Proof of Ira Gessel's Lattice Path Conjecture

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(joint work with M. Kauers and D. Zeilberger)

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Gessel walks

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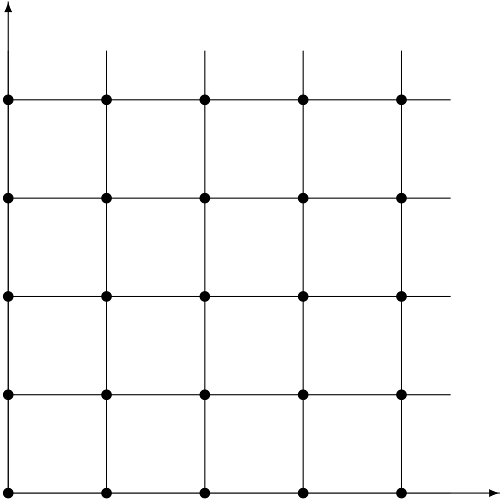
- only certain steps are allowed:

$$\begin{aligned} G &:= \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \\ &= \{\leftarrow, \rightarrow, \swarrow, \nearrow\} \end{aligned}$$

Gessel walks — Example



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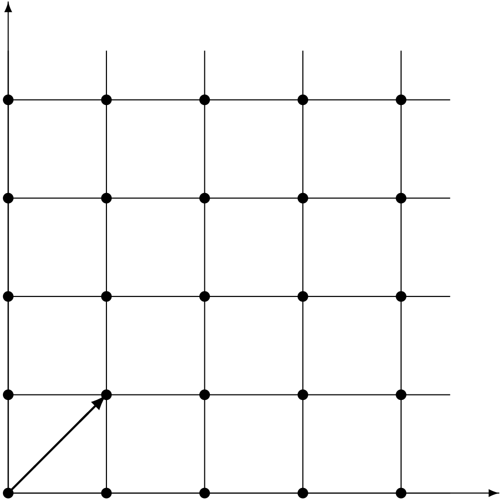

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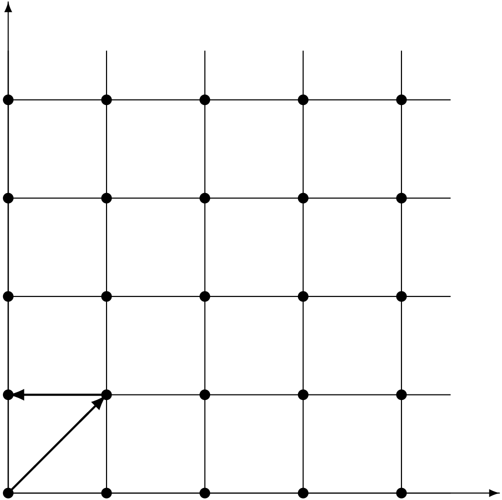

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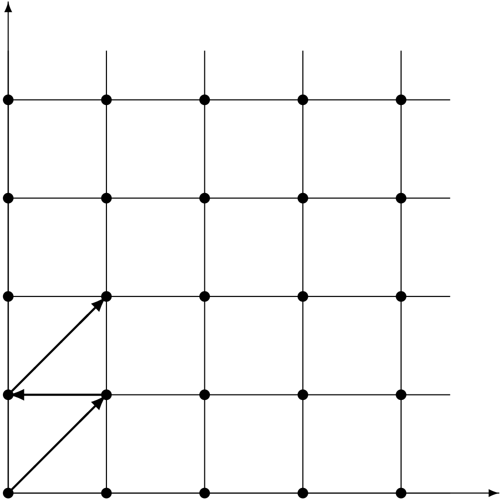

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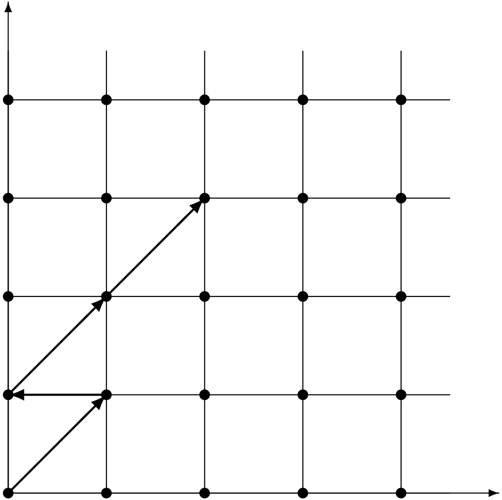

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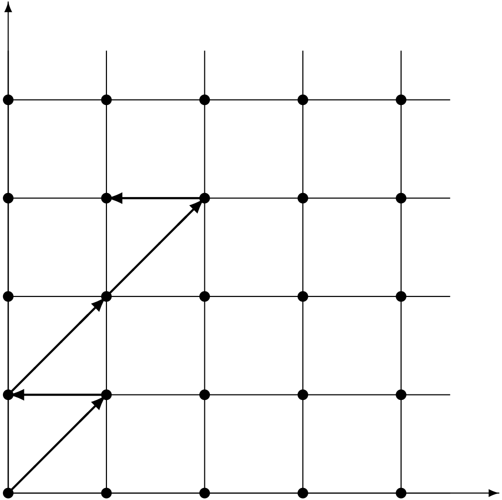

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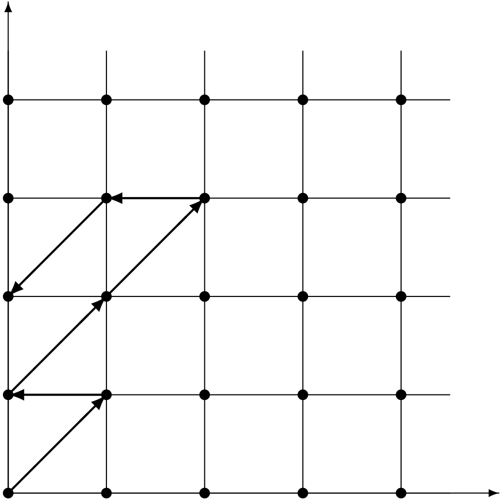

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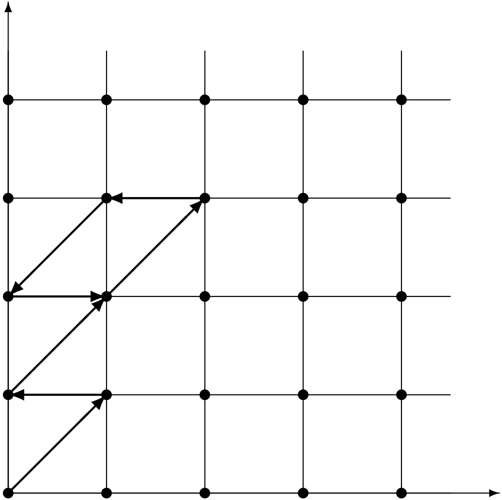

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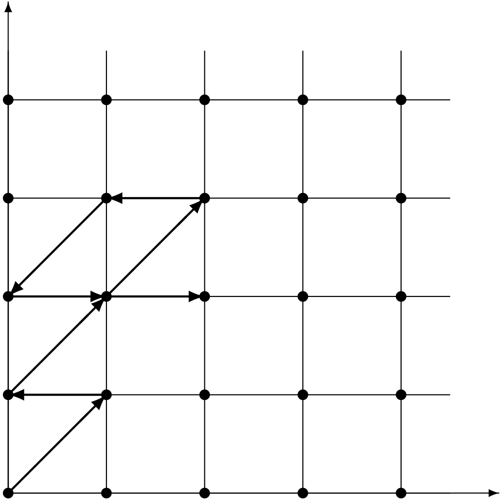

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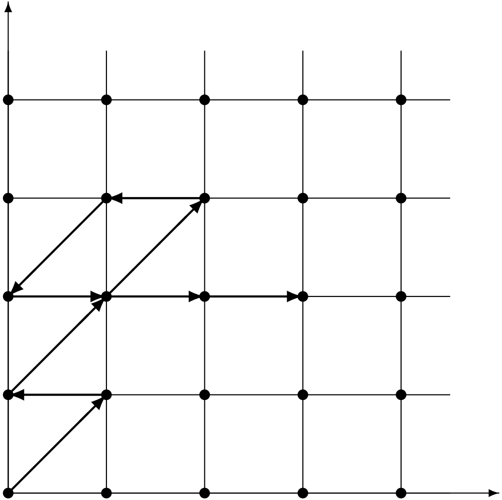

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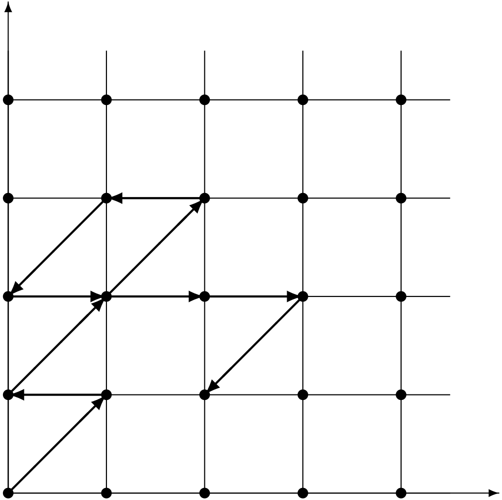

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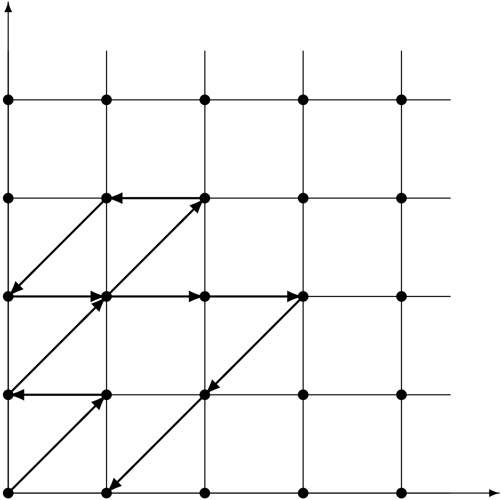

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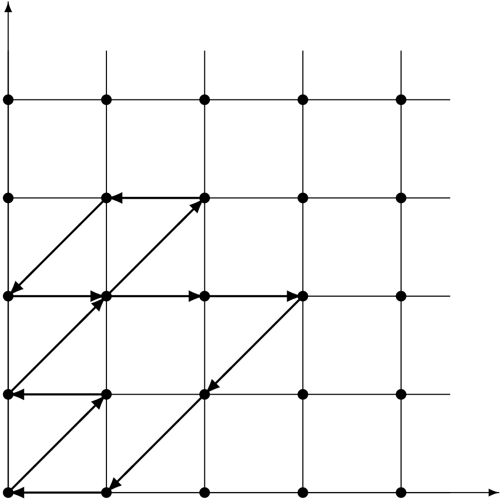

Definition

Let $f(n;i,j)$ denote the number of Gessel walks

- with exactly n steps
- starting at the origin $(0,0)$
- ending at the point (i,j)

Ira Gessel's conjecture

Ira Gessel in 2001 conjectured that

$$f(n;0,0) = \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

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The function $f(n;0,0)$ counts the number of closed Gessel walks.

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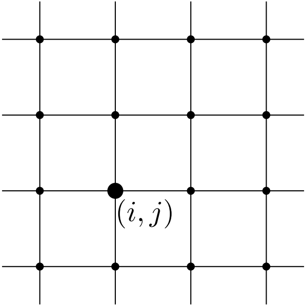

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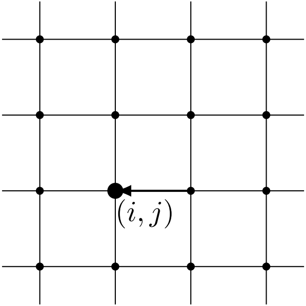

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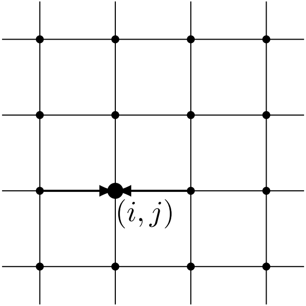

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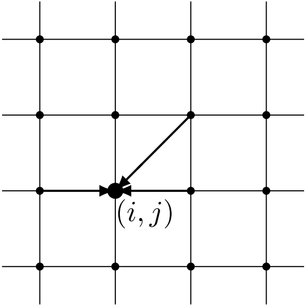

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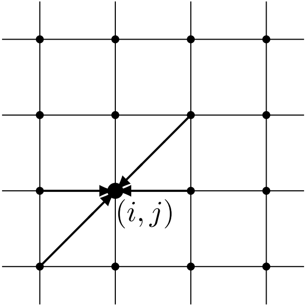

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Remark: We have to prove that the guessed recurrences are indeed correct!

Ore operators

The recurrence

$$\begin{aligned} f(n+1;i,j) &= f(n;i+1,j) + f(n;i-1,j) \\ &\quad + f(n;i+1,j+1) + f(n;i-1,j-1) \end{aligned}$$

translates to the annihilating Ore operator

$$S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$$

in the Ore algebra $\mathbb{O} = \mathbb{Q}(i,j,n)[S_i; S_i, 0][S_j; S_j, 0][S_n; S_n, 0]$.

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$\Big($ S_i denotes the shift operator w.r.t. i, i.e.,

$$S_i \bullet f(n;i,j) = f(n;i+1,j).$$

In $\mathbb{O}$ it does not commute with i, namely $S_i \cdot i = (i+1)\cdot S_i$. $\Big)$

Our guessing resulted in a set A of 68 operators:

$$
A = \{S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1, (i+1)(i-2j-3n-20)(i-2j-n-12) S_i S_n^3 S_j^4 - 2(i-2j-
7)(2i-4j-3n-26)(i-2j-n-12) S_n^2 S_j^4 - 32(i-2j-7)(i-2j-3n-13)(n+1) S_j^4 + 16(i+
1)(i^2-4ji-4ni-22i+4j^2-3n^2+44j+8jn+14n+89) S_i S_n S_j^4 - (i-n-4)(i-2j-n-12)(i-
j-n-7) S_n^4 S_j^3 + (i+1)(11i^2-12ji-4ni-36i+12j^2+21n^2+104j+8jn+204n+596) S_i S_n^3 S_j^3 -
4(6i^3-24ji^2+2ni^2-70i^2+32j^2 i-9n^2 i+256ji+16jni+19ni+478i-16j^3+8n^3-176j^2+
6jn^2+93n^2-544j+58jn+451n-126) S_n^2 S_j^3 - 64(n+1)(2i^2-8ji-3ni-30i+8j^2-4n^2+60j+
6jn+3n+96) S_j^3 + 16(i+1)(3i^2-12ji-4ni-42i+12j^2-21n^2+84j+8jn-66n+51) S_i S_n S_j^3 -
(i-n-4)(5i^2-4ji-7ni-29i+4j^2+2n^2+20j+5n+16) S_n^4 S_j^2 + (i+1)(11i^2-12ji+4ni+8i+
12j^2+21n^2+16j-8jn+164n+376) S_i S_n^3 S_j^2 - 4(4i^3-16ji^2+33ni^2+38i^2+16j^2 i-36n^2 i+
56ji-20jni-154ni+8i+16n^3+24j^2+90n^2+120j+20j^2 n+100jn+379n+494) S_n^2 S_j^2 - 64(n+
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21n^2+36j-8jn-106n-69) S_i S_n S_j^2 + (i-n-4)(j-n-2)(i-2j+n+2) S_n^4 S_j + (i+1)(i-2j+
n+2)(i-2j+3n+10) S_i S_n^3 S_j + 4(2i^3-8ji^2-18ni^2-50i^2+16j^2 i+3n^2 i+64ji+16jni+3ni-
14i-16j^3-8n^3-64j^2+6jn^2-63n^2+16j+58jn-161n-194) S_n^2 S_j - 64(n+1)(2i^2-8ji+3ni-
10i+8j^2-4n^2+20j-6jn-27n-4) S_j + 16(i+1)(i^2-4ji+4ni+2i+4j^2-3n^2-4j-8jn-26n-
31) S_i S_n S_j + 2(i-2j-3)(i-2j+n+2)(2i-4j+3n+6) S_n^2 - 32(i-2j-3)(n+1)(i-2j+3n+3), \ldots \}
$$

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$$\begin{aligned} R(n,i,j,S_n,S_i,S_j) = P(n,S_n) \quad &+ \quad iQ_1(n,i,j,S_n,S_i,S_j) \\ &+ \quad jQ_2(n,i,j,S_n,S_i,S_j) \end{aligned}$$

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Problem: $R(n,i,j,S_n,S_i,S_j)$ is too big to be computed.

Takayama, Chyzak, and Salvy enter the game

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Remark: The result will be $P(n,S_n)$ as above, but Q_1 and Q_2 are not computed at all.

$\longrightarrow$ Computation becomes feasible!

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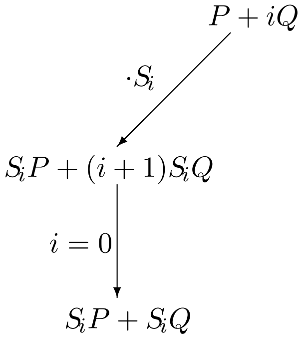

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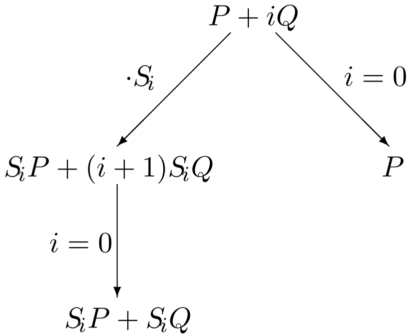

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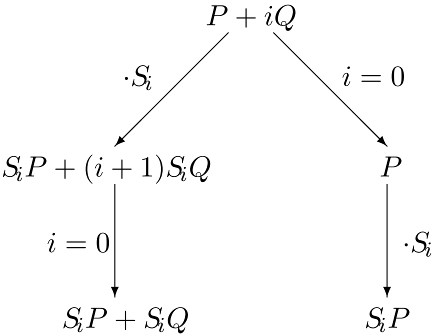

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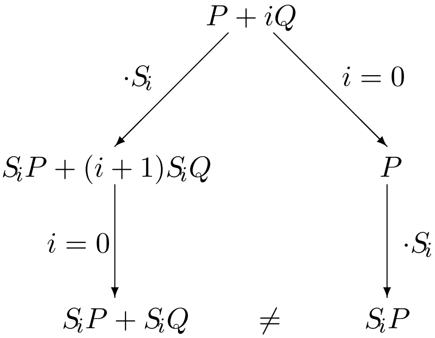

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4. $A := A|_{i \to 0, j \to 0}$
5. translate the elements of A to vectors w.r.t. the basis $\{S_i^\alpha S_j^\beta \mid 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\}$

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2. set $A := A \cup \bigcup_{k=1}^{m} \{S_i^\alpha A_k \,|\, 1 \leq \alpha \leq d_i - \deg_{S_i} A_k\}$
3. do the same for j
4. $A := A|_{i \to 0, j \to 0}$
5. translate the elements of A to vectors w.r.t. the basis $\{S_i^\alpha S_j^\beta \,|\, 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\}$, e.g., $S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$ translates to $(-1, -1, 0, 0, S_n, 0, 0, -1, -1)$

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6. compute a Gröbner basis of A in this module
7. if no $(P(n, S_n), 0, \dots, 0)$ is found, increase d_i and d_j

Result

The operator $P(n, S_n)$ annihilating $f(n; 0, 0)$ has

- order 32
- polynomial coefficients of degree 172
- and integer coefficients up to 385 digits.

The computation was done with CK's implementation of noncommutative Gröbner bases and Takayama's algorithm; it took 7 hours.

Make the proof rigorous!

Verify that $P(n, S_n)$ also annihilates $g(n; 0, 0)$ for

$$g(n;0,0) := \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Compare initial values, i.e., $f(n; 0, 0) = g(n; 0, 0)$ for $0 \leq n \leq 31$.

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Make sure that the leading coefficient of $P(n, S_n)$ (and all contents that have been cancelled out during the computation) do not have positive integer roots (= poles).

Don't forget: Prove correctness of guessed recurrences!

How to prove that $R \bullet f = R(n, i, j, S_n, S_i, S_j) \bullet f(n; i, j) = 0$?

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$$TR = UT + V$$

where $T = S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$.

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Once we know that $(TR) \bullet f = 0$, it can be algorithmically decided whether $R \bullet f = 0$.

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- Initial values are $(R \bullet f)(0; i, j)$
- $f(n; i, j) = 0$ for $i > n$ or $j > n$
- only finitely many values to check!

More conjectures, more proofs (1)

Marko Petkovšek and Herb Wilf conjectured that

$$f(2n;0,1) = 16^n \frac{\left(\frac{1}{2}\right)_n}{(3)_n} \left(\frac{\left(111n^2 + 183n - 50\right)\left(\frac{5}{6}\right)_n}{270\left(\frac{8}{3}\right)_n} + \frac{5\left(\frac{7}{6}\right)_n}{27\left(\frac{7}{3}\right)_n} \right)$$

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This conjecture is proven in the same way!

More conjectures, more proofs (2)

Marko Petkovšek and Herb Wilf conjectured that
$g(n) := f(2n+1;1,0)$ satisfies the second order recurrence

$$\begin{aligned} &(n+3)(3n+7)(3n+8)\ g(n+1) \\ -8&(2n+3)(18n^2+54n+35)\ g(n) \\ &+256n(3n+1)(3n+2)\ g(n-1) = 0 \end{aligned}$$

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This conjecture is **dis**proven in the same way!

More conjectures, more proofs (4)

In fact, $h(n) = f(2n; 2, 0)$ satisfies the recurrence

$4096(n+1)(2n+1)(2n+3)(3n+4)(3n+5)(6n+5)(6n+7)(6144n^7 + 130560n^6 + 1169216n^5 + 5718720n^4 + 16490716n^3 + 28015035n^2 + 25933899n + 10077210)h(n) - 128(2n+3)(31850496n^{13} + 1043103744n^{12} + 15528112128n^{11} + 139066675200n^{10} + 835537836288n^9 + 3554184658752n^8 + 11003992594864n^7 + 25083927328960n^6 + 42052581871616n^5 + 51138759649954n^4 + 43770815405708n^3 + 24915467579665n^2 + 8429189779675n + 1274964941250)h(n+1) + 48(n+4)(15925248n^{13} + 561364992n^{12} + 9001764864n^{11} + 86874808320n^{10} + 562452019584n^9 + 2576877461856n^8 + 8584177057392n^7 + 21020268432120n^6 + 37767656881868n^5 + 49065078284877n^4 + 44671143917844n^3 + 26891118085035n^2 + 9545234776900n + 1498120123500)h(n+2) - 8(n+4)(n+5)(3n+13)(3n+14)(442368n^{10} + 11612160n^9 + 133731840n^8 + 888142080n^7 + 3758533024n^6 + 10562908440n^5 + 19901273510n^4 + 24718969695n^3 + 19263730233n^2 + 8437822050n + 1558180800)h(n+3) + (n+4)(n+5)(n+6)(3n+13)(3n+14)(3n+16)(3n+17)(6144n^7 + 87552n^6 + 514880n^5 + 1616000n^4 + 2911836n^3 + 2992423n^2 + 1606825n + 341550)h(n+4)$

More recent results

In August 2008, Alin Bostan and Manuel Kauers proved that the trivariate generating function of $f(n; i, j)$ is not only holonomic but even algebraic!

Doron Zeilberger's bet

"I offer a prize of one hundred (100) US-dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel's conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel's conjecture is indeed beyond the scope of humankind."