Proof of Ira Gessel’s Lattice Path Conjecture

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Gessel walks

- walks in the integer lattice $\mathbb{N}^2$
Gessel walks

- walks in the integer lattice $\mathbb{N}^2$
- start at $(0, 0)$
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- do not leave $\mathbb{N}^2$
Gessel walks

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- start at $(0, 0)$
- do not leave $\mathbb{N}^2$
- only certain steps are allowed:

$$G := \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$= \{\leftarrow, \rightarrow, \downarrow, \uparrow\}$$
Gessel walks — Example
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Definition

Let $f(n; i, j)$ denote the number of Gessel walks

- with exactly $n$ steps
- starting at the origin $(0, 0)$
- ending at the point $(i, j)$
Ira Gessel in 2001 conjectured that

\[ f(n; 0, 0) = \begin{cases} 
16k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\
0 & \text{if } n \text{ is odd}
\end{cases} \]
Ira Gessel’s conjecture

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The function \( f(n; 0, 0) \) counts the number of closed Gessel walks.
Get ready for the proof!

**Need:** relations (linear recurrences with polynomial coefficients) for \( f(n; i, j) \)
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The step set $\{\leftarrow, \rightarrow, \uparrow, \downarrow\}$ gives readily rise to the recurrence

$$f(n + 1; i, j) =$$
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\[
f(n + 1; i, j) = f(n; i + 1, j)
\]
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$$f(n; i + 1, j)$$

$$+ f(n; i - 1, j)$$
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$$f(n + 1; i, j) = f(n; i + 1, j) + f(n; i - 1, j) + f(n; i + 1, j + 1)$$
Get ready for the proof!

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$$+ f(n; i - 1, j)$$

$$+ f(n; i + 1, j + 1)$$

$$+ f(n; i - 1, j - 1)$$
More recurrences

**Question:** How to find more such recurrences?

**Answer:**

• ansatz with unspecified coefficients
• plug in small values for \( n, i, j \)
• solve the corresponding linear system

**Remark:** We have to prove that the guessed recurrences are indeed correct!
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Ore operators

The recurrence

\[ f(n + 1; i, j) = f(n; i + 1, j) + f(n; i - 1, j) \]

\[ + f(n; i + 1, j + 1) + f(n; i - 1, j - 1) \]

translates to the annihilating Ore operator

\[ S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1 \]

in the Ore algebra \( \mathbb{O} = \mathbb{Q}(i, j, n)[S_i; S_i, 0][S_j; S_j, 0][S_n; S_n, 0] \).
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\( S_i \) denotes the shift operator w.r.t. \( i \), i.e.,

\[ S_i \cdot f(n; i, j) = f(n; i + 1, j). \]

In \( \mathbb{O} \) it does not commute with \( i \), namely \( S_i \cdot i = (i + 1) \cdot S_i \).
Our guessing resulted in a set $A$ of 68 operators:

$$A = \{ S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1, (i + 1)(i - 2j - 3n - 20)(i - 2j - n - 12)S_i S_n^3 S_j^4 - 2(i - 2j - 7)(2i - 4j - 3n - 26)(i - 2j - n - 12)S_n^2 S_j^4 - 32(i - 2j - 7)(i - 2j - 3n - 13)(n + 1)S_j^4 + 16(i + 1)(i^2 - 4ji - 4ni - 22i + 4j^2 - 3n^2 + 44j + 8jn + 14n + 89)S_i S_n S_j^4 - (i - n - 4)(i - 2j - n - 12)(i - j - n - 7)S_n^4 S_j^3 + (i + 1)(11i^2 - 12ji - 4ni - 36i + 12j^2 + 21n^2 + 104j + 8jn + 204n + 596)S_i S_n S_j^3 - 4(6i^3 - 24ji^2 + 2ni^2 - 70i^2 + 32j^2 i - 9n^2 i + 256ji + 16jni + 19ni + 478i - 16j^3 + 8n^3 - 176j^2 + 6jn^2 + 93n^2 - 544j + 58jn + 451n - 126)S_n^2 S_j^3 - 64(n + 1)(2i^2 - 8ji - 3ni - 30i + 8j^2 - 4n^2 + 60j + 6jn + 3n + 96)S_j^3 + 16(i + 1)(3i^2 - 12ji - 4ni - 42i + 12j^2 - 21n^2 + 84j + 8jn - 66n + 51)S_i S_n S_j^3 - (i - n - 4)(5i^2 - 4ji - 7ni - 29i + 4j^2 + 2n^2 + 20j + 5n + 16)S_n^4 S_j^2 + (i + 1)(11i^2 - 12ji + 4ni + 8i + 12j^2 + 21n^2 + 16j - 8jn + 164n + 376)S_i S_n S_j^2 - 4(4i^3 - 16ji^2 + 33ni^2 + 38i^2 + 16j^2 i - 36n^2 i + 56ji - 20jni - 154ni + 8i + 16n^3 + 24j^2 + 90n^2 + 120j + 20j^2 n + 100jn + 379n + 494)S_n^2 S_j^2 - 64(n + 1)(3i^2 - 12ji - 30i + 12j^2 - 8n^2 + 60j - 30n + 51)S_j^2 + 16(i + 1)(3i^2 - 12ji + 4ni - 18i + 12j^2 - 21n^2 + 36j - 8jn - 106n - 69)S_i S_n S_j^2 + (i - n - 4)(j - n - 2)(i - 2j + n + 2)S_n^4 S_j + (i + 1)(i - 2j + n + 2)(i - 2j + 3n + 10)S_i S_n^3 S_j + 4(2i^3 - 8ji^2 - 18ni^2 - 50i^2 + 16j^2 i + 3n^2 i + 64ji + 16jni + 3ni - 14i - 16j^3 - 8n^3 - 64j^2 + 6jn^2 - 63n^2 + 16j + 58jn - 161n - 194)S_n^2 S_j - 64(n + 1)(2i^2 - 8ji + 3ni - 10i + 8j^2 - 4n^2 + 20j - 6jn - 27n - 4)S_j + 16(i + 1)(i^2 - 4ji + 4ni + 2i + 4j^2 - 3n^2 - 4j - 8jn - 26n - 31)S_i S_n S_j + 2(i - 2j - 3)(i - 2j + n + 2)(2i - 4j + 3n + 6)S_n^2 - 32(i - 2j - 3)(n + 1)(i - 2j + 3n + 3), \ldots \}$
Note: The operators in $A$ generate a left ideal, namely $\circ\langle A \rangle$, all of whose elements annihilate $f(n; i, j)$.
Zeilberger’s quasi-holonomic ansatz

**Note:** The operators in $A$ generate a left ideal, namely $\mathfrak{o}\langle A \rangle$, all of whose elements annihilate $f(n; i, j)$.

**Idea:** Find an operator $R \in \mathfrak{o}\langle A \rangle$ of the form

$$R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j)$$

*Problem:* $R(n, i, j, S_n, S_i, S_j)$ is too big to be computed.
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1. substitute \( i \to 0 \) and \( j \to 0 \) for all operators \( T \in A \)
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We apply the following trick:

1. substitute \( i \rightarrow 0 \) and \( j \rightarrow 0 \) for all operators \( T \in A \)
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Remark: The result will be \( P(n, S_n) \) as above, but \( Q_1 \) and \( Q_2 \) are not computed at all.

\[ \rightarrow \] Computation becomes feasible!
How to eliminate?

Problem: After setting $i = 0$, no multiplication by $S_i$ is allowed!
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$$P + iQ$$
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\[
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\[
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\]  

\[
i = 0 \\
P
\]  

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- $\cdot S_i$
- $i = 0$

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- $i = 0$
- $\neq$

$$S_i P + S_i Q$$

$$P$$

- $\cdot S_i$

$$S_i P$$
A variant of a variant of Takayama’s algorithm

Let $A = \{A_1, \ldots, A_m\}$ be a set of annihilating operators.
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5. translate the elements of $A$ to vectors w.r.t. the basis $
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   $S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$ translates to
   $(-1, -1, 0, 0, S_n, 0, 0, -1, -1)$
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   \]
6. compute a Gröbner basis of $A$ in this module
7. if no $(P(n, S_n), 0, \ldots, 0)$ is found, increase $d_i$ and $d_j$
The operator $P(n, S_n)$ annihilating $f(n; 0, 0)$ has

- order 32
- polynomial coefficients of degree 172
- and integer coefficients up to 385 digits.

The computation was done with CK’s implementation of noncommutative Gröbner bases and Takayama’s algorithm; it took 7 hours.
Make the proof rigorous!

Verify that $P(n, S_n)$ also annihilates $g(n; 0, 0)$ for

$$g(n; 0, 0) := \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Compare initial values, i.e., $f(n; 0, 0) = g(n; 0, 0)$ for $0 \leq n \leq 31$. 
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Compare initial values, i.e., $f(n; 0, 0) = g(n; 0, 0)$ for $0 \leq n \leq 31$.

Make sure that the leading coefficient of $P(n, S_n)$ (and all contents that have been cancelled out during the computation) do not have positive integer roots (≡ poles).
Don’t forget: Prove correctness of guessed recurrences!

How to prove that $R \cdot f = R(n, i, j, S_n, S_i, S_j) \cdot f(n; i, j) = 0$?
Don’t forget: Prove correctness of guessed recurrences!

How to prove that \( R \cdot f = R(n, i, j, S_n, S_i, S_j) \cdot f(n; i, j) = 0 \)?

By division with remainder computation we get

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TR = UT + V
\]

where \( T = S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1 \).
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Since \((UT) \cdot f = 0\) for sure, we reduced the problem: We have to show that \( V \cdot f = 0 \) (which is of smaller degree in \( n, i, j \)).
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Since $(UT) \cdot f = 0$ for sure, we reduced the problem: We have to show that $V \cdot f = 0$ (which is of smaller degree in $n, i, j$).

Once we know that $(TR) \cdot f = 0$, it can be algorithmically decided whether $R \cdot f = 0$. 
Prove correctness of guessed recurrences

How to decide whether $R \cdot f = 0$, provided that $(TR) \cdot f = 0$?
Prove correctness of guessed recurrences

How to decide whether $R \cdot f = 0$, provided that $(TR) \cdot f = 0$?

- $R \cdot f$ satisfies the recurrence $T$
Prove correctness of guessed recurrences

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- only finitely many values to check!
Marko Petkovšek and Herb Wilf conjectured that

\[ f(2n; 0, 1) = 16^n \frac{\left(\frac{1}{2}\right)_n}{(3)_n} \left( \frac{(111n^2 + 183n - 50) \left(\frac{5}{6}\right)_n}{270 \left(\frac{8}{3}\right)_n} + \frac{5 \left(\frac{7}{6}\right)_n}{27 \left(\frac{7}{3}\right)_n} \right) \]
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This conjecture is proven in the same way!
Marko Petkovšek and Herb Wilf conjectured that
\[ g(n) := f(2n + 1; 1, 0) \]
satisfies the second order recurrence
\[
(n + 3)(3n + 7)(3n + 8)g(n + 1) - 8(2n + 3)(18n^2 + 54n + 35)g(n) + 256n(3n + 1)(3n + 2)g(n - 1) = 0
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This conjecture is disproven in the same way!
In fact, \( h(n) = f(2n; 2, 0) \) satisfies the recurrence

\[
4096(n + 1)(2n + 1)(2n + 3)(3n + 4)(3n + 5)(6n + 5)(6n + 7)(6144n^7 + 130560n^6 + 1169216n^5 + 5718720n^4 + 16490716n^3 + 28015035n^2 + 25933899n + 10077210)h(n) - 128(2n + 3)(31850496n^{13} + 1043103744n^{12} + 15528112128n^{11} + 139066675200n^{10} + 835537836288n^9 + 3554184658752n^8 + 11003992594864n^7 + 25083927328960n^6 + 42052581871616n^5 + 51138759649954n^4 + 43770815405708n^3 + 24915467579665n^2 + 8429189779675n + 1274964941250)h(n + 1) + 48(n + 4)(15925248n^{13} + 561364992n^{12} + 9001764864n^{11} + 86874808320n^{10} + 562452019584n^9 + 2576877461856n^8 + 8584177057392n^7 + 21020268432120n^6 + 37767656881868n^5 + 49065078284877n^4 + 44671143917844n^3 + 2689118085035n^2 + 9545234776900n + 1498120123500)h(n + 2) - 8(n + 4)(n + 5)(3n + 13)(3n + 14)(3n + 15)h(n + 3) + (n + 4)(n + 5)(n + 6)(3n + 13)(3n + 14)(3n + 16)(3n + 17)(6144n^7 + 87552n^6 + 514880n^5 + 1616000n^4 + 2911836n^3 + 2992423n^2 + 1606825n + 341550)h(n + 4).
\]
More recent results

In August 2008, Alin Bostan and Manuel Kauers proved that the trivariate generating function of $f(n; i, j)$ is not only holonomic but even algebraic!
Doron Zeilberger’s bet

“I offer a prize of one hundred (100) US-dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel’s conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel’s conjecture is indeed beyond the scope of humankind.”