Proof of Ira Gessel’s Lattice Path Conjecture

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Gessel walks

- walks in the integer lattice $\mathbb{N}^2$
- start at $(0, 0)$
- do not leave $\mathbb{N}^2$
- only certain steps are allowed:

$$G := \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{\leftarrow, \rightarrow, \downarrow, \uparrow\}$$
Gessel walks — Example
Definition

Let $f(n; i, j)$ denote the number of walks ("Gessel walks")
- in the integer lattice $\mathbb{N}^2$
- with exactly $n$ steps
- starting at the origin $(0, 0)$
- ending at the point $(i, j)$
- using only steps from $G$. 
Ira Gessel (2001) conjectured that

$$f(n; 0, 0) = \begin{cases} 
16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\
0 & \text{if } n \text{ is odd}
\end{cases}$$

The function $f(n; 0, 0)$ counts the number of closed Gessel walks.
Get ready for the proof!

Need: relations (linear recurrences with polynomial coefficients) for $f(n; i, j)$ The step set $\{\leftarrow, \rightarrow, \uparrow, \downarrow\}$ gives readily rise to the recurrence

$$f(n + 1; i, j) =$$

$$f(n; i + 1, j) + f(n; i - 1, j) + f(n; i + 1, j + 1) + f(n; i - 1, j - 1)$$
More recurrences

**Question:** How to find more such recurrences?

**Answer:** With guessing!

- ansatz with unspecified coefficients
- plug in small values for $n, i, j$
- solve the corresponding linear system

**Remark:** We have to prove that the guessed recurrences are indeed correct!
Ore operators

The recurrence

\[ f(n + 1; i, j) = f(n; i + 1, j) + f(n; i - 1, j) \]
\[ + f(n; i + 1, j + 1) + f(n; i - 1, j - 1) \]

translates to the annihilating Ore operator

\[ S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1 \]

in the Ore algebra \( \mathbb{O} = \mathbb{Q}(i, j, n)[S_i; S_i, 0][S_j; S_j, 0][S_n; S_n, 0] \).

\((S_i\) denotes the shift operator w.r.t. \(i\), i.e.,

\[ S_i \cdot f(n; i, j) = f(n; i + 1, j). \]

In \( \mathbb{O} \) it does not commute with \(i\), namely \( S_i i = (i + 1) S_i \).\]
Our guessing resulted in a set $A$ of 68 operators:

$$A = \{ S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1, (i + 1)(i - 2(j - 3n - 20)(i - 2j - n - 12)S_i S_n^3 S_j^4 - 2(i - 2j - 7)(2i - 4j - 3n - 26)(i - 2j - n - 12)S_n^2 S_j^4 - 32(i - 2j - 7)(i - 2j - 3n - 13)(n + 1)S_j^4 + 16(i + 1)(i - 2j - n - 7)S_n^4 S_j^3 + (i + 1)(11i^2 - 12ji - 4ni - 36i + 12j^2 + 21n^2 + 104j + 8jn + 204n + 596)S_i S_n S_j^4 - (i - n - 4)(i - 2j - n - 12)(i - j - n - 7)S_n^4 S_j^3 + (i + 1)(11i^2 - 12ji - 4ni - 36i + 12j^2 + 21n^2 + 104j + 8jn + 204n + 596)S_i S_n S_j^3 - 4(6i^3 - 24ji^2 + 2ni^2 - 70i^2 + 32j^2i - 9n^2i + 256j + 16j + 9ni + 19ni + 478i - 16j^3 + 8n^3 - 176j^2 + 6jn^2 + 93n^2 - 544j + 58jn + 451n - 126)S_n^2 S_j^3 - 64(n + 1)(2i^2 - 8ji - 3ni - 30i + 8j^2 - 4n^2 + 60j + 6jn + 3n + 96)S_j^3 + 16(i + 1)(3i^2 - 12ji - 4ni - 42i + 12j^2 - 21n^2 + 84j + 8jn - 66n + 51)S_i S_n S_j^3 - 4(4i^3 - 16ji^2 + 33ni^2 + 38i^2 + 16j^2i - 36n^2i + 56ji - 20j + 7ni - 154ni + 8i + 16n^3 + 24j^2 + 90n^2 + 120j + 20j^2n + 100jn + 379n + 494)S_n^2 S_j^2 - 64(n + 1)(3i^2 - 12ji - 30i + 12j^2 - 8n^2 + 60j - 30n + 51)S_j^2 + 16(i + 1)(3i^2 - 12ji + 4ni - 18i + 12j^2 - 21n^2 + 36j + 8jn - 106n - 69)S_i S_n S_j^2 + (i - n - 4)(j - n - 2)(i - 2j + n + 2)S_n^4 S_j + (i + 1)(i - 2j + n + 2)(i - 2j + 3n + 10)S_i S_n^3 S_j + 4(2i^3 - 8ji^2 - 18ni^2 - 50i^2 + 16j^2i + 3n^2i + 64ji + 16jni + 3ni - 14i - 16j^3 - 8n^3 - 64j^2 + 6jn^2 - 63n^2 + 16j + 58jn - 161n - 194)S_n^2 S_j - 64(n + 1)(2i^2 - 8ji + 3ni - 10i + 8j^2 - 4n^2 + 20j - 6jn - 27n - 4)S_j + 16(i + 1)(i^2 - 4ji + 4ni + 2i + 4j^2 - 3n^2 - 4j - 8jn - 26n - 31)S_i S_n S_j + 2(i - 2j - 3)(i - 2j + n + 2)(2i - 4j + 3n + 6)S_n^2 - 32(i - 2j - 3)(n + 1)(i - 2j + 3n + 3), \ldots \} \)
Zeilberger’s quasi-holonomic ansatz

**Note:** The operators in $A$ generate a left ideal, namely $\mathbb{O}\langle A \rangle$, all of whose elements are annihilating $f(n; i, j)$. **Idea:** Find an operator $R \in \mathbb{O}\langle A \rangle$ of the form

$$R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j)$$

- $R(n, i, j, S_n, S_i, S_j)$ annihilates $f(n; i, j)$
- set $i = j = 0$
- $P(n, S_n)$ annihilates $f(n; 0, 0)$

**Problem:** $R(n, i, j, S_n, S_i, S_j)$ is too big to be computed.
Takayama enters the game

\[ R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j) \]

We use Takayama’s trick:

1. substitute \( i \rightarrow 0 \) and \( j \rightarrow 0 \) for all operators \( T \in A \)
2. eliminate \( S_i \) and \( S_j \)

Remark: The result will be \( P(n, S_n) \) as above, but \( Q_1 \) and \( Q_2 \) are not computed at all.

\[ \rightarrow \] Computation becomes feasible!
How to eliminate?

**Problem:** After setting $i = 0$, no multiplication by $S_i$ is allowed!

**Example:**

\[
\begin{align*}
&P + iQ \\
\cdot S_i & \quad \quad \quad \quad i = 0 \\
S_iP + (i + 1)S_iQ & \quad \quad \quad \quad S_iP + S_iQ \\
\neq & \quad \quad \quad \quad \neq \\
\end{align*}
\]
Let $A = \{A_1, \ldots, A_m\}$ be a set of annihilating operators.

1. $d_i := \max_{1 \leq k \leq m} \deg_{S_i} A_k$

2. set $A := A \cup \bigcup_{k=1}^{m} \{S_i^\alpha A_k \mid 1 \leq \alpha \leq d_i - \deg_{S_i} A_k\}$

3. do the same for $j$

4. $A := A|_{i \rightarrow 0, j \rightarrow 0}$

5. translate the elements of $A$ to vectors w.r.t. the basis
$\{S_i^\alpha S_j^\beta \mid 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\}$
A variant of Takayama’s algorithm

Let \( A = \{A_1, \ldots, A_m\} \) be a set of annihilating operators.

1. \( d_i := \max_{1 \leq k \leq m} \deg_{S_i} A_k \)
2. set \( A := A \cup \bigcup_{k=1}^{m} \{S_i^\alpha A_k \mid 1 \leq \alpha \leq d_i - \deg_{S_i} A_k\} \)
3. do the same for \( j \)
4. \( A := A|_{i \rightarrow 0, j \rightarrow 0} \)
5. translate the elements of \( A \) to vectors w.r.t. the basis \( \{S_i^\alpha S_j^\beta \mid 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\} \), e.g.,
   \( S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1 \) translates to
   \((-1, -1, 0, 0, S_n, 0, 0, -1, -1)\)
6. compute a Gröbner basis of \( A \) in this module
7. if no \((P(n, S_n), 0, \ldots, 0)\) is found, increase \( d \)
Result

The operator $P(n, S_n)$ annihilating $f(n; 0, 0)$ has

- order 32
- polynomial coefficients of degree 172
- and integer coefficients up to 385 digits.

The computation was done with CK’s implementation of noncommutative Gröbner bases and Takayama, and took 7 hours.
Make the proof rigorous!

Verify that $P(n, S_n)$ also annihilates $g(n; 0, 0)$ for

$$g(n; 0, 0) := \begin{cases} 
16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\
0 & \text{if } n \text{ is odd}
\end{cases}$$

Compare initial values, i.e., $f(n; 0, 0) = g(n; 0, 0)$ for $0 \leq n \leq 31$. Make sure that the leading coefficient of $P(n, S_n)$ (and all contents that have been cancelled out during the computation) does not have positive integer roots (≡ poles).
Doron Zeilberger’s bet

“I offer a prize of one hundred (100) US-dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel’s conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel’s conjecture is indeed beyond the scope of humankind.”