Puiseux Series and Integral Bases of Algebraic Functions

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Outline of the Talk

1. Introduction
2. Main Theorem
3. Conclusion

References

- Ford and Zassenhaus, 1978
  $\longrightarrow$ algorithm for algebraic number fields
- Barry Trager: *Integration of algebraic functions*, Ph.D. thesis, 1984.
  $\longrightarrow$ generalization of Ford-Zassenhaus to algebraic function fields
- Mark van Hoeij: *An Algorithm for Computing an Integral Basis in an Algebraic Function Field*, Journal of Symbolic Computation **18**, 353–363, 1994.
  $\longrightarrow$ faster algorithm for algebraic function fields

Notation

We employ the following notation:

- L is an algebraically closed field of characteristic 0
- x is transcendental over L
- y is algebraic over $L(x)$ with minimal polynomial f
- n is the degree of f
- $K \leqslant L$ denotes the field of coefficients of f, i.e., $f \in K[x, y]$
- $\overline{L[x]}$ is the integral closure of $L[x]$ in $L(x, y)$

Vector Space Basis

We know that $L(x, y)$ is a $L(x)$-vector space of dimension n.

Elements $b_0, \ldots, b_{n-1} \in L(x, y)$ are a vector space basis if

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$$b_0 = 1, \quad b_1 = y, \quad b_2 = y^2, \ldots, \quad b_{n-1} = y^{n-1}.$$

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Example: $f = y^3 - x^2$, $b_0 = 1$, $b_1 = x^{2/3}$, $b_2 = x^{4/3}$

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W.l.o.g. we assume that y is an integral element.

Hence, every element in $L[x, y]$ is integral. We have

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Goal: Find an integral basis $b_0, \ldots, b_{n-1}$ with $\deg(b_i) = i$ and with $b_i \in K(x, y)$ for all i.

Caveat: Note that Puiseux series expansions may require coefficients in a larger field than K.

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- Compute b_d with $\deg(b_d) = d$ such that

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- Iterate to obtain an integral basis $b_0, \ldots, b_{n-1}$

One Step of the Algorithm

Task: We have to find the next element b_d of the integral basis.

- Start with $b_d = y^d$ (optimization: use $b_d = yb_{d-1}$)
- $V := \{a \in \overline{L[x]} \mid \deg(a) \leqslant d\} \setminus \big(L[x]\, b_0 + \cdots + L[x]\, b_d\big)$

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While $V \neq 0$ do the following:

1. Choose $a \in V$ such that a can be written as

$$a = \frac{1}{k}\big(a_0 b_0 + \cdots + a_d b_d\big)$$

with $a_0, \ldots, a_d, k \in K[x]$ and with $a_d = 1$.

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with $a_0, \ldots, a_d, k \in K[x]$ and with $a_d = 1$.
2. Since
$$L[x]\, b_0 + \cdots + L[x]\, b_{d-1} + L[x]\, b_d \subset$$
$$L[x]\, b_0 + \cdots + L[x]\, b_{d-1} + L[x]\, a \subset \overline{L[x]}$$
we can replace b_d by a in our basis and get a smaller V.

Problems

The strategy described before rises the following questions:

1. How can we ensure termination of the algorithm?
2. We have to show that in the case $V \neq 0$ the element a can be chosen in the described form.
3. How can we decide whether $V \neq 0$ and how can we compute $a_0, \ldots, a_d, k$?

Problem 1: Termination

Look at the discriminant (Trager's idea):

$$D := \mathrm{disc}(1, y, \ldots, y^{n-1}) = \mathrm{Res}_y\Big(f, \frac{\partial f}{\partial y}\Big) \in K[x]$$

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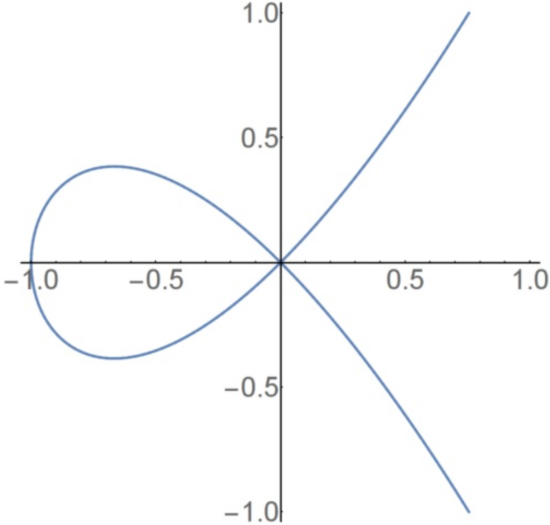
Bonus: This reasoning tells us that the candidates for k are exactly the factors of D.

Picture



$$f = y^2 - x^3 - x^2$$

$$D = \mathrm{Res}_y\left(f, \frac{\partial f}{\partial y}\right) = -4x^2(x+1)$$

Problem 2: The Particular Form of a

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To do:

1. Show that we can choose $a_0, \ldots, a_d$ such that $a_d = 1$.
2. Show that we can choose $a_0, \ldots, a_d, k \in K[x]$ instead of $L[x]$.

$a_d = 1$

We may multiply a by some element in $L[x]$ such that

- the result is still in V,
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But then, we can write $a_i = q_i \cdot (x - \alpha) + a'_i$ with $a'_i \in L$.

Still, $a'_d \neq 0$, so we can divide by a'_d, obtaining

$$\frac{a_0 b_0 + \cdots + a_d b_d}{x - \alpha} \quad \text{with } a_i \in L \text{ and } a_d = 1.$$

Next step: Argue that the a_i are actually in $K(\alpha)$.

$a_i \in K(\alpha)$

Lemma: Under the previous assumptions the $a_i \in L$ are unique.
Proof: Assume to the contrary, that there were two different sequences $a_0, \ldots, a_d$. Then the difference would be an element in V of degree less than d. Contradiction.

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- If a_i was transcendental over $K(\alpha, a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d)$, then it could be replaced by another element from K. Contradiction to the lemma.

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- Hence we conclude that $a_i \in K(\alpha)$ for all i.

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We have also $v_P(a_0 b_0 + \cdots + a_d b_d) \geqslant v_P(x - \alpha_i)$, where the α_i are the conjugates of α, because $a_0 b_0 + \cdots + a_d b_d \in K(x, y)$

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Conclusion: We can find $a \in V$ of the form

$$a = \frac{1}{k}\big(a_0 b_0 + \cdots + a_d b_d\big) \quad \text{with } a_0, \ldots, a_d, k \in K[x],$$

where $k \in K[x]$ is the minimal polynomial of α.

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Since $b_i \in K[x, y]$, this yields, for each b_i, a set of Puiseux series expansions. Hence we can write down the Puiseux expansions of

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The ansatz a is integral if and only if the coefficients of all negative powers in all Puiseux expansions vanish. This yields a linear system of equations for the a_i over $K(\alpha)$.

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Instead, one could also use lazy series evaluation.