Exact lower bounds for monochromatic Schur triples

Christoph Koutschan
(joint work Elaine Wong)

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

June 12, 2019
SFB Meeting, RICAM, Linz

Basic definitions and notations

Schur triple:

$$(x, y, z) \in \mathbb{N}^3 \quad \text{with} \quad x + y = z$$

Basic definitions and notations

Schur triple:

$$(x, y, z) \in \mathbb{N}^3 \quad \text{with} \quad x + y = z$$

Finite range of positive integers:

$$[n] := \{1, \ldots, n\} \subset \mathbb{N}$$

Basic definitions and notations

Schur triple:

$$(x, y, z) \in \mathbb{N}^3 \quad \text{with} \quad x + y = z$$

Finite range of positive integers:

$$[n] := \{1, \dots, n\} \subset \mathbb{N}$$

2-Coloring of $[n]$:

$$\chi\colon [n] \to \{\text{red}, \text{blue}\}$$

Basic definitions and notations

Schur triple:

$$(x, y, z) \in \mathbb{N}^3 \quad \text{with} \quad x + y = z$$

Finite range of positive integers:

$$[n] := \{1, \dots, n\} \subset \mathbb{N}$$

2-Coloring of $[n]$:

$$\chi\colon [n] \to \{\text{red}, \text{blue}\}$$

Number of monochromatic Schur triples (MSTs):

$$\mathcal{M}(n, \chi) := \left|\left\{(x, y, z) \in [n]^3 : z = x + y \wedge \chi(x) = \chi(y) = \chi(z)\right\}\right|$$

Basic definitions and notations

Schur triple:

$$(x, y, z) \in \mathbb{N}^3 \quad \text{with} \quad x + y = z$$

Finite range of positive integers:

$$[n] := \{1, \ldots, n\} \subset \mathbb{N}$$

2-Coloring of $[n]$:

$$\chi \colon [n] \to \{\text{red}, \text{blue}\}$$

Number of monochromatic Schur triples (MSTs):

$$\mathcal{M}(n, \chi) := \left|\left\{(x, y, z) \in [n]^3 : z = x + y \,\wedge\, \chi(x) = \chi(y) = \chi(z)\right\}\right|$$

Schur triples $(x, y, x + y)$ and $(y, x, x + y)$ are distinct if $x \neq y$!

Example

Consider the 2-coloring χ of $[6] = \{1, 2, 3, 4, 5, 6\}$:

$$\chi(2) = \chi(4) = \text{red}, \qquad \chi(1) = \chi(3) = \chi(5) = \chi(6) = \text{blue}$$

Example

Consider the 2-coloring χ of $[6] = \{1, 2, 3, 4, 5, 6\}$:

$$\chi(2) = \chi(4) = \text{red}, \qquad \chi(1) = \chi(3) = \chi(5) = \chi(6) = \text{blue}$$

Short notation: BRBRBB, or graphically:

$$\{ \boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6} \}$$

Example

Consider the 2-coloring χ of $[6] = \{1, 2, 3, 4, 5, 6\}$:

$$\chi(2) = \chi(4) = \text{red}, \qquad \chi(1) = \chi(3) = \chi(5) = \chi(6) = \text{blue}$$

Short notation: BRBRBB, or graphically:

$$\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}\}$$

We have $\mathcal{M}(6, \chi) = 4$, i.e., there are exactly 4 MSTs:

$$(\boxed{1}, \boxed{5}, \boxed{6}), \quad (\boxed{2}, \boxed{2}, \boxed{4}), \quad (\boxed{3}, \boxed{3}, \boxed{6}), \quad (\boxed{5}, \boxed{1}, \boxed{6}).$$

Bounds on the number of MSTs

What can be said about $\mathcal{M}(n, \chi)$ if the coloring χ is not fixed?

Bounds on the number of MSTs

What can be said about $\mathcal{M}(n,\chi)$ if the coloring χ is not fixed?

Upper bound: There are $\sum_{i=1}^{n-1} i = \binom{n}{2}$ Schur triples on $[n]$, thus

$$\mathcal{M}(n,R^n) = \mathcal{M}(n,B^n) = \binom{n}{2}$$

is a (sharp!) upper bound for the number of MSTs.

Bounds on the number of MSTs

What can be said about $\mathcal{M}(n,\chi)$ if the coloring χ is not fixed?

Upper bound: There are $\sum_{i=1}^{n-1} i = \binom{n}{2}$ Schur triples on $[n]$, thus

$$\mathcal{M}(n,R^n) = \mathcal{M}(n,B^n) = \binom{n}{2}$$

is a (sharp!) upper bound for the number of MSTs. Boring!

Bounds on the number of MSTs

What can be said about $\mathcal{M}(n, \chi)$ if the coloring χ is not fixed?

Upper bound: There are $\sum_{i=1}^{n-1} i = \binom{n}{2}$ Schur triples on $[n]$, thus

$$\mathcal{M}(n, R^n) = \mathcal{M}(n, B^n) = \binom{n}{2}$$

is a (sharp!) upper bound for the number of MSTs. Boring!

Lower bound: Find a function $b(n)$ such that $\mathcal{M}(n, \chi) \geqslant b(n)$ for all $n \in \mathbb{N}$ and for all $\chi \in \{R, B\}^{[n]}$.

Bounds on the number of MSTs

What can be said about $\mathcal{M}(n,\chi)$ if the coloring χ is not fixed?

Upper bound: There are $\sum_{i=1}^{n-1} i = \binom{n}{2}$ Schur triples on $[n]$, thus

$$\mathcal{M}(n,R^n) = \mathcal{M}(n,B^n) = \binom{n}{2}$$

is a (sharp!) upper bound for the number of MSTs. Boring!

Lower bound: Find a function $b(n)$ such that $\mathcal{M}(n,\chi) \geqslant b(n)$ for all $n \in \mathbb{N}$ and for all $\chi \in \{R,B\}^{[n]}$. Define $b(n) = 0$. Boring!

Bounds on the number of MSTs

What can be said about $\mathcal{M}(n, \chi)$ if the coloring χ is not fixed?

Upper bound: There are $\sum_{i=1}^{n-1} i = \binom{n}{2}$ Schur triples on $[n]$, thus

$$\mathcal{M}(n, R^n) = \mathcal{M}(n, B^n) = \binom{n}{2}$$

is a (sharp!) upper bound for the number of MSTs. Boring!

Lower bound: Find a function $b(n)$ such that $\mathcal{M}(n, \chi) \geqslant b(n)$ for all $n \in \mathbb{N}$ and for all $\chi \in \{R, B\}^{[n]}$. Define $b(n) = 0$. Boring!

Minimal number: Determine the minimal number $\mathcal{M}(n)$ of MSTs among all possible 2-colorings of $[n]$

$$\mathcal{M}(n) := \min_{\chi\colon [n] \to \{R,B\}} \mathcal{M}(n, \chi)$$

and the / a corresponding coloring χ.

Bounds on the number of MSTs

What can be said about $\mathcal{M}(n,\chi)$ if the coloring χ is not fixed?

Upper bound: There are $\sum_{i=1}^{n-1} i = \binom{n}{2}$ Schur triples on $[n]$, thus

$$\mathcal{M}(n,R^n) = \mathcal{M}(n,B^n) = \binom{n}{2}$$

is a (sharp!) upper bound for the number of MSTs. Boring!

Lower bound: Find a function $b(n)$ such that $\mathcal{M}(n,\chi) \geqslant b(n)$ for all $n \in \mathbb{N}$ and for all $\chi \in \{R,B\}^{[n]}$. Define $b(n) = 0$. Boring!

Minimal number: Determine the minimal number $\mathcal{M}(n)$ of MSTs among all possible 2-colorings of $[n]$

$$\mathcal{M}(n) := \min_{\chi\colon [n]\to\{R,B\}} \mathcal{M}(n,\chi)$$

and the / a corresponding coloring χ. Sounds interesting!

Example

Consider again $[6] = \{1, 2, 3, 4, 5, 6\}$.

- What is $\mathcal{M}(6)$?
- Which coloring $\chi \colon [6] \to \{R, B\}$ yields the least number of monochromatic Schur triples (MSTs)?

Example

Consider again $[6] = \{1, 2, 3, 4, 5, 6\}$.

- What is $\mathcal{M}(6)$?
- Which coloring $\chi\colon [6] \to \{R, B\}$ yields the least number of monochromatic Schur triples (MSTs)?

Answer: Choose the coloring $\chi = R^2B^3R = RRBBBR$:

$$\{1, 2, 3, 4, 5, 6\}$$

Then there exists only one single MST, namely $(1, 1, 2)$, hence $\mathcal{M}(6) = 1$.

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs
- Erdős and Szekeres (1935): rediscovery of Ramsey's theorem and simpler proof of Schur's theorem

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs
- Erdős and Szekeres (1935): rediscovery of Ramsey's theorem and simpler proof of Schur's theorem
- Alan Goodman (1959): minimum number of monochromatic triangles under a 2-coloring of a complete graph

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs
- Erdős and Szekeres (1935): rediscovery of Ramsey's theorem and simpler proof of Schur's theorem
- Alan Goodman (1959): minimum number of monochromatic triangles under a 2-coloring of a complete graph
- Graham, Rödl, Ruciński (1996) asked about the minimum number of Schur triples under any 2-coloring (100 USD prize!)

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs
- Erdős and Szekeres (1935): rediscovery of Ramsey's theorem and simpler proof of Schur's theorem
- Alan Goodman (1959): minimum number of monochromatic triangles under a 2-coloring of a complete graph
- Graham, Rödl, Ruciński (1996) asked about the minimum number of Schur triples under any 2-coloring (100 USD prize!)
- Robertson, Zeilberger (1998) answer this question asymptotically.

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs
- Erdős and Szekeres (1935): rediscovery of Ramsey's theorem and simpler proof of Schur's theorem
- Alan Goodman (1959): minimum number of monochromatic triangles under a 2-coloring of a complete graph
- Graham, Rödl, Ruciński (1996) asked about the minimum number of Schur triples under any 2-coloring (100 USD prize!)
- Robertson, Zeilberger (1998) answer this question asymptotically.
- Their result was independently confirmed by Schoen (1999), Datskovsky (2003), and Thanatipanonda (2009).

Historical remarks

- Schur (1917) studied a modular version of Fermat's last theorem
- Hilfssatz: $\exists n = n(m)$ s.t. for any m-coloring of $[n]$ an MST exists (nowadays known as Schur's theorem)
- Van der Waerden (1927): generalization to monochromatic arithmetic progressions of length k
- Ramsey (1928): same for monochromatic subgraphs
- Erdős and Szekeres (1935): rediscovery of Ramsey's theorem and simpler proof of Schur's theorem
- Alan Goodman (1959): minimum number of monochromatic triangles under a 2-coloring of a complete graph
- Graham, Rödl, Ruciński (1996) asked about the minimum number of Schur triples under any 2-coloring (100 USD prize!)
- Robertson, Zeilberger (1998) answer this question asymptotically.
- Their result was independently confirmed by Schoen (1999), Datskovsky (2003), and Thanatipanonda (2009).
- Recent work on generalized Schur triples by Butler, Costello, Graham (2010), and Thanatipanonda, Wong (2017).

Open problems session at SFB Strobl meeting

On December 3, 2018, Elaine posed some conjectures on generalized Schur triples:

Conjectures of Asymptotic Bounds

To Do	Equation	Conditions	Coloring	Min/Max	Conjecture
☒	$x+y=z$	2 colors	$\left[R^{\frac{4n}{11}}, B^{\frac{6n}{11}}, R^{\frac{n}{11}}\right]$	Min	$\frac{n^2}{22}+\mathcal{O}(n)$
☒	$x+ay=z$	2 colors $a \geq 2$	$\left[1 : a+\frac{1}{a+1} : \frac{1}{a+1}\right]$ Ratio of $R:B:R$	Min	$\frac{n^2}{2a(a^2+2a+3)}+\mathcal{O}(n)$

Puzzling observation: the case for general a does not specialize to the $a=1$ case:

$$\left.\frac{n^2}{2a(a^2+2a+3)}\right|_{a=1} = \frac{n^2}{12}$$

Three blocks

It has been shown previously (RobertsonZeilberger 98, Schoen 99) that the number $\mathcal{M}(n,\chi)$ is minimized when χ is of the form

$$R^s B^{t-s} R^{n-t},$$

where $s \approx \frac{4}{11}n$ and $t \approx \frac{10}{11}n$.

Three blocks

It has been shown previously (RobertsonZeilberger 98, Schoen 99) that the number $\mathcal{M}(n,\chi)$ is minimized when χ is of the form

$$R^s B^{t-s} R^{n-t},$$

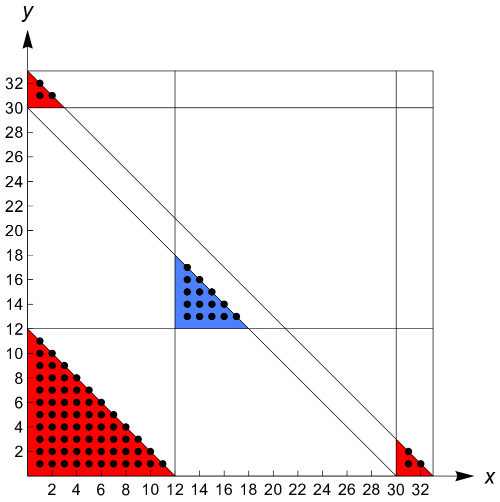
where $s \approx \frac{4}{11}n$ and $t \approx \frac{10}{11}n$.

Lemma. Let $n, s, t \in \mathbb{N}$ be such that $1 \leqslant s \leqslant t \leqslant n$. Moreover, assume that the inequalities $t \geqslant 2s$ and $s \geqslant n-t$ hold. Then the number of monochromatic Schur triples on $[n]$ under the coloring $R^s B^{t-s} R^{n-t}$ is exactly

$$\mathcal{M}(n,s,t) = \frac{s(s-1)}{2} + \frac{(t-2s)(t-2s-1)}{2} + (n-t)(n-t-1).$$

Proof (by example)

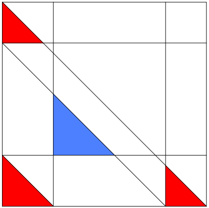
$$\mathcal{M}(n,s,t) = \frac{s(s-1)}{2} + \frac{(t-2s)(t-2s-1)}{2} + (n-t)(n-t-1).$$



- $\chi = R^{12}B^{18}R^{3}$
- $s = 12$, $t = 30$
- $\mathcal{M}(33, 12, 30) = 66 + 15 + 6 = 87$
- Actually we have $\mathcal{M}(33) = 87$

Optimal values for s and t

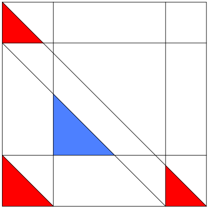
The optimal values for s and t are easily derived using the techniques of multivariable calculus:



- let n go to infinity

Optimal values for s and t

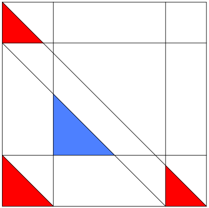
The optimal values for s and t are easily derived using the techniques of multivariable calculus:



- let n go to infinity
- scale the square $[0, n]^2 \subset \mathbb{R}^2$ to the unit square $[0, 1]^2$

Optimal values for s and t

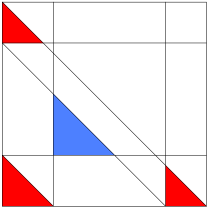
The optimal values for s and t are easily derived using the techniques of multivariable calculus:



- let n go to infinity
- scale the square $[0, n]^2 \subset \mathbb{R}^2$ to the unit square $[0, 1]^2$
- integers s and t turn into real numbers satisfying $0 \leqslant s \leqslant t \leqslant 1$

Optimal values for s and t

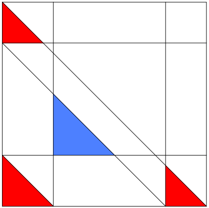
The optimal values for s and t are easily derived using the techniques of multivariable calculus:



- let n go to infinity
- scale the square $[0, n]^2 \subset \mathbb{R}^2$ to the unit square $[0, 1]^2$
- integers s and t turn into real numbers satisfying $0 \leqslant s \leqslant t \leqslant 1$
- portion of pairs $(x, y) \in [n]^2$ for which $(x, y, x + y)$ is an MST equals the area of a certain region in the unit square

Optimal values for s and t

The optimal values for s and t are easily derived using the techniques of multivariable calculus:



- let n go to infinity
- scale the square $[0, n]^2 \subset \mathbb{R}^2$ to the unit square $[0, 1]^2$
- integers s and t turn into real numbers satisfying $0 \leqslant s \leqslant t \leqslant 1$
- portion of pairs $(x, y) \in [n]^2$ for which $(x, y, x + y)$ is an MST equals the area of a certain region in the unit square
- This area is calculated by the formula

$$A(s,t) = \frac{s^2}{2} + \frac{(t-2s)^2}{2} + 2 \cdot \frac{(1-t)^2}{2}$$
$$= \frac{5s^2}{2} + \frac{3t^2}{2} - 2st - 2t + 1.$$

Optimal values for s and t

Recall:

$$A(s,t) = \frac{5s^2}{2} + \frac{3t^2}{2} - 2st - 2t + 1.$$

Equating the gradient

$$\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right) = (5s - 2t, 3t - 2s - 2)$$

to zero, one immediately gets the location of the minimum

$$(s,t) = \left(\tfrac{4}{11}, \tfrac{10}{11}\right).$$

Optimal values for discrete s and t

Lemma. For fixed $n \in \mathbb{N}$, the integers s_0 and t_0 that minimize the function $\mathcal{M}(n,s,t)$ are given by

$$s_0 = \left\lfloor \frac{4n+2}{11} \right\rfloor \qquad \text{and} \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor.$$

Optimal values for discrete s and t

Lemma. For fixed $n \in \mathbb{N}$, the integers s_0 and t_0 that minimize the function $\mathcal{M}(n, s, t)$ are given by

$$s_0 = \left\lfloor \frac{4n+2}{11} \right\rfloor \qquad \text{and} \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor.$$

Proof. By case distinction, according to n mod 11.

Optimal values for discrete s and t

Lemma. For fixed $n \in \mathbb{N}$, the integers s_0 and t_0 that minimize the function $\mathcal{M}(n, s, t)$ are given by

$$s_0 = \left\lfloor \frac{4n+2}{11} \right\rfloor \qquad \text{and} \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor.$$

Proof. By case distinction, according to n mod 11.

For example, assume $n = 11k + 5$ for $k \in \mathbb{N}$. Then

$$s_0 = \left\lfloor \frac{44k+22}{11} \right\rfloor = 4k+2 \quad \text{and} \quad t_0 = \left\lfloor \frac{110k+50}{11} \right\rfloor = 10k+4.$$

Optimal values for discrete s and t

Lemma. For fixed $n \in \mathbb{N}$, the integers s_0 and t_0 that minimize the function $\mathcal{M}(n, s, t)$ are given by

$$s_0 = \left\lfloor \frac{4n+2}{11} \right\rfloor \qquad \text{and} \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor.$$

Proof. By case distinction, according to $n \bmod 11$.

For example, assume $n = 11k + 5$ for $k \in \mathbb{N}$. Then

$$s_0 = \left\lfloor \frac{44k+22}{11} \right\rfloor = 4k+2 \quad \text{and} \quad t_0 = \left\lfloor \frac{110k+50}{11} \right\rfloor = 10k+4.$$

We want to show that among all integers $i, j \in \mathbb{Z}$ the expression $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$.

Proof (cont.)

Show that $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$:

$$\mathcal{M}(11k + 5, 4k + 2 + i, 10k + 4 + j) = \frac{1}{2}\big(2 + 5i + 5i^2 - 3j - 4ij + 3j^2 + 12k + 22k^2\big).$$

Proof (cont.)

Show that $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$:

$$\mathcal{M}(11k + 5, 4k + 2 + i, 10k + 4 + j) = \frac{1}{2}\left(2 + 5i + 5i^2 - 3j - 4ij + 3j^2 + 12k + 22k^2\right).$$

This is equivalent to showing that the polynomial

$$p(i, j) = 5i + 5i^2 - 3j - 4ij + 3j^2$$

is nonnegative for all $(i, j) \in \mathbb{Z}^2$.

Proof (cont.)

Show that $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$:

$$\mathcal{M}(11k + 5, 4k + 2 + i, 10k + 4 + j) = \frac{1}{2}\big(2 + 5i + 5i^2 - 3j - 4ij + 3j^2 + 12k + 22k^2\big).$$

This is equivalent to showing that the polynomial

$$p(i, j) = 5i + 5i^2 - 3j - 4ij + 3j^2$$

is nonnegative for all $(i, j) \in \mathbb{Z}^2$.

- Such a task can, in principle, be routinely executed by cylindrical algebraic decomposition (CAD).

Proof (cont.)

Show that $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$:

$$\mathcal{M}(11k + 5, 4k + 2 + i, 10k + 4 + j) = \frac{1}{2}\left(2 + 5i + 5i^2 - 3j - 4ij + 3j^2 + 12k + 22k^2\right).$$

This is equivalent to showing that the polynomial

$$p(i, j) = 5i + 5i^2 - 3j - 4ij + 3j^2$$

is nonnegative for all $(i, j) \in \mathbb{Z}^2$.

- Such a task can, in principle, be routinely executed by cylindrical algebraic decomposition (CAD).
- In this method, the variables i and j are treated as real variables, which causes some problems here...

CAD

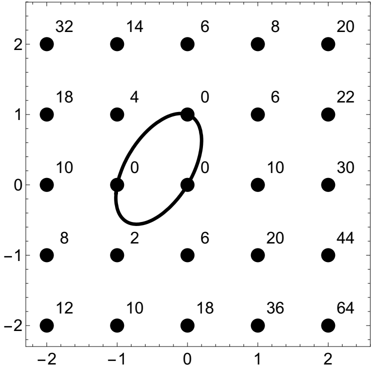
`CylindricalDecomposition[`$p(i,j) \geqslant 0$, $\{i,j\}$`]` does not yield

`True`

CAD

`CylindricalDecomposition[`$p(i,j) \geqslant 0$, $\{i,j\}$`]` does not yield

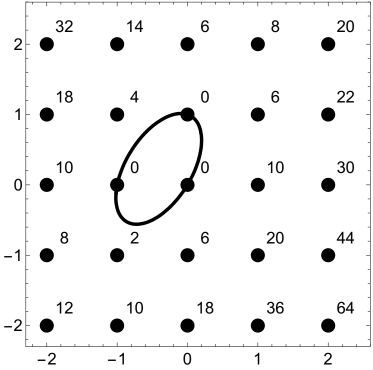
`True`



CAD

`CylindricalDecomposition[`$p(i,j) \geqslant 0$, $\{i,j\}$`]` does not yield

`True`

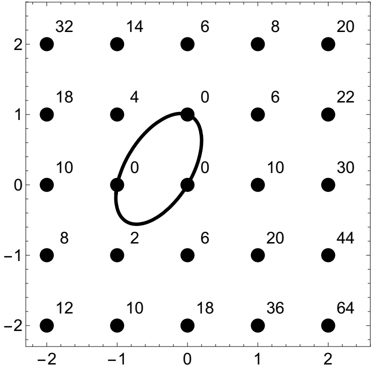


- Show that $p(i,j) \geqslant 0$ for all integer points that are close to $(0,0)$, e.g., for all (i,j) with $-2 \leqslant i \leqslant 2$ and $-2 \leqslant j \leqslant 2$.

CAD

`CylindricalDecomposition`$[p(i,j) \geqslant 0,\ \{i,j\}]$ does not yield

`True`



- Show that $p(i,j) \geqslant 0$ for all integer points that are close to $(0,0)$, e.g., for all (i,j) with $-2 \leqslant i \leqslant 2$ and $-2 \leqslant j \leqslant 2$.
- Invoke cylindrical algebraic decomposition on the formula

$$\forall i,j \in \mathbb{R}\colon (-2 \leqslant i \leqslant 2 \wedge -2 \leqslant j \leqslant 2) \vee p(i,j) \geqslant 0,$$

Exact lower bound

Theorem. The minimal number of monochromatic Schur triples that can be attained under any 2-coloring of $[n]$ is

$$\mathcal{M}(n) = \left\lfloor \frac{n^2 - 4n + 6}{11} \right\rfloor.$$

Exact lower bound

Theorem. The minimal number of monochromatic Schur triples that can be attained under any 2-coloring of $[n]$ is

$$\mathcal{M}(n) = \left\lfloor \frac{n^2 - 4n + 6}{11} \right\rfloor.$$

Proof.

$$\begin{aligned}
\ell = 0\colon \mathcal{M}(11k, 4k, 10k) &= 11k^2 - 4k &&= \tfrac{1}{11}(n^2 - 4n) \\
\ell = 1\colon \mathcal{M}(11k+1, 4k, 10k) &= 11k^2 - 2k &&= \tfrac{1}{11}(n^2 - 4n + 3) \\
\ell = 2\colon \mathcal{M}(11k+2, 4k, 10k+1) &= 11k^2 &&= \tfrac{1}{11}(n^2 - 4n + 4) \\
\ell = 3\colon \mathcal{M}(11k+3, 4k+1, 10k+2) &= 11k^2 + 2k &&= \tfrac{1}{11}(n^2 - 4n + 3) \\
\ell = 4\colon \mathcal{M}(11k+4, 4k+1, 10k+3) &= 11k^2 + 4k &&= \tfrac{1}{11}(n^2 - 4n) \\
&\ \vdots && \ \vdots \\
\ell = 9\colon \mathcal{M}(11k+9, 4k+3, 10k+8) &= 11k^2 + 14k + 4 &&= \tfrac{1}{11}(n^2 - 4n - 1) \\
\ell = 10\colon \mathcal{M}(11k+10, 4k+3, 10k+9) &= 11k^2 + 16k + 6 &&= \tfrac{1}{11}(n^2 - 4n + 6)
\end{aligned}$$

A new entry in the OEIS

The first 22 terms of the sequence $(\mathcal{M}(n))_{n\geqslant 1}$ are

$$0, 0, 0, 0, 1, 1, 2, 3, 4, 6, 7, 9, 11, 13, 15, 18, 20, 23, 26, 29, 33, 36, \ldots$$

A321195 Minimum number of monochromatic Schur triples over all 2-colorings of [n]. 1

0, 0, 0, 0, 1, 1, 2, 3, 4, 6, 7, 9, 11, 13, 15, 18, 20, 23, 26, 29, 33, 36, 40, 44, 48, 52, 57, 61, 66, 71, 76, 82, 87, 93, 99, 105, 111, 118, 124, 131, 138, 145, 153, 160, 168, 176, 184, 192, 201, 209, 218, 227, 236, 246, 255, 265, 275, 285, 295, 306, 316, 327, 338 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,7

COMMENTS A Schur triple is an integer triple (x,y,x+y). A 2-coloring on [n] = {1,...,n} is a surjective map [n] --> {Red, Blue}. A triple (x,y,z) is monochromatic if the set {x,y,z} is mapped to one color. a(n) is exactly the minimum number of monochromatic Schur triples over all possible 2-colorings of [n]. The "optimal" coloring for all n (i.e. the coloring that produces the minimum number of monochromatic Schur triples) was shown by Robertson and Zeilberger in 1998 and later with a greedy algorithm by Thanatipanonda in 2009 (see the example for the optimal coloring of [11]). The exact formula for a(n) is computed based on this optimal coloring. One might notice in the literature that there is an additional assumption x<=y. We do not impose this constraint.

LINKS Elaine Wong, Table of n, a(n) for n = 1..1000

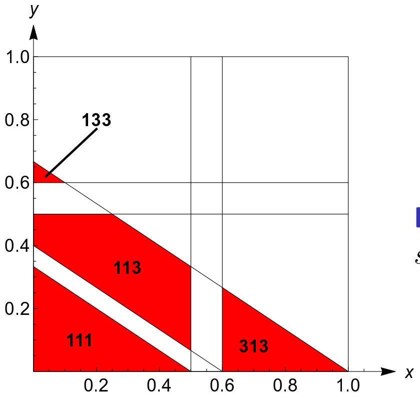
B. Datskovsky, On the number of monochromatic Schur triples, Advances in Applied Math, 31 (2003), 193-198.

A. Robertson and D. Zeilberger, A 2-coloring of [1,n] can have n^2/22+O(n) monochromatic Schur triples, but not less!, Electronic Journal of Combinatorics, 5 #R19 (1998).

T. Schoen, The number of monochromatic Schur Triples, European Journal of Combinatorics, 20 (1999), 855-866.

Generalized Schur triples

- For $a \in \mathbb{N}$, a generalized Schur triple $(x, y, z) \in \mathbb{N}^3$ satisfies $x + ay = z$.
- Extend this to $a \in \mathbb{R}^+$ by imposing $x + \lfloor ay \rfloor = z$.



Example:
$s = \frac{1}{2}$, $t = \frac{3}{5}$, $a = \frac{3}{2}$

Case distinctions for polygon 133

Polygon 133 corresponds to triples (x, y, z) that satisfy $x \leqslant s$ and $y, z \geqslant t$.

Case distinctions for polygon 133

Polygon 133 corresponds to triples (x, y, z) that satisfy $x \leqslant s$ and $y, z \geqslant t$.

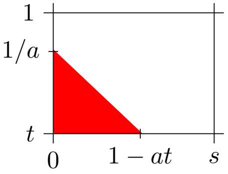
- Depending on the values of s, t, a, polygon 133 is a triangle, a quadrilateral, or a polygon with five or six vertices.
- For some values of s, t, a it is not present at all.
- Hence its area is given by a piecewise defined function.

Case distinctions for polygon 133

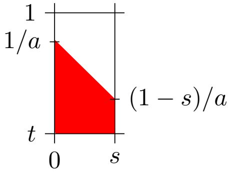
Polygon 133 corresponds to triples (x, y, z) that satisfy $x \leqslant s$ and $y, z \geqslant t$.

- Depending on the values of s, t, a, polygon 133 is a triangle, a quadrilateral, or a polygon with five or six vertices.
- For some values of s, t, a it is not present at all.
- Hence its area is given by a piecewise defined function.

(1) $t < 1/a \leqslant 1 \;\wedge\; 1 - at \leqslant s$



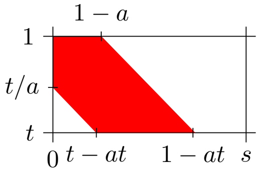
(2) $t < 1/a \leqslant 1 \;\wedge\; (1 - s)/a > t$



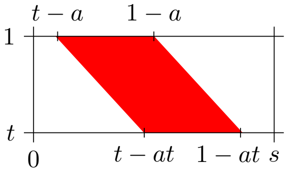
Case distinctions for polygon 133

$$(3) \quad 0 < 1 - a \leqslant s \;\wedge\; 1 - at \leqslant s$$

$(3.1) \quad t/a \leqslant 1$

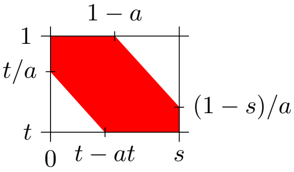


$(3.2) \quad t/a > 1$

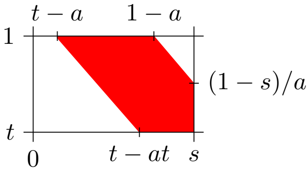


$$(4) \quad 0 < 1 - a \leqslant s \;\wedge\; (1 - s)/a > t$$

$(4.1) \quad t/a \leqslant 1$



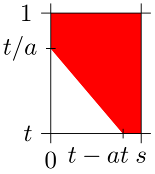
$(4.2) \quad t/a > 1$



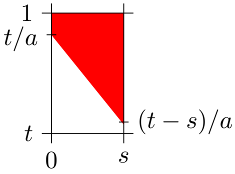
Case distinctions for polygon 133

$$(5) \quad 1 - a > s$$

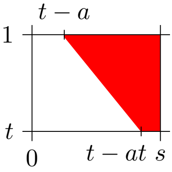
$$(5.1) \quad t/a \leqslant 1 \ \wedge \ t - at \leqslant s$$



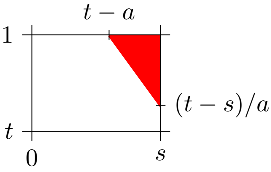
$$(5.2) \quad t/a \leqslant 1 \ \wedge \ (t - s)/a > t$$



$$(5.3) \quad t/a > 1 \ \wedge \ t - at \leqslant s$$



$$(5.4) \quad t/a > 1 \ \wedge \ t < (t - s)/a < 1$$



Case distinctions for all polygons

Performing a similar case analysis for all possible seven polygons

$$111,\ 222,\ 113,\ 131,\ 133,\ 313,\ 333$$

we encounter a set of 16 "atomic" conditions:

Case distinctions for all polygons

Performing a similar case analysis for all possible seven polygons

111, 222, 113, 131, 133, 313, 333

we encounter a set of 16 "atomic" conditions:

$$
\begin{aligned}
C_1 &\equiv 1 - as \geqslant 0, & C_2 &\equiv 1 - as - s \geqslant 0, \\
C_3 &\equiv 1 - as - t \geqslant 0, & C_4 &\equiv t - as \geqslant 0, \\
C_5 &\equiv t - as - s \geqslant 0, & C_6 &\equiv 1 - at \geqslant 0, \\
C_7 &\equiv 1 - at - s \geqslant 0, & C_8 &\equiv 1 - at - t \geqslant 0, \\
C_9 &\equiv 1 - a \geqslant 0, & C_{10} &\equiv 1 - a - s \geqslant 0, \\
C_{11} &\equiv s - a \geqslant 0, & C_{12} &\equiv 1 - a - t \geqslant 0, \\
C_{13} &\equiv t - a \geqslant 0, & C_{14} &\equiv t - a - s \geqslant 0, \\
C_{15} &\equiv s - at \geqslant 0, & C_{16} &\equiv t - at - s \geqslant 0.
\end{aligned}
$$

Case distinctions for all polygons

Performing a similar case analysis for all possible seven polygons

111, 222, 113, 131, 133, 313, 333

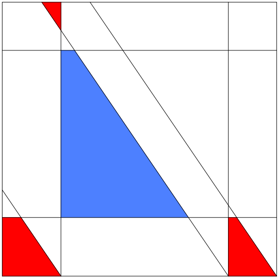
we encounter a set of 16 "atomic" conditions:

$$
\begin{aligned}
C_1 &\equiv 1 - as \geqslant 0, & C_2 &\equiv 1 - as - s \geqslant 0, \\
C_3 &\equiv 1 - as - t \geqslant 0, & C_4 &\equiv t - as \geqslant 0, \\
C_5 &\equiv t - as - s \geqslant 0, & C_6 &\equiv 1 - at \geqslant 0, \\
C_7 &\equiv 1 - at - s \geqslant 0, & C_8 &\equiv 1 - at - t \geqslant 0, \\
C_9 &\equiv 1 - a \geqslant 0, & C_{10} &\equiv 1 - a - s \geqslant 0, \\
C_{11} &\equiv s - a \geqslant 0, & C_{12} &\equiv 1 - a - t \geqslant 0, \\
C_{13} &\equiv t - a \geqslant 0, & C_{14} &\equiv t - a - s \geqslant 0, \\
C_{15} &\equiv s - at \geqslant 0, & C_{16} &\equiv t - at - s \geqslant 0.
\end{aligned}
$$

The area of each polygon is given by a piecewise function, whose definition involves logical combinations of $C_1, \ldots, C_{16}$.

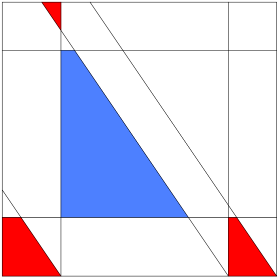
Combining piecewise functions

Recall that we are interested in the total area A of the shaded regions, as a function of s, t, a. The area of each polygon is given by a piecewise (rational) function in s, t, a. Hence, we have to express the sum of seven piecewise functions as a single (piecewise) function.



Combining piecewise functions

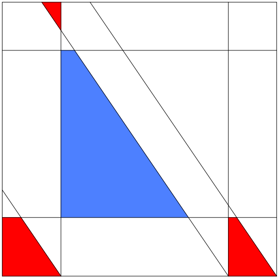
Recall that we are interested in the total area A of the shaded regions, as a function of s, t, a. The area of each polygon is given by a piecewise (rational) function in s, t, a. Hence, we have to express the sum of seven piecewise functions as a single (piecewise) function.



- Need a common refinement of the regions on which the seven area functions are defined.

Combining piecewise functions

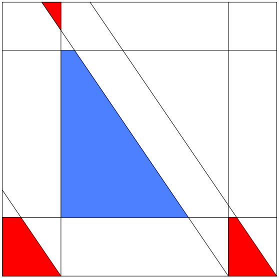
Recall that we are interested in the total area A of the shaded regions, as a function of s, t, a. The area of each polygon is given by a piecewise (rational) function in s, t, a. Hence, we have to express the sum of seven piecewise functions as a single (piecewise) function.



- Need a common refinement of the regions on which the seven area functions are defined.
- Start with the finest possible refinement, which is obtained by considering all $2^{16} = 65536$ logical combinations of C_i and $\overline{C_i}$.

Combining piecewise functions

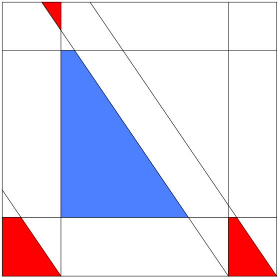
Recall that we are interested in the total area A of the shaded regions, as a function of s, t, a. The area of each polygon is given by a piecewise (rational) function in s, t, a. Hence, we have to express the sum of seven piecewise functions as a single (piecewise) function.



- Need a common refinement of the regions on which the seven area functions are defined.
- Start with the finest possible refinement, which is obtained by considering all $2^{16} = 65536$ logical combinations of C_i and $\overline{C_i}$.
- Remove those cases that contain contradictory combinations of conditions.

Combining piecewise functions

Recall that we are interested in the total area A of the shaded regions, as a function of s, t, a. The area of each polygon is given by a piecewise (rational) function in s, t, a. Hence, we have to express the sum of seven piecewise functions as a single (piecewise) function.



- Need a common refinement of the regions on which the seven area functions are defined.
- Start with the finest possible refinement, which is obtained by considering all $2^{16} = 65536$ logical combinations of C_i and $\overline{C_i}$.
- Remove those cases that contain contradictory combinations of conditions.
- Merge regions on which $A(s, t, a)$ is defined by the same expression into a single region.

The area function $A(s,t,a)$

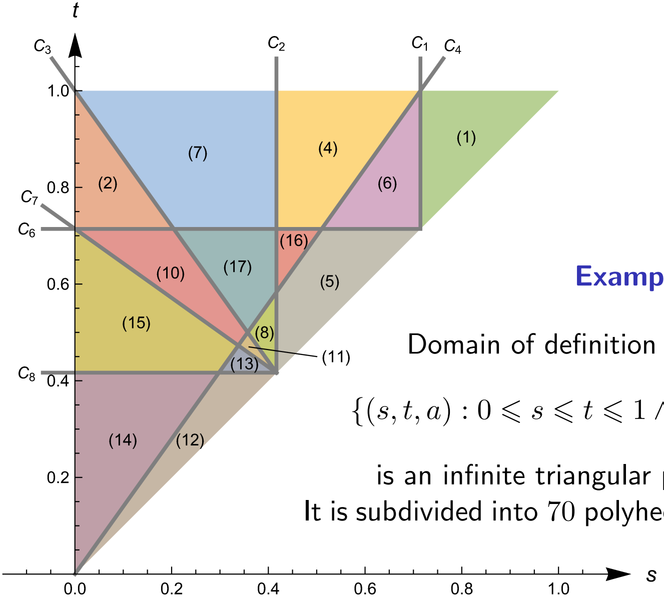
Lemma. Let $a, s, t \in \mathbb{R}$ with $a > 0$ and $0 \leqslant s \leqslant t \leqslant 1$. Then the area $A(s,t,a)$ of the region

$$\left\{(x,y) \in \mathbb{R}^2 : (x, y, x+ay) \in \left([0,s] \cup (t,1]\right)^3 \cup (s,t]^3\right\}$$

is given by the following piecewise defined function (70 cases!):

	conditions on a, s, t	$A(s,t,a)$
(R_1)	$\overline{C_1}$	$\frac{s^2-2ts+2s+t^2-2t+1}{2a}$
(R_2)	$C_3 \wedge C_4 \wedge \overline{C_6}$	$\frac{2as^2+2s^2+2as-4ats-2ts+t^2}{2a}$
(R_3)	$C_3 \wedge \overline{C_4} \wedge \overline{C_6}$	$\frac{-a^2s^2+2as^2+2s^2+2as-2ats-2ts}{2a}$
(R_4)	$\overline{C_2} \wedge C_4 \wedge \overline{C_6}$	$\frac{s^2+2as-2ats-2ts+2s+2t^2-2t}{2a}$
(R_5)	$\overline{C_2} \wedge \overline{C_4} \wedge C_6$	$\frac{-a^2s^2+s^2+2as-2ts+2s+a^2t^2+t^2-2at-2t+1}{2a}$
(R_6)	$C_1 \wedge \overline{C_2} \wedge \overline{C_4} \wedge \overline{C_6}$	$\frac{-a^2s^2+s^2+2as-2ts+2s+t^2-2t}{2a}$
(R_7)	$C_2 \wedge \overline{C_3} \wedge C_4 \wedge \overline{C_6}$	$\frac{a^2s^2+2as^2+2s^2-2ats-2ts+2t^2-2t+1}{2a}$
(R_8)	$C_2 \wedge \overline{C_3} \wedge \overline{C_4} \wedge C_6$	$\frac{2as^2+2s^2-2ts+a^2t^2+t^2-2at-2t+2}{2a}$
$\vdots$	$\vdots$	$\vdots$

Polyhedral subdivision



Example: $a = 1.4$

Domain of definition of $A(s, t, a)$

$$\{(s,t,a) : 0 \leqslant s \leqslant t \leqslant 1 \wedge a \geqslant 0\}$$

is an infinite triangular prism in $\mathbb{R}^3$.
It is subdivided into 70 polyhedral regions.

Minimize the area function $A(s,t,a)$

Lemma. For $a>0$, the minimum of the function $A(s,t,a)$

$$m(a) := \min_{0\leqslant s\leqslant t\leqslant 1} A(s,t,a)$$

is given by a piecewise rational function, depending on a:

	s_0	t_0	$m(a)$
$0\leqslant a\leqslant \alpha_1$	$\frac{(a-4)a}{a^3-a-4}$	$\frac{-2a^2+4a+2}{-a^3+a+4}$	$\frac{-a^4+2a^3-2a^2+6a-4}{2\left(a^3-a-4\right)}$
$\alpha_1\leqslant a\leqslant \alpha_2$	$\frac{a\left(a^2-3\right)}{a^4-8a-1}$	$\frac{a^3+a^2-5a-1}{a^4-8a-1}$	$\frac{a^3-2a^2+a-2}{2\left(a^4-8a-1\right)}$
$\alpha_2\leqslant a\leqslant \alpha_3$	$\frac{-2a^3+2a+1}{-a^4+8a+3}$	$\frac{2a^3+a^2-6a-2}{a^4-8a-3}$	$\frac{a^6+a^4-12a^3+4a^2-1}{2a\left(a^4-8a-3\right)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\alpha_7\leqslant a\leqslant 1$	$\frac{(a+1)^2}{a(7a+4)}$	$\frac{(a+1)(4a+1)}{a(7a+4)}$	$\frac{-7a^4+6a^3+6a^2-2a-1}{2a^2(7a+4)}$
$1\leqslant a\leqslant \alpha_8$	$\frac{(a+1)^2}{a^4+2a^3+3a^2+2a+3}$	$\frac{(a+1)\left(a^2+2a+2\right)}{a^4+2a^3+3a^2+2a+3}$	$\frac{a^4-a^2-2a+4}{2a\left(a^4+2a^3+3a^2+2a+3\right)}$
$\alpha_8\leqslant a$	$\frac{a+1}{a^2+2a+3}$	$\frac{a^2+2a+2}{a^2+2a+3}$	$\frac{1}{2a\left(a^2+2a+3\right)}$

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s,t,a)$ is defined:

- View $A(s,t,a)$ as a function in s,t with a parameter a.

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s,t,a)$ is defined:

- View $A(s,t,a)$ as a function in s, t with a parameter a.
- Compute the gradient $\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right)$.

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s,t,a)$ is defined:

- View $A(s,t,a)$ as a function in s,t with a parameter a.
- Compute the gradient $\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right)$.
- Find all points $(s,t) \in \mathbb{R}^2$ where the gradient is zero.

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s,t,a)$ is defined:

- View $A(s,t,a)$ as a function in s, t with a parameter a.
- Compute the gradient $\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right)$.
- Find all points $(s,t) \in \mathbb{R}^2$ where the gradient is zero.
- For each such point determine for which values of a it actually lies in (R_i).

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s,t,a)$ is defined:

- View $A(s,t,a)$ as a function in s,t with a parameter a.
- Compute the gradient $\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right)$.
- Find all points $(s,t) \in \mathbb{R}^2$ where the gradient is zero.
- For each such point determine for which values of a it actually lies in (R_i).

Example: On (R_2) the gradient is $\left(\frac{2as-2at+2s-t+a}{a}, \frac{t-2as-s}{a}\right)$, which is zero for

$$(s,t) = \left(\frac{a}{4a^2+2a-1}, \frac{a(2a+1)}{4a^2+2a-1}\right).$$

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s,t,a)$ is defined:

- View $A(s,t,a)$ as a function in s,t with a parameter a.
- Compute the gradient $\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right)$.
- Find all points $(s,t) \in \mathbb{R}^2$ where the gradient is zero.
- For each such point determine for which values of a it actually lies in (R_i).

Example: On (R_2) the gradient is $\left(\frac{2as-2at+2s-t+a}{a}, \frac{t-2as-s}{a}\right)$, which is zero for

$$(s,t) = \left(\frac{a}{4a^2+2a-1}, \frac{a(2a+1)}{4a^2+2a-1}\right).$$

Definition of (R_2): $as + t \leqslant 1 \ \wedge \ t \geqslant as \ \wedge \ at > 1 \ \wedge \ 0 < s < t < 1$.

Find the local minima

For each region (R_i), $1 \leqslant i \leqslant 70$, on which $A(s,t,a)$ is defined:

- View $A(s,t,a)$ as a function in s,t with a parameter a.
- Compute the gradient $\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right)$.
- Find all points $(s,t) \in \mathbb{R}^2$ where the gradient is zero.
- For each such point determine for which values of a it actually lies in (R_i).

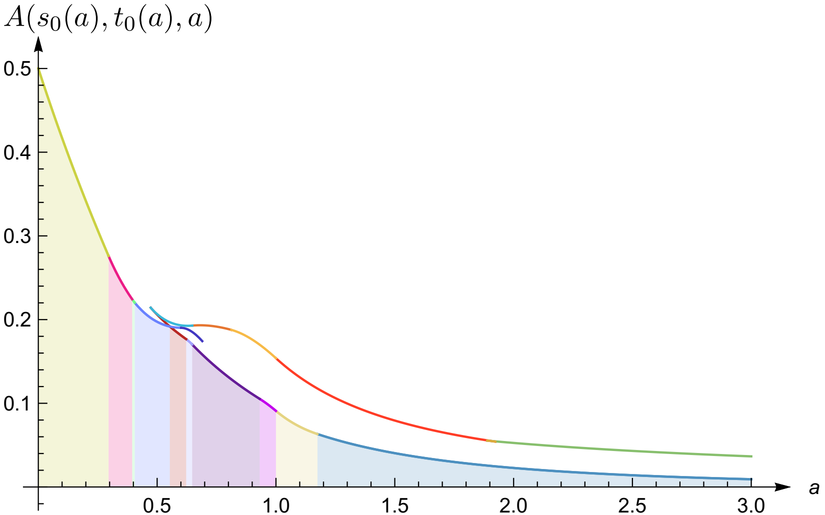
Example: On (R_2) the gradient is $\left(\frac{2as-2at+2s-t+a}{a}, \frac{t-2as-s}{a}\right)$, which is zero for

$$(s,t) = \left(\frac{a}{4a^2+2a-1}, \frac{a(2a+1)}{4a^2+2a-1}\right).$$

Definition of (R_2): $as+t \leqslant 1 \;\wedge\; t \geqslant as \;\wedge\; at > 1 \;\wedge\; 0 < s < t < 1$.
Using CAD one finds the admissible range for a:

$$a \geqslant \mathrm{Root}\big(2a^3 - 3a^2 - 2a + 1, [1,2]\big) = 1.889228559...$$

Plot of the local minima



Find the global minimum

This way we identify 17 local minima, each occurring only for a in a certain interval.

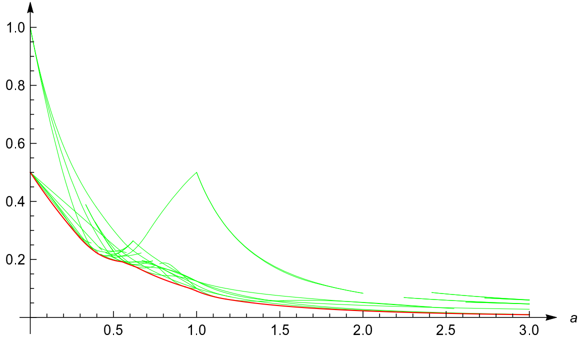
Find the global minimum

This way we identify 17 local minima, each occurring only for a in a certain interval. Once again, CAD is employed to identify the global minimum where these intervals overlap.

Find the global minimum

This way we identify 17 local minima, each occurring only for a in a certain interval. Once again, CAD is employed to identify the global minimum where these intervals overlap.

Note that $A(s, t, a)$ is defined piecewise and therefore may not be differentiable (it is, however, obvious from construction that it is continuous). $\rightarrow$ Search for minima along boundaries of regions.



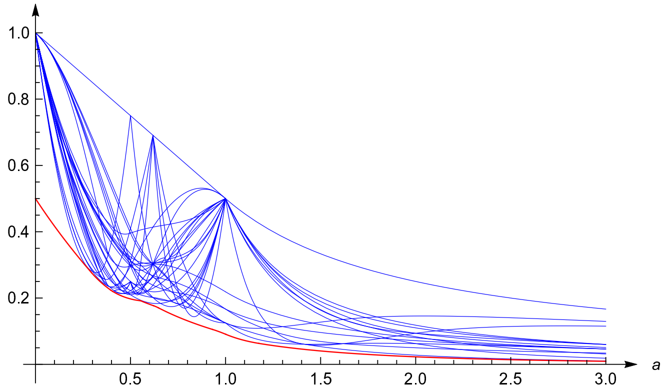
Find the global minimum

Similarly, we consider the function values of $A(s,t,a)$ at all intersections of the lines defined by the C_i (depending on a, 348 cases).

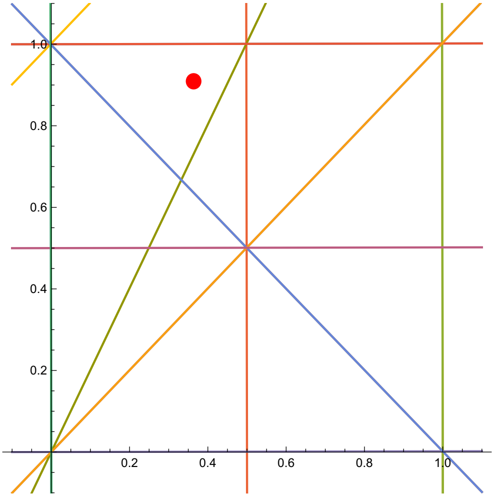
Find the global minimum

Similarly, we consider the function values of $A(s,t,a)$ at all intersections of the lines defined by the C_i (depending on a, 348 cases).

For each of them, CAD confirms (rigorously!) that the value of $A(s,t,a)$ does not go below the claimed minimum $m(a)$.

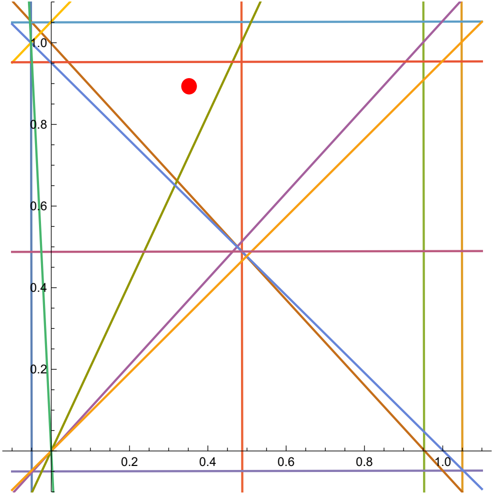


Location of the global minimum



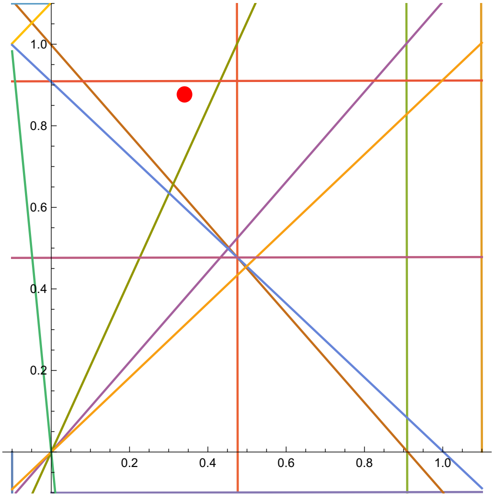
$a = 1.00$

Location of the global minimum



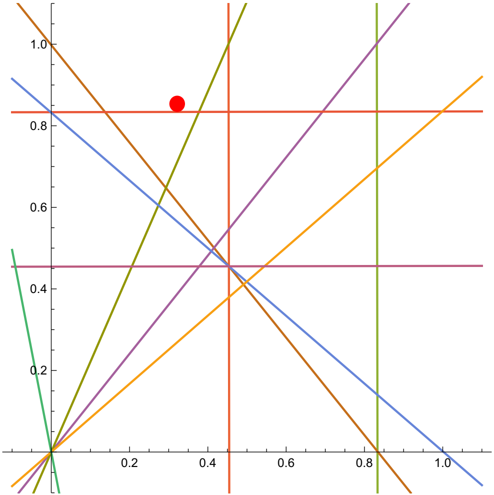
$a = 1.05$

Location of the global minimum



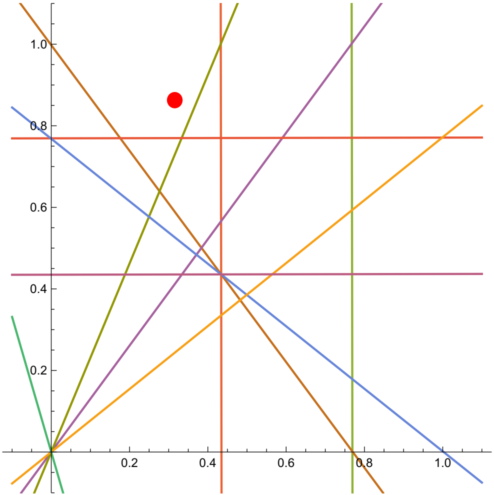
$a = 1.10$

Location of the global minimum



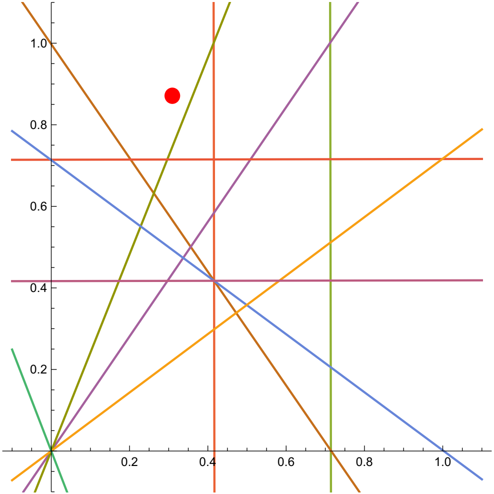
$$a = 1.20$$

Location of the global minimum



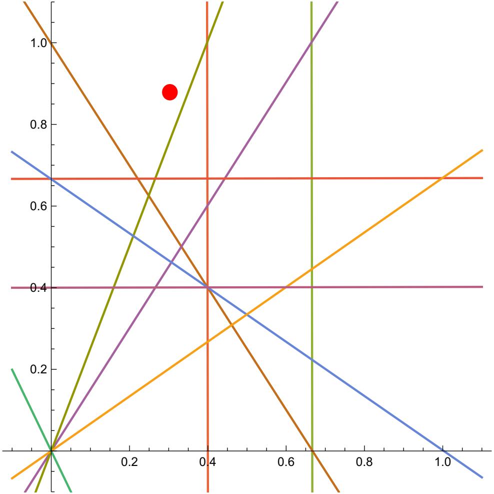
$a = 1.30$

Location of the global minimum



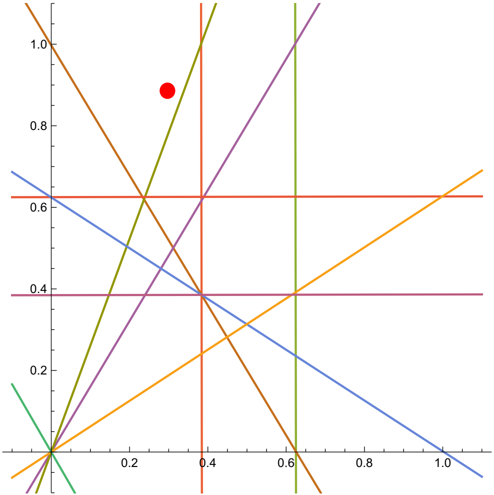
$$a = 1.40$$

Location of the global minimum



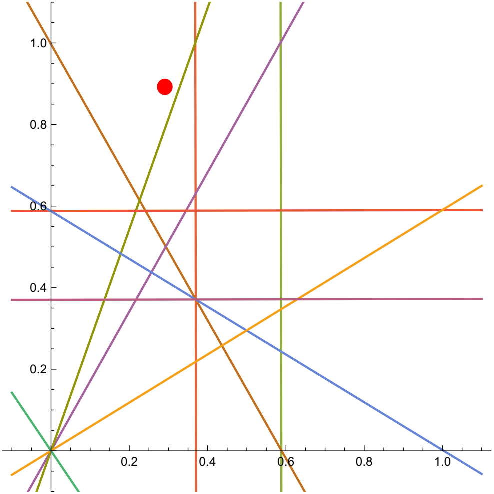
$$a = 1.50$$

Location of the global minimum



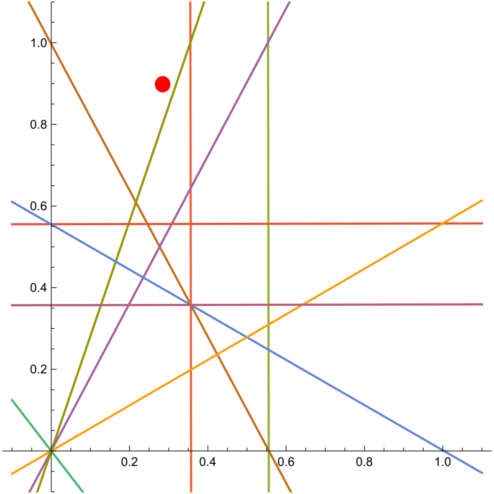
$a = 1.60$

Location of the global minimum



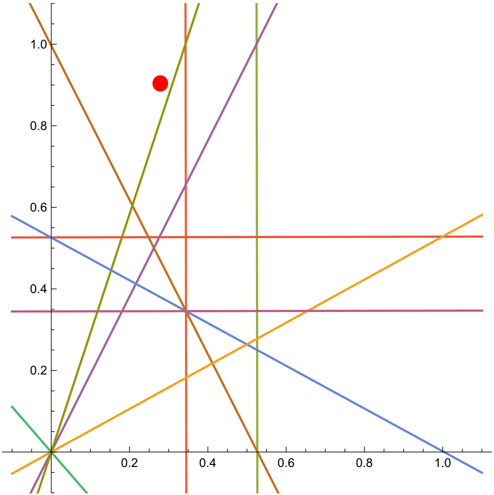
$a = 1.70$

Location of the global minimum



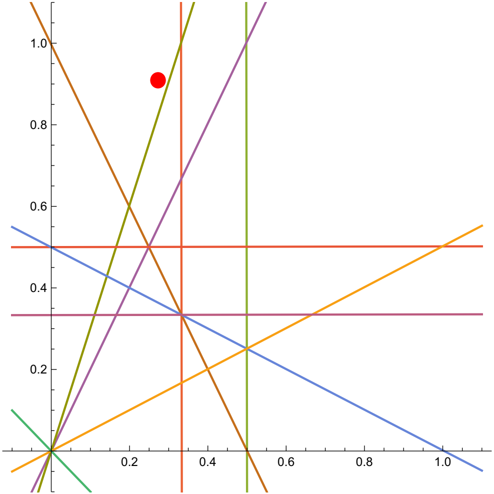
$a = 1.80$

Location of the global minimum



$a = 1.90$

Location of the global minimum



$a = 2.00$

Exact number of MSTs for a 3-block coloring

Lemma. Let $a \in \mathbb{R}$ with $a \geqslant 1$ and let $n, s, t \in \mathbb{N}$ with $1 \leqslant s \leqslant t \leqslant n$. Furthermore, assume that the inequalities $as + t \geqslant n$, $t \geqslant as$, and $s + as \leqslant t$ hold.

Then the number $\mathcal{M}^{(a)}(n, s, t)$ of monochromatic generalized Schur triples of $[n]$ under the coloring $R^s B^{t-s} R^{n-t}$ is given by

$$\sum_{y=1}^{\lfloor s/a \rfloor} \sum_{x=1}^{s-\lfloor ay \rfloor} 1 + \sum_{y=s+1}^{\lfloor (t-s)/a \rfloor} \sum_{x=s+1}^{t-\lfloor ay \rfloor} 1 + \sum_{y=1}^{\lfloor (n-t)/a \rfloor} \sum_{x=t+1}^{n-\lfloor ay \rfloor} 1 + \sum_{y=t+1}^{\lfloor n/a \rfloor} \sum_{x=1}^{n-\lfloor ay \rfloor} 1.$$

Moreover, the explicit list of these MGSTs $(x, y, x + \lfloor ay \rfloor)$ can be directly read off from the above formula.

Exact lower bound for $a = 2$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 2y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(2)}(n) = \left\lfloor \frac{n^2 - 10n + 33}{44} \right\rfloor .$$

Exact lower bound for $a = 2$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 2y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(2)}(n) = \left\lfloor \frac{n^2 - 10n + 33}{44} \right\rfloor .$$

Proof. Specialize the general formula from the previous lemma:

$$\mathcal{M}^{(2)}(n, s, t) = \left\lfloor \tfrac{s}{2} \right\rfloor \left\lfloor \tfrac{s-1}{2} \right\rfloor + \left\lfloor \tfrac{n-t}{2} \right\rfloor \left\lfloor \tfrac{n-t-1}{2} \right\rfloor + \left\lfloor \tfrac{t-s}{2} \right\rfloor \left\lfloor \tfrac{t-s-1}{2} \right\rfloor + 2s^2 - st + s.$$

Exact lower bound for $a = 2$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 2y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(2)}(n) = \left\lfloor \frac{n^2 - 10n + 33}{44} \right\rfloor.$$

Proof. Specialize the general formula from the previous lemma:

$$\mathcal{M}^{(2)}(n, s, t) = \left\lfloor \tfrac{s}{2} \right\rfloor \left\lfloor \tfrac{s-1}{2} \right\rfloor + \left\lfloor \tfrac{n-t}{2} \right\rfloor \left\lfloor \tfrac{n-t-1}{2} \right\rfloor + \left\lfloor \tfrac{t-s}{2} \right\rfloor \left\lfloor \tfrac{t-s-1}{2} \right\rfloor + 2s^2 - st + s.$$

Using our knowledge of $A(s, t, a)$, we find (empirically) that the minimum of $\mathcal{M}^{(2)}(n, s, t)$ is attained at

$$s_0 = \left\lfloor \frac{3n+1}{11} \right\rfloor, \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor + \begin{cases} -1, & \text{if } n = 22k + 10, \\ 0, & \text{otherwise.} \end{cases}$$

Exact lower bound for $a = 2$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 2y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(2)}(n) = \left\lfloor \frac{n^2 - 10n + 33}{44} \right\rfloor.$$

Proof. Specialize the general formula from the previous lemma:

$$\mathcal{M}^{(2)}(n,s,t) = \left\lfloor \tfrac{s}{2} \right\rfloor \left\lfloor \tfrac{s-1}{2} \right\rfloor + \left\lfloor \tfrac{n-t}{2} \right\rfloor \left\lfloor \tfrac{n-t-1}{2} \right\rfloor + \left\lfloor \tfrac{t-s}{2} \right\rfloor \left\lfloor \tfrac{t-s-1}{2} \right\rfloor + 2s^2 - st + s.$$

Using our knowledge of $A(s, t, a)$, we find (empirically) that the minimum of $\mathcal{M}^{(2)}(n, s, t)$ is attained at

$$s_0 = \left\lfloor \frac{3n+1}{11} \right\rfloor, \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor + \begin{cases} -1, & \text{if } n = 22k + 10, \\ 0, & \text{otherwise.} \end{cases}$$

Analogous to the $a = 1$ case, make a case distinction according to $n = 22k + \ell$ and apply CAD in each case.

Exact lower bound for $a = 3$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 3y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(3)}(n) = \left\lfloor \frac{n^2 - 18n + 101}{108} \right\rfloor + \begin{cases} 1, & \text{if } n = 54k + 36, \\ -1, & \text{if } n = 54k + 30 \\ & \text{or } n = 54k + 42 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Analogous to previous theorem, but 54 case distinctions.

Exact lower bound for $a = 4$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 4y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(4)}(n) = \left\lfloor \frac{n^2 - 28n + 245}{216} \right\rfloor - \begin{cases} 1, & \text{if } n = 108k + i \text{ for } i \in I, \\ 0, & \text{otherwise,} \end{cases}$$

where the set I is given by

$$\{0, 1, 27, 28, 43, 47, 48, 53, 58, 63, 67, 68, 69, 73, 78, 83, 88, 89, 93\}.$$

Proof. Analogous to previous theorem, but 108 case distinctions.

Exact lower bound for $a = \frac{1}{2}$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $\left(x, y, x + \lfloor \frac{1}{2} y \rfloor\right)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is given by

$$\mathcal{M}^{(1/2)}(n) = \left\lfloor \frac{15n^2 + 72}{76} \right\rfloor + \begin{cases} 1, & \text{if } n = 38k + 18 \text{ or } n = 38k + 20 \\ -1, & \text{if } n = 38k + 19, \\ 0, & \text{otherwise.} \end{cases}$$

Exact lower bound for $a = \frac{1}{2}$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $\left(x, y, x + \lfloor \frac{1}{2} y \rfloor\right)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is given by

$$\mathcal{M}^{(1/2)}(n) = \left\lfloor \frac{15n^2 + 72}{76} \right\rfloor + \begin{cases} 1, & \text{if } n = 38k + 18 \text{ or } n = 38k + 20 \\ -1, & \text{if } n = 38k + 19, \\ 0, & \text{otherwise.} \end{cases}$$

Counter-example. For $n = 4$ the theorem predicts a minimum of four MSTs, under the coloring $\{1, 2, 3, 4\}$, namely

$$(1, 1, 1), (4, 1, 4), (2, 2, 3), (2, 3, 3).$$

Exact lower bound for $a = \frac{1}{2}$

Theorem. The minimal number of monochromatic generalized Schur triples of the form $\left(x, y, x + \lfloor \frac{1}{2} y \rfloor\right)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is given by

$$\mathcal{M}^{(1/2)}(n) = \left\lfloor \frac{15n^2 + 72}{76} \right\rfloor + \begin{cases} 1, & \text{if } n = 38k + 18 \text{ or } n = 38k + 20 \\ -1, & \text{if } n = 38k + 19, \\ 0, & \text{otherwise.} \end{cases}$$

Counter-example. For $n = 4$ the theorem predicts a minimum of four MSTs, under the coloring $\{1, 2, 3, 4\}$ (1 and 4 red, 2 and 3 blue), namely

$$(1, 1, 1), (4, 1, 4), (2, 2, 3), (2, 3, 3).$$

However, for the coloring $\{1, 2, 3, 4\}$ (1 and 3 red, 2 and 4 blue) we get only three MSTs:

$$(1, 1, 1), (3, 1, 3), (2, 4, 4).$$

True minimum for $a = \frac{1}{2}$

Conjecture. For $n \geqslant 12$, the minimal number of monochromatic generalized Schur triples of the form $\left(x, y, x + \lfloor \frac{1}{2} y \rfloor\right)$ that can be attained under any 2-coloring of $[n]$ is given by

$$\left\lfloor \frac{n^2+5}{6} \right\rfloor,$$

and it occurs at the coloring $R^s B^{t-s} R^{u-t} B^{n-u}$ for

$$s = \left\lfloor \frac{n+3}{6} \right\rfloor, \qquad t = \left\lfloor \frac{n+1}{2} \right\rfloor, \qquad u = \left\lfloor \frac{5n+3}{6} \right\rfloor.$$