Computer-Algebra-Based MIMO Performance Analysis

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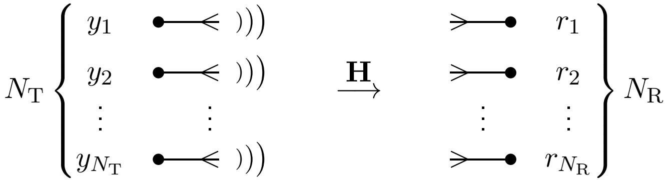
MIMO Wireless Communication Systems

MIMO = Multiple Input + Multiple Output:



Notation:

- N_{T}: number of transmitting antennas
- N_{R}: number of receiving antennas
- $\mathbf{y} = (y_1, y_2, \ldots, y_{N_{\mathrm{T}}})^{\mathcal{T}} \in \mathbb{C}^{N_{\mathrm{T}}}$: transmitted signal vector
- $\mathbf{H}$: the $N_{\mathrm{R}} \times N_{\mathrm{T}}$ channel matrix
- $\mathbf{r} = (r_1, r_2, \ldots, r_{N_{\mathrm{R}}})^{\mathcal{T}} = \mathbf{H}\mathbf{y} + \mathbf{n}$: received signal vector, where $\mathbf{n}$ is some additive zero-mean Gaussian noise

Channel Matrix

The channel matrix is modeled as a complex-valued Gaussian random matrix, written as

$$\mathbf{H} = \mathbf{H}_{\mathrm{d}} + \mathbf{H}_{\mathrm{r}}$$

where $\mathbf{H}_{\mathrm{d}}$ denotes the deterministic component ("mean") and $\mathbf{H}_{\mathrm{r}}$ the random component.

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Fading:

- Rayleigh fading, i.e., $\mathbf{H}_{\mathrm{d}} = 0$ (previous work)
- Rician fading, i.e., $\mathbf{H}_{\mathrm{d}} \neq 0$ (current work)

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For sake of simplicity (not w.l.o.g.!), certain assumptions on $\mathbf{H}$:

- $\mathbf{H}_{\mathrm{d}}$ has rank 1
- further assumptions (zero row correlation, etc.)

Zero-Forcing Detection

Recall:

$$\mathbf{r} = \mathbf{H}\mathbf{y} + \mathbf{n}.$$

Zero-Forcing means finding the (modulation constellation) symbols closest to each element of vector

$$\left(\mathbf{H}^{\mathcal{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathcal{H}}\mathbf{r} = \mathbf{y} + \left(\mathbf{H}^{\mathcal{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathcal{H}}\mathbf{n}.$$

Goal of the analysis: say something about the quality of the connection, i.e., how many symbols are transmitted correctly in average.

The following parameters will be used:

- $N = N_{\mathrm{R}} - N_{\mathrm{T}} + 1$
- x_1, x_2: related to $\|\mathbf{H}_{\mathrm{d}}\|^2 / \mathbb{E}\{\|\mathbf{H}_{\mathrm{r}}\|^2\}$
- Γ_1: related to the additive noise

Signal-to-Noise Ratio (SNR)

The SNR is the ultimate performance measure (determines the quality of the connection).

Theorem. The moment generating function $M(s; x_1, x_2)$ of the SNR for zero-forcing under full-Rician fading with $r = 1$ is

$$M(s; x_1, x_2) = \frac{e^{-x_2}}{(1-\Gamma_1 s)^N} \sum_{n_2=0}^{\infty} \frac{x_2^{n_2}}{n_2!} \, {}_1F_1\!\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s x_1}{1-\Gamma_1 s}\right).$$

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Definition. The **hypergeometric function** ${}_1F_1$ is defined by

$${}_1F_1(a; b; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \quad \text{where}$$

$$(a)_k := a \cdot (a+1) \cdots (a+k-1), \quad (a)_0 := 1$$

is the **Pochhammer symbol** (or rising factorial).

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$$e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2+N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1}}{(1-s\Gamma_1)^{N+n_1-m_1}}.$$

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Obtain the SNR probability density function by Laplace transform:

$$\frac{1}{(1-s\Gamma_1)^{N+n_1-m_1}}\quad\overset{\text{Laplace}}{\longleftrightarrow}\quad\frac{t^{N+n_1-m_1-1}e^{-t/\Gamma_1}}{(N+n_1-m_1-1)!\,\Gamma_1^{N+n_1-m_1}}$$

SNR Probability Density Function

Thus we obtain for the SNR probability density function $p(t; x_1, x_2)$:

$$\begin{aligned} p(t; x_1, x_2) &= \int_0^\infty e^{-st} M(s; x_1, x_2)\,\mathrm{d}s \\ &= e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2 + N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \\ &\qquad \times \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1} t^{N+n_1-m_1-1} e^{-t/\Gamma_1}}{(N+n_1-m_1-1)!\,\Gamma_1^{N+n_1-m_1}}. \end{aligned}$$

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Definition. Using this, we define the main object of interest, the **outage probability** $P_\mathrm{o}(x_1, x_2)$:

$$P_\mathrm{o}(x_1, x_2) = \int_0^\tau p(t; x_1, x_2)\,\mathrm{d}t$$

where τ is a certain prescribed SNR threshold.

Evaluate

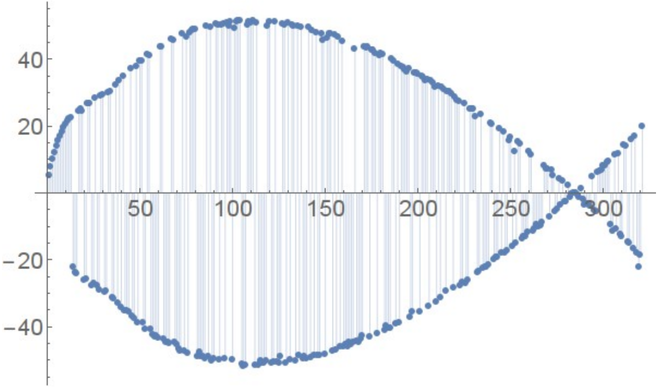
Now, for certain choices of the parameters $N_{\mathrm{R}}, N, x_1, x_2, \Gamma_1, \tau$, we want to "compute" (i.e., evaluate numerically) the outage probability.

First try: truncate the infinite series

$$P_{\mathrm{o}}(x_1, x_2) = e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2 + N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \times \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1} \gamma(N + n_1 - m_1, \tau/\Gamma_1)}{(N + n_1 - m_1 - 1)!}$$

$\longrightarrow$ Problem: slow convergence.

Difficulties in the Evaluation



- Accuracy problems with standard floating-point arithmetic.

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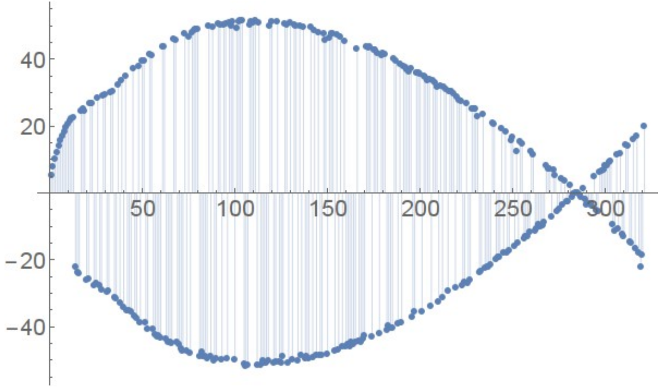

- Accuracy problems with standard floating-point arithmetic.
- Use arbitrary-precision in a computer algebra system.
  But this makes computations even slower.

Holonomic Gradient Method (HGM)

$\longrightarrow$ Methods for evaluating and optimizing certain expressions.
(Nakayama, Nishiyama, Noro, Ohara, Sei, Takayama, Takemura)

Input: $f(x_1, \ldots, x_s)$ holonomic, $(a_1, \ldots, a_s) \in \mathbb{R}^s$
Output: an approximation of $f(a_1, \ldots, a_s)$

1. Determine a holonomic system (set of differential equations) to which f is a solution, and let r be its holonomic rank.
2. Determine a suitable "basis" of derivatives $\mathbf{f} = \left(f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}\right)$ of $f(x_1, \ldots, x_s)$.
3. Convert the holonomic system into a set of Pfaffian systems, i.e., $\frac{\mathrm{d}}{\mathrm{d}x_i}\mathbf{f} = \mathbf{A}_i\mathbf{f}$ for each x_i.
4. Compute $f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}$ at a suitably chosen point $(b_1, \ldots, b_s) \in \mathbb{R}^s$, for which this is easy to achieve.
5. Use your favourite numerical integration procedure (e.g., Euler, Runge-Kutta) to obtain $\mathbf{f}(a_1, \ldots, a_s)$.

Holonomic Functions

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$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_s=0}^{\infty} f(n_1, \ldots, n_s)\, x_1^{n_1} \cdots x_s^{n_s}$$

is holonomic in the above sense.

$\longrightarrow$ Definition applies also to the mixed case.

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$\longrightarrow$ Definition applies also to the mixed case.

Fact. Holonomic functions are closed under addition, multiplication, (certain) substitutions, taking sums and integrals.

Example

The function $f(x) = {}_1F_1(a;b;x)$ is holonomic in x since it satisfies the differential equation

$$xf''(x) + (b-x)f'(x) - af(x) = 0.$$

In operator notation:

$$P(f) = 0 \quad \text{with} \quad P = xD_x^2 + (b-x)D_x - a.$$

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Likewise, ${}_1F_1(a; b; x)$ is holonomic w.r.t. the discrete variables a and b since

$$G(y, z) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} {}_1F_1(a; b; x) y^a z^b$$

is holonomic in y and z.

Closure Properties (Example)

We have seen that the following expression is holonomic:

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$$ {}_1F_1\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s}{1 - \Gamma_1 s x_1}\right) $$

Substitution $a \to N, b \to n_2 + N_{\mathrm{R}}, x \to \dfrac{\Gamma_1 s}{1 - \Gamma_1 s x_1}$

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$\dfrac{x_2^{n_2}}{n_2!}$ is holonomic (the generating function is $e^{x_2 y}$).

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Multiplication

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Summation

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e^{-x_2} is holonomic.

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$(1-\Gamma_1 s)^N$ is holonomic.

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Division

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$(1-\Gamma_1 s)^{-N}$ is holonomic as well!

Closure Properties (Example)

We have seen that the following expression is holonomic:

$$M(s;x_1,x_2) = \frac{e^{-x_2}}{(1-\Gamma_1 s)^N} \sum_{n_2=0}^{\infty} \frac{x_2^{n_2}}{n_2!} \, {}_1F_1\!\left(N; n_2+N_{\mathrm{R}}; \frac{\Gamma_1 s}{1-\Gamma_1 s x_1}\right)$$

Hence, by inspection, our SNR moment generating function is holonomic. Likewise, $p(t;x_1,x_2)$ and $P_{\mathrm{o}}(x_1,x_2)$ are holonomic.

Annihilator and Gröbner Bases

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Example. The annihilator of ${}_1F_1(a;b;x)$ in $\mathbb{C}(a,b,x)[S_a,S_b,D_x]$ is generated by the three operators

$$\begin{aligned}&(b-a)S_b + bD_x - b,\\&aS_a - xD_x - a,\\&xD_x^2 + (b-x)D_x - a.\end{aligned}$$

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Nota bene. We use (left) Gröbner bases to deal with annihilators.

Pfaffian Systems

Fix $f(x_1, \ldots, x_s)$.

A suitable "basis of derivatives" $\mathbf{f} = \left(f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}\right)$ for HGM step 2 is given by the (finite!) list of monomials that are irreducible modulo the annihilator ideal.

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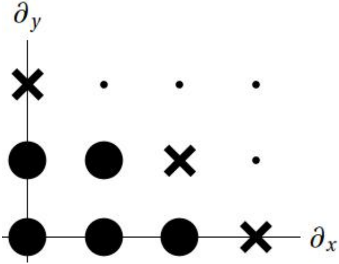

"monomials under the staircase" $(r = 5)$

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The **Pfaffian system** (given by the matrix $\mathbf{A}_i$) for x_i

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\mathbf{f} = \mathbf{A}_i\mathbf{f}$$

is obtained by reduction with the Gröbner basis.

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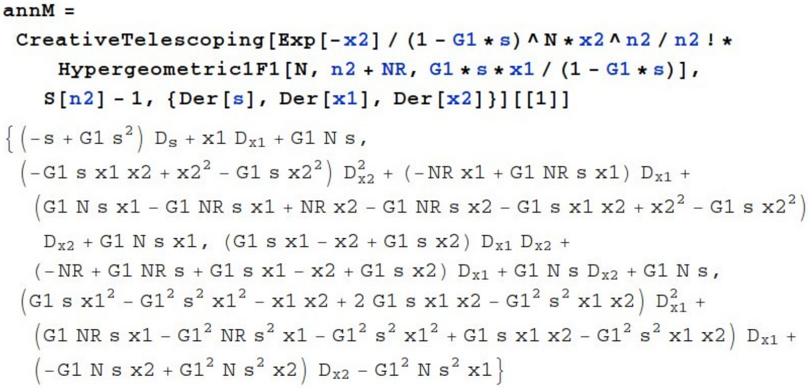
Nota bene. For $s = 1$ (ODE case) the matrix $\mathbf{A}$ is a companion matrix.

Annihilator for $M(s; x_1, x_2)$

Apply creative telescoping (HolonomicFunctions package) to

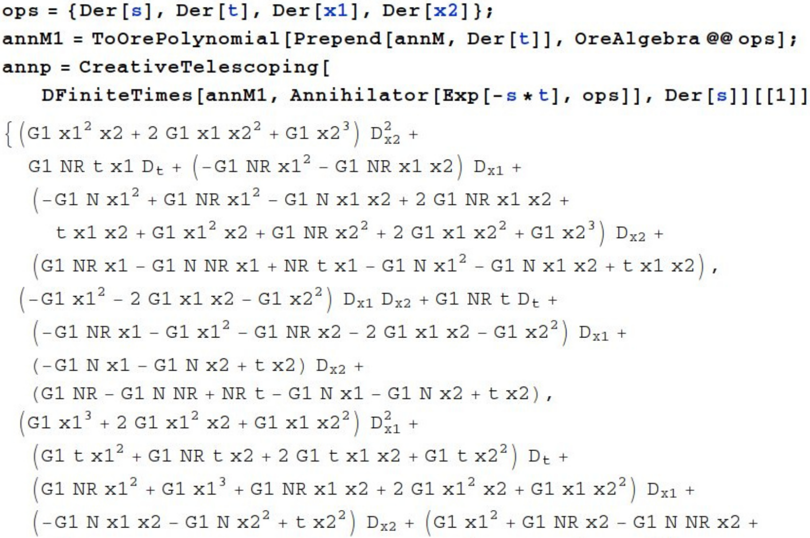
$$\sum_{n_2=0}^{\infty} \frac{e^{-x_2}}{(1-\Gamma_1 s)^N} \frac{x_2^{n_2}}{n_2!} \, {}_1F_1\!\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s x_1}{1-\Gamma_1 s}\right)$$



```
annM =
 CreativeTelescoping[Exp[-x2] / (1 - G1 * s) ^ N * x2 ^ n2 / n2 ! *
    Hypergeometric1F1[N, n2 + NR, G1 * s * x1 / (1 - G1 * s)],
   S[n2] - 1, {Der[s], Der[x1], Der[x2]}][[1]]
```

$$\begin{aligned}
&\{(-s + G1\ s^2)\ D_s + x1\ D_{x1} + G1\ N\ s,\\
&\ (-G1\ s\ x1\ x2 + x2^2 - G1\ s\ x2^2)\ D_{x2}^2 + (-NR\ x1 + G1\ NR\ s\ x1)\ D_{x1} +\\
&\quad (G1\ N\ s\ x1 - G1\ NR\ s\ x1 + NR\ x2 - G1\ NR\ s\ x2 - G1\ s\ x1\ x2 + x2^2 - G1\ s\ x2^2)\\
&\quad D_{x2} + G1\ N\ s\ x1,\ (G1\ s\ x1 - x2 + G1\ s\ x2)\ D_{x1}\ D_{x2} +\\
&\quad (-NR + G1\ NR\ s + G1\ s\ x1 - x2 + G1\ s\ x2)\ D_{x1} + G1\ N\ s\ D_{x2} + G1\ N\ s,\\
&\ (G1\ s\ x1^2 - G1^2\ s^2\ x1^2 - x1\ x2 + 2\ G1\ s\ x1\ x2 - G1^2\ s^2\ x1\ x2)\ D_{x1}^2 +\\
&\quad (G1\ NR\ s\ x1 - G1^2\ NR\ s^2\ x1 - G1^2\ s^2\ x1^2 + G1\ s\ x1\ x2 - G1^2\ s^2\ x1\ x2)\ D_{x1} +\\
&\quad (-G1\ N\ s\ x2 + G1^2\ N\ s^2\ x2)\ D_{x2} - G1^2\ N\ s^2\ x1\}
\end{aligned}$$

Annihilator for $p(t; x_1, x_2)$



```
ops = {Der[s], Der[t], Der[x1], Der[x2]};
annM1 = ToOrePolynomial[Prepend[annM, Der[t]], OreAlgebra @@ ops];
annp = CreativeTelescoping[
   DFiniteTimes[annM1, Annihilator[Exp[-s * t], ops]], Der[s]][[1]]
```

```
{ (G1 x1^2 x2 + 2 G1 x1 x2^2 + G1 x2^3) D_{x2}^2 +
   G1 NR t x1 D_t + (-G1 NR x1^2 - G1 NR x1 x2) D_{x1} +
   (-G1 N x1^2 + G1 NR x1^2 - G1 N x1 x2 + 2 G1 NR x1 x2 +
     t x1 x2 + G1 x1^2 x2 + G1 NR x2^2 + 2 G1 x1 x2^2 + G1 x2^3) D_{x2} +
   (G1 NR x1 - G1 N NR x1 + NR t x1 - G1 N x1^2 - G1 N x1 x2 + t x1 x2),
  (-G1 x1^2 - 2 G1 x1 x2 - G1 x2^2) D_{x1} D_{x2} + G1 NR t D_t +
   (-G1 NR x1 - G1 x1^2 - G1 NR x2 - 2 G1 x1 x2 - G1 x2^2) D_{x1} +
   (-G1 N x1 - G1 N x2 + t x2) D_{x2} +
   (G1 NR - G1 N NR + NR t - G1 N x1 - G1 N x2 + t x2),
  (G1 x1^3 + 2 G1 x1^2 x2 + G1 x1 x2^2) D_{x1}^2 +
   (G1 t x1^2 + G1 NR t x2 + 2 G1 t x1 x2 + G1 t x2^2) D_t +
   (G1 NR x1^2 + G1 x1^3 + G1 NR x1 x2 + 2 G1 x1^2 x2 + G1 x1 x2^2) D_{x1} +
   (-G1 N x1 x2 - G1 N x2^2 + t x2^2) D_{x2} + (G1 x1^2 + G1 NR x2 - G1 N NR x2 +
```

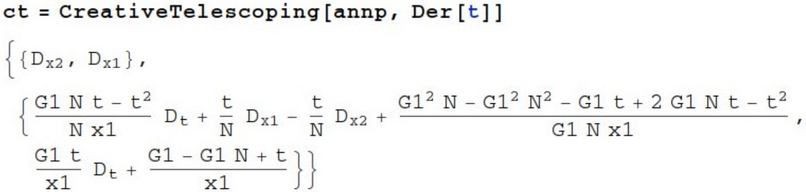
⋮

Annihilator for $P_{\mathrm{o}}(x_1, x_2)$

Recall:

$$P_{\mathrm{o}}(x_1, x_2) = \int_0^{\tau} p(t; x_1, x_2)\,\mathrm{d}t$$

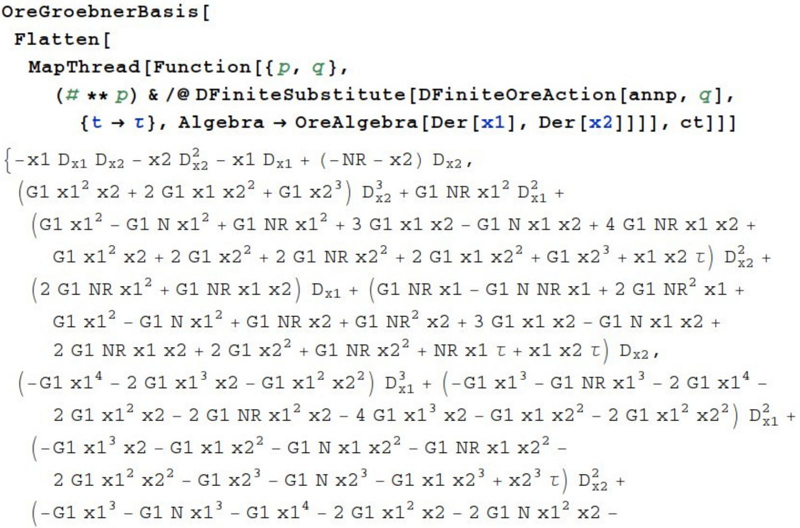
Hence we apply creative telescoping to $p(t; x_1, x_2)$:



```
ct = CreativeTelescoping[annp, Der[t]]
```

$$\left\{\{D_{x2},\ D_{x1}\},\ \left\{\frac{\text{G1 N t} - \text{t}^2}{\text{N x1}}\,D_t + \frac{\text{t}}{\text{N}}\,D_{x1} - \frac{\text{t}}{\text{N}}\,D_{x2} + \frac{\text{G1}^2\,\text{N} - \text{G1}^2\,\text{N}^2 - \text{G1 t} + 2\,\text{G1 N t} - \text{t}^2}{\text{G1 N x1}},\ \frac{\text{G1 t}}{\text{x1}}\,D_t + \frac{\text{G1} - \text{G1 N} + \text{t}}{\text{x1}}\right\}\right\}$$

Annihilator for $P_{\mathrm{o}}(x_1, x_2)$



```
OreGroebnerBasis[
 Flatten[
  MapThread[Function[{p, q},
    (# ** p) & /@ DFiniteSubstitute[DFiniteOreAction[annp, q],
      {t → τ}, Algebra → OreAlgebra[Der[x1], Der[x2]]]], ct]]
```

$$
\begin{aligned}
&\{-\mathrm{x1}\,D_{\mathrm{x1}}\,D_{\mathrm{x2}} - \mathrm{x2}\,D_{\mathrm{x2}}^2 - \mathrm{x1}\,D_{\mathrm{x1}} + (-\mathrm{NR} - \mathrm{x2})\,D_{\mathrm{x2}},\\
&\ (\mathrm{G1}\,\mathrm{x1}^2\,\mathrm{x2} + 2\,\mathrm{G1}\,\mathrm{x1}\,\mathrm{x2}^2 + \mathrm{G1}\,\mathrm{x2}^3)\,D_{\mathrm{x2}}^3 + \mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}^2\,D_{\mathrm{x1}}^2 +\\
&\quad (\mathrm{G1}\,\mathrm{x1}^2 - \mathrm{G1}\,\mathrm{N}\,\mathrm{x1}^2 + \mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}^2 + 3\,\mathrm{G1}\,\mathrm{x1}\,\mathrm{x2} - \mathrm{G1}\,\mathrm{N}\,\mathrm{x1}\,\mathrm{x2} + 4\,\mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}\,\mathrm{x2} +\\
&\qquad \mathrm{G1}\,\mathrm{x1}^2\,\mathrm{x2} + 2\,\mathrm{G1}\,\mathrm{x2}^2 + 2\,\mathrm{G1}\,\mathrm{NR}\,\mathrm{x2}^2 + 2\,\mathrm{G1}\,\mathrm{x1}\,\mathrm{x2}^2 + \mathrm{G1}\,\mathrm{x2}^3 + \mathrm{x1}\,\mathrm{x2}\,\tau)\,D_{\mathrm{x2}}^2 +\\
&\quad (2\,\mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}^2 + \mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}\,\mathrm{x2})\,D_{\mathrm{x1}} + (\mathrm{G1}\,\mathrm{NR}\,\mathrm{x1} - \mathrm{G1}\,\mathrm{N}\,\mathrm{NR}\,\mathrm{x1} + 2\,\mathrm{G1}\,\mathrm{NR}^2\,\mathrm{x1} +\\
&\qquad \mathrm{G1}\,\mathrm{x1}^2 - \mathrm{G1}\,\mathrm{N}\,\mathrm{x1}^2 + \mathrm{G1}\,\mathrm{NR}\,\mathrm{x2} + \mathrm{G1}\,\mathrm{NR}^2\,\mathrm{x2} + 3\,\mathrm{G1}\,\mathrm{x1}\,\mathrm{x2} - \mathrm{G1}\,\mathrm{N}\,\mathrm{x1}\,\mathrm{x2} +\\
&\qquad 2\,\mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}\,\mathrm{x2} + 2\,\mathrm{G1}\,\mathrm{x2}^2 + \mathrm{G1}\,\mathrm{NR}\,\mathrm{x2}^2 + \mathrm{NR}\,\mathrm{x1}\,\tau + \mathrm{x1}\,\mathrm{x2}\,\tau)\,D_{\mathrm{x2}},\\
&\ (-\mathrm{G1}\,\mathrm{x1}^4 - 2\,\mathrm{G1}\,\mathrm{x1}^3\,\mathrm{x2} - \mathrm{G1}\,\mathrm{x1}^2\,\mathrm{x2}^2)\,D_{\mathrm{x1}}^3 + (-\mathrm{G1}\,\mathrm{x1}^3 - \mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}^3 - 2\,\mathrm{G1}\,\mathrm{x1}^4 -\\
&\qquad 2\,\mathrm{G1}\,\mathrm{x1}^2\,\mathrm{x2} - 2\,\mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}^2\,\mathrm{x2} - 4\,\mathrm{G1}\,\mathrm{x1}^3\,\mathrm{x2} - \mathrm{G1}\,\mathrm{x1}\,\mathrm{x2}^2 - 2\,\mathrm{G1}\,\mathrm{x1}^2\,\mathrm{x2}^2)\,D_{\mathrm{x1}}^2 +\\
&\quad (-\mathrm{G1}\,\mathrm{x1}^3\,\mathrm{x2} - \mathrm{G1}\,\mathrm{x1}\,\mathrm{x2}^2 - \mathrm{G1}\,\mathrm{N}\,\mathrm{x1}\,\mathrm{x2}^2 - \mathrm{G1}\,\mathrm{NR}\,\mathrm{x1}\,\mathrm{x2}^2 -\\
&\qquad 2\,\mathrm{G1}\,\mathrm{x1}^2\,\mathrm{x2}^2 - \mathrm{G1}\,\mathrm{x2}^3 - \mathrm{G1}\,\mathrm{N}\,\mathrm{x2}^3 - \mathrm{G1}\,\mathrm{x1}\,\mathrm{x2}^3 + \mathrm{x2}^3\,\tau)\,D_{\mathrm{x2}}^2 +\\
&\quad (-\mathrm{G1}\,\mathrm{x1}^3 - \mathrm{G1}\,\mathrm{N}\,\mathrm{x1}^3 - \mathrm{G1}\,\mathrm{x1}^4 - 2\,\mathrm{G1}\,\mathrm{x1}^2\,\mathrm{x2} - 2\,\mathrm{G1}\,\mathrm{N}\,\mathrm{x1}^2\,\mathrm{x2} -
\end{aligned}
$$

$$\vdots$$

HGM computation

The irreducible monomials of the annihilator of $P_{\mathrm{o}}(x_1, x_2)$ are

$$1, D_1, D_2, D_1^2, D_2^2.$$

Hence, we take the following basis:

$$\mathbf{f} = \left(P_{\mathrm{o}}, P_{\mathrm{o}}^{(0,1)}, P_{\mathrm{o}}^{(1,0)}, P_{\mathrm{o}}^{(2,0)}, P_{\mathrm{o}}^{(0,2)}\right).$$

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The matrix $\mathbf{A}_1$ of the Pfaffian system $D_1\mathbf{f} = \mathbf{A}_1\mathbf{f}$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{N_{\mathrm{R}}+x_2}{x_1} & -1 & -\frac{x_2}{x_1} & 0 \\ 0 & \langle\cdots\rangle & -\frac{N_{\mathrm{R}}x_1(2x_1+x_2)}{x_2(x_1+x_2)^2} & \langle\cdots\rangle & -\frac{N_{\mathrm{R}}x_1^2}{x_2(x_1+x_2)^2} \\ 0 & \langle\cdots\rangle & \frac{N_{\mathrm{R}}x_1}{(x_1+x_2)^2} & \langle\cdots\rangle & -\frac{(x_1+x_2)^2+N_{\mathrm{R}}x_2}{(x_1+x_2)^2} \end{pmatrix}.$$

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Similar for $D_2\mathbf{f} = \mathbf{A}_2\mathbf{f}$.

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The irreducible monomials of the annihilator of $P_{\mathrm{o}}(x_1, x_2)$ are

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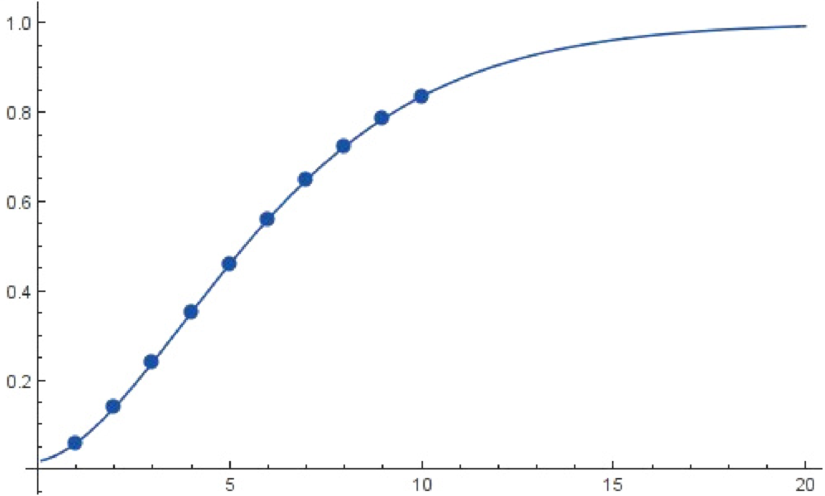
Similar for $D_2\mathbf{f} = \mathbf{A}_2\mathbf{f}$.

$\mathbf{A}_1$ and $\mathbf{A}_2$ allow to propagate the initial values along both coordinate axes.

HGM computation



- dots: computed with truncated series $(167s)$
- line: computed with HGM $(< 1s)$

Application

"From an engineering perspective, Fig. 6 reveals that even at Γ_b as low as 0 dB (i.e., low transmit power, or strong receiver noise), the low-complexity ZF can sustain a rate of almost 5 bits per channel use (bpcu) for each of the 20 streams, if the base station has 100 antennas. If a WiFi system would implement this, then, within only 20 MHz of bandwidth it would support 20 users each of them downloading at 20 MHz $\times$ 5 bpcu $=$ 100 Mbps. But nowadays WiFi routers already can use 100 MHz channels (in the 5GHz band), which would bring the download speed for each user at 0.5 Gbps. This would be very useful in a conference hall ;-) "

Outlook

- consider more complicated models
- choice of direction for numerical integration
- certified numerical evaluations