Diagonals, determinants, and rigidity

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Austrian Academy of Sciences

July 12, 2019
SIAM-AAG, Bern, Switzerland

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$$\Delta\left(\frac{1}{1\ +x\ +y\ +z\ \ +x\,y\ +y\,z\ \ -x^3\,y\,z}\right) = \dots$$

$$Heun\Big(\frac{1}{2} - \frac{i\sqrt{3}}{2},\ \frac{1}{2} - \frac{i\sqrt{3}}{6},\ 1,\ 1,\ 1,\ 1,\ \ \frac{3}{2}\cdot\left(-3 + i\sqrt{3}\right)\cdot x\Big)$$

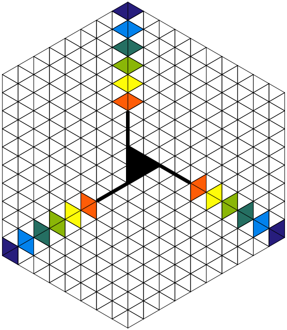
$$= \ \frac{1}{1\ +\ 3\,x}\cdot {}_2F_1\Big([\frac{1}{3},\frac{2}{3}],\ [1],\ \frac{27\cdot x^3}{(1\ +\ 3\,x)^3}\Big)$$

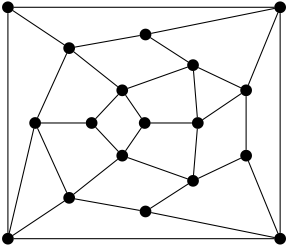
$$= \ \Big(\frac{1}{1\ +9\,x\ +27\,x^2\ -27\,x^3}\Big)^{1/3}$$

$$\times\ {}_2F_1\Big([\frac{1}{6},\frac{2}{3}],\ [1],\ -\frac{108\cdot x^3\cdot(1\ +9\,x\ +27\,x^2)}{(1\ +9\,x\ +27\,x^2\ -27\,x^3)^2}\Big)$$

$$= \ \Big(\frac{1}{1\ +3\,x}\Big)^{1/4}\cdot\Big(\frac{1}{1\ +9\,x\ +27\,x^2\ +3\,x^3}\Big)^{1/4}$$

$$\times\ {}_2F_1\Big([\frac{1}{12},\frac{5}{12}],\ [1],\ \frac{1728\cdot x^9\cdot(1\ +9\,x\ +27\,x^2)}{(1\ +3\,x)^3\cdot(1\ +9\,x\ +27\,x^2\ +3\,x^3)^3}\Big)$$





Exact lower bounds for monochromatic Schur triples

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(joint work Elaine Wong)

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Basic definitions and notations

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$$(x, y, z) \in \mathbb{N}^3 \quad \text{with} \quad x + y = z$$

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Number of monochromatic Schur triples (MSTs):

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Schur triples $(x, y, x + y)$ and $(y, x, x + y)$ are distinct if $x \neq y$!

Example

Consider the 2-coloring χ of $[6] = \{1, 2, 3, 4, 5, 6\}$:

$$\chi(2) = \chi(4) = \text{red}, \qquad \chi(1) = \chi(3) = \chi(5) = \chi(6) = \text{blue}$$

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We have $\mathcal{M}(6, \chi) = 4$, i.e., there are exactly 4 MSTs:

$$(1, 5, 6), \quad (2, 2, 4), \quad (3, 3, 6), \quad (5, 1, 6).$$

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Minimal number: Determine the minimal number $\mathcal{M}(n)$ of MSTs among all possible 2-colorings of $[n]$

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Example

Consider again $[6] = \{1, 2, 3, 4, 5, 6\}$.

- What is $\mathcal{M}(6)$?
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- What is $\mathcal{M}(6)$?
- Which coloring $\chi\colon [6] \to \{R, B\}$ yields the least number of monochromatic Schur triples (MSTs)?

Answer: Choose the coloring $\chi = R^2B^3R = RRBBBR$:

$$\{1, 2, 3, 4, 5, 6\}$$

Then there exists only one single MST, namely $(1, 1, 2)$, hence $\mathcal{M}(6) = 1$.

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs
- Erdős and Szekeres (1935): rediscovery of Ramsey's theorem and simpler proof of Schur's theorem
- Alan Goodman (1959): minimum number of monochromatic triangles under a 2-coloring of a complete graph
- Graham, Rödl, Ruciński (1996) asked about the minimum number of Schur triples under any 2-coloring (100 USD prize!)
- Robertson, Zeilberger (1998) answer this question asymptotically.
- Their result was independently confirmed by Schoen (1999), Datskovsky (2003), and Thanatipanonda (2009).
- Recent work on generalized Schur triples by Butler, Costello, Graham (2010), and Thanatipanonda, Wong (2017).

Three blocks

It has been shown previously (RobertsonZeilberger 98, Schoen 99) that the number $\mathcal{M}(n, \chi)$ is minimized when χ is of the form

$$R^s B^{t-s} R^{n-t},$$

where $s \approx \frac{4}{11}n$ and $t \approx \frac{10}{11}n$.

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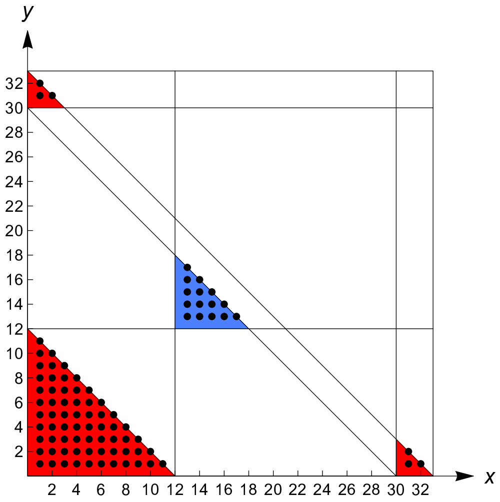
Lemma. Let $n, s, t \in \mathbb{N}$ be such that $1 \leqslant s \leqslant t \leqslant n$. Moreover, assume that the inequalities $t \geqslant 2s$ and $s \geqslant n - t$ hold. Then the number of monochromatic Schur triples on $[n]$ under the coloring $R^s B^{t-s} R^{n-t}$ is exactly

$$\mathcal{M}(n, s, t) = \frac{s(s-1)}{2} + \frac{(t-2s)(t-2s-1)}{2} + (n-t)(n-t-1).$$

Proof (by example)

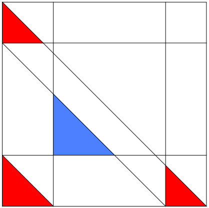
$$\mathcal{M}(n,s,t) = \frac{s(s-1)}{2} + \frac{(t-2s)(t-2s-1)}{2} + (n-t)(n-t-1).$$



- $\chi = R^{12}B^{18}R^{3}$
- $s = 12,\ t = 30$
- $\mathcal{M}(33,12,30) = 66 + 15 + 6 = 87$
- Actually we have $\mathcal{M}(33) = 87$

Optimal values for s and t

The optimal values for s and t are easily derived using the techniques of multivariable calculus:



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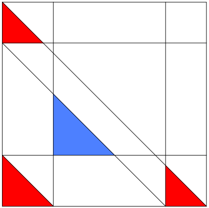

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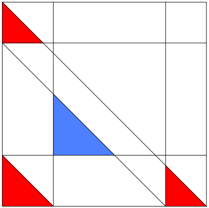

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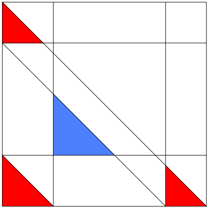

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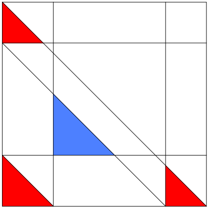

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- portion of pairs $(x, y) \in [n]^2$ for which $(x, y, x + y)$ is an MST equals the area of a certain region in the unit square
- This area is calculated by the formula

$$A(s, t) = \frac{s^2}{2} + \frac{(t - 2s)^2}{2} + 2 \cdot \frac{(1 - t)^2}{2}$$
$$= \frac{5s^2}{2} + \frac{3t^2}{2} - 2st - 2t + 1.$$

Optimal values for s and t

Recall:

$$A(s,t) = \frac{5s^2}{2} + \frac{3t^2}{2} - 2st - 2t + 1.$$

Equating the gradient

$$\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right) = (5s - 2t, 3t - 2s - 2)$$

to zero, one immediately gets the location of the minimum

$$(s,t) = \left(\frac{4}{11}, \frac{10}{11}\right).$$

Optimal values for discrete s and t

Lemma. For fixed $n \in \mathbb{N}$, the integers s_0 and t_0 that minimize the function $\mathcal{M}(n, s, t)$ are given by

$$s_0 = \left\lfloor \frac{4n+2}{11} \right\rfloor \qquad \text{and} \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor.$$

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- We want to show that among all integers $i, j \in \mathbb{Z}$ the expression $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$.

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- We want to show that among all integers $i, j \in \mathbb{Z}$ the expression $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$.
- Such a task can, in principle, be routinely executed by cylindrical algebraic decomposition (CAD).
- Small adaptions to take into account that i, j are integers.

Exact lower bound

Theorem 1. The minimal number of monochromatic Schur triples that can be attained under any 2-coloring of $[n]$ is

$$\mathcal{M}(n) = \left\lfloor \frac{n^2 - 4n + 6}{11} \right\rfloor.$$

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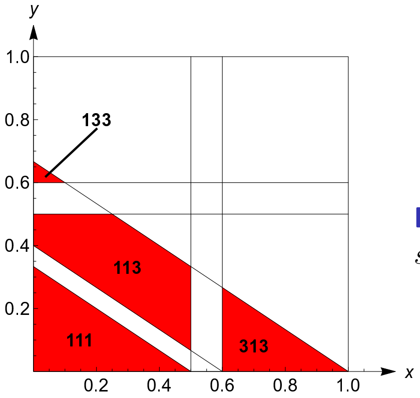
Proof.

$$\begin{aligned}
\ell = 0\colon \mathcal{M}(11k, 4k, 10k) &= 11k^2 - 4k &&= \tfrac{1}{11}(n^2 - 4n) \\
\ell = 1\colon \mathcal{M}(11k+1, 4k, 10k) &= 11k^2 - 2k &&= \tfrac{1}{11}(n^2 - 4n + 3) \\
\ell = 2\colon \mathcal{M}(11k+2, 4k, 10k+1) &= 11k^2 &&= \tfrac{1}{11}(n^2 - 4n + 4) \\
\ell = 3\colon \mathcal{M}(11k+3, 4k+1, 10k+2) &= 11k^2 + 2k &&= \tfrac{1}{11}(n^2 - 4n + 3) \\
\ell = 4\colon \mathcal{M}(11k+4, 4k+1, 10k+3) &= 11k^2 + 4k &&= \tfrac{1}{11}(n^2 - 4n) \\
&\vdots && \vdots \\
\ell = 9\colon \mathcal{M}(11k+9, 4k+3, 10k+8) &= 11k^2 + 14k + 4 &&= \tfrac{1}{11}(n^2 - 4n - 1) \\
\ell = 10\colon \mathcal{M}(11k+10, 4k+3, 10k+9) &= 11k^2 + 16k + 6 &&= \tfrac{1}{11}(n^2 - 4n + 6)
\end{aligned}$$

Generalized Schur triples

- For $a \in \mathbb{N}$, a generalized Schur triple $(x, y, z) \in \mathbb{N}^3$ satisfies $x + ay = z$.
- Extend this to $a \in \mathbb{R}^+$ by imposing $x + \lfloor ay \rfloor = z$.



Example:
$s = \frac{1}{2}$, $t = \frac{3}{5}$, $a = \frac{3}{2}$

Case distinctions for polygon 133

Polygon 133 corresponds to triples (x, y, z) that satisfy $x \leqslant s$ and $y, z \geqslant t$.

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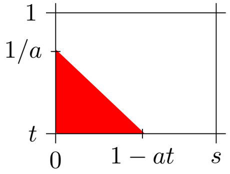
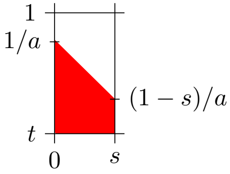


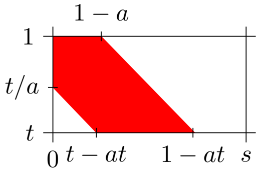
(2) $t < 1/a \leqslant 1 \ \wedge \ (1 - s)/a > t$



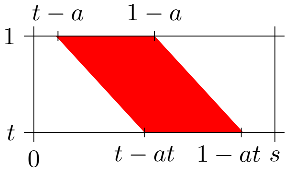
Case distinctions for polygon 133

$(3) \quad 0 < 1-a \leqslant s \ \wedge \ 1-at \leqslant s$

$(3.1) \quad t/a \leqslant 1$



$(3.2) \quad t/a > 1$



$(4) \quad 0 < 1-a \leqslant s \ \wedge \ (1-s)/a > t$

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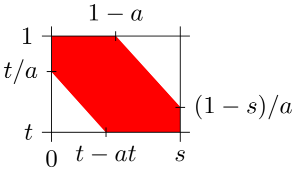

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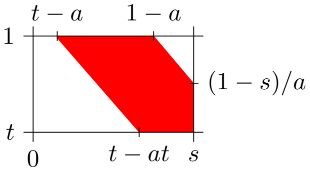
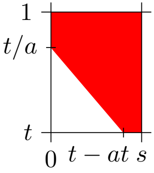


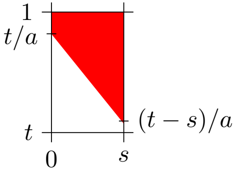
Case distinctions for polygon 133

$$(5) \quad 1 - a > s$$

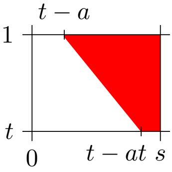
$(5.1) \quad t/a \leqslant 1 \ \wedge\ t - at \leqslant s$



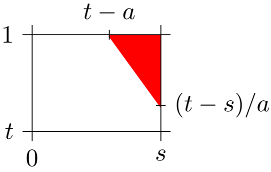
$(5.2) \quad t/a \leqslant 1 \ \wedge\ (t-s)/a > t$



$(5.3) \quad t/a > 1 \ \wedge\ t - at \leqslant s$



$(5.4) \quad t/a > 1 \ \wedge\ t < (t-s)/a < 1$



Case distinctions for all polygons

Performing a similar case analysis for all possible seven polygons

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C_7 &\equiv 1 - at - s \geqslant 0, & C_8 &\equiv 1 - at - t \geqslant 0,\\
C_9 &\equiv 1 - a \geqslant 0, & C_{10} &\equiv 1 - a - s \geqslant 0,\\
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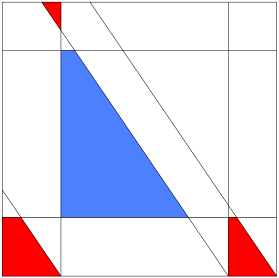
The area of each polygon is given by a piecewise function, whose definition involves logical combinations of $C_1, \ldots, C_{16}$.

Combining piecewise functions

Recall that we are interested in the total area A of the shaded regions, as a function of s, t, a. The area of each polygon is given by a piecewise (rational) function in s, t, a. Hence, we have to express the sum of seven piecewise functions as a single (piecewise) function.



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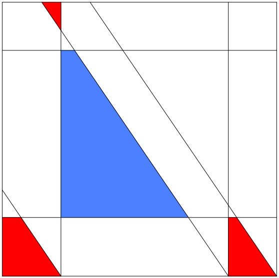

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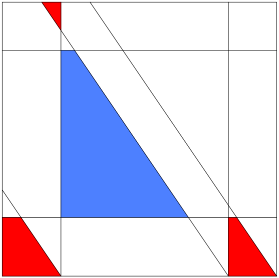

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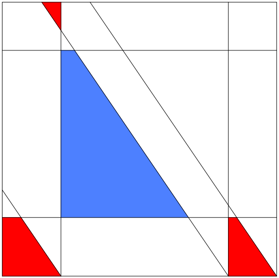

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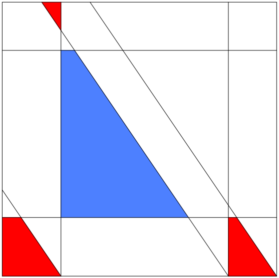

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- Start with the finest possible refinement, which is obtained by considering all $2^{16} = 65536$ logical combinations of C_i and $\overline{C_i}$.
- Remove those cases that contain contradictory combinations of conditions.
- Merge regions on which $A(s, t, a)$ is defined by the same expression into a single region.

The area function $A(s,t,a)$

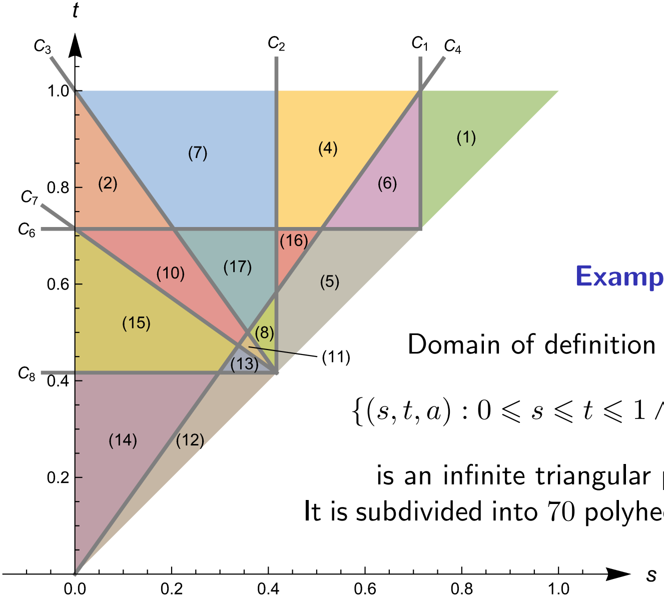
Lemma. Let $a, s, t \in \mathbb{R}$ with $a > 0$ and $0 \leqslant s \leqslant t \leqslant 1$. Then the area $A(s,t,a)$ of the region

$$\left\{(x,y) \in \mathbb{R}^2 : (x, y, x+ay) \in \left([0,s] \cup (t,1]\right)^3 \cup (s,t]^3\right\}$$

is given by the following piecewise defined function (70 cases):

	conditions on a, s, t	$A(s,t,a)$
(R_1)	$\overline{C_1}$	$\frac{s^2-2ts+2s+t^2-2t+1}{2a}$
(R_2)	$C_3 \wedge C_4 \wedge \overline{C_6}$	$\frac{2as^2+2s^2+2as-4ats-2ts+t^2}{2a}$
(R_3)	$C_3 \wedge \overline{C_4} \wedge \overline{C_6}$	$\frac{-a^2s^2+2as^2+2s^2+2as-2ats-2ts}{2a}$
(R_4)	$\overline{C_2} \wedge C_4 \wedge \overline{C_6}$	$\frac{s^2+2as-2ats-2ts+2s+2t^2-2t}{2a}$
(R_5)	$\overline{C_2} \wedge \overline{C_4} \wedge C_6$	$\frac{-a^2s^2+s^2+2as-2ts+2s+a^2t^2+t^2-2at-2t+1}{2a}$
(R_6)	$C_1 \wedge \overline{C_2} \wedge \overline{C_4} \wedge \overline{C_6}$	$\frac{-a^2s^2+s^2+2as-2ts+2s+t^2-2t}{2a}$
(R_7)	$C_2 \wedge \overline{C_3} \wedge C_4 \wedge \overline{C_6}$	$\frac{a^2s^2+2as^2+2s^2-2ats-2ts+2t^2-2t+1}{2a}$
(R_8)	$C_2 \wedge \overline{C_3} \wedge \overline{C_4} \wedge C_6$	$\frac{2as^2+2s^2-2ts+a^2t^2+t^2-2at-2t+2}{2a}$
$\vdots$	$\vdots$	$\vdots$

Polyhedral subdivision



Example: $a = 1.4$

Domain of definition of $A(s,t,a)$

$$\{(s,t,a) : 0 \leqslant s \leqslant t \leqslant 1 \wedge a \geqslant 0\}$$

is an infinite triangular prism in $\mathbb{R}^3$.
It is subdivided into 70 polyhedral regions.

Minimize the area function $A(s,t,a)$

Lemma. For $a>0$, the minimum of the function $A(s,t,a)$

$$m(a) := \min_{0\leqslant s\leqslant t\leqslant 1} A(s,t,a)$$

is given by a piecewise rational function, depending on a:

	s_0	t_0	$m(a)$
$0\leqslant a\leqslant \alpha_1$	$\frac{(a-4)a}{a^3-a-4}$	$\frac{-2a^2+4a+2}{-a^3+a+4}$	$\frac{-a^4+2a^3-2a^2+6a-4}{2\left(a^3-a-4\right)}$
$\alpha_1\leqslant a\leqslant \alpha_2$	$\frac{a\left(a^2-3\right)}{a^4-8a-1}$	$\frac{a^3+a^2-5a-1}{a^4-8a-1}$	$\frac{a^3-2a^2+a-2}{2\left(a^4-8a-1\right)}$
$\alpha_2\leqslant a\leqslant \alpha_3$	$\frac{-2a^3+2a+1}{-a^4+8a+3}$	$\frac{2a^3+a^2-6a-2}{a^4-8a-3}$	$\frac{a^6+a^4-12a^3+4a^2-1}{2a\left(a^4-8a-3\right)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\alpha_7\leqslant a\leqslant 1$	$\frac{(a+1)^2}{a(7a+4)}$	$\frac{(a+1)(4a+1)}{a(7a+4)}$	$\frac{-7a^4+6a^3+6a^2-2a-1}{2a^2(7a+4)}$
$1\leqslant a\leqslant \alpha_8$	$\frac{(a+1)^2}{a^4+2a^3+3a^2+2a+3}$	$\frac{(a+1)\left(a^2+2a+2\right)}{a^4+2a^3+3a^2+2a+3}$	$\frac{a^4-a^2-2a+4}{2a\left(a^4+2a^3+3a^2+2a+3\right)}$
$\alpha_8\leqslant a$	$\frac{a+1}{a^2+2a+3}$	$\frac{a^2+2a+2}{a^2+2a+3}$	$\frac{1}{2a\left(a^2+2a+3\right)}$

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s, t, a)$ is defined:

- View $A(s, t, a)$ as a function in s, t with a parameter a.

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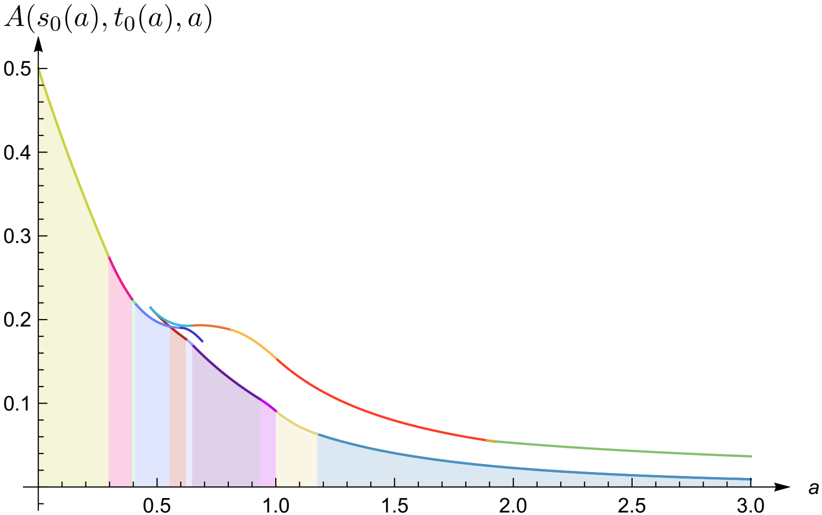
Using CAD one finds the admissible range for a:

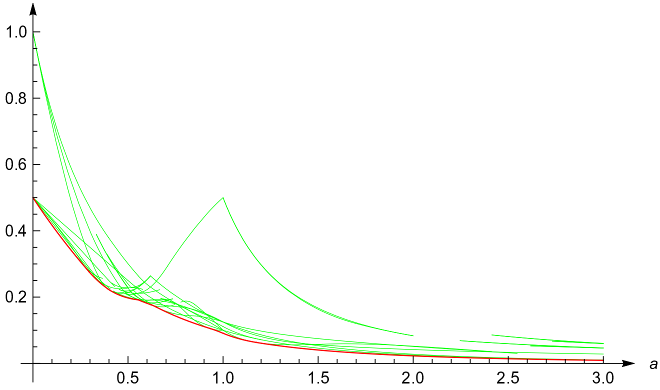
$$a \geqslant \mathrm{Root}\big(2a^3 - 3a^2 - 2a + 1, [1,2]\big) = 1.889228559...$$

Plot of the local minima



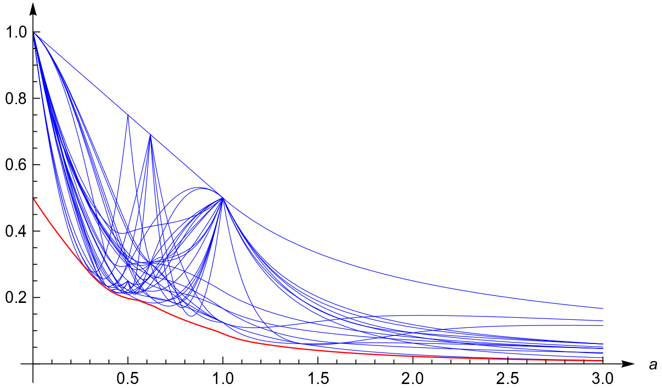
Find the global minimum

Note that $A(s, t, a)$ is defined piecewise and therefore may not be differentiable (it is, however, obvious from construction that it is continuous). $\rightarrow$ Search for minima along boundaries of regions.

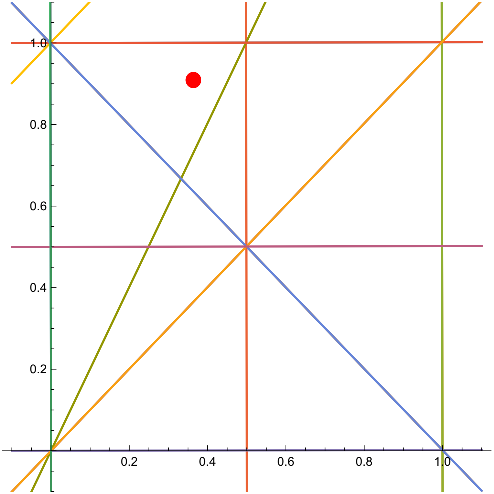


Find the global minimum

Similarly, we consider the function values of $A(s,t,a)$ at all intersections of the lines defined by the C_i (these points depend on a, and we get 348 cases to check).



Location of the global minimum



$a = 1.00$

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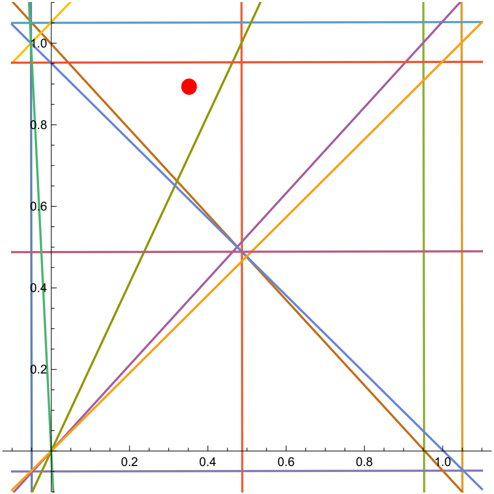

$a = 1.05$

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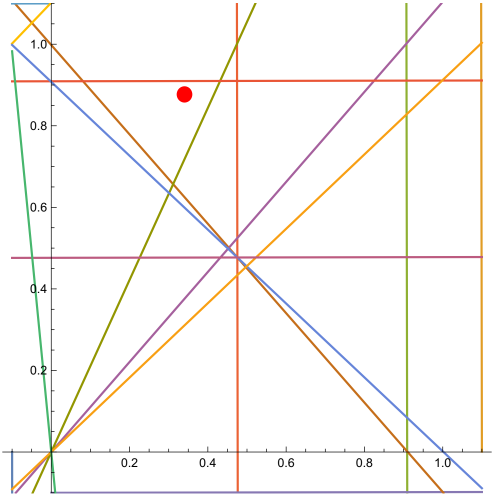
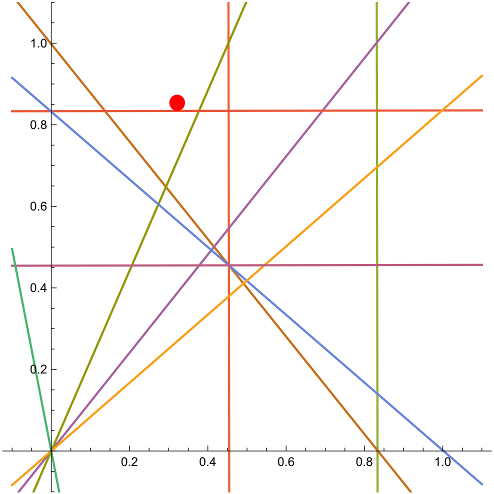


$a = 1.10$

Location of the global minimum



$a = 1.20$

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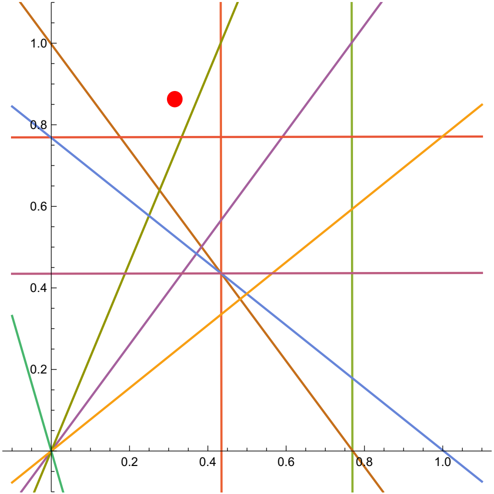
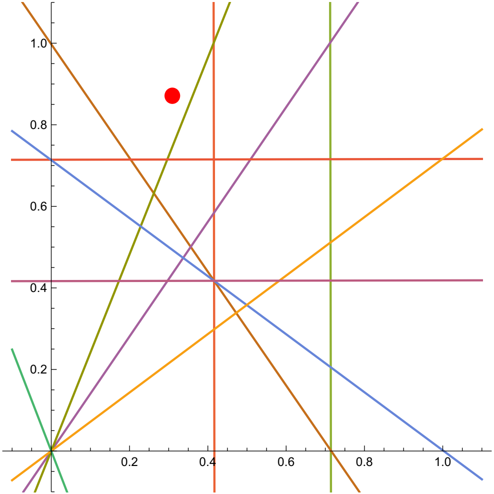


$$a = 1.30$$

Location of the global minimum



$$a = 1.40$$

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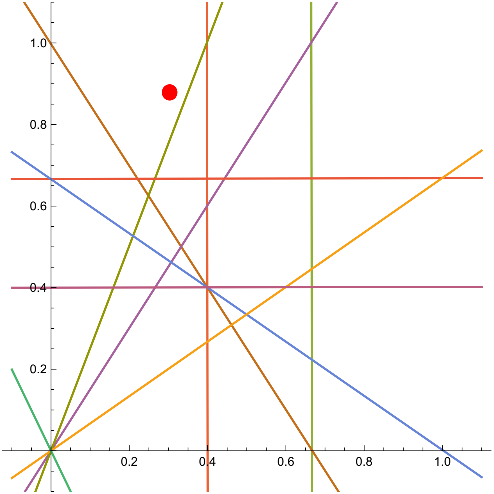

$$a = 1.50$$

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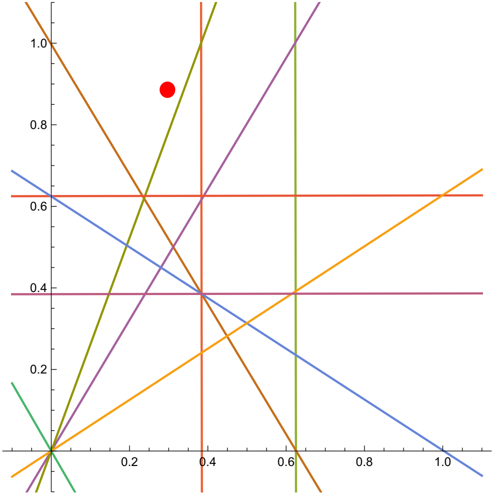

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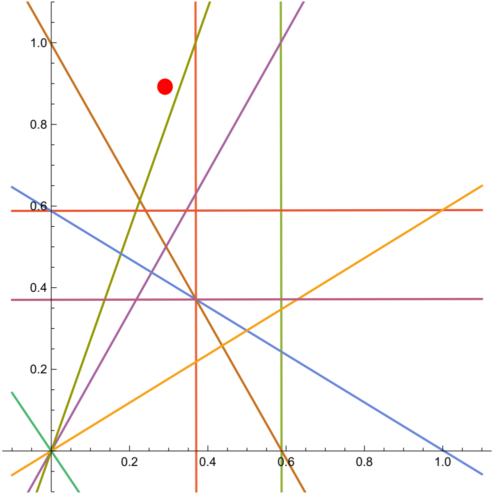

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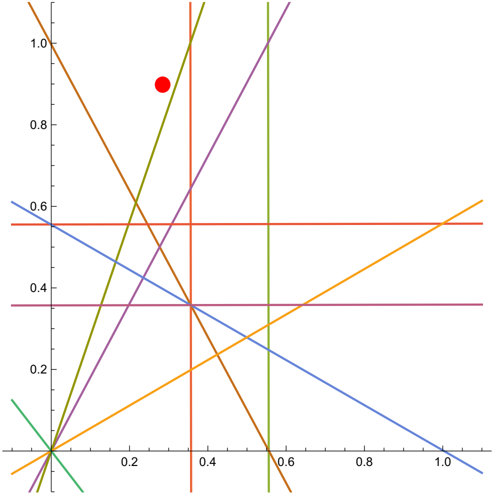
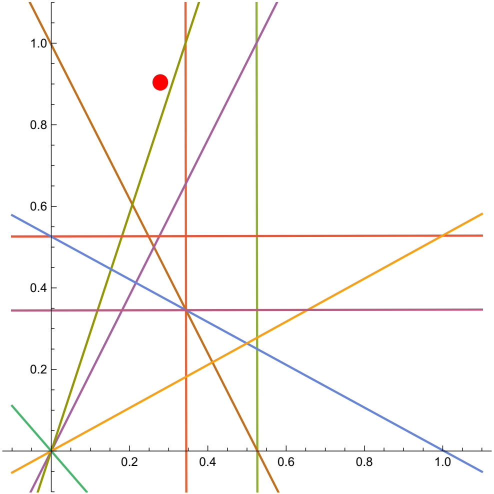
Location of the global minimum



$a = 1.80$

Location of the global minimum



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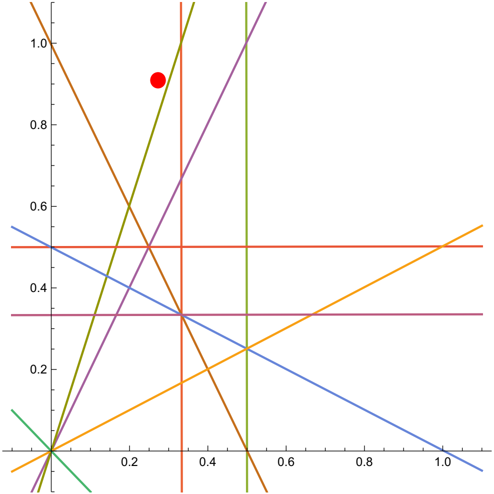

$$a = 2.00$$

Exact lower bound for $a = 2$

Theorem 2. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 2y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(2)}(n) = \left\lfloor \frac{n^2 - 10n + 33}{44} \right\rfloor .$$

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Proof. Using the knowledge of $A(s, t, a)$, we find (empirically) that the minimum of $\mathcal{M}^{(2)}(n, s, t)$ is attained at

$$s_0 = \left\lfloor \frac{3n+1}{11} \right\rfloor, \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor + \begin{cases} -1, & \text{if } n = 22k + 10, \\ 0, & \text{otherwise.} \end{cases}$$

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Analogous to the $a = 1$ case, make a case distinction according to $n = 22k + \ell$ and apply CAD in each case.

Exact lower bound for $a = 3$

Theorem 3. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 3y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(3)}(n) = \left\lfloor \frac{n^2 - 18n + 101}{108} \right\rfloor + \begin{cases} 1, & \text{if } n = 54k + 36, \\ -1, & \text{if } n = 54k + 30 \\ & \text{or } n = 54k + 42 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Analogous to previous theorem, but 54 case distinctions.

Exact lower bound for $a = 4$

Theorem 4. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 4y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(4)}(n) = \left\lfloor \frac{n^2 - 28n + 245}{216} \right\rfloor - \begin{cases} 1, & \text{if } n = 108k + i \text{ for } i \in I, \\ 0, & \text{otherwise,} \end{cases}$$

where the set I is given by

$$\{0, 1, 27, 28, 43, 47, 48, 53, 58, 63, 67, 68, 69, 73, 78, 83, 88, 89, 93\}.$$

Proof. Analogous to previous theorem, but 108 case distinctions.

Exact lower bound for $a = \frac{1}{2}$

Theorem $\frac{1}{2}$. The minimal number of monochromatic generalized Schur triples of the form $\left(x, y, x + \lfloor \frac{1}{2} y \rfloor\right)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is given by

$$\mathcal{M}^{(1/2)}(n) = \left\lfloor \frac{15n^2 + 72}{76} \right\rfloor + \begin{cases} 1, & \text{if } n = 38k + 18 \text{ or } n = 38k + 20 \\ -1, & \text{if } n = 38k + 19, \\ 0, & \text{otherwise.} \end{cases}$$

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Counter-example. For $n = 4$ the theorem predicts a minimum of four MSTs, under the coloring $\{1, 2, 3, 4\}$, namely

$$(1, 1, 1), (4, 1, 4), (2, 2, 3), (2, 3, 3).$$

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However, for the coloring $\{1, 2, 3, 4\}$ we get only three MSTs:

$$(1, 1, 1), (3, 1, 3), (2, 4, 4).$$

True minimum for $a = \frac{1}{2}$

Conjecture. For $n \geqslant 12$, the minimal number of monochromatic generalized Schur triples of the form $\left(x, y, x + \lfloor \frac{1}{2} y \rfloor\right)$ that can be attained under any 2-coloring of $[n]$ is given by

$$\left\lfloor \frac{n^2 + 5}{6} \right\rfloor,$$

and it occurs at the coloring $R^s B^{t-s} R^{u-t} B^{n-u}$ for

$$s = \left\lfloor \frac{n+3}{6} \right\rfloor, \qquad t = \left\lfloor \frac{n+1}{2} \right\rfloor, \qquad u = \left\lfloor \frac{5n+3}{6} \right\rfloor.$$