Planar Linkages Following a Prescribed Motion

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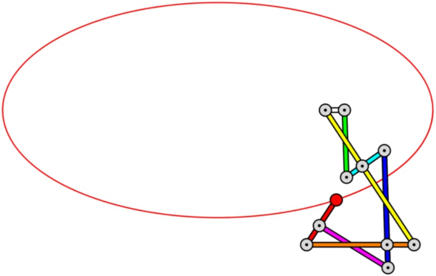
JOHANN · RADON · INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Motivation



Material (paper, pictures, movies) is available at

`http://www.koutschan.de/data/link/`

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Definition. A **linkage** is a mechanism which consists of

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Restriction: We consider only **planar linkages**, i.e., all links move in parallel planes.

There are two different types of joints:

1. revolute (rotational) joints
2. prismatic (translational) joints

Kempe's Universality Theorem

Goal: For a given planar curve, construct a linkage that draws it.

- Motivation from engineering, dates back to 18th century
- Example: Watt's linkage ("one of the most ingenious simple pieces of mechanics I have invented")

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Theorem. (Kempe 1876)
Let $f \in \mathbb{R}[x, y]$ be a polynomial, and let $B \subseteq \mathbb{R}^2$ be a closed disk. Then there exists a planar linkage which draws the curve

$$B \cap \left\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\right\}.$$

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- Proof of the theorem is constructive.
- "parsing algorithm" with input $f(x, y)$
- Kempe's constructions yield very complicated linkages.
- Can be applied to any algebraic curve.

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3. Answer to the collision problem.
4. A prototype implementation.

Mathematical model for linkages

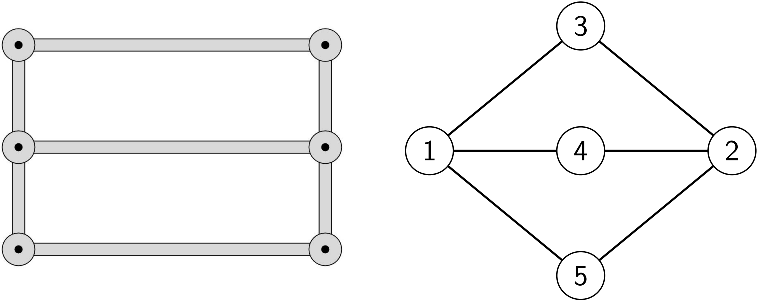
1. Self-collisions of the links are not taken into account, i.e., the joints are the only constraints for the motion of the links.
2. Thus the actual shape of the links doesn't matter, just the position of the joints.
3. Not a single frame of reference for the configuration of a linkage, but each link has its own frame of reference.

Linkages

Link graph: encodes the "topological information" of a linkage.

- Each link corresponds to a vertex.
- Each joint corresponds to an edge.

Example:



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Definition. A **planar linkage** with revolute joints is a connected undirected graph $G = (V, E)$ without self-loops, together with a map $\rho\colon E \to \mathbb{R}^2$.

- The point $\rho(e)$ is the position of the joint e in the "initial configuration" of the linkage.
- W.l.o.g. assume that V is of the form $\{1, \ldots, n\}$.
- Elements of E are given by unordered pairs $\{i, j\} \subseteq V$.

In the following let $L = (G, \rho)$ with $G = (V, E)$ be a linkage.

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We then have $\sigma_{i,j} = \sigma_{i,k} \circ \sigma_{k,j}$.

Configurations

Definition. Let $\{i, j\} \in E$. The set of **virtual relative positions** of link i w.r.t. link j, denoted $\mathrm{VRP}(i, j)$, is the subgroup of SE_2 of rotations around the point $\rho(i, j)$.

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Definition. A **configuration** of L is a collection of relative positions $\sigma_{i,j} \in \mathrm{VRP}(i, j)$, subject to the constraints:

- If $(i, h_1), (h_1, h_2), \dots, (h_s, i)$ is a **directed cycle** in G i.e., $\{i, h_1\} \in E$, $\{h_1, h_2\} \in E$ etc.,
- then $\sigma_{i,h_1} \circ \cdots \circ \sigma_{h_s,i} = \mathrm{id}$.
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$\longrightarrow$ The cycle condition also implies that $\sigma_{i,j} = \sigma_{j,i}^{-1}$.

Configuration space

Definition. The **configuration space** of a linkage L is the set of all its configurations:

$$\mathrm{Conf}(L) = \left\{ (\sigma_{k,l}) \in \prod_{\{i,j\}\in E} \mathrm{VRP}(i,j) \times \mathrm{VRP}(j,i) \;:\; \text{cycle conds} \right\}$$

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By identifying $\mathrm{VRP}(i,j)$ with the real projective line $\mathbb{P}^1_{\mathbb{R}}$, $\mathrm{Conf}(L)$ acquires the structure of a projective subvariety of $\left(\mathbb{P}^1_{\mathbb{R}}\right)^{2|E|}$.

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Definition. The **mobility** of L is the dimension of $\mathrm{Conf}(L)$.

Recapitulation and outlook

Have: Mathematical model for linkages that uses direct isometries in an essential way.

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Outline of solution:

- Embed SE_2 in the real projective space $\mathbb{P}^3_{\mathbb{R}}$.
- Interpret the points in $\mathbb{P}^3_{\mathbb{R}}$ as elements of some ring $\mathbb{K}$.
- The multiplication in $\mathbb{K}$ will correspond to the group operation $\circ$ in SE_2.
- Employ the polynomial ring $\mathbb{K}[t]$ to describe **motions**.

Embedding of SE_2 in $\mathbb{P}^3_{\mathbb{R}}$

Definition. The n-dimensional **real projective space** is the set

$$\mathbb{P}^n_{\mathbb{R}} := \big(\mathbb{R}^{n+1} \setminus \{(0,\dots,0)\}\big)/\sim,$$

where

$$(x_0,\dots,x_n) \sim (y_0,\dots,y_n) \;:\Longleftrightarrow \\ \exists\, c \in \mathbb{R}^*\colon (x_0,\dots,x_n) = c \cdot (y_0,\dots,y_n).$$

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Write a point in $\mathbb{P}^3_{\mathbb{R}}$ with the coordinates $(x_1 : x_2 : y_1 : y_2)$.

Embedding: We embed SE_2 in $\mathbb{P}^3_{\mathbb{R}}$ as the open subset

$$\mathcal{U} = \mathbb{P}^3_{\mathbb{R}} \setminus \big\{(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_{\mathbb{R}} \mid x_1^2 + x_2^2 = 0\big\}.$$

Geometric interpretation: The set $\mathcal{U}$ is the complement of the line $x_1 = x_2 = 0$.

Action

Let $\sigma \in \mathrm{SE}_2$ be given by the point $(x_1 : x_2 : y_1 : y_2) \in \mathcal{U} \subset \mathbb{P}^3_{\mathbb{R}}$.

The action of σ on a point $(x, y) \in \mathbb{R}^2$ is given by

$$\frac{1}{x_1^2 + x_2^2}\left[\begin{pmatrix} x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix}\right].$$

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Exercise. What kind of isometry is given by $(x_1 : x_2 : 0 : 0)$?

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Exercise. What kind of isometry is given by $(1 : 0 : y_1 : y_2)$?

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- If $x_2 = 0$ then we have a pure translation.

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Exercise. Which points correspond to the identity isometry?

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- The rotational part depends only on x_1 and x_2.
- If $x_2 = 0$ then we have a pure translation.
- The identity isometry is given by $(x_1 : 0 : 0 : 0)$.

Product

With this action the product in SE_2 becomes a bilinear map:

$$(x_1 : x_2 : y_1 : y_2) \cdot (x_1' : x_2' : y_1' : y_2') =$$

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Notation 1. Write a point $(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_{\mathbb{R}}$ as a pair of complex numbers $(z, w) = (x_1 + \imath\, x_2, y_1 + \imath\, y_2) \in \mathbb{C}^2$.

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Using this notation, the product in SE_2 can be rewritten as

$$(z, w) \cdot (z', w') = \big(z\, z', \overline{z}\, w' + z' w\big)$$

where the bar $\overline{(\cdot)}$ denotes complex conjugation.

Product (continued)

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Rational motions and motion polynomials

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$\longrightarrow$ The multiplication of $P \in \mathbb{K}[t]$ by a real polynomial $R \in \mathbb{R}[t]$ gives a new motion polynomial $RP = PR$, which however describes the **same** rational motion.

Connection to rational curves

Proposition. Let $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ be a rational parametrization,

$$\varphi(t) = \left(\frac{f(t)}{h(t)}, \frac{g(t)}{h(t)}\right), \quad \text{for some } f, g, h \in \mathbb{R}[t],$$

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In many considerations, we restrict ourselves to monic motion polynomials (justification will be given later).

Definition. We say that $P = Z + \eta\, W \in \mathbb{K}[t]$ is **monic** if its leading coefficient is 1, i.e.: $Z \in \mathbb{C}[t]$ is monic and $\deg W < \deg Z$.

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$\longrightarrow$ If P is monic then $\lim_{t\to+\infty} P(t) = (1 : 0 : 0 : 0) \in \mathbb{P}^3_{\mathbb{R}}$, which corresponds to the identity element in SE_2.

Characterization of simple motions

Lemma. Let $\ell \subseteq \mathbb{P}^3_{\mathbb{R}}$ be a projective line passing through the point $(1:0:0:0)$, and let $X = \ell \cap \{x_1 = x_2 = 0\}$. Then:

1. if $|X| = 1$, then $\ell \setminus X$ corresponds to a subgroup of SE_2 that consists of all translations along a fixed common direction;
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$\longrightarrow$ Linear motion polynomials describe exactly those motions that are realized by joints.

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3. Multiplication of motion polynomials corresponds to the composition of motions, e.g.,

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Weak and strong realization

Let $L = \big((V,E),\rho\big)$ be a linkage, $\phi\colon \mathbb{R} \to \mathbb{P}^3_{\mathbb{R}}$ a rational motion.

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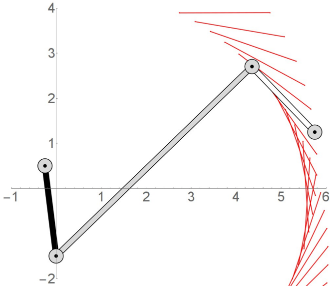

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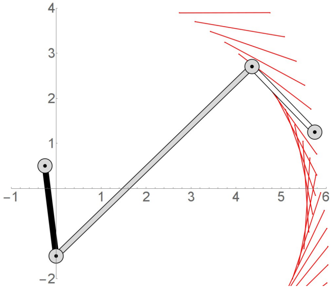

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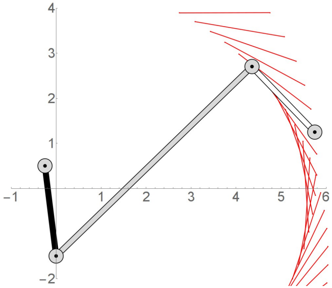

Overview

Task: Construct a linkage that realizes a given rational motion ϕ.

Solution strategy:

1. The motion ϕ is described by a motion polynomial $P \in \mathbb{K}[t]$.
2. Factor P into linear factors.
3. Each linear factor represents an "elementary" motion (revolution, translational motion), which can be realized by a single joint.
4. A factorization of P gives rise to an open chain of links, which weakly realizes the motion ϕ.
5. Insert more links in order to restrain the mobility of the linkage so that it strongly realizes the motion ϕ.

Factorization into linear factors

Let $P = Z + \eta W \in \mathbb{K}[t]$ be a monic motion polynomial of degree n.

Goal: Factor P into monic linear motion polynomials, i.e.,

$$P = P_1 \cdot P_2 \cdots P_n, \quad P_i = t - z_i + \eta\, w_i, \quad w_i, z_i \in \mathbb{C}.$$

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Recall: $(z + \eta\, w) \cdot (z' + \eta\, w') = z\, z' + \eta\, (\overline{z}\, w' + z'\, w)$.

By expanding the ansatz we obtain:

$$P = (t - z_1 + \eta\, w_1) \cdot (t - z_2 + \eta\, w_2) \cdots (t - z_n + \eta\, w_n) =$$

$$= \underbrace{\prod_{i=1}^{n} (t - z_i)}_{Z(t)} + \eta \underbrace{\sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} (t - \overline{z_j}) \right) \left(\prod_{j=k+1}^{n} (t - z_j) \right) w_k}_{W(t)} .$$

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$\longrightarrow$ The w_i can be found by ansatz and solving a linear system.

$\longrightarrow$ The order of $z_1, \ldots, z_n$ matters!

How to compute the w_i

Fix a permutation $\mathbf{z} = (z_1, \ldots, z_n)$ of the complex roots of Z.

$$P = Z(t) + \eta \sum_{k=1}^{n} \underbrace{\left(\prod_{j=1}^{k-1} (t - \overline{z_j}) \right) \left(\prod_{j=k+1}^{n} (t - z_j) \right)}_{=:\, Q_k(\mathbf{z})} w_k.$$

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Lemma. Let $P = Z + \eta W$ be monic and let $\mathbf{z} = (z_1, \ldots, z_n)$ be a fixed permutation of the roots of Z over $\mathbb{C}$. Then P admits a factorization $P = P_1 \cdots P_n$ where $P_i(t) = (t - z_i) + \eta\, w_i$, $w_i \in \mathbb{C}$, if and only if W lies in the linear span $\big\langle Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z}) \big\rangle_{\mathbb{C}}$.

Sufficient condition for factorization

$P = Z + \eta W$, monic, $\deg P = n$, admits a factorization

$\iff W \in \langle Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}}$

$\iff \gcd\big(Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z})\big) = 1$ (note that $\deg Q_k = n - 1$).

Clearly, this is the case (for arbitrary W with $\deg W < n$) if the following matrix $M_n \in \mathbb{C}^{n \times n}$ is non-singular:

$$M_n = \begin{pmatrix} \langle t^0 \rangle Q_1 & \cdots & \langle t^0 \rangle Q_n \\ \langle t^1 \rangle Q_1 & \cdots & \langle t^1 \rangle Q_n \\ \vdots & & \vdots \\ \langle t^{n-1} \rangle Q_1 & \cdots & \langle t^{n-1} \rangle Q_n \end{pmatrix}$$

(here $\langle t^i \rangle Q_k$ denotes the coefficient of t^i in Q_k).

$\longrightarrow$ The matrix entries are, up to sign, elementary symmetric polynomials in the z_i and $\overline{z_i}$.

Evaluating the determinant

We have $\det(M_n) = \prod_{1 \leqslant i < j \leqslant n} (\overline{z_i} - z_j)$.

- This is very much reminiscent of the Vandermonde determinant, and it can be proved in a similar fashion.
- A similar determinant evaluation is given in (Lascoux/Pragacz 2002) where the z_i appear without conjugation.
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Lemma. Let $P = Z + \eta W \in \mathbb{K}[t]$, monic, be such that Z has no pair of complex-conjugate roots (i.e., $Z(\alpha) = 0 \implies Z(\overline{\alpha}) \neq 0$). Then for every permutation $(z_1, \ldots, z_n)$ of the roots of Z, the polynomial P admits a factorization into linear factors.

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$\longrightarrow$ This condition is only sufficient, but not necessary, for the existence of a factorization.

Characterization of factorizable polynomials

Proposition. Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $W \in \mathbb{C}[t]$, $\deg W < n$. Then

$$W \in \big\langle Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z}) \big\rangle_{\mathbb{C}} \iff W \in \big(Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z})\big) \cdot \mathbb{C}[t]$$

where $\big(Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z})\big) \cdot \mathbb{C}[t]$ is the ideal generated by the $Q_k(\mathbf{z})$.

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where $\big(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})\big) \cdot \mathbb{C}[t]$ is the ideal generated by the $Q_k(\mathbf{z})$.

Remarks:

- The ideal on the right-hand side is generated by a single polynomial $G := \gcd(Q_1, \dots, Q_n)$.
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Characterization of factorizable polynomials

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- The ideal on the right-hand side is generated by a single polynomial $G := \gcd(Q_1, \dots, Q_n)$.
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Corollary. A monic motion polynomial $Z + \eta W \in \mathbb{K}[t]$ can be factored completely if there exists a permutation $\mathbf{z}$ such that $G \mid W$.

Determine the gcd G

$G = \gcd\big(Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z})\big)$ for some $\mathbf{z} \in \mathbb{C}^n$.

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\begin{aligned}
Q_1 &= \phantom{(t-\overline{z_1})}(t-z_2)(t-z_3)(t-z_4)\cdots(t-z_{n-1})(t-z_n)\\
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Computation of G

Definition. Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. A set

$$M \subseteq \left\{(i,j) : 1 \leqslant i < j \leqslant n \wedge z_i = \overline{z_j}\right\}$$

is called a **matching** of $\mathbf{z}$ if for all $(i_1, j_1), (i_2, j_2) \in M$ we have $i_1 \neq i_2$ and $j_1 \neq j_2$.

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$$\gcd\big(Q_1(\mathbf{z}), \ldots, Q_n(\mathbf{z})\big) = \prod_{(i,j) \in M} (t - z_j)$$

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Exercise: What is the gcd in the above example?

Some examples

Exercise: Let $Z = (t-\alpha)^r(t-\overline{\alpha})^{r+1}$. Find the permutations $\mathbf{z}$ of the roots of Z that give the following G's.

$\mathbf{z}$	G	M
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The cases discussed here are the extreme ones:

- It is easy to see that $r \leqslant \deg(G) \leqslant 2r$.
- For any $G = (t-\alpha)^i(t-\overline{\alpha})^j$ with $0 \leqslant i,j \leqslant r$ and $i+j \geqslant r$ there exists a permutation $\mathbf{z}$ which produces this gcd G.

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Recall: $P = Z + \eta W$ factors iff there exists $\mathbf{z}$ such that $G \mid W$.

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$\longrightarrow$ Caveat: this trick works only for $\alpha \notin \mathbb{R}$!

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Theorem. Let $P \in \mathbb{K}[t]$ be a bounded motion polynomial. Then there exists a real polynomial $R \in \mathbb{R}[t]$ such that RP can be factored into linear polynomials.

Factorization algorithm

Input: $P = Z + \eta W \in \mathbb{K}[t]$ a bounded motion polynomial such that Z and W have no common factor in $\mathbb{R}[t] \setminus \mathbb{R}$.

Output: a polynomial $R \in \mathbb{R}[t]$ and a tuple $(k_1, \ldots, k_n)$ of elements of $\mathbb{K}$ such that $(t - k_1) \cdots (t - k_n) = R(t) \cdot P(t)$.

Factor $Z(t)$ over $\mathbb{C}$: $Z = \prod_{i=1}^{h} (t - \alpha_i)^{r_i} (t - \overline{\alpha_i})^{s_i}$ with $r_i \geqslant s_i \geqslant 0$.
Initialize $q = \texttt{empty}$ (the empty tuple).
For $i = 1, \ldots, h$ **Do**
    Set $u_i = \max_j \big((t - \alpha_i)^j \mid W \big)$ and $v_i = \max_j \big((t - \overline{\alpha_i})^j \mid W \big)$.
    Set $m_i = \min\{s_i, u_i + v_i\}$
    Set $\omega = \big(\underbrace{\overline{\alpha_i}, \ldots, \overline{\alpha_i}}_{s_i - \min\{s_i, v_i\}}, \underbrace{\alpha_i, \ldots, \alpha_i}_{r_i + s_i - m_i}, \underbrace{\overline{\alpha_i}, \ldots, \overline{\alpha_i}}_{s_i - \min\{s_i, u_i\}} \big)$.
    Set $q = \texttt{concatenate}(q, \omega)$.
End For
Set $R = \prod_{i=1}^{h} \big((t - \alpha_i)(t - \overline{\alpha_i}) \big)^{s_i - m_i}$.
Set $n = \texttt{length}(q)$.
Set $Q_j = \prod_{l=1}^{j-1} (t - q_l) \prod_{l=j+1}^{n} (t - \overline{q_l})$ for all $j \in \{1, \ldots, n\}$.
Compute $\{w_j\}_{j=1}^{n}$ s.t. $RW = \sum_{j=1}^{n} w_j Q_j$ using linear algebra.
Set $k_j = q_j - \eta\, w_j$ for all $j \in \{1, \ldots, n\}$.
Return $\big(R, (k_1, \ldots, k_n) \big)$.

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We want to construct a linkage drawing an ellipse with radii 1 and $\frac{1}{2}$.

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Construction of weak linkages

Recall: $L = \big((V, E), \rho\big)$ weakly realizes a motion $\phi\colon \mathbb{R} \to \mathrm{SE}_2$ if $\phi(\mathbb{R}) \subseteq \mathrm{RP}(i, j)$ for some $i, j \in V$.

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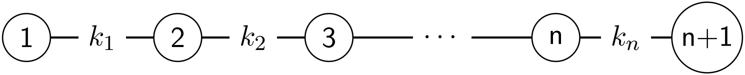

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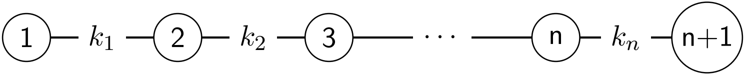

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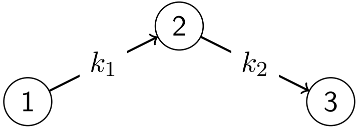

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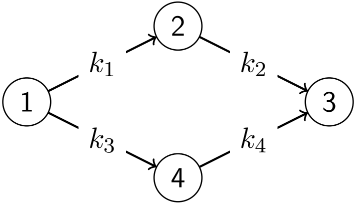

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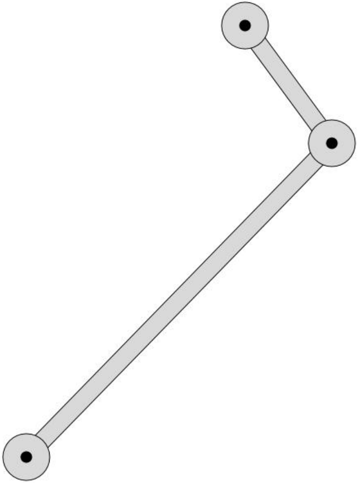

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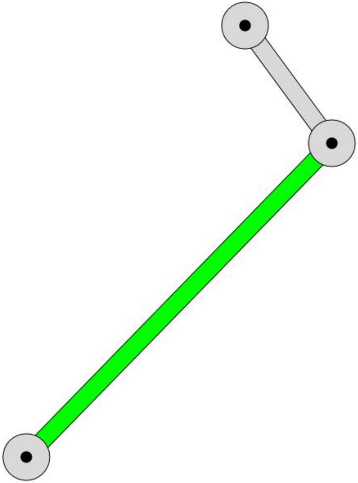

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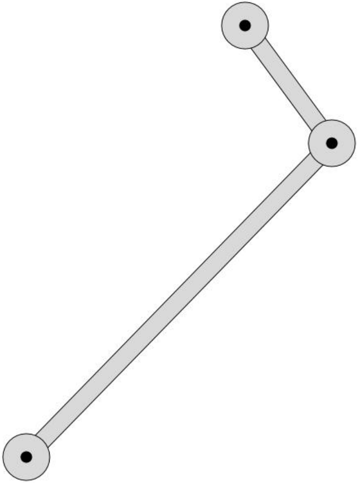

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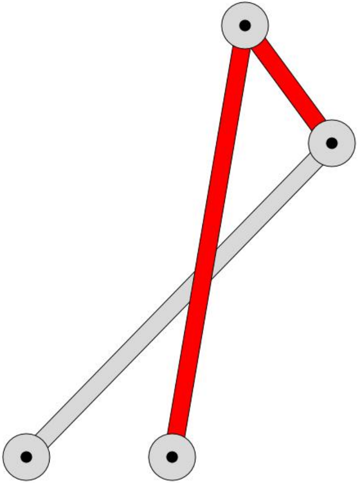

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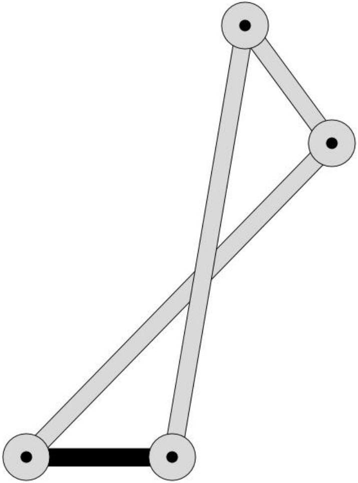
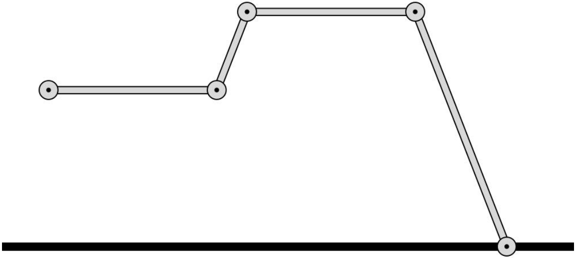


Iterated flips

$$1 \xrightarrow{k_1} 2 \xrightarrow{k_2} 3 \xrightarrow{k_3} 4 \xrightarrow{k_4} 5$$



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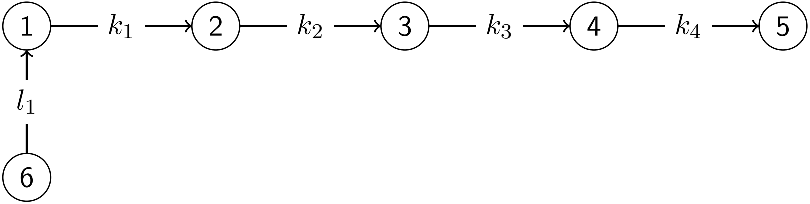

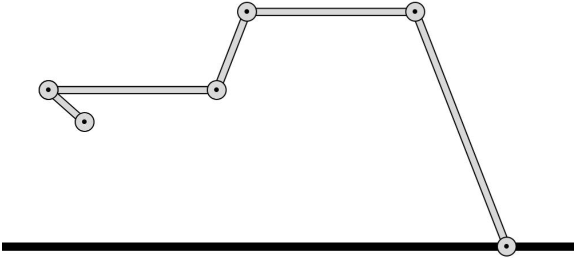

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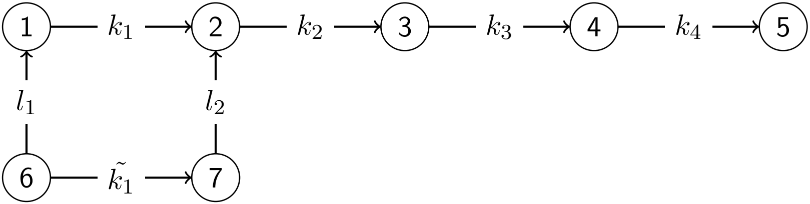

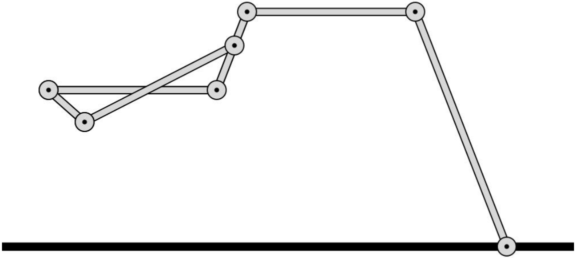

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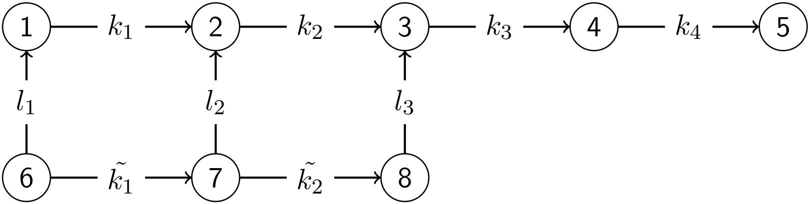

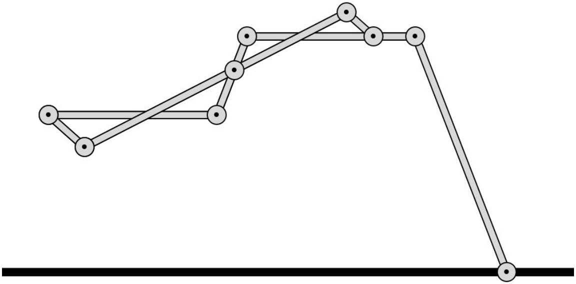

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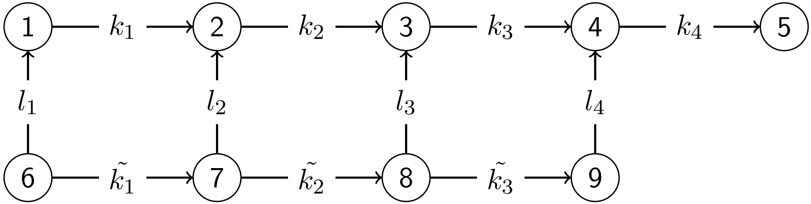

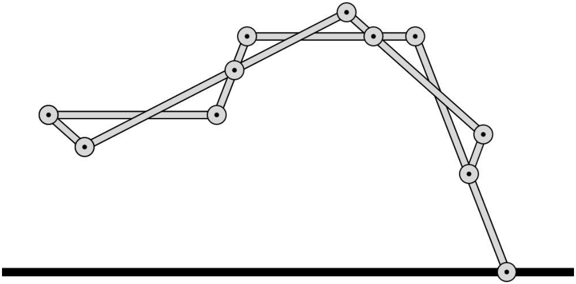

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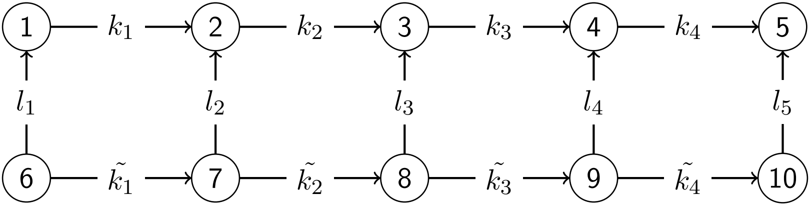

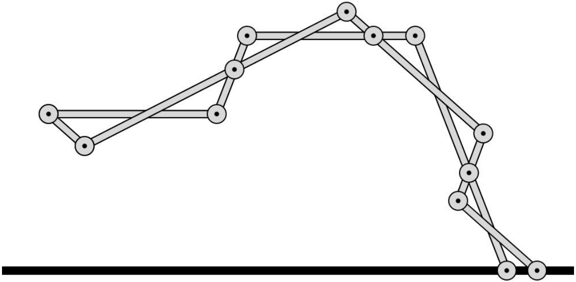

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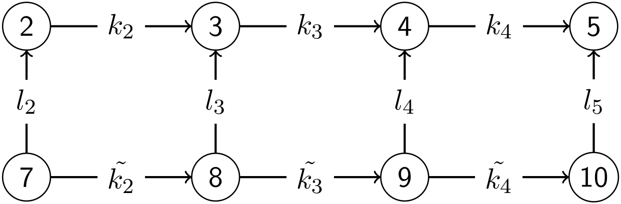

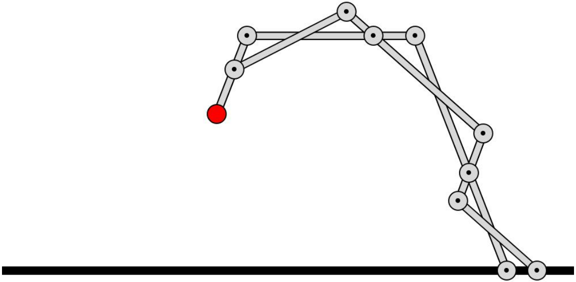

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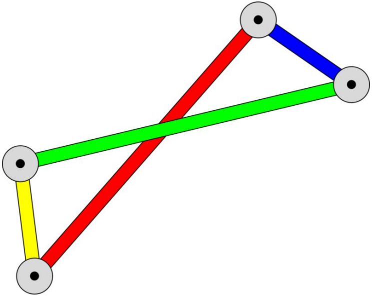
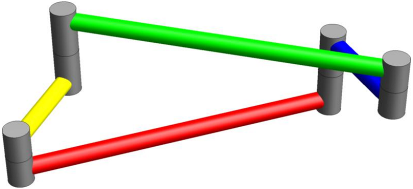
Corollary. A bounded rational curve given by $(f/h, g/h)$ with $f, g, h \in \mathbb{R}[t]$ such that $\deg h > \max\{\deg f, \deg g\}$ can be drawn by a linkage with at most $4d$ links and $6d - 2$ joints ($d = \deg h$).

Physical realization

Standard way of realizing planar linkages:

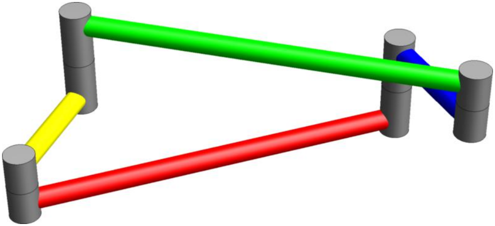
- Each link is realized as a polygon (convex hull of the positions of its joints)
- Each link moves parallel to the horizontal (x, y)-plane at a certain height z.
- The joints are realized as vertical connections between links.





Self-collisions

Vertical arrangement of links is crucial when studying collisions!



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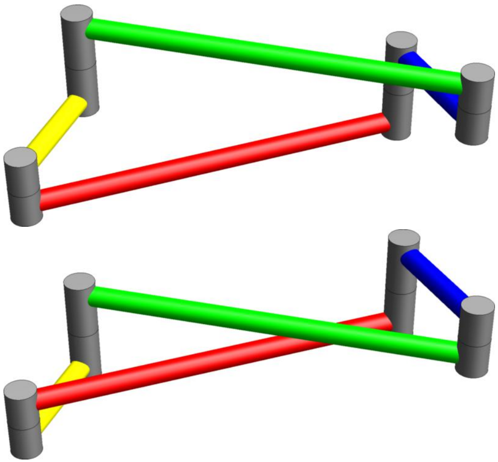

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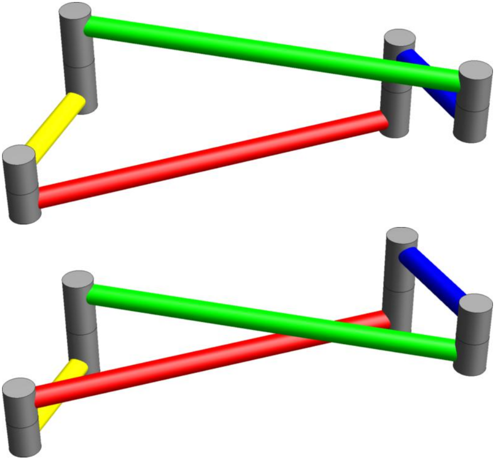

$\longrightarrow$ It suffices to study link–joint collisions.

Self-collisions

W.l.o.g. assume that the link labels $\{1, \ldots, n\}$ correspond to their heights, i.e., their z-coordinates.

Collision:

- links $i < k < j$
- $\big(x_1(t), y_1(t)\big)$ = position of joint (i, j)
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- a collision happens if

$$x_1(t) = s \cdot x_2(t) + (1 - s) \cdot x_3(t)$$
$$y_1(t) = s \cdot y_2(t) + (1 - s) \cdot y_3(t)$$

for some $t \in \mathbb{R} \cup \{\infty\}$ and $0 \leqslant s \leqslant 1$.

Detect collisions

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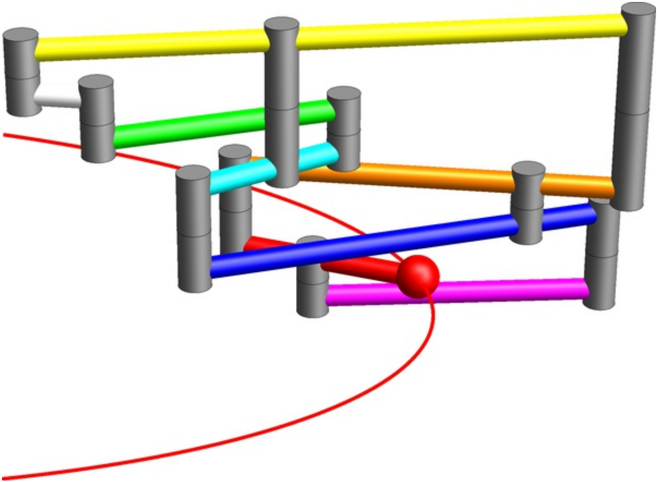
$\longrightarrow$ In contrast to general linkages, our construction allows for a relatively simple collision detection!

Detect collisions

Example: For our linkage drawing an ellipse we can find a spatial arrangement of the links s.t. only 2 collisions occur (both at $t = \infty$).



Avoid collisions

If we consider links of a more complicated 3-dimensional shape, we can completely avoid collisions:

- W.l.o.g. assume that the layers correspond to consecutive integer numbers.
- F-links ("flat" links): as before, located in a single layer $a \in \mathbb{N}$
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$\longrightarrow$ This solves an open problem, posed by O'Rourke, concerning the existence of collision-free linkages.

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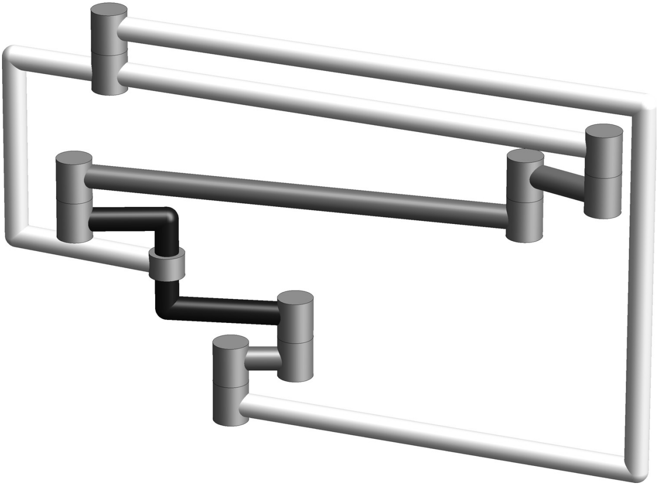
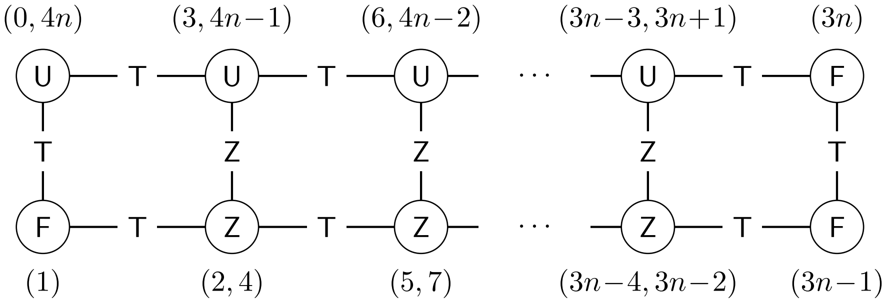


Collision-free linkages

The following scheme allows to realize a planar linkage with ladder-shaped link graph without collisions:



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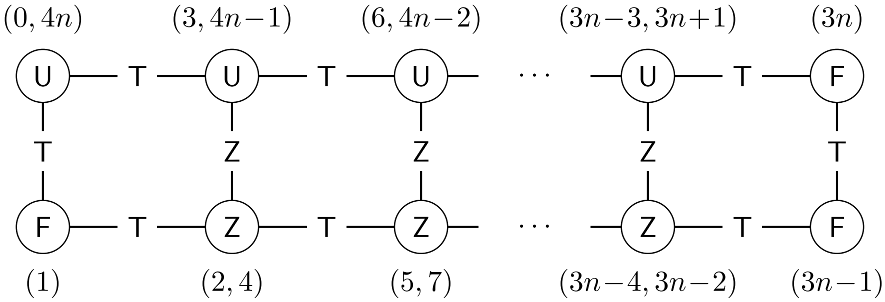

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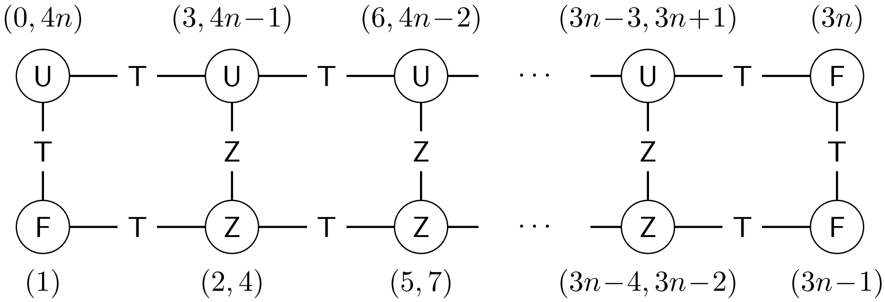

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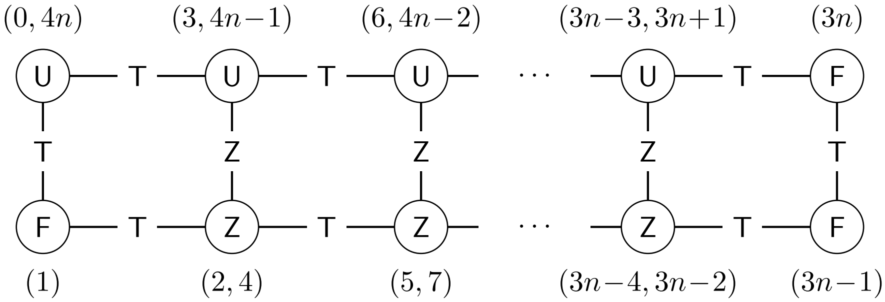

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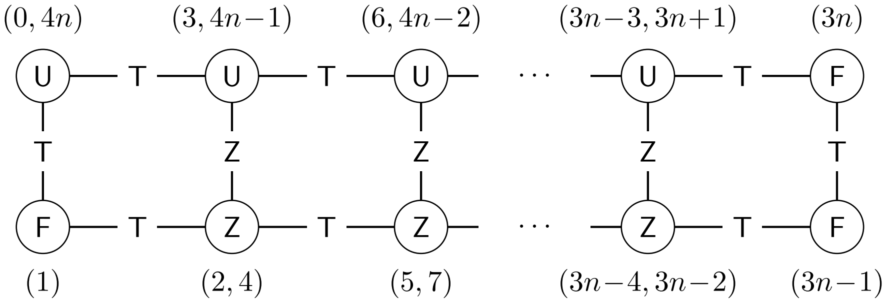

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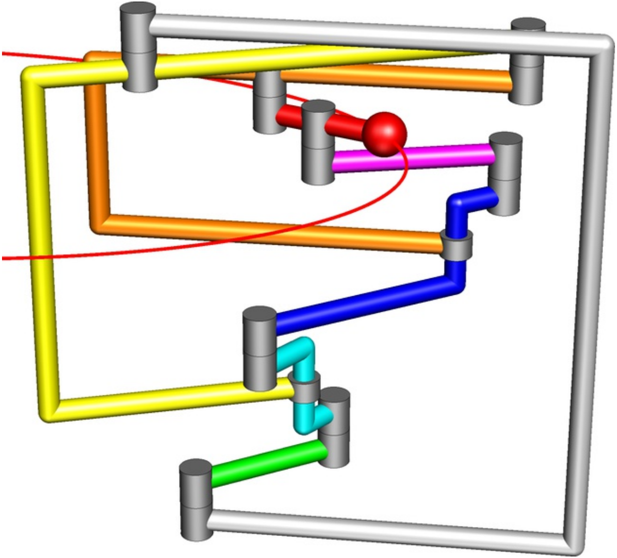
- Collisions between a U-link and an F- or Z-link: similarly.

Collision-free linkage for the ellipse



Final Example

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Linkages drawing a full signature would be too complex, hence previous attempts have focused on the "J" only.

Final Example

Approximate the "J" by a rational curve $(f/h, g/h)$:

$$f(t) = -321880t^5 - 436132t^4 - 237449t^3 - 64488t^2 - 8666t - 451,$$
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Let $d = \deg h = 6$; we obtain a linkage with

- $26 = 2 \cdot (2d + 1)$ links,
- $37 = 2 \cdot (2d) + (2d + 1)$ joints.

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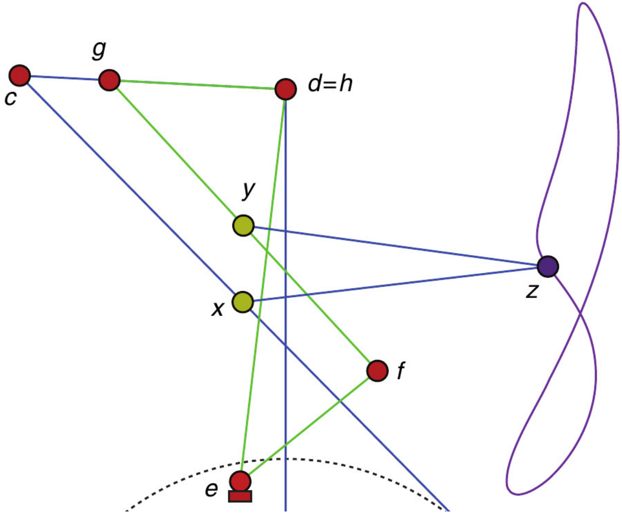

[J. O'Rourke: *How to fold it.* Cambridge University Press, 2011]

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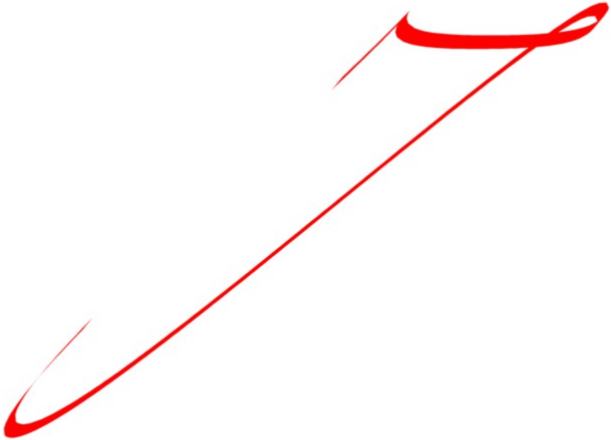

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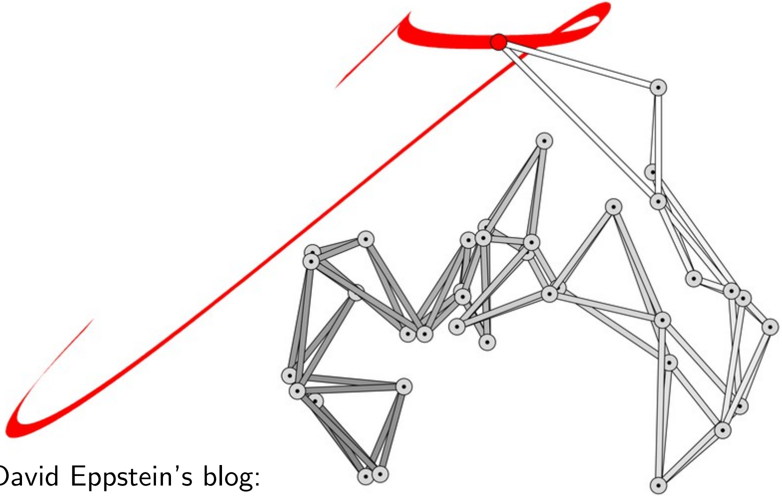

From David Eppstein's blog:
"It's been long known that you can make a linkage that can draw any algebraic curve but these people are trying to make it actually work — my contacts in mechanical engineering sound quite excited about it."