Planar Linkages Following a Prescribed Motion

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Material (paper, pictures, movies) is available at

http://www.koutschan.de/data/link/
**Definition.** A *linkage* is a mechanism which consists of

- several rigid bodies, called *links*;
- the links are connected by *joints*.

The joints restrict the relative positions of the links.
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Restriction: We consider only planar linkages, i.e., all links move in parallel planes.

There are two different types of joints:
1. revolute (rotational) joints
2. prismatic (translational) joints
Kempe’s Universality Theorem

**Goal:** For a given planar curve, construct a linkage that draws it.

- Motivation from engineering, dates back to 18th century
- Example: Watt’s linkage ("one of the most ingenious simple pieces of mechanics I have invented")
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**Theorem.** (Kempe 1876)

Let $f \in \mathbb{R}[x, y]$ be a polynomial, and let $B \subseteq \mathbb{R}^2$ be a closed disk. Then there exists a planar linkage which draws the curve

$$B \cap \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}.$$
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▶ Proof of the theorem is constructive.
▶ “parsing algorithm” with input $f(x, y)$
▶ Kempe’s constructions yield very complicated linkages.
▶ Can be applied to any algebraic curve.
Our contribution

1. A new algorithm for constructing linkages.

More general than Kempe's algorithm: Construction of linkages that realize a prescribed motion (drawing a curve is a special case).

Less general than Kempe's algorithm: Our algorithm can only be applied to rational curves.

"Better" than Kempe's algorithm: Our algorithm usually gives much simpler linkages. (158 links/235 joints versus 8 links/10 joints for the ellipse)

2. An algebraic framework to represent motions.

3. Answer to the collision problem.

4. A prototype implementation.
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Mathematical model for linkages

1. Self-collisions of the links are not taken into account, i.e., the joints are the only constraints for the motion of the links.

2. Thus the actual shape of the links doesn’t matter, just the position of the joints.

3. Not a single frame of reference for the configuration of a linkage, but each link has its own frame of reference.
Linkages

Link graph: encodes the “topological information” of a linkage.
- Each link corresponds to a vertex.
- Each joint corresponds to an edge.

Example:
Linkages

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**Definition.** A **planar linkage** with revolute joints is a connected undirected graph $G = (V, E)$ without self-loops, together with a map $\rho : E \rightarrow \mathbb{R}^2$.

- The point $\rho(e)$ is the position of the joint $e$ in the “initial configuration” of the linkage.
- W.l.o.g. assume that $V$ is of the form $\{1, \ldots, n\}$.
- Elements of $E$ are given by unordered pairs $\{i, j\} \subseteq V$.

In the following let $L = (G, \rho)$ with $G = (V, E)$ be a linkage.
Positions of links

**Notation.** We denote by $\text{SE}_2$ the **Special Euclidean group**, i.e., the group of **direct isometries** of the plane.
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$$\sigma \in SE_2 \iff \sigma(x) = Ax + b, \ A \text{ orthogonal, } \det A = 1, \ b \in \mathbb{R}^2.$$
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**Relative position:** Let $\sigma_i, \sigma_j \in \text{SE}_2$ describe the absolute positions of the links $i$ and $j$. Then $\sigma_{i,j} := \sigma_i \circ \sigma_j^{-1}$ gives the **relative position** of link $i$ w.r.t. link $j$. 


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We then have $\sigma_{i,j} = \sigma_{i,k} \circ \sigma_{k,j}$. 


Configurations

**Definition.** Let \( \{i, j\} \in E \). The set of **virtual relative positions** of link \( i \) w.r.t. link \( j \), denoted \( \text{VRP}(i, j) \), is the subgroup of \( \text{SE}_2 \) of rotations around the point \( \rho(i, j) \).

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**Definition.** A **configuration** of \( L \) is a collection of relative positions \( \sigma_{i,j} \in \text{VRP}(i, j) \), subject to the constraints:

- If \( (i, h_1), (h_1, h_2), \ldots, (h_s, i) \) is a **directed cycle** in \( G \) i.e., \( \{i, h_1\} \in E, \{h_1, h_2\} \in E \) etc.,
- then \( \sigma_{i,h_1} \circ \cdots \circ \sigma_{h_s,i} = \text{id} \).
- (This is the relative position of link \( i \) w.r.t. itself.)
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\[ \rightarrow \text{The cycle condition also implies that } \sigma_{i,j} = \sigma_{j,i}^{-1}. \]
**Configuration space**

**Definition.** The configuration space of a linkage $L$ is the set of all its configurations:

$$\text{Conf}(L) = \left\{ (\sigma_{k,l}) \in \prod_{\{i,j\} \in E} \text{VRP}(i,j) \times \text{VRP}(j,i) : \text{cycle conds} \right\}$$

where $\prod$ is the Cartesian product so that $(\sigma_{k,l})$ is a $2|E|$-tuple.
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By identifying $\text{VRP}(i,j)$ with the real projective line $\mathbb{P}_{\mathbb{R}}^1$, $\text{Conf}(L)$ acquires the structure of a projective subvariety of $(\mathbb{P}_{\mathbb{R}}^1)^{2|E|}$. 
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Definition. The mobility of $L$ is the dimension of $\text{Conf}(L)$.

Configuration space
Recapitulation and outlook

**Have:** Mathematical model for linkages that uses direct isometries in an essential way.

**Want:** An *algebraic* representation of direct isometries that allows for simple *computations*. 

Outline of solution:

1. Embed $SE_2$ in the real projective space $P^3_R$.
2. Interpret the points in $P^3_R$ as elements of some ring $K$.
3. The multiplication in $K$ will correspond to the group operation $\circ$ in $SE_2$.
4. Employ the polynomial ring $K[t]$ to describe motions.
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- Interpret the points in $\mathbb{P}^3_R$ as elements of some ring $\mathbb{K}$.
- The multiplication in $\mathbb{K}$ will correspond to the group operation $\circ$ in $\mathbb{SE}_2$.
- Employ the polynomial ring $\mathbb{K}[t]$ to describe *motions*. 
**Embedding of $\text{SE}_2$ in $\mathbb{P}^3_{\mathbb{R}}$**

**Definition.** The $n$-dimensional **real projective space** is the set

$$\mathbb{P}^n_{\mathbb{R}} := \left( \mathbb{R}^{n+1} \setminus \{(0, \ldots, 0)\} \right) / \sim,$$

where

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) :\iff \exists c \in \mathbb{R}^*: (x_0, \ldots, x_n) = c \cdot (y_0, \ldots, y_n).$$
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Write a point in $\mathbb{P}^3_\mathbb{R}$ with the coordinates $(x_1 : x_2 : y_1 : y_2)$.

**Embedding:** We embed SE$_2$ in $\mathbb{P}^3_\mathbb{R}$ as the open subset

$$\mathcal{U} = \mathbb{P}^3_\mathbb{R} \setminus \{(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_\mathbb{R} \mid x_1^2 + x_2^2 = 0\}.$$

**Geometric interpretation:** The set $\mathcal{U}$ is the complement of the line $x_1 = x_2 = 0$. 
Action

Let $\sigma \in SE_2$ be given by the point $(x_1 : x_2 : y_1 : y_2) \in U \subset \mathbb{P}_\mathbb{R}^3$.

The action of $\sigma$ on a point $(x, y) \in \mathbb{R}^2$ is given by

$$\frac{1}{x_1^2 + x_2^2} \begin{pmatrix} x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix}.$$

Remarks:

▶ Note that the points for which $x_1^2 + x_2^2 = 0$ were excluded.

▶ Note that this action is compatible with $\sim$.

▶ The translational part vanishes if $y_1 = y_2 = 0$.

▶ The rotational part depends only on $x_1$ and $x_2$.

▶ If $x_2 = 0$ then we have a pure translation.

▶ The identity isometry is given by $(x_1 : 0 : 0 : 0)$.

Exercise. What kind of isometry is given by $(x_1 : x_2 : 0 : 0)$?

Exercise. What kind of isometry is given by $(1 : 0 : y_1 : y_2)$?

Exercise. Which points correspond to the identity isometry?
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- If $x_2 = 0$ then we have a pure translation.
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**Exercise.** Which points correspond to the identity isometry?
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Product

With this action the product in $\text{SE}_2$ becomes a bilinear map:

$$(x_1 : x_2 : y_1 : y_2) \cdot (x'_1 : x'_2 : y'_1 : y'_2) =$$

$$= (x_1 x'_1 - x_2 x'_2 : x_1 x'_2 + x_2 x'_1 : x_1 y'_1 + x_2 y'_2 + y_1 x'_1 - y_2 x'_2 : x_1 y'_2 - x_2 y'_1 + y_1 x'_2 + y_2 x'_1)$$
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Using this notation, the product in $\text{SE}_2$ can be rewritten as

$$(z, w) \cdot (z', w') = (z z', \bar{z} w' + z' w)$$

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\[z \eta = \eta \overline{z} \quad \text{for all} \quad z \in \mathbb{C} \quad \text{and} \quad \eta^2 = 0.\]

Hence: \((z + \eta w) \cdot (z' + \eta w') = zz' + \eta \overline{z}w' + \eta w z' + \eta w \eta w' = zz' + \eta(\overline{z}w' + z'w).\)
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\[(z + \eta w) \cdot (z' + \eta w') = z z' + z \eta w' + \eta w z' + \eta w \eta w' = z z' + \eta(\bar{z} w' + z' w).\]

**Notation 3.** Denote by \(\mathbb{K}\) the \(\mathbb{R}\)-algebra \(\mathbb{C}[\eta]/\langle \imath \eta + \eta \imath, \eta^2 \rangle\), i.e., the ring of dual complex numbers.
Rational motions and motion polynomials

**Intuition:** A motion is a family of direct isometries, more precisely, a continuous function $\mathbb{R} \rightarrow \mathbb{SE}_2$. 
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**Definition.** A rational motion is a map $\mathbb{R} \to \mathbb{P}^3_\mathbb{R}$ given by four real polynomials $X_1, X_2, Y_1, Y_2 \in \mathbb{R}[t]$ such that $X_1^2 + X_2^2 \neq 0$.

Hence for almost every $t$ this yields a direct isometry in $\text{SE}_2$. 
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A polynomial in $\mathbb{K}[t]$ is called a **motion polynomial**.
**Rational motions and motion polynomials**

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where \( Z = X_1 + \imath X_2 \) and \( W = Y_1 + \imath Y_2 \).

\[ \Rightarrow \] A polynomial in \( \mathbb{K}[t] \) is called a **motion polynomial**.

\[ \Rightarrow \] The multiplication of \( P \in \mathbb{K}[t] \) by a real polynomial \( R \in \mathbb{R}[t] \) gives a new motion polynomial \( RP = PR \), which however describes the **same** rational motion.
Connection to rational curves

**Proposition.** Let \( \varphi : \mathbb{R} \to \mathbb{R}^2 \) be a rational parametrization,

\[
\varphi(t) = \left( \frac{f(t)}{h(t)}, \frac{g(t)}{h(t)} \right), \quad \text{for some } f, g, h \in \mathbb{R}[t],
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of a real curve. Then the orbit of \((0, 0)\) under the motion given by \( P = h + \eta (f + \imath g) \) is exactly the image of \( \varphi \).
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In many considerations, we restrict ourselves to monic motion polynomials (justification will be given later).

**Definition.** We say that \( P = Z + \eta W \in \mathbb{K}[t] \) is **monic** if its leading coefficient is 1, i.e.: \( Z \in \mathbb{C}[t] \) is monic and \( \deg W < \deg Z \).
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\[ \implies \text{If } P \text{ is monic then } \lim_{t \to +\infty} P(t) = (1 : 0 : 0 : 0) \in \mathbb{P}_\mathbb{R}^3, \text{ which corresponds to the identity element in } SE_2. \]
Characterization of simple motions

**Lemma.** Let \( \ell \subseteq \mathbb{P}^3_{\mathbb{R}} \) be a projective line passing through the point \((1 : 0 : 0 : 0)\), and let \( X = \ell \cap \{x_1 = x_2 = 0\} \). Then:

1. if \( |X| = 1 \), then \( \ell \setminus X \) corresponds to a subgroup of \( \text{SE}_2 \) that consists of all translations along a fixed common direction;

2. if \( X = \emptyset \), then \( \ell \) corresponds to a subgroup of \( \text{SE}_2 \) that consists of all rotations around a fixed common point.

**Corollary.** Let \( P \in K[t] \) be a monic motion polynomial of degree 1, i.e., \( P(t) = t + ix_2 + \eta(y_1 + iy_2) \) with \( x_2, y_1, y_2 \in \mathbb{R} \). Then:

1. if \( x_2 = 0 \) then \( P \) gives a translational motion in direction \((y_1, y_2)\);

2. if \( x_2 \neq 0 \) then \( P \) gives a revolution around the point \((1/2x_2, -y_2, y_1)\).

Linear motion polynomials describe exactly those motions that are realized by joints.
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**Corollary.** Let \( P \in \mathbb{K}[t] \) be a monic motion polynomial of degree 1, i.e., \( P(t) = t + i x_2 + \eta (y_1 + i y_2) \) with \( x_2, y_1, y_2 \in \mathbb{R} \). Then:

1. if \( x_2 = 0 \) then \( P \) gives a translational motion in direction \( (y_1) \).
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Characterization of simple motions

Lemma. Let $\mathcal{L} \subseteq \mathbb{P}_R^3$ be a projective line passing through the point $(1 : 0 : 0 : 0)$, and let $X = \mathcal{L} \cap \{x_1 = x_2 = 0\}$. Then:

1. if $|X| = 1$, then $\mathcal{L} \setminus X$ corresponds to a subgroup of $\text{SE}_2$ that consists of all translations along a fixed common direction;
2. if $X = \emptyset$, then $\mathcal{L}$ corresponds to a subgroup of $\text{SE}_2$ that consists of all rotations around a fixed common point.

Corollary. Let $P \in \mathbb{K}[t]$ be a monic motion polynomial of degree 1, i.e., $P(t) = t + \imath x_2 + \eta (y_1 + \imath y_2)$ with $x_2, y_1, y_2 \in \mathbb{R}$. Then:

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Examples

1. Which motion does the motion polynomial \( t + \eta \) describe?
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   \[ \rightarrow \text{It gives a revolution around the origin } (0, 0). \]
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1. Which motion does the motion polynomial $t + \iota$ describe?
   \[\rightarrow\] It gives a revolution around the origin $(0, 0)$.

2. What about $t + \eta$?
   
   \[\rightarrow\] Translational motion since $t^2 + 1$ is a real polynomial.
   
   \[\rightarrow\] The translational vector is given by $\frac{1}{t^2 + 1}(t - 1)$.
   
   \[\rightarrow\] It parametrizes the circle with radius $\frac{1}{2}$ and center $\frac{1}{2}(0 - 1)$.
   
   \[\rightarrow\] Hence we get a circular translation.
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2. What about \( t + \eta \)?
   \( \rightarrow \) This gives a translational motion along the line \( y = 0 \).

3. Multiplication of motion polynomials corresponds to the composition of motions, e.g.,
   \[
   (t + \imath) \cdot (t - \imath + \eta) = (t^2 + 1) + \eta(t - \imath).
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Weak and strong realization

Let $L = ((V, E), \rho)$ be a linkage, $\phi : \mathbb{R} \to \mathbb{P}^3_\mathbb{R}$ a rational motion.

Let $RP(i, j) \subseteq SE_2$ denote the set of **relative positions** of link $j$ with respect to the link $i$. 
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**Weak realization:** $\phi(\mathbb{R}) \subseteq \text{RP}(i, j)$ for some $i, j \in V$. 
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**Strong realization:** weak realization, plus $L$ has mobility one.
Task: Construct a linkage that realizes a given rational motion \( \phi \).

Solution strategy:

1. The motion \( \phi \) is described by a motion polynomial \( P \in \mathbb{K}[t] \).
2. Factor \( P \) into linear factors.
3. Each linear factor represents an “elementary” motion (revolution, translational motion), which can be realized by a single joint.
4. A factorization of \( P \) gives rise to an open chain of links, which weakly realizes the motion \( \phi \).
5. Insert more links in order to restrain the mobility of the linkage so that it strongly realizes the motion \( \phi \).
Factorization into linear factors

Let $P = Z + \eta W \in \mathbb{K}[t]$ be a monic motion polynomial of degree $n$.

**Goal:** Factor $P$ into monic linear motion polynomials, i.e.,

$$P = P_1 \cdot P_2 \cdots P_n, \quad P_i = t - z_i + \eta w_i, \quad w_i, z_i \in \mathbb{C}.$$
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**Recall:**

$$(z + \eta w) \cdot (z' + \eta w') = z z' + \eta (\bar{z} w' + z' w).$$

By expanding the ansatz we obtain:

$$P = (t - z_1 + \eta w_1) \cdot (t - z_2 + \eta w_2) \cdots (t - z_n + \eta w_n) =$$

$$= \prod_{i=1}^{n} (t - z_i) + \eta \sum_{k=1}^{n} \left( \prod_{j=1}^{k-1} (t - \bar{z}_j) \right) \left( \prod_{j=k+1}^{n} (t - z_j) \right) w_k.$$

$Z(t)$  $W(t)$
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\[
\begin{align*}
\underbrace{Z(t)}_{\text{The } z_i \text{ are precisely the complex roots of } Z(t).} & \quad \underbrace{W(t)}_{\text{The } w_i \text{ can be found by ansatz and solving a linear system.}}
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$\overset{Z(t)}{\underbrace{\prod_{i=1}^{n} (t - z_i)}} + \overset{W(t)}{\underbrace{\sum_{k=1}^{n} \left( \prod_{j=1}^{k-1} (t - \bar{z}_j) \right) \left( \prod_{j=k+1}^{n} (t - z_j) \right) w_k}}$

$\rightarrow$ The $z_i$ are precisely the complex roots of $Z(t)$.

$\rightarrow$ The $w_i$ can be found by ansatz and solving a linear system.

$\rightarrow$ The order of $z_1, \ldots, z_n$ matters!
How to compute the $w_i$

Fix a permutation $z = (z_1, \ldots, z_n)$ of the complex roots of $Z$.

$$P = Z(t) + \eta \sum_{k=1}^{n} \left( \prod_{j=1}^{k-1} (t - z_j) \right) \left( \prod_{j=k+1}^{n} (t - z_j) \right) w_k.$$  

\[\equiv: Q_k(z)\]

The undetermined coefficients $w_1, \ldots, w_n \in \mathbb{C}$ have to satisfy

$$\sum_{k=1}^{n} w_k Q_k(z) = W.$$
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**Lemma.** Let $P = Z + \eta W$ be monic and let $\mathbf{z} = (z_1, \ldots, z_n)$ be a fixed permutation of the roots of $Z$ over $\mathbb{C}$. Then $P$ admits a factorization $P = P_1 \cdots P_n$ where $P_i(t) = (t - z_i) + \eta w_i$, $w_i \in \mathbb{C}$, if and only if $W$ lies in the linear span $\langle Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}}$. 
Sufficient condition for factorization

\[ P = Z + \eta W, \text{ monic, } \deg P = n, \text{ admits a factorization} \]

\[ \iff W \in \langle Q_1(z), \ldots, Q_n(z) \rangle_{\mathbb{C}} \]

\[ \iff \gcd (Q_1(z), \ldots, Q_n(z)) = 1 \quad (\text{note that } \deg Q_k = n - 1). \]

Clearly, this is the case (for arbitrary \( W \) with \( \deg W < n \)) if the following matrix \( M_n \in \mathbb{C}^{n \times n} \) is non-singular:

\[
M_n = \begin{pmatrix}
\langle t^0 \rangle Q_1 & \ldots & \langle t^0 \rangle Q_n \\
\langle t^1 \rangle Q_1 & \ldots & \langle t^1 \rangle Q_n \\
\vdots & & \vdots \\
\langle t^{n-1} \rangle Q_1 & \ldots & \langle t^{n-1} \rangle Q_n
\end{pmatrix}
\]

(here \( \langle t^i \rangle Q_k \) denotes the coefficient of \( t^i \) in \( Q_k \)).

\[ \rightarrow \text{ The matrix entries are, up to sign, elementary symmetric polynomials in the } z_i \text{ and } \overline{z}_i. \]
Evaluating the determinant

We have \( \text{det}(M_n) = \prod_{1 \leq i < j \leq n} (z_i - z_j) \).

- This is very much reminiscent of the Vandermonde determinant, and it can be proved in a similar fashion.
- A similar determinant evaluation is given in (Lascoux/Pragacz 2002) where the \( z_i \) appear without conjugation.
- The above formula is a special case of a determinant evaluation that appears in (Krattenthaler 1999).
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**Lemma.** Let \( P = Z + \eta W \in \mathbb{K}[t] \), monic, be such that \( Z \) has no pair of complex-conjugate roots (i.e., \( Z(\alpha) = 0 \implies Z(\overline{\alpha}) \neq 0 \)). Then for every permutation \( (z_1, \ldots, z_n) \) of the roots of \( Z \), the polynomial \( P \) admits a factorization into linear factors.
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\( \rightarrow \) This condition is only sufficient, but not necessary, for the existence of a factorization.
Characterization of factorizable polynomials

**Proposition.** Let \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and \( W \in \mathbb{C}[t], \deg W < n \). Then

\[
W \in \langle Q_1(z), \ldots, Q_n(z) \rangle_{\mathbb{C}} \iff W \in \left( Q_1(z), \ldots, Q_n(z) \right) \cdot \mathbb{C}[t]
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where \( \left( Q_1(z), \ldots, Q_n(z) \right) \cdot \mathbb{C}[t] \) is the ideal generated by the \( Q_k(z) \).
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**Remarks:**

- The ideal on the right-hand side is generated by a single polynomial \( G := \gcd(Q_1, \ldots, Q_n) \).
- Note that \( G \) depends on the permutation \( z \).
**Characterization of factorizable polynomials**

**Proposition.** Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $W \in \mathbb{C}[t]$, $\deg W < n$. Then

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where $(Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z})) \cdot \mathbb{C}[t]$ is the ideal generated by the $Q_k(\mathbf{z})$.

**Remarks:**

- The ideal on the right-hand side is generated by a single polynomial $G := \gcd(Q_1, \ldots, Q_n)$.
- Note that $G$ depends on the permutation $\mathbf{z}$.

**Corollary.** A monic motion polynomial $Z + \eta W \in \mathbb{K}[t]$ can be factored completely if there exists a permutation $\mathbf{z}$ such that $G \mid W$. 
Determine the gcd $G$

$$G = \gcd\left(Q_1(z), \ldots, Q_n(z)\right) \text{ for some } z \in \mathbb{C}^n.$$ 

$$Q_1 = (t - z_2)(t - z_3)(t - z_4) \cdots (t - z_{n-1})(t - z_n)$$ 

$$Q_2 = (t - \overline{z}_1)(t - z_3)(t - z_4) \cdots (t - z_{n-1})(t - z_n)$$ 

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- Assume $\overline{z_2} = z_3$. Then $(t - z_3) \mid G$. 

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Computation of $G$

**Definition.** Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. A set

$$M \subseteq \{(i, j) : 1 \leq i < j \leq n \land z_i = \overline{z_j}\}$$

is called a **matching** of $z$ if for all $(i_1, j_1), (i_2, j_2) \in M$ we have $i_1 \neq i_2$ and $j_1 \neq j_2$. 

Example: $(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha)$

**Proposition.** Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and let $M$ be a matching of $z$ of maximal size. Then we have

$$\gcd(Q_1(z), \ldots, Q_n(z)) = \prod_{(i, j) \in M} (t - \overline{z_j})$$

(where the $\gcd$ is assumed to be a monic polynomial).

**Exercise:** What is the $\gcd$ in the above example?
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Some examples

**Exercise:** Let \( Z = (t - \alpha)^r (t - \overline{\alpha})^{r+1} \). Find the permutations \( z \) of the roots of \( Z \) that give the following \( G' \)’s.

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The cases discussed here are the extreme ones:
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The cases discussed here are the extreme ones:

- It is easy to see that $r \leq \deg(G) \leq 2r$.
- For any $G = (t - \alpha)^i(t - \overline{\alpha})^j$ with $0 \leq i, j \leq r$ and $i + j \geq r$ there exists a permutation $z$ which produces this gcd $G$. 
No factorization?

Recall: \( P = Z + \eta W \) factors iff there exists \( z \) such that \( G \mid W \).

Problem: Consider the motion polynomial \( t^2 + 1 + \eta \).
No factorization?

**Recall:** $P = Z + \eta W$ factors iff there exists $z$ such that $G \mid W$.

**Problem:** Consider the motion polynomial $t^2 + 1 + \eta$.

- For the two permutations $(\iota, -\iota)$ and $(-\iota, \iota)$ we get the gcd $G = t + \iota$ resp. $G = t - \iota$. 

$\rightarrow$ Caveat: this trick works only for $\alpha \not\in \mathbb{R}$!
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- In both cases \( G \nmid W \) since \( W = 1 \).

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- Note that this doesn’t change the motion itself.
- Consider e.g. $R = (t - \alpha)(t - \overline{\alpha})$ and put $P' = Z' + \eta W' = PR$.
- Clearly, $W' = WR$, so we add two roots to $W$. 

No factorization?

Recall: \( P = Z + \eta W \) factors iff there exists \( z \) such that \( G \mid W \).

Problem: Consider the motion polynomial \( t^2 + 1 + \eta \).

- For the two permutations \((i, -i)\) and \((-i, i)\) we get the gcd \( G = t + i \) resp. \( G = t - i \).
- In both cases \( G \nmid W \) since \( W = 1 \).

\( \Rightarrow \) This polynomial cannot be factored!

Solution: Multiply \( P \) by some real polynomial \( R \in \mathbb{R}[t] \)!

- Note that this doesn’t change the motion itself.
- Consider e.g. \( R = (t - \alpha)(t - \overline{\alpha}) \) and put \( P' = Z' + \eta W' = PR \).
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\[ \implies \text{Caveat: this trick works only for } \alpha \not\in \mathbb{R}! \]
Bounded motions

Recall: The origin under the motion $h + \eta (f + \imath g)$ traces the curve $(f/h, g/h)$.

▶ We are interested in linkages with revolute joints.
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**Definition.** Let \( P = Z + \eta W \) be a motion polynomial. We say that \( P \) is **bounded** if it is monic and if \( Z \) does not have real roots.

**Theorem.** Let \( P \in \mathbb{K}[t] \) be a bounded motion polynomial. Then there exists a real polynomial \( R \in \mathbb{R}[t] \) such that \( RP \) can be factored into linear polynomials.
Factorization algorithm

**Input:** $P = Z + \eta W \in \mathbb{K}[t]$ a bounded motion polynomial such that $Z$ and $W$ have no common factor in $\mathbb{R}[t] \setminus \mathbb{R}$.

**Output:** a polynomial $R \in \mathbb{R}[t]$ and a tuple $(k_1, \ldots, k_n)$ of elements of $\mathbb{K}$ such that $(t - k_1) \cdots (t - k_n) = R(t) \cdot P(t)$.

1: Factor $Z(t)$ over $\mathbb{C}$: $Z = \prod_{i=1}^{h} (t - \alpha_i)^{r_i} (t - \overline{\alpha_i})^{s_i}$ with $r_i \geq s_i \geq 0$.

2: Initialize $q = \text{empty}$ (the empty tuple).

3: For $i = 1, \ldots, h$ Do

4: Set $u_i = \max_j ((t - \alpha_i)^j | W)$ and $v_i = \max_j ((t - \overline{\alpha_i})^j | W)$.

5: Set $m_i = \min \{s_i, u_i + v_i\}$

6: Set $\omega = \left(\overline{\alpha_i}, \ldots, \overline{\alpha_i}, \alpha_i, \ldots, \alpha_i\right)$, $s_i - \min \{s_i, v_i\}$, $r_i + s_i - m_i$, $s_i - \min \{s_i, u_i\}$.

7: Set $q = \text{concatenate}(q, \omega)$.

8: End For

9: Set $R = \prod_{i=1}^{h} ((t - \alpha_i)(t - \overline{\alpha_i}))^{s_i - m_i}$.

10: Set $n = \text{length}(q)$.

11: Set $Q_j = \prod_{l=1}^{j-1} (t - q_l) \prod_{l=j+1}^{n} (t - \overline{q_l})$ for all $j \in \{1, \ldots, n\}$.

12: Compute $\{w_j\}_{j=1}^{n}$ s.t. $RW = \sum_{j=1}^{n} w_j Q_j$ using linear algebra.

13: Set $k_j = q_j - \eta w_j$ for all $j \in \{1, \ldots, n\}$.

14: Return $(R, (k_1, \ldots, k_n))$. 
Example: ellipse

We want to construct a linkage drawing an ellipse with radii 1 and $\frac{1}{2}$. 

---

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- Hence the motion polynomial $t^2 + 1 + \eta (-2 + \nu t)$ describes a motion under which the origin traces the ellipse.
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▸ Hence the motion polynomial $t^2 + 1 + \eta(-2 + \eta t)$ describes a motion under which the origin traces the ellipse.

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Construction of weak linkages

Recall: $L = ((V, E), \rho)$ weakly realizes a motion $\phi: \mathbb{R} \to SE_2$ if $\phi(\mathbb{R}) \subseteq \text{RP}(i, j)$ for some $i, j \in V$.

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- Let \( G = (V, E) \) with
  
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  V = \{1, \ldots, n + 1\}
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- The link graph looks as follows:

```
1 -- k_1 -- 2 -- k_2 -- 3 -- \cdots -- n -- k_n -- n+1
```
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\[
\begin{array}{cccccccc}
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\end{array}
\]

\( \longrightarrow \) Show Mathematica demo!
The flip procedure

**Goal:** “Rigidify” the weak linkage in order to get a strong realization (mobility one).
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Simplest case: $P = (t - z_1 - \eta w_1) \cdot (t - z_2 - \eta w_2)$, bounded:

- If $z_1 \neq \overline{z_2}$ then there exist $w_3, w_4 \in \mathbb{C}$ such that 
  $$(t - z_1 - \eta w_1) \cdot (t - z_2 - \eta w_2) = (t - z_2 - \eta w_3) \cdot (t - z_1 - \eta w_4).$$
- If $z_1 \neq z_2$ then these two factorizations are different.
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Let $k_1 = z_1 + \eta w_1$ and $k_2 = z_2 + \eta w_2$ ($z_i, w_i \in \mathbb{C}$).

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Combining these two linkages yields a linkage with mobility one, strongly realizing \( P \).
Iterated flips
Iterated flips

1 \rightarrow k_1 \rightarrow 2 \rightarrow k_2 \rightarrow 3 \rightarrow k_3 \rightarrow 4 \rightarrow k_4 \rightarrow 5

\begin{align*}
l_1 & \downarrow \\
6 & \end{align*}
Iterated flips
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1 \xrightarrow{k_1} 2 \xrightarrow{k_2} 3 \xrightarrow{k_3} 4 \xrightarrow{k_4} 5

\[ \begin{align*}
& 6 \xrightarrow{\tilde{k}_1} 7 \xrightarrow{\tilde{k}_2} 8
\end{align*} \]
Iterated flips
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1 \rightarrow \tilde{k}_1 \rightarrow 2 \rightarrow k_2 \rightarrow 3 \rightarrow k_3 \rightarrow 4 \rightarrow k_4 \rightarrow 5

6 \rightarrow l_1 \rightarrow \tilde{k}_1 \rightarrow 7 \rightarrow \tilde{k}_2 \rightarrow 8 \rightarrow \tilde{k}_3 \rightarrow 9 \rightarrow \tilde{k}_4 \rightarrow 10
Iterated flips

\[ \begin{align*}
2 & \xrightarrow{k_2} 3 & k_3 & \xrightarrow{k_4} 4 & \xrightarrow{k_5} 5 \\
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\end{align*} \]
Construction of strong linkages

Algorithm (sketch).

1. Given a motion $\phi: \mathbb{R} \rightarrow SE_2$ via a polynomial $P \in K[t]$.

Theorem. The linkage obtained in this way has mobility one and strongly realizes the motion $\phi$.

However, $Conf(L)$ has two components, i.e., this linkage can jump between two different “modes”.

Corollary. A bounded rational curve given by $(f/h, g/h)$ with $f, g, h \in \mathbb{R}[t]$ such that $\deg h > \max\{\deg f, \deg g\}$ can be drawn by a linkage with at most $4d$ links and $6d - 2$ joints ($d = \deg h$).
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**Theorem.** The linkage obtained in this way has mobility one and strongly realizes the motion $\phi$.

However, $Conf(L)$ has two components. i.e., this linkage can jump between two different “modes”.

**Corollary.** A bounded rational curve given by $(f/h, g/h)$ with $f, g, h \in \mathbb{R}[t]$ such that $\deg h > \max\{\deg f, \deg g\}$ can be drawn by a linkage with at most $4d$ links and $6d - 2$ joints ($d = \deg h$).
Physical realization

Standard way of realizing planar linkages:

▶ Each link is realized as a polygon (convex hull of the positions of its joints)
▶ Each link moves parallel to the horizontal \((x, y)\)-plane at a certain height \(z\).
▶ The joints are realized as vertical connections between links.
Self-collisions

Vertical arrangement of links is crucial when studying collisions!
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→ It suffices to study link–joint collisions.
Self-collisions

W.l.o.g. assume that the link labels \( \{1, \ldots, n\} \) correspond to their heights, i.e., their \( z \)-coordinates.

**Collision:**

- links \( i < k < j \)
- \( (x_1(t), y_1(t)) = \) position of joint \((i, j)\)
- \( (x_2(t), y_2(t)) = \) position of some joint connected to \( k \)
- \( (x_3(t), y_3(t)) = \) position of some other joint of \( k \)

\[
\begin{align*}
\text{for some } t & \in \mathbb{R} \cup \{\infty\} \\
0 & \leq s \leq 1
\end{align*}
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- \((x_3(t), y_3(t)) = \) position of some other joint of \(k\)
- a collision happens if
  
  \[
  x_1(t) = s \cdot x_2(t) + (1 - s) \cdot x_3(t) \\
  y_1(t) = s \cdot y_2(t) + (1 - s) \cdot y_3(t)
  \]

  for some \( t \in \mathbb{R} \cup \{\infty\} \) and \( 0 \leq s \leq 1 \).
Detect collisions

\[ x_1(t) = s \cdot x_2(t) + (1 - s) \cdot x_3(t) \]
\[ y_1(t) = s \cdot y_2(t) + (1 - s) \cdot y_3(t) \]
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- Because of our construction, \( x_1(t), y_1(t), \ldots \) are rational functions.
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- For each joint \( (i, j) \) and each line of each link \( i < k < j \) such a system has to be solved.
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- This can be done with reasonably small effort (note that \(s\) appears only linearly).
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This can be done with reasonably small effort (note that \(s\) appears only linearly).

\[ \rightarrow \text{In contrast to general linkages, our construction allows for a relatively simple collision detection!} \]
Detect collisions

**Example:** For our linkage drawing an ellipse we can find a spatial arrangement of the links s.t. only 2 collisions occur (both at $t = \infty$).
Avoid collisions

If we consider links of a more complicated 3-dimensional shape, we can completely avoid collisions:

- W.l.o.g. assume that the layers correspond to consecutive integer numbers.
- F-links ("flat" links): as before, located in a single layer $a \in \mathbb{N}$
- U-links, Z-links: stretch over two layers $(a, b) \in \mathbb{N}^2$, $a < b$
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← This solves an open problem, posed by O'Rourke, concerning the existence of collision-free linkages.
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Avoid collisions

Example:
Collision-free linkages

The following scheme allows to realize a planar linkage with ladder-shaped link graph without collisions:

(0, 4n)        (3, 4n−1)        (6, 4n−2)        (3n−3, 3n+1)        (3n)

U              U              U              …              U

F              Z              Z              …              Z

(1)            (2, 4)         (5, 7)         (3n−4, 3n−2)       (3n−1)

No collisions with joints can happen.

A Z-link can never collide with an F-link or another Z-link.

Collisions between nested U-links can be avoided by moving the vertical part of the outer U-link sufficiently far away.

Collisions between a U-link and an F- or Z-link: similarly.
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\[\begin{array}{c}
\text{U} \quad \text{T} \quad \text{U} \quad \text{T} \quad \text{U} \\
\text{T} \quad \text{Z} \quad \text{T} \quad \text{Z} \quad \text{T} \\
\text{F} \quad \text{T} \quad \text{Z} \quad \text{T} \quad \text{F} \\
\text{(1)} \quad \text{(2, 4)} \quad \text{(5, 7)} \quad \text{(3n-4, 3n-2)} \quad \text{(3n-1)}
\end{array}\]

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Collision-free linkage for the ellipse
Final Example

Popular formulation (by W. Thurston) of Kempe’s theorem:

There is a linkage that signs your name.
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Here is a famous signature (US Declaration of Independence):

» John Hancock (1737–1793)
» merchant, smuggler, statesman, and prominent patriot of the American revolution
» “John Hancock” has become a synonym for a signature
Final Example

Popular formulation (by W. Thurston) of Kempe’s theorem:

*There is a linkage that signs your name.*

Here is a famous signature (US Declaration of Independence):

![Signature of John Hancock](image)

- John Hancock (1737–1793)
- merchant, smuggler, statesman, and prominent patriot of the American revolution
- “John Hancock” has become a synonym for a signature

Linkages drawing a full signature would be too complex, hence previous attempts have focused on the “J” only.
Approximate the “J” by a rational curve \((f/h, g/h)\):

\[
\begin{align*}
  f(t) &= -321880t^5 - 436132t^4 - 237449t^3 - 64488t^2 - 8666t - 451, \\
  g(t) &= -336018t^5 - 472949t^4 - 270569t^3 - 78158t^2 - 11325t - 651, \\
  h(t) &= 170 \left( 7225t^6 + 13770t^5 + 11187t^4 + 4908t^3 + 1219t^2 + 162t + 9 \right).
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\end{align*}
\]

Let \(d = \deg h = 6\); we obtain a linkage with

- \(26 = 2 \cdot (2d + 1)\) links,
- \(37 = 2 \cdot (2d) + (2d + 1)\) joints.
Final Example

[J. O’Rourke: *How to fold it*. Cambridge University Press, 2011]
Final Example
From David Eppstein’s blog:
“It’s been long known that you can make a linkage that can draw any algebraic curve but these people are trying to make it actually work — my contacts in mechanical engineering sound quite excited about it.”