Symbolic Computation in Knot Theory

Christoph Koutschan
(joint work with Stavros Garoufalidis)

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

27 August 2014
Sozopol, Bulgaria

Overview

Knot Theory

- AJ Conjecture
  - **A**-polynomial
  - Colored **J**ones polynomial

Computer Algebra

- Guessing
- Symbolic Summation
  - Holonomic Systems Approach
  - Creative Telescoping
- Factorization of q-shift operators

Computer algebra matters for knot theory!

Overview

Knot Theory

- AJ Conjecture
  - **A**-polynomial
  - Colored **J**ones polynomial

Computer Algebra

- Guessing
- Symbolic Summation
  - Holonomic Systems Approach
  - Creative Telescoping
- Factorization of q-shift operators

Computer algebra matters for knot theory!

…and for many other things!

Basics of knot theory

Knot:

- embedding of the circle S^1 in S^3 (or in Euclidean space $\mathbb{R}^3$)
- "knotted (closed) string"
- oriented or non-oriented

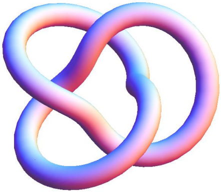
Basics of knot theory

Knot:

- embedding of the circle S^1 in S^3 (or in Euclidean space $\mathbb{R}^3$)
- "knotted (closed) string"
- oriented or non-oriented

Examples:

- unknot: ◯
- trefoil knot 3_1:



Basics of knot theory

Knot:

- embedding of the circle S^1 in S^3 (or in Euclidean space $\mathbb{R}^3$)
- "knotted (closed) string"
- oriented or non-oriented

Equivalence of knots:

- equivalence relation: ambient isotopy
- "two knots are the same if they can be transformed into each other without cutting the string"

Basics of knot theory

Link:

- disjoint union of one or several knots ("components")
- may be entangled with each other
- equivalence is defined as for knots

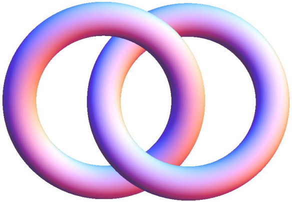
Basics of knot theory

Link:

- disjoint union of one or several knots ("components")
- may be entangled with each other
- equivalence is defined as for knots

Examples:

- unlink: 
- Hopf link:



Basics of knot theory

Tame knot:

- polygonal knot: union of non-intersecting line segments
- exists a projection with finitely many crossings
- from now on: consider only tame knots

Basics of knot theory

Tame knot:

- polygonal knot: union of non-intersecting line segments
- exists a projection with finitely many crossings
- from now on: consider only tame knots

Wild knot:

- no projection with finitely many crossings exists

Basics of knot theory

Tame knot:

- polygonal knot: union of non-intersecting line segments
- exists a projection with finitely many crossings
- from now on: consider only tame knots

Wild knot:

- no projection with finitely many crossings exists

Knot diagram:

- obtained by projecting the knot into a plane
- planar graph with over-/underpass information at the vertices

Basics of knot theory

Fundamental problem:

Determine whether two descriptions (e.g., knot diagrams) represent the same knot.

Basics of knot theory

Fundamental problem:

Determine whether two descriptions (e.g., knot diagrams) represent the same knot.

Knot invariants:

- combinatorial invariants
- knot polynomials
- quantum invariants

Basics of knot theory

Fundamental problem:

Determine whether two descriptions (e.g., knot diagrams) represent the same knot.

Knot invariants:

- combinatorial invariants
- knot polynomials
- quantum invariants

Knot polynomials:

- Alexander polynomial (1928)
- Jones polynomial (1984)
- A-polynomial
- HOMFLY polynomial

The A-polynomial

The A-polynomial $A_K(M, L)$ of a knot K parametrizes the affine variety of $\mathrm{SL}(2, \mathbb{C})$ representations of the knot complement, viewed from the boundary torus:

- $M_K := S^3$ minus a tubular neighborhood of K ("knot complement")
- character variety: $X_{M_K} = \mathrm{Hom}(\pi_1(M_K), \mathrm{SL}(2, \mathbb{C}))$
- boundary: $X_{\partial(M_K)} = \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$
- consider the restriction map $\phi : X_{M_K} \to X_{\partial(M_K)}$
- its image is defined by a bivariate polynomial, $A_K(M, L)$
- difficult to compute (e.g., using elimination)
- even unknown for some knots with 9 Xings.

Example: trefoil

A finite presentation of the fundamental group of the trefoil knot is given by:

$$\pi_1(S^3 \setminus 3_1) = \langle a, b \mid aabbb \rangle.$$

Study the $\mathrm{SL}(2, \mathbb{C})$ representations:

$$a \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} =: A \qquad \text{(w.l.o.g.)}$$

$$b \to \begin{pmatrix} v & w \\ x & y \end{pmatrix} =: B \quad \text{with } \det B = 1$$

Example: trefoil

There are two distinguished elements in $\pi_1(S^3 \setminus K)$, the meridian μ and the longitude λ, which live on the boundary torus.

For the trefoil knot we get:

$$\mu = bab$$
$$\lambda = ba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}ab$$

Example: trefoil

Impose the following conditions:

$$\operatorname{tr}\left(\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} - \mathcal{M}\right) = \operatorname{tr}\left(\begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} - \Lambda\right) = 0$$

where

$$\mathcal{M} = BAB,$$
$$\Lambda = BA^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}AB^{-1}A^{-1}B^{-1}AB.$$

Example: trefoil

Putting things together, we have to consider the ideal

$$\Big\langle vy - wx - 1, AABBB - \mathrm{Id}_2, \\ M + M^{-1} - \mathrm{tr}(\mathcal{M}), L + L^{-1} - \mathrm{tr}(\Lambda) \Big\rangle$$

and intersect it with $\mathbb{Q}[M, L]$, e.g., by Gröbner basis elimination.

In this case, we obtain $A_{3_1}(M, L) = L + M^6$.

The Jones polynomial

Skein relation:

- a means to define polynomial invariants
- three-term relation connecting the polynomials of knots which differ only locally:

The Jones polynomial

Skein relation:

- a means to define polynomial invariants
- three-term relation connecting the polynomials of knots which differ only locally:

Definition. The skein relation for the Jones polynomial $J(K)$ is

$$q^{-1}J(L_+) - qJ(L_-) = (q^{1/2} - q^{-1/2})J(L_0)$$

Initial condition: $J(\bigcirc) = 1$.

The colored Jones function

The colored Jones function $J_{K,n}(q)$ of a knot K is a generalization of the classical Jones polynomial.
It is a sequence of Laurent polynomials:

$$J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}}.$$

It can be defined using the n-th parallels of K:

$$J_{K,n}(q) = \sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} J(K^{(k)})$$

where $J(K^{(k)})$ denotes the Jones polynomial of $K^{(k)}$, the k-th parallel of K.

The colored Jones function

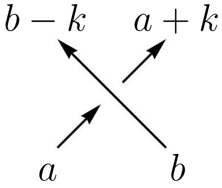
Alternative definition via state sums using an oriented diagram of K:

- label the m crossings with variables $\boldsymbol{k} = k_1, \ldots, k_m$
- label the arcs

The colored Jones function

Alternative definition via state sums using an oriented diagram of K:

- label the m crossings with variables $\boldsymbol{k} = k_1, \dots, k_m$
- label the arcs: at a left-hand crossing k_i
  - add k_i to the label $a(\boldsymbol{k})$ of the underpass
  - subtract k_i from the label $b(\boldsymbol{k})$ of the overpass



The colored Jones function

Alternative definition via state sums using an oriented diagram of K:

- label the m crossings with variables $\boldsymbol{k} = k_1, \dots, k_m$
- label the arcs
- associate to each crossing k_i a proper q-hypergeometric expression R_i

$$R_i(n, \boldsymbol{k}) = q^{-n/2 - a(\boldsymbol{k})(n + k_i - b(\boldsymbol{k}))} \left(q^{a(\boldsymbol{k}) - n}; q\right)_{k_i} \begin{bmatrix} b(\boldsymbol{k}) \\ k_i \end{bmatrix}_q$$

- the colored Jones function of K is given by an m-fold sum:

$$J_{K,n}(q) = \sum_{0 \leq \boldsymbol{k} \leq n} R_1 \cdots R_m$$

q-calculus

Recall some notation from q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$$

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

$$[n]! = \prod_{k=1}^{n}[k]$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}$$

q-calculus

Recall some notation from q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

$$[n] = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$$

$$[n]! = \prod_{k=1}^{n}[k]$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}$$

$\longrightarrow$ All these terms are (proper) q-hypergeometric.

Wilf-Zeilberger theory

Theorem. ("fundamental theorem of WZ theory")
Every (multi-) sum over a proper q-hypergeometric term is q-holonomic.

Wilf-Zeilberger theory

Theorem. ("fundamental theorem of WZ theory")
Every (multi-) sum over a proper q-hypergeometric term is q-holonomic.

$\longrightarrow$ The colored Jones function is a q-holonomic sequence.

q-holonomic sequences

Notation.

- $\mathbb{K}$: field of characteristic zero
- q: indeterminate, transcendental over $\mathbb{K}$

Definition.

A univariate sequence $\big(f_n(q)\big)_{n\in\mathbb{N}}$ is called q-holonomic if it satisfies a nontrivial linear recurrence with coefficients that are polynomials in q and q^n:

$$\sum_{j=0}^{d} c_j(q, q^n) f_{n+j}(q) = 0 \qquad (n \in \mathbb{N})$$

where the $c_j(x, y) \in \mathbb{K}[x, y]$ are bivariate polynomials for $j = 0, \ldots, d$ with $c_d(x, y) \neq 0$.

The noncommutative A-polynomial

Notation.

Introduce operator notation:

$$(Lf)_n(q) = f_{n+1}(q), \qquad (Mf)_n(q) = q^n f_n(q)$$

and let

$$\mathbb{O} = \mathbb{K}(q, M)\langle L\rangle/(LM - qML).$$

The noncommutative A-polynomial

Notation.

Introduce operator notation:

$$(Lf)_n(q) = f_{n+1}(q), \qquad (Mf)_n(q) = q^n f_n(q)$$

and let

$$\mathbb{O} = \mathbb{K}(q, M)\langle L \rangle / (LM - qML).$$

Definition.

The noncommutative A-polynomial $A_K(q, M, L) \in \mathbb{O}$ of a knot K is the minimal-order operator (denominator- and content-free) that annihilates $J_{K,n}(q)$.

The AJ conjecture

There is a close relation between the A-polynomial $A_K(M, L)$ and the annihilator $A_K(q, M, L)$ of the colored Jones function:

AJ Conjecture:

For every knot K the following identity holds:

$$A_K(1, M, L) = \text{poly}(M) \cdot A_K(M^{1/2}, L).$$

The AJ conjecture

There is a close relation between the A-polynomial $A_K(M, L)$ and the annihilator $A_K(q, M, L)$ of the colored Jones function:

AJ Conjecture:

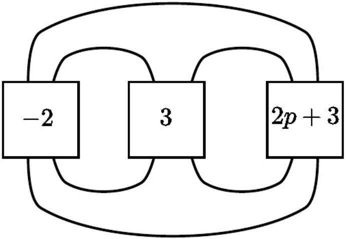
For every knot K the following identity holds:

$$A_K(1, M, L) = \mathrm{poly}(M) \cdot A_K(M^{1/2}, L).$$

The AJ conjecture has been verified (rigorously / non-rigorously) for some knots with few crossings, by explicit computations, as well as for some special families of knots.

Pretzel knots

Consider 1-parameter family of pretzel knots $K_p = (-2, 3, 2p+3)$:



Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by “guessing”:

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.
2. For the recurrence equation make an ansatz:

$$A(n) = \sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j}(q) q^{jn} J_{K,n+i}(q)$$

with undetermined coefficients $c_{i,j} \in \mathbb{K}(q)$.

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.
2. For the recurrence equation make an ansatz:

$$A(n) = \sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j}(q) q^{jn} J_{K,n+i}(q)$$

   with undetermined coefficients $c_{i,j} \in \mathbb{K}(q)$.
3. Solve the linear system $A(1) = \cdots = A(N-r) = 0$ for the $c_{i,j}$.

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.
2. For the recurrence equation make an ansatz:

$$A(n) = \sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j}(q) q^{jn} J_{K,n+i}(q)$$

   with undetermined coefficients $c_{i,j} \in \mathbb{K}(q)$.
3. Solve the linear system $A(1) = \cdots = A(N-r) = 0$ for the $c_{i,j}$.
4. If there is a solution for $N - r \geq (r+1)(d+1)$, then this is a very plausible candidate.

Palindromicity

We say that an operator P is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

Palindromicity

We say that an operator P is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

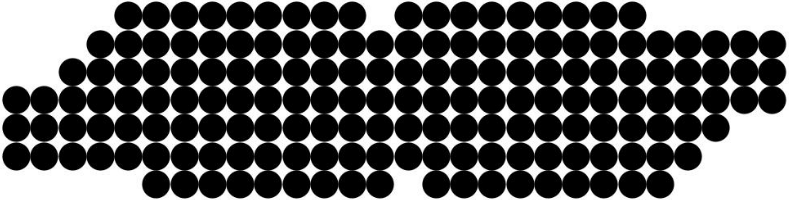
where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

Palindromicity

We say that an operator P is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

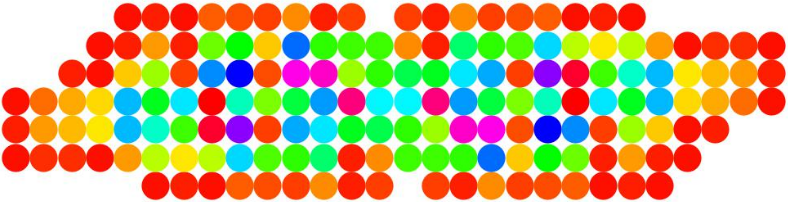


Palindromicity

We say that an operator P is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.



Palindromicity

We say that an operator P is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

Palindromicity implies that this operator has some palindromic bi-infinite sequences $f_n(q), n \in \mathbb{Z}$ as solutions, i.e., either $f_n(q) = f_{-n}(q)$ for all integers n, or $f_n(q) = -f_{-n}(q)$ for all integers n.

Palindromicity

We say that an operator P is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

Palindromicity implies that this operator has some palindromic bi-infinite sequences $f_n(q), n \in \mathbb{Z}$ as solutions, i.e., either $f_n(q) = f_{-n}(q)$ for all integers n, or $f_n(q) = -f_{-n}(q)$ for all integers n.

$\longrightarrow$ All operators here are palindromic!

Guessed recurrences

p	L-degree	M-degree	q-degree	largest cf.
-5	12	125	946	3.0×10^8
-4	9	66	392	12345
-3	6	27	85	33
-2	3	12	19	4
-1	1	6	3	1
0	2	13	13	2
1	2	16	16	2
2	6	58	233	6
3	9	114	514	118
4	12	191	1151	386444
5	15	288	2174	2.2×10^{11}

Verification of AJ conjecture

1. The A-polynomials of $K_{-5}, \ldots, K_5$ were known.
2. Compute the $q = 1$ images of the guessed recurrence operators.
3. The results are in accordance with the AJ conjecture.
4. Assuming that the guessed operators are correct, how can we know that they are of minimal order?

Verification of AJ conjecture

1. The A-polynomials of $K_{-5}, \ldots, K_5$ were known.
2. Compute the $q = 1$ images of the guessed recurrence operators.
3. The results are in accordance with the AJ conjecture.
4. Assuming that the guessed operators are correct, how can we know that they are of minimal order?
5. Try to show irreducibility, which implies minimality.

An easy sufficient criterion for irreducibility

Consider

$$A(q, M, L) = \sum_{j=0}^{d} a_j(q, M) L^j \in \mathbb{O}$$

with $d > 1$ and assume

- $A(1, M, L) \in \mathbb{K}(M)[L]$ is well-defined,
- irreducible,
- and $a_0(1, M) a_d(1, M) \neq 0$.

Then $A(q, M, L)$ is irreducible in $\mathbb{O}$.

An easy sufficient criterion for irreducibility

Consider

$$A(q,M,L) = \sum_{j=0}^{d} a_j(q,M)L^j \in \mathbb{O}$$

with $d > 1$ and assume

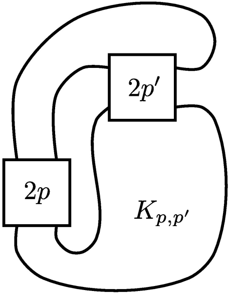
- $A(1,M,L) \in \mathbb{K}(M)[L]$ is well-defined,
- irreducible,
- and $a_0(1,M)a_d(1,M) \neq 0$.

Then $A(q,M,L)$ is irreducible in $\mathbb{O}$.

$\longrightarrow$ Most of the guessed operators are irreducible by this criterion and therefore of minimal order.

Double twist knots

Consider the family of double twist knots $K_{p,p'}$:



$\longrightarrow$ Interesting family because their A-polynomials are reducible.

Colored Jones function of $K_{p,p'}$

Using the Habiro theory of the colored Jones function, we get for $J_{K_{p,p'},n}(q)$ the following expression:

$$\sum_{k=0}^{n-1}(-1)^k c_{p,k}(q)c_{p',k}(q)q^{-kn-\frac{k(k+3)}{2}}(q^{n-1};q^{-1})_k(q^{n+1};q)_k$$

where the sequence $c_{p,n}(q)$ is defined by

$$\sum_{k=0}^{n}(-1)^{k+n}q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2p}\frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

Colored Jones function of $K_{p,p'}$

Using the Habiro theory of the colored Jones function, we get for $J_{K_{p,p'},n}(q)$ the following expression:

$$\sum_{k=0}^{n-1}(-1)^k c_{p,k}(q)c_{p',k}(q)q^{-kn-\frac{k(k+3)}{2}}(q^{n-1};q^{-1})_k(q^{n+1};q)_k$$

where the sequence $c_{p,n}(q)$ is defined by

$$\sum_{k=0}^{n}(-1)^{k+n}q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2p}\frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

$\longrightarrow$ Apply Wilf-Zeilberger Theory!

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ (which is 7_4).

Result:

- inhomogeneous recurrence of order 5
- M-degree 24 and q-degree 65
- corresponds to 4 printed pages

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ (which is 7_4).

Result:

- inhomogeneous recurrence of order 5
- M-degree 24 and q-degree 65
- corresponds to 4 printed pages

Problem:

Creative telescoping doesn't necessarily give the minimal-order recurrence (same problem as before).

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ (which is 7_4).

Result:

- inhomogeneous recurrence of order 5
- M-degree 24 and q-degree 65
- corresponds to 4 printed pages

Problem:

Creative telescoping doesn't necessarily give the minimal-order recurrence (same problem as before).

Strategy:

Again, we would like to show that the corresponding operator is irreducible.

How to show irreducibility?

Unfortunately, we cannot apply the previous criterion, since $A(1, M, L)$ now is reducible.

- $A(1, M, L)$ predicts orders of possible factors.
- Proceed by investigating right factors of order k.
- Compute the k-th exterior power $\bigwedge^k P$.
- Verify that $\bigwedge^k P$ has no linear right factors.
- This can be done by the algorithm qHyper.
- It follows that P has no right factor of order k.

Results for double twist knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order

Results for double twist knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order

$K_{3,3}$:

- rigorous computation of $A(q, M, L)$
- (q, M, L)-degree $= (458, 74, 11)$
- minimality proof out of scope

Results for double twist knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order

$K_{3,3}$:

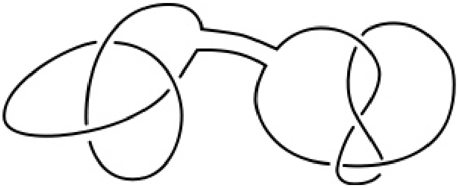
- rigorous computation of $A(q, M, L)$
- (q, M, L)-degree $= (458, 74, 11)$
- minimality proof out of scope

$K_{4,4}$, $K_{5,5}$:

- $A(q, M, L)$ guessed
- (q, M, L)-degree $= (2045, 184, 19)$ resp. $(6922, 396, 29)$ (ByteCount = 8GB)

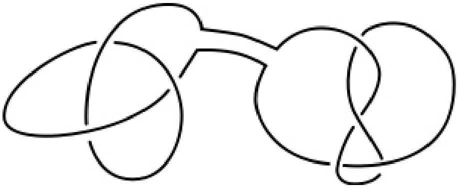
Connected sum of knots

Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



Connected sum of knots

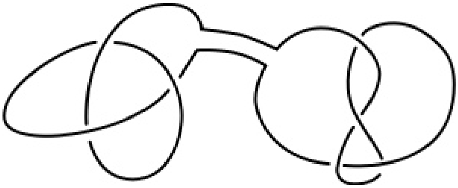
Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



- A knot is irreducible if it cannot be written as connected sum of two nontrivial knots.

Connected sum of knots

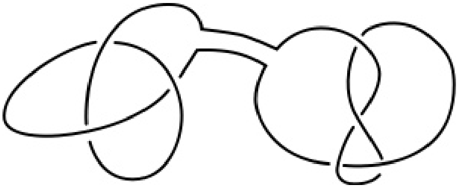
Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



- A knot is irreducible if it cannot be written as connected sum of two nontrivial knots.
- Each knot has a "unique factorization".

Connected sum of knots

Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



- A knot is irreducible if it cannot be written as connected sum of two nontrivial knots.
- Each knot has a "unique factorization".
- Rolfsen's table contains only irreducible knots.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the symmetric product $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the symmetric product $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Remark 1: P is unique up to multiplication by elements from $\mathbb{K}(q, M) \setminus \{0\}$.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the symmetric product $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Remark 1: P is unique up to multiplication by elements from $\mathbb{K}(q, M) \setminus \{0\}$.

Remark 2: The definition does not imply that the symmetric product gives the shortest recurrence for the product of two sequences.

Colored Jones for connected sum of knots

Fact: Let K_1 and K_2 be two knots in 3-space. Then the colored Jones function of their connected sum is given by

$$J_{K_1\#K_2,n}(q) = J_{K_1,n}(q)\,J_{K_2,n}(q) \quad \text{for all } n \in \mathbb{N}.$$

$\longrightarrow$ Like for the classical Jones polynomial.

Colored Jones for connected sum of knots

Fact: Let K_1 and K_2 be two knots in 3-space. Then the colored Jones function of their connected sum is given by

$$J_{K_1\#K_2,n}(q) = J_{K_1,n}(q)\,J_{K_2,n}(q) \quad \text{for all } n \in \mathbb{N}.$$

$\longrightarrow$ Like for the classical Jones polynomial.

Corollary: Let K_1 and K_2 be two knots and let $P_1, P_2 \in \mathbb{O}$ be annihilating operators of their colored Jones functions, respectively. Then the symmetric product $P_1 \star P_2$ annihilates $J_{K_1\#K_2,n}(q)$.

A-polynomial for connected sums

Definition.

For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the "A-product" $A_1 \diamond A_2$:

A-polynomial for connected sums

Definition.

For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the "A-product" $A_1 \diamond A_2$:

- Let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal $\langle A_1(M, L_1), A_2(M, L_2), L - L_1 L_2 \rangle$.

A-polynomial for connected sums

Definition.

For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the "A-product" $A_1 \diamond A_2$:

- Let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal $\langle A_1(M, L_1), A_2(M, L_2), L - L_1 L_2 \rangle$.
- Then $A_1 \diamond A_2$ is defined to be the generator of the elimination ideal $I \cap \mathbb{K}(M)[L]$.

A-polynomial for connected sums

Definition.
For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the "A-product" $A_1 \diamond A_2$:

- Let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal $\langle A_1(M, L_1), A_2(M, L_2), L - L_1 L_2 \rangle$.
- Then $A_1 \diamond A_2$ is defined to be the generator of the elimination ideal $I \cap \mathbb{K}(M)[L]$.

Fact: Let K_1 and K_2 be two knots and $A_1(M, L)$ and $A_2(M, L)$ their respective A-polynomials. Then the A-polynomial of $K_1 \# K_2$ is given by $A_1 \diamond A_2$.

Theorem

Notation: We introduce the map ψ by

$$\psi\colon \mathbb{O} \to \mathbb{K}(M)[L],\ P(q,M,L) \mapsto P(1,M,L).$$

Theorem

Notation: We introduce the map ψ by

$$\psi\colon \mathbb{O} \to \mathbb{K}(M)[L],\ P(q,M,L) \mapsto P(1,M,L).$$

Theorem.
Let $P_1(q,M,L)$ and $P_2(q,M,L)$ be two operators in the algebra $\mathbb{O}$. Then the following divisibility condition holds:

$$\psi(P_1) \diamond \psi(P_2) \mid \psi(P_1 \star P_2)$$

as polynomials in $\mathbb{K}(M)[L]$, provided that the above quantities are defined.

Results for connected sums

Consider connected sums of 3_1 and 4_1:

- $3_1\#3_1$: $\deg_L(P) = 2$, reducible into $1 + 1$
- $3_1\#4_1$: $\deg_L(P) = 5$, reducible into $2 + 1 + 2$ and $1 + 2 + 2$
- $4_1\#4_1$: $\deg_L(P) = 5$, reducible into $2 + 3$

$\longrightarrow$ In all cases the operators are reducible.

$\longrightarrow$ Nevertheless, in all cases they are minimal, but this requires a full factorization of the operator.

$\longrightarrow$ All these results confirm the AJ conjecture.

Appendix:

Introduction to Holonomic Functions

Holonomic systems approach

1. Functions and sequences are represented by their annihilating left ideals (and initial values).
2. Holonomic functions are closed under certain operations, e.g., addition, multiplication, but **not** division.
3. An annihilating ideal is given by its Gröbner basis (i.e., a finite set of generators that allows to decide ideal membership and equality of ideals).
4. Integrals and sums are treated by the method of creative telescoping.
5. The output is always given as an annihilating ideal, not as a closed form.

Definitions

1. The **annihilator** of a function f w.r.t. an Ore algebra $\mathbb{O}$:

$$\operatorname{ann}_{\mathbb{O}}(f) = \left\{ P \in \mathbb{O} \mid P \bullet f = 0 \right\}$$

Definitions

1. The **annihilator** of a function f w.r.t. an Ore algebra $\mathbb{O}$:

$$\operatorname{ann}_{\mathbb{O}}(f) = \left\{ P \in \mathbb{O} \mid P \bullet f = 0 \right\}$$

2. A function is called **∂-finite w.r.t. $\mathbb{O}$** ("holonomic") if

$$\dim_{\mathbb{K}(\boldsymbol{v})} \left(\mathbb{O} / \operatorname{ann}_{\mathbb{O}}(f) \right) < \infty$$

$\left(\operatorname{ann}_{\mathbb{O}}(f)\right.$ is a zero-dimensional left ideal in $\left.\mathbb{O}\right)$

Definitions

1. The **annihilator** of a function f w.r.t. an Ore algebra $\mathbb{O}$:

$$\operatorname{ann}_{\mathbb{O}}(f) = \left\{P \in \mathbb{O} \mid P \bullet f = 0\right\}$$

2. A function is called **$\boldsymbol{\partial}$-finite w.r.t. $\mathbb{O}$** ("holonomic") if

$$\dim_{\mathbb{K}(\boldsymbol{v})}\left(\mathbb{O}/\operatorname{ann}_{\mathbb{O}}(f)\right) < \infty$$

$\big($ $\operatorname{ann}_{\mathbb{O}}(f)$ is a zero-dimensional left ideal in $\mathbb{O}$ $\big)$

3. The **holonomic rank** of a ∂-finite function f is the integer

$$r = \dim_{\mathbb{K}(\boldsymbol{v})}\left(\mathbb{O}/\operatorname{ann}_{\mathbb{O}}(f)\right)$$

Definitions

1. The **annihilator** of a function f w.r.t. an Ore algebra $\mathbb{O}$:

$$\operatorname{ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} \mid P \bullet f = 0\}$$

2. A function is called **∂-finite w.r.t. $\mathbb{O}$** ("holonomic") if

$$\dim_{\mathbb{K}(\boldsymbol{v})} \left(\mathbb{O} / \operatorname{ann}_{\mathbb{O}}(f)\right) < \infty$$

($\operatorname{ann}_{\mathbb{O}}(f)$ is a zero-dimensional left ideal in $\mathbb{O}$)

3. The **holonomic rank** of a ∂-finite function f is the integer

$$r = \dim_{\mathbb{K}(\boldsymbol{v})} \left(\mathbb{O} / \operatorname{ann}_{\mathbb{O}}(f)\right)$$

4. The definitions **∂-finite** and **holonomic** differ only by some technical conditions.

Example: Legendre polynomials $P_n(x)$

Important family of orthogonal polynomials $P_0(x), P_1(x), \ldots$:

$$\deg(P_n(x)) = n, \qquad \text{and} \qquad \frac{2n+1}{2}\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \delta_{m,n}.$$

Example: Legendre polynomials $P_n(x)$

Important family of orthogonal polynomials $P_0(x), P_1(x), \ldots$:

$$\deg(P_n(x)) = n, \qquad \text{and} \qquad \frac{2n+1}{2} \int_{-1}^{1} P_m(x) P_n(x) \, \mathrm{d}x = \delta_{m,n}.$$

They are a particular solution of the Legendre differential equation:

$$(x^2 - 1) P_n''(x) + 2x P_n'(x) - n(n+1) P_n(x) = 0.$$

Corresponding operator: $(x^2 - 1) D_x^2 + 2x D_x - n(n+1)$.

Example: Legendre polynomials $P_n(x)$

Important family of orthogonal polynomials $P_0(x), P_1(x), \ldots$:

$$\deg(P_n(x)) = n, \qquad \text{and} \qquad \frac{2n+1}{2}\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \delta_{m,n}.$$

They are a particular solution of the Legendre differential equation:

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Corresponding operator: $(x^2-1)D_x^2 + 2xD_x - n(n+1)$.

Legendre polynomials also satisfy the three-term recurrence

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Corresponding operator: $(n+2)S_n^2 - (2n+3)xS_n + (n+1)$.

Example: Legendre polynomials $P_n(x)$

Important family of orthogonal polynomials $P_0(x), P_1(x), \ldots$:

$$\deg(P_n(x)) = n, \qquad \text{and} \qquad \frac{2n+1}{2}\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \delta_{m,n}.$$

They are a particular solution of the Legendre differential equation:

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Corresponding operator: $(x^2-1)D_x^2 + 2xD_x - n(n+1)$.

Legendre polynomials also satisfy the three-term recurrence

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Corresponding operator: $(n+2)S_n^2 - (2n+3)xS_n + (n+1)$.

These operators live in the Ore algebra $\mathbb{K}(x,n)\langle D_x, S_n\rangle$.

(Incomplete) list of ∂-finite functions

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, BesselI, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, BesselJ, StruveL, ArcSec, Factorial2, KelvinBer, BesselK, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin,

Creative telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Creative telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$.

Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

Creative telescoping

Method for doing integrals and sums

(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$.

Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

Creative Telescoping: write

$$c_r(n) f(n+r,k) + \cdots + c_0(n) f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_r(n) F(n+r) + \cdots + c_0(n) F(n) = g(n,b+1) - g(n,a).$$

Creative telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following integration problem: $F(x) = \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: write $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right)\mathrm{d}y \qquad = g(x,b) - g(x,a)$.

Creative Telescoping: write

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} f(x,y) + \cdots + c_0(x) f(x,y) \;=\; \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} F(x) + \cdots + c_0(x) F(x) \;=\; g(x,b) - g(x,a)$$

The right-hand side

$$\begin{aligned} c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1) \cdot g(n,k). \end{aligned}$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_r(n)f(n+r,i) + \cdots + c_0(n)f(n,i)\big)$$

The right-hand side

$$\begin{aligned} c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1)\cdot g(n,k). \end{aligned}$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_r(n)f(n+r,i) + \cdots + c_0(n)f(n,i)\big)$$

A reasonable choice for where to look for g is $\mathbb{O} \cdot f$.

Then the task is to find $P(n,S_n) = c_r(n)S_n^r + \cdots + c_0(n)$ and $Q \in \mathbb{O}$ such that

$$\big(P - (S_k - 1)Q\big)\cdot f = 0 \quad \Longleftrightarrow \quad P - (S_k - 1)Q \in \operatorname{ann}_{\mathbb{O}}(f).$$

$\longrightarrow$ There are algorithms and implementations for that.

Computer proof of a special function identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t$$

Computer proof of a special function identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t$$

```
<< RISC`HolonomicFunctions`
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
  {S[a],S[n],Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

Computer proof of a special function identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t$$

```
<< RISC`HolonomicFunctions`
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
  {S[a],S[n],Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*
  BesselJ[a,2*Sqrt[t*x]], Der[t], {S[a],S[n],Der[x]}]
```

$$\{\{-2S_n + 2xD_x + (a+2n+2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},$$
$$\{-2t, -4tx, -2tx\}\}$$

$\longrightarrow$ The annihilating ideals agree; check a few initial values.

Examples: more special function identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{1}$$

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{(m+\frac{1}{2}, -m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}} (a+1)^{m+\frac{1}{2}}} \tag{2}$$

$$e^{-x} x^{a/2} n! L_n^a(x) = \int_0^\infty e^{-t} t^{\frac{a}{2}+n} J_a\left(2\sqrt{tx}\right) \mathrm{d}t \tag{3}$$

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x) H_n(x) r^m s^n e^{-x^2}}{m! n!} \, \mathrm{d}x = \sqrt{\pi} e^{2rs} \tag{4}$$

$$\int_{-1}^{1} \left(1 - x^2\right)^{\nu - \frac{1}{2}} e^{iax} C_n^{(\nu)}(x) \, \mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu) J_{n+\nu}(a)}{2^{\nu-1} a^\nu n! \Gamma(\nu)} \tag{5}$$

$$\frac{\sin\left(\sqrt{z^2 + 2tz}\right)}{z} = \sum_{n=0}^{\infty} \frac{(-t)^n y_{n-1}(z)}{n!} \tag{6}$$