Proof of the q-TSPP conjecture

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- \$1000 prize!

Plane Partitions

A plane partition π is:

- two-dimensional array $\pi = (\pi_{i,j})_{1 \leq i,j}$
- $\pi_{i,j} \in \mathbb{N}$ with finite sum $|\pi| = \sum \pi_{i,j}$
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A plane partition π is identified with its 3D Ferrers diagram:
$\longrightarrow$ stack $\pi_{i,j}$ unit cubes on top of the location (i, j)!

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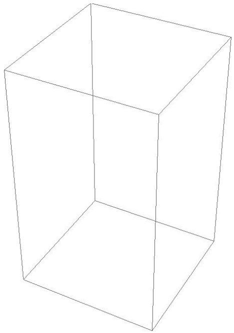

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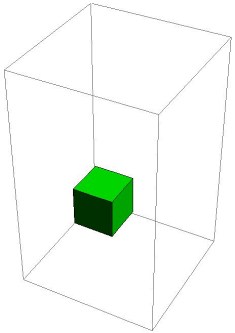

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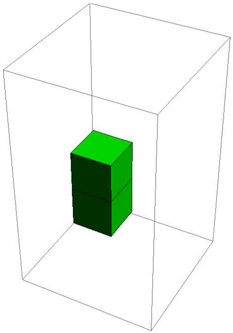

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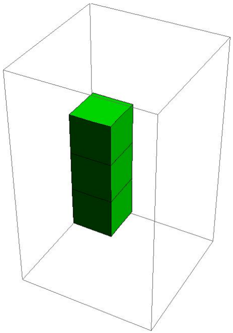

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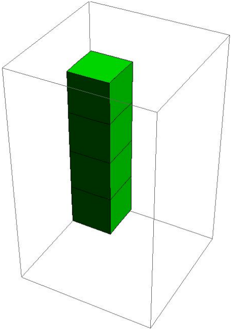

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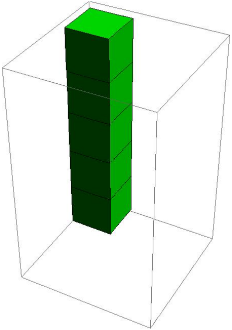

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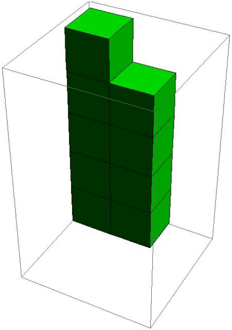

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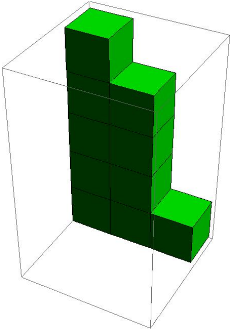

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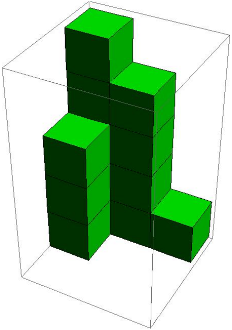

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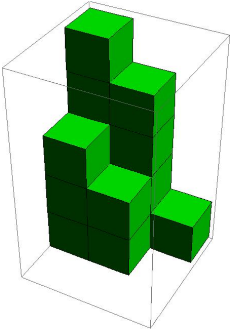

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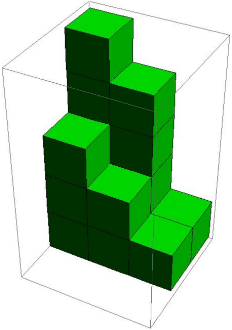

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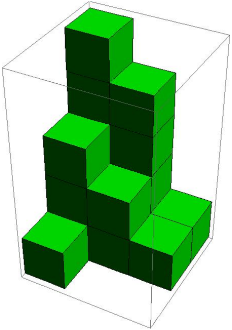

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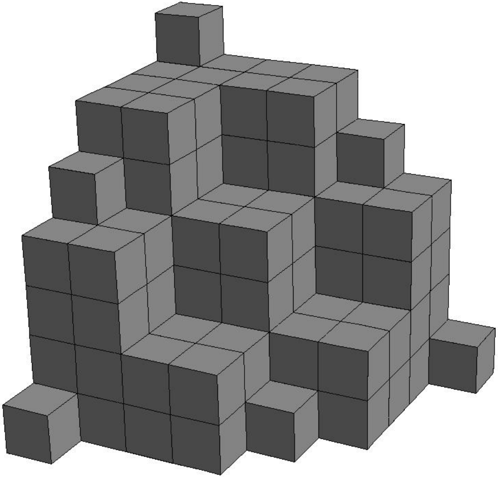
Example: TSPP



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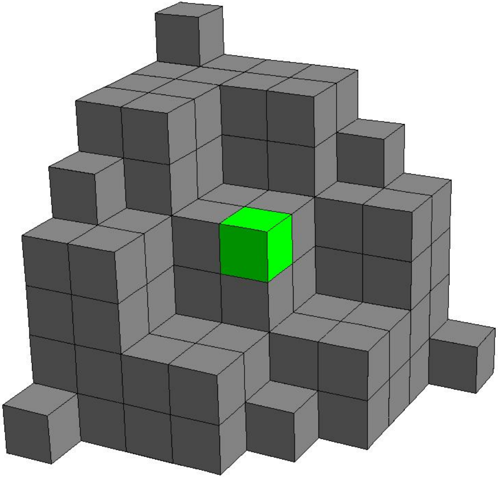

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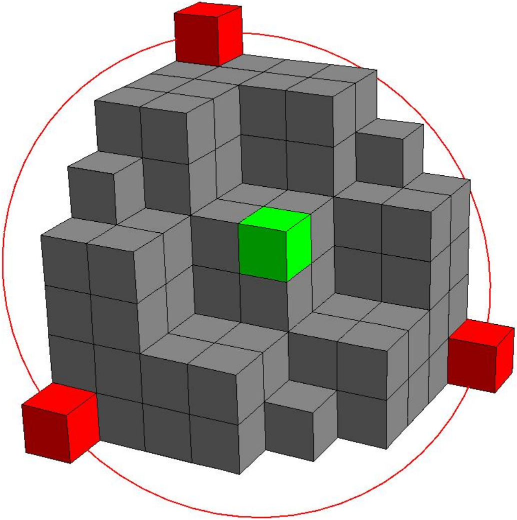

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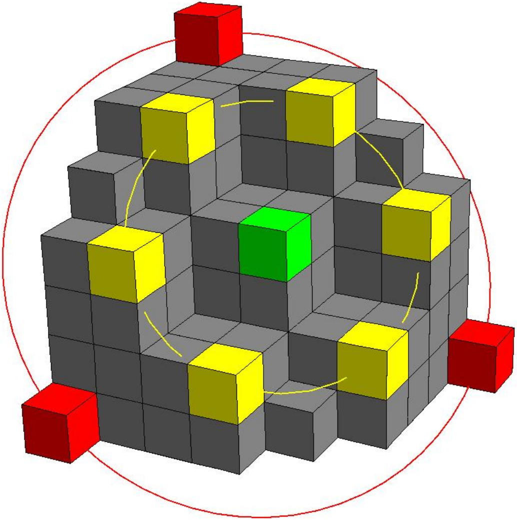

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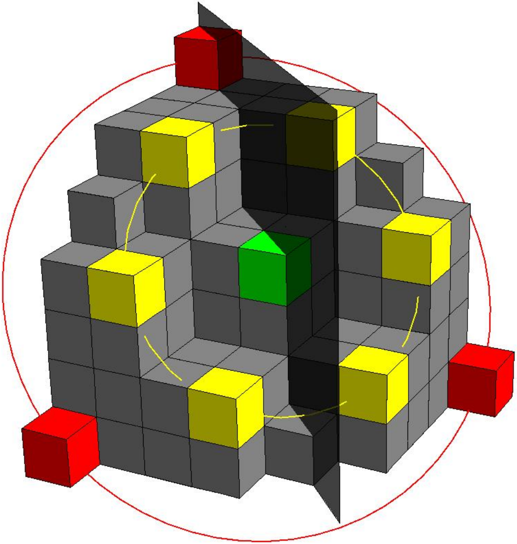

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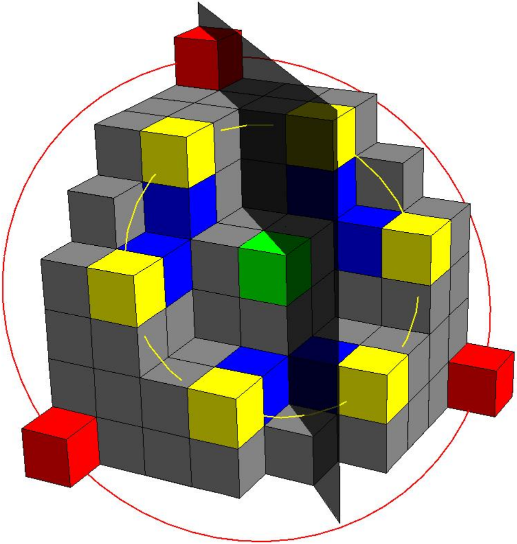
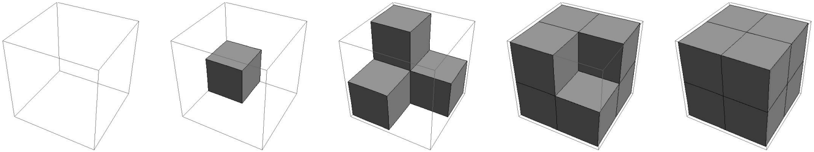


Let $T(n)$ denote set of TSPPs with largest part at most n.



Stembridge's Theorem: $$|T(n)| = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}$$

q-TSPP conjecture: $$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

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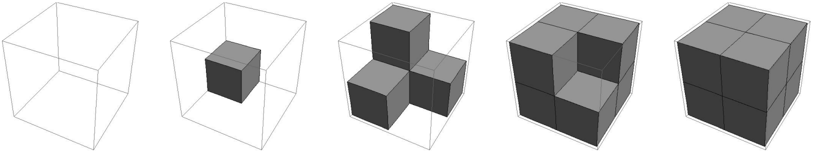

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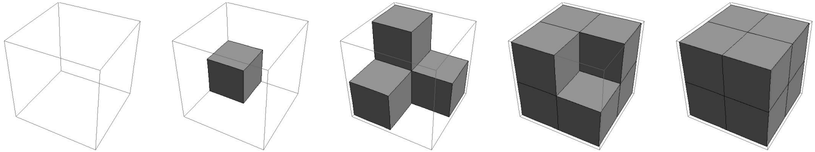

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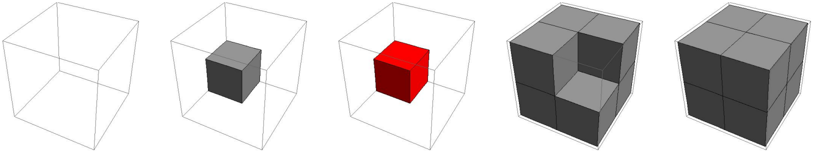

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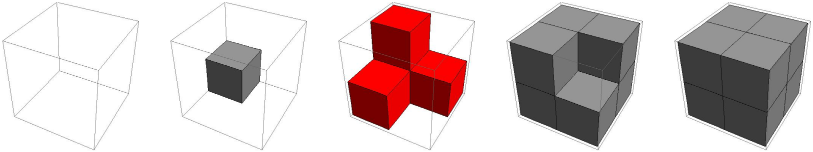

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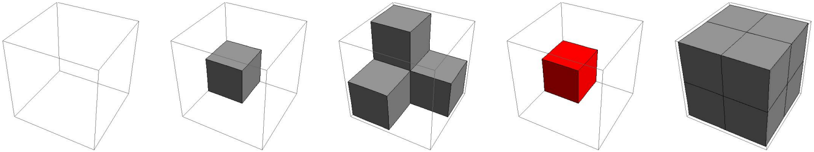

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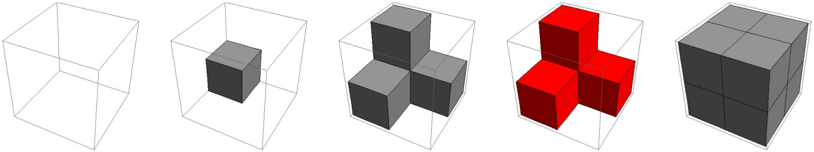

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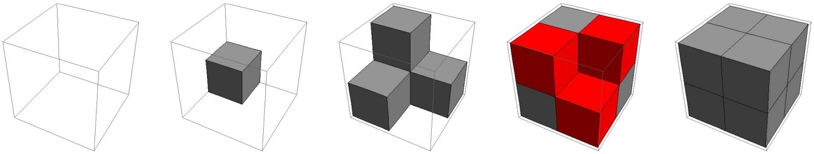

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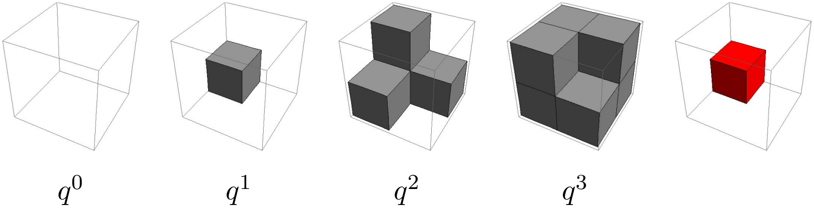

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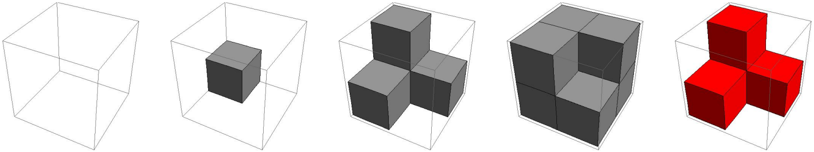

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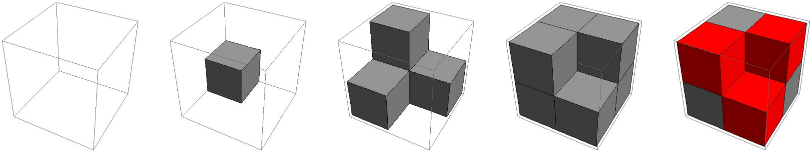

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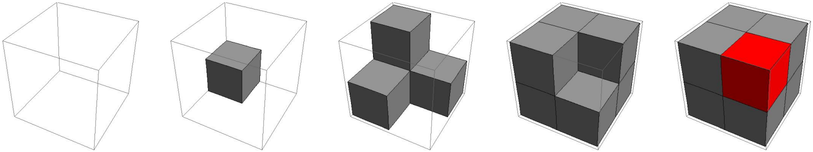

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A parallel universe

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q-binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

(for $q = 1$ we obtain the binomial coefficient $\binom{n}{k}$).

The determinant

We employ an elegant reduction by  (   ):

The q-TSPP conjecture is true if

$$\det(a_{i,j})_{1\le i,j\le n} = \prod_{1\le i\le j\le k\le n} \left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 =: b_n.$$

where

$$a_{i,j} := q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q\right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}.$$

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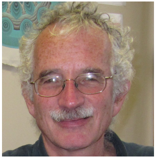
For $q = 1$: Stembridge's theorem!

The holonomic ansatz

We pursue an approach by  ( ):

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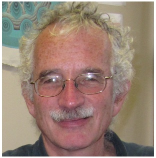
We pursue an approach by  (Doron Zeilberger):

"Pull out of the hat" a discrete function $c_{n,j}$ and prove

$$c_{n,n} = 1 \qquad (n \geq 1),$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0 \qquad (1 \leq i < n),$$

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1).$$

Then $\det(a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds.

Why???

Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!

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Example: The binomial coefficient $f_{n,k} = \binom{n}{k}$ can be described by

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Analogously, we get for the q-binomial coefficient $\bar{f}_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$:

$$\begin{aligned} (q^{n+1}-q^k)\bar{f}_{n+1,k} &= (q^{k+n+1}-q^k)\bar{f}_{n,k} \\ (q^{2k+1}-q^k)\bar{f}_{n,k+1} &= (q^n-q^k)\bar{f}_{n,k} \\ \bar{f}_{0,0} &= 1 \end{aligned}$$

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All linear combinations of shifts are again valid recurrences:

$$\begin{array}{l} (n-k)f_{n+1,k+1} - (n+1)f_{n,k+1} = 0 \\ (k+1)f_{n+1,k+1} - (n-k+1)f_{n+1,k} = 0 \\ \hline (n+1)f_{n+1,k+1} - (n-k+1)f_{n+1,k} - (n+1)f_{n,k+1} = 0 \end{array}$$

They form a left ideal in some noncommutative operator algebra.

Ideals of recurrences

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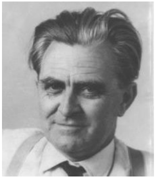 (   ) bases, for example.

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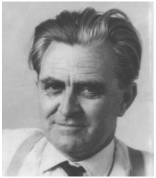

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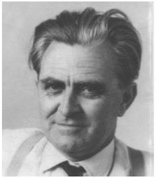

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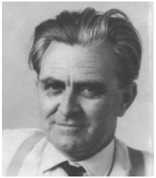

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Plug in concrete values:

$n = 1,\ k = 1$: $\quad x_{0,1} + x_{0,2} + x_{0,3} + 2\,(x_{1,1} + x_{1,2} + x_{1,3}) = 0$

$n = 2,\ k = 1$: $\quad 2\,(x_{0,1} + x_{0,2} + 2x_{0,3}) + 3\,(x_{1,1} + x_{1,2} + 2x_{1,3}) = 0$

$n = 3,\ k = 1$: $\quad 3\,(x_{0,1} + x_{0,2} + 3x_{0,3}) + 4\,(x_{1,1} + x_{1,2} + 3x_{1,3}) = 0$

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Solve this linear system:
$x_{0,1} = -C, x_{0,2} = 0, x_{0,3} = -C, x_{1,1} = C, x_{1,2} = -C, x_{1,3} = C$

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Found the recurrence $(-C - nC)f_{n,k} + (C - kC + Cn)f_{n+1,k} = 0$.

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(Manuel Kauers) guessed some recurrences for $c_{n,j}$.

Their Gröbner basis has the form (without their coefficients which are polynomials in $\mathbb{Q}[q, q^j, q^n]$ with degree ≤ 100):

$$\begin{aligned}
c_{n,j+4} &= c_{n,j} + c_{n,j+1} + c_{n,j+2} + c_{n,j+3} + c_{n+2,j} + c_{n+2,j+1} \\
c_{n+1,j+3} &= c_{n,j} + c_{n,j+1} + c_{n,j+2} + c_{n,j+3} + c_{n+1,j} + c_{n+1,j+1} + \\
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Let's have a closer look at it!

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Let's have a closer look at it! Its total size is 244MB.

The first identity (normalization)

With this implicit description of $c_{n,j}$ we are going to prove the necessary identities (there are algorithms for automatically proving such identities, and they are implemented in our package HolonomicFunctions).

How to prove $c_{n,n} = 1$ for all $n \geq 1$?

The first identity (normalization)

With this implicit description of $c_{n,j}$ we are going to prove the necessary identities (there are algorithms for automatically proving such identities, and they are implemented in our package HolonomicFunctions).

How to prove $c_{n,n} = 1$ for all $n \geq 1$?

- We find an element in the annihilating ideal of $c_{n,j}$ of the form

$$p_7 c_{n+7,j+7} = p_6 c_{n+6,j+6} + \cdots + p_1 c_{n+1,j+1} + p_0 c_{n,j}$$

  with $p_i \in \mathbb{Q}[q, q^j, q^n]$.
- Substituting $j \to n$ yields a recurrence for the diagonal sequence $c_{n,n}$.
- Show that the corresponding operator factors into $P_1 P_2$ where P_2 corresponds to $c_{n+1,n+1} = c_{n,n}$.
- Show that $c_{1,1} = \cdots = c_{7,7} = 1$.

The third identity

How to prove $$(1+q^n) - c_{n,n-1} + \sum_{j=1}^{n} c'_{n,j} = \frac{b_n}{b_{n-1}}$$

with $$c'_{n,j} = q^{n+j-1} \left(\begin{bmatrix} n+j-2 \\ n-1 \end{bmatrix}_q + q \begin{bmatrix} n+j-1 \\ n \end{bmatrix}_q \right) c_{n,j}?$$

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- Compute an annihilating ideal for $c'_{n,j}$ via closure properties.
- Find a relation in this ideal of the form

$$p_7 c'_{n+7,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

  where the $p_7, \ldots, p_0$ are rational functions in $\mathbb{Q}(q, q^n)$ and $t_{n,j}$ is a $\mathbb{Q}(q, q^j, q^n)$-linear combination of certain shifts of $c'_{n,j}$.
- Creative telescoping yields a recurrence for the sum.

Creative telescoping?

Creative telescoping?

We consider $\sum_{j=1}^{n} c'_{n,j}$ and have

$$p_7 c'_{n+7,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

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We show that $c'_{n,j} = 0$ for $j \leq 0$ and for $j > n$.
Now just sum over both sides:

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \left(p_7 c'_{n+7,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j}\right) &= \sum_{j=-\infty}^{\infty} (t_{n,j+1} - t_{n,j}) \\
\sum_{j=0}^{n+7} p_7 c'_{n+7,j} + \cdots + \sum_{j=0}^{n+1} p_1 c'_{n+1,j} + \sum_{j=0}^{n} p_0 c'_{n,j} &= 0 \\
p_7 \sum_{j=0}^{n+7} c'_{n+7,j} + \cdots + p_1 \sum_{j=0}^{n+1} c'_{n+1,j} + p_0 \sum_{j=0}^{n} c'_{n,j} &= 0
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- Compute an annihilating ideal for $c'_{n,j}$ via closure properties.
- Find a relation (certificate!) in this ideal of the form

$$p_7 c'_{n+7,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

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- Creative telescoping yields a recurrence for the sum.
- Closure properties yield a recurrence for the left-hand side.
- Recurrence for right-hand side is a right factor.
- Compare finitely many initial values.

A computational challenge

How to find the certificate

$$p_7 c'_{n+7,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(n,j)c'_{n+3,j} + \cdots + r_9(n,j)c'_{n,j+1} + r_{10}(n,j)c'_{n,j}$?

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- The default algorithm by  (Frédéric Chyzak) leads to a coupled first-order parametrized linear system of difference equations. Unfeasible in this instance!
- Refined ansatz with $r_i(n,j) = \dfrac{s_{i,43}(q^j)^{43} + \cdots + s_{i,1}q^j + s_{i,0}}{d_i(n,j)}$ where the denominators d_i can be "guessed", and $s_{i,m} \in \mathbb{Q}(q,q^n)$. Leads to a linear system over $\mathbb{Q}(q,q^n)$.

A computational challenge

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Homomorphic images (modular computations):

- plug in concrete integral values for q and q^n
- do all computations modulo some prime, e.g., 2147483629

Doing the computation for sufficiently many values and primes allows to reconstruct the solution. We need:

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- each case takes about a minute (after lots of optimizations)
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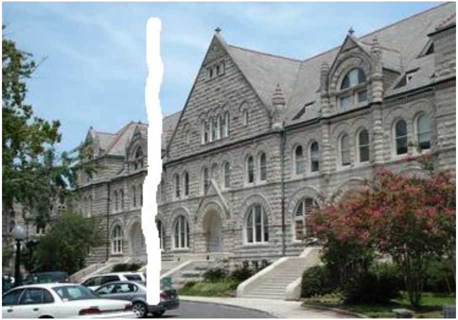
Applying further tricks reduced the actual time to 35 days.

A computational challenge

The certificate for the third identity has size 7GB (with factored coefficients):



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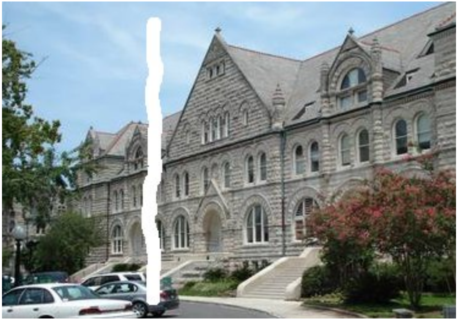

Further technical problems:

- check singularities of leading coefficients
- initial values
- cleaning ladies

Quod erat demonstrandum.

THEOREM. Let π/S_3 denote the set of orbits of a plane partition π under the action of the symmetric group S_3. Then the orbit-counting generating function is given by

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \le i \le j \le k \le n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

where $T(n)$ denotes the set of totally symmetric plane partitions with largest part at most n.