (less computations).

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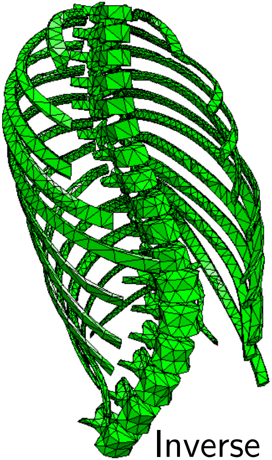
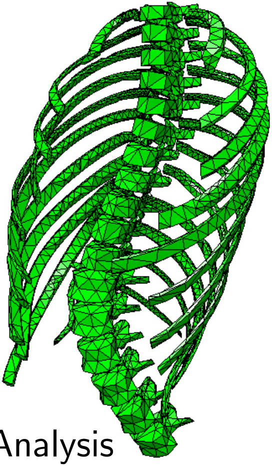
Idea: Apply the holonomic ansatz for Pfaffians to this one and hope that this yields a simpler proof (less computations).

Some preliminary results:

- Annihilating ideal for $c_{2n,i}$ has holonomic rank 3 only.
- But the matrix entries are more complicated, and involve some "exceptional" term which has to be treated separately.



Part IV



Evaluating Determinants that yield an Inverse Inequality in Numerical Analysis

(joint work with M. Neumüller and S. Radu)

Problem Statement

The interest in numerical analysis (e.g., finite element methods) in so-called inverse inequalities yields to the following problem:

Find the largest eigenvalue λ_n of the generalized eigenvalue problem

$$B_n \vec{x}_n = \lambda_n A_n \vec{x}_n$$

where A_n and B_n are certain $n \times n$ matrices.

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Equivalent formulation:

$$\lambda_n := \max_{\lambda} \det(B_n - \lambda A_n) = 0.$$

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Equivalent formulation:

$$\lambda_n := \max_{\lambda} \det(B_n - \lambda A_n) = 0.$$

Relaxed problem: find expressions $b_1(n)$ and $b_2(n)$ such that

$$b_1(n) < \lambda_n < b_2(n)$$

("as accurate as possible").

The entries of the matrices A_n and B_n in our case are:

$$a_{i,j} := \frac{1-(-1)^{i+j-1}}{i+j-1}$$

$$b_{i,j} := (i-1)(j-1)\frac{1-(-1)^{i+j-3}}{i+j-3}$$

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$$B_n - \lambda A_n = \begin{pmatrix} -2\lambda & 0 & -\frac{2}{3}\lambda & 0 & -\frac{2}{5}\lambda & 0 \\ 0 & 2-\frac{2}{3}\lambda & 0 & 2-\frac{2}{5}\lambda & 0 & 2-\frac{2}{7}\lambda \\ -\frac{2}{3}\lambda & 0 & \frac{8}{3}-\frac{2}{5}\lambda & 0 & \frac{16}{5}-\frac{2}{7}\lambda & 0 \\ 0 & 2-\frac{2}{5}\lambda & 0 & \frac{18}{5}-\frac{2}{7}\lambda & 0 & \frac{30}{7}-\frac{2}{9}\lambda \\ -\frac{2}{5}\lambda & 0 & \frac{16}{5}-\frac{2}{7}\lambda & 0 & \frac{32}{7}-\frac{2}{9}\lambda & 0 \\ 0 & 2-\frac{2}{7}\lambda & 0 & \frac{30}{7}-\frac{2}{9}\lambda & 0 & \frac{50}{9}-\frac{2}{11}\lambda \end{pmatrix}$$

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Hence we get

$$\det(B_n - \lambda A_n) = 2^n \det\left(A^{(1)}_{\lceil n/2 \rceil}\right) \cdot \det\left(A^{(0)}_{\lfloor n/2 \rfloor}\right).$$

$$A_n^{(0)} = (a_{i,j}^{(0)})_{1 \leqslant i,j \leqslant n} \quad \text{with} \quad a_{i,j}^{(0)} := \frac{(2i-1)(2j-1)}{2i+2j-3} - \frac{\lambda}{2i+2j-1}$$

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$$\det A_1^{(0)} = 1 - \frac{\lambda}{3}$$

$$\det A_2^{(0)} = \frac{4\lambda^2}{525} - \frac{12\lambda}{35} + \frac{4}{5}$$

$$\det A_3^{(0)} = -\frac{256\lambda^3}{22920975} + \frac{512\lambda^2}{218295} - \frac{256\lambda}{4851} + \frac{256}{2205}$$

$$\det A_4^{(0)} = \frac{65536\lambda^4}{632759873999625} - \frac{131072\lambda^3}{200876150475} + \frac{65536\lambda^2}{1217431215} - \frac{65536\lambda}{6689182}$$

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\end{aligned}$$

- These polynomials are irreducible.
- Hence $\det\big(A_n^{(0)}\big)/\det\big(A_{n-1}^{(0)}\big)$ is (probably) not holonomic.
- Neither is $\det\big(A_n^{(0)}\big)$ a holonomic sequence.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} (-1)^{n+1}\dfrac{M_{n,1}}{M_{n,n}} \\ (-1)^{n+2}\dfrac{M_{n,2}}{M_{n,n}} \\ (-1)^{n+3}\dfrac{M_{n,3}}{M_{n,n}} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \dfrac{\det A_n^{(0)}}{\det A_{n-1}^{(0)}} \end{pmatrix}$$

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$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} (-1)^{n+1} M_{n,1} \\ (-1)^{n+2} M_{n,2} \\ (-1)^{n+3} M_{n,3} \\ \vdots \\ M_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \det A_n^{(0)} \end{pmatrix}$$

- This normalization could be used if $\det A^{(0)}$ was holonomic.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} (-1)^{n+1}\dfrac{M_{n,1}}{\ell_n} \\ (-1)^{n+2}\dfrac{M_{n,2}}{\ell_n} \\ (-1)^{n+3}\dfrac{M_{n,3}}{\ell_n} \\ \vdots \\ \dfrac{M_{n,n}}{\ell_n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \dfrac{\det A_n^{(0)}}{\ell_n} \end{pmatrix}$$

- ℓ_n is the leading coefficient of $\det A_n^{(0)}$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} c_{n,1}^{(0)} \\ c_{n,2}^{(0)} \\ c_{n,3}^{(0)} \\ \vdots \\ c_{n,n}^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \dfrac{\det A_n^{(0)}}{\ell_n} \end{pmatrix}$$

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- Define $c_{n,j}^{(0)} := (-1)^{n+j} \dfrac{M_{n,j}}{\ell_n}$.

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- ℓ_n is the leading coefficient of $\det A_n^{(0)}$.
- Define $c_{n,j}^{(0)} := (-1)^{n+j} \frac{M_{n,j}}{\ell_n}$.
- Thanks to the parameter λ this normalization is easy to achieve.

We conjecture

$$c_{n,j}^{(0)} = \frac{2^{2n+2j-3}\left(\frac{3}{2}\right)_{2n-1}\left(n+\frac{1}{2}\right)_{j-1}}{(n-1)!\,(2j-1)!} \times \sum_{m=0}^{n-1}\sum_{k=0}^{2n-2m-2}\frac{(-1)^{j+m}\,(2m+1)_{2k}\,\lambda^{m}}{4^{m+k}k!\,(2m+k-n-j+2)!}$$

We conjecture

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Then we prove

$$\sum_{j=1}^{n} a_{i,j}^{(0)} c_{n,j}^{(0)} = \delta_{i,n}\sum_{j=0}^{n}(-4)^{j-n}\frac{(2n-2j+1)_{2n}}{(2j)!}\lambda^j$$

from which we can conclude that

$$\det A_n^{(0)} = c_n \cdot \sum_{j=0}^{n}(-4)^{j-n}\frac{(2n-2j+1)_{2n}}{(2j)!}\lambda^j$$

for some (yet unknown) constant c_n.

With the original version of the holonomic ansatz, we prove

$$c_n = \det\Big(\lim_{\lambda\to\infty} \tfrac{1}{\lambda} A_n^{(0)}\Big) = \big(-\tfrac{1}{2}\big)^n \prod_{i=1}^{n} \frac{\big((i-1)!\big)^2}{\big(i+\frac{1}{2}\big)_n}$$

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And hence we obtain:

Theorem.

$$\det A_n^{(0)} = \underbrace{\left(-\tfrac{1}{2}\right)^n \prod_{i=1}^{n} \frac{\bigl((i-1)!\bigr)^2}{\bigl(i+\tfrac{1}{2}\bigr)_n}}_{\text{“hyperholonomic” part}} \underbrace{\sum_{j=0}^{n} (-4)^{j-n} \frac{(2n-2j+1)_{2n}}{(2j)!} \lambda^j}_{\text{holonomic part}} .$$

For all $n \in \mathbb{N}$ we have the estimate $b_1(n) < \lambda_n < b_2(n)$ with

$$b_1(n) := \frac{m_1(n)}{2}\left(1 + \sqrt{1 - \frac{2}{3}\frac{(n-2)(n-3)(n+3)(n+4)}{n(n-1)(n+1)(n+2)}}\right),$$

$$b_2(n) := m_1(n)\left(\frac{1}{3} + \left(r_1(n) + \sqrt{r_2(n)}\right)^{1/3} + \left(r_1(n) - \sqrt{r_2(n)}\right)^{1/3}\right),$$

where m_1, r_1, and r_2 are given by

$$\begin{aligned} m_1(n) &:= \frac{n(n-1)(n+1)(n+2)}{8}, \\ r_1(n) &:= \frac{2(n^8 + 4n^7 + 8n^6 + \cdots - 4733n^2 - 5130n + 16200)}{135n^2(n-1)^2(n+1)^2(n+2)^2}, \\ r_2(n) &:= \frac{(n-2)(n-3)(n+4)(n+3)\,q(n)}{145800n^4(n-1)^4(n+1)^4(n+2)^4}, \end{aligned}$$

and the polynomial q in r_2 is given by

$$7n^{12} + 42n^{11} - 641n^{10} + \cdots - 44971200n + 116640000.$$

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Conclusion

We have demonstrated that the holonomic ansatz is a very flexible tool for evaluating families of determinants symbolically.

bla bla bla