The Number of Realizations of Laman Graphs

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(joint work with Jose Capco, Matteo Gallet, Georg Grasegger,
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Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences

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JOHANN·RADON·INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Rigid and Non-Rigid Graphs

Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

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Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1,2), (2,3), (3,4), (1,4)\}$$

and

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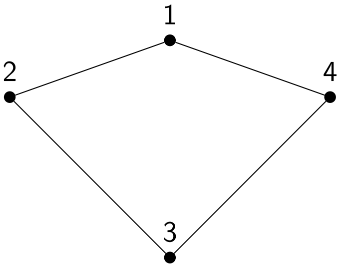

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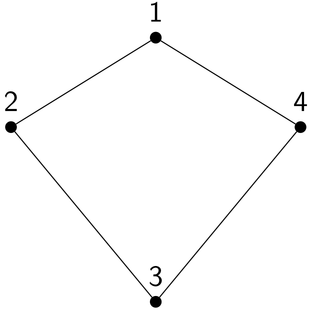

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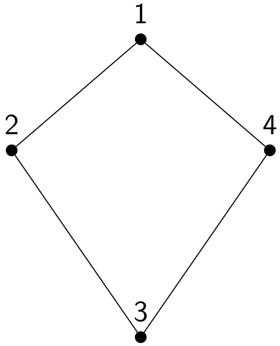

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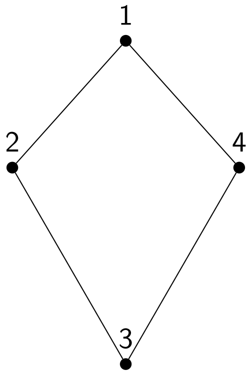

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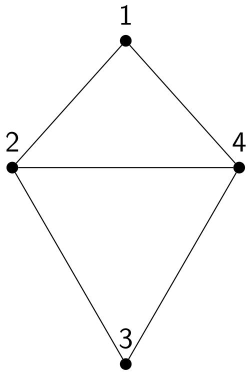

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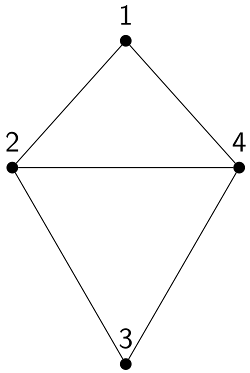

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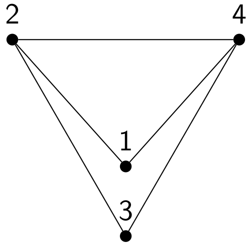

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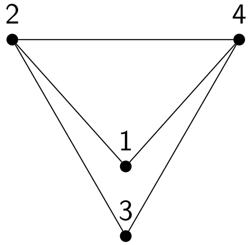

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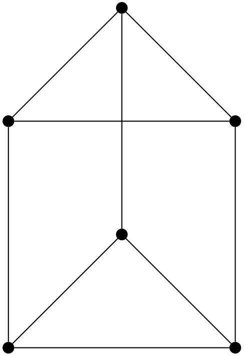
Three-Prism Graph

Is this graph rigid?



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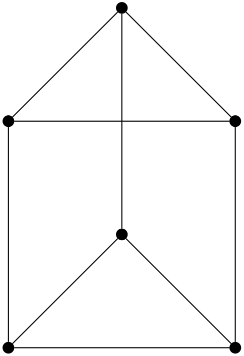

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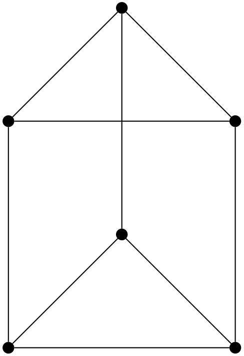

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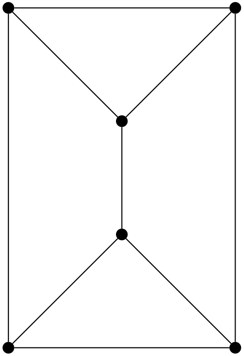

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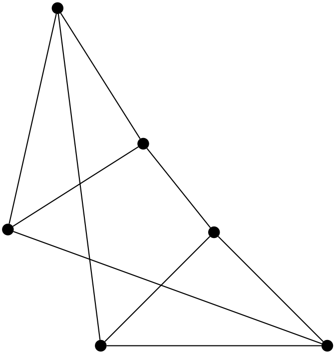

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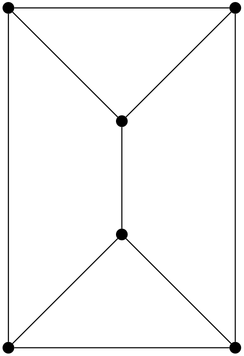

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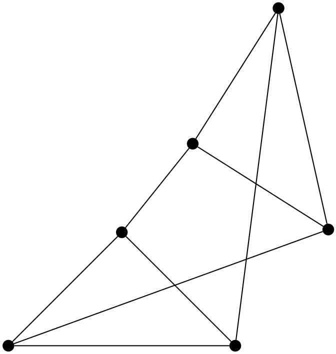

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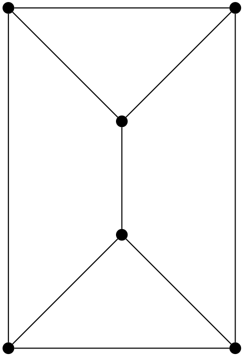

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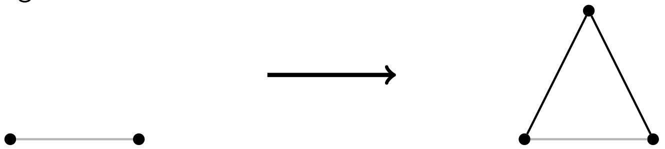
Theorem. (Laman, 1970)
A graph $G = (V, E)$ is minimally rigid if and only if

1. $|E| = 2|V| - 3$,
2. $|E'| \leqslant 2|V'| - 3$ for each subgraph $G' = (V', E')$ of G.

Henneberg Steps

Theorem. (Henneberg, 1911) The following two simple rules allow to construct any Laman graph, starting from a single edge.

Henneberg step (type I): add a new vertex and connect it to two existing vertices.



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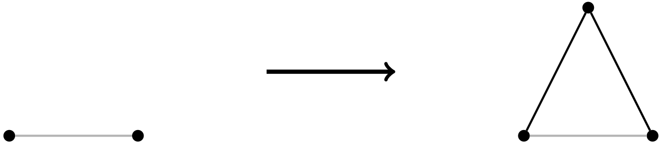
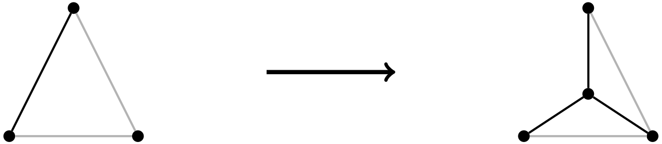


Henneberg step (type II): select three vertices of the graph, at least two of which are connected by an edge e; delete the edge e; add a new vertex and connect it to the three chosen ones.



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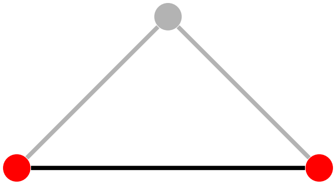

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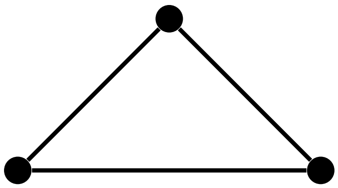

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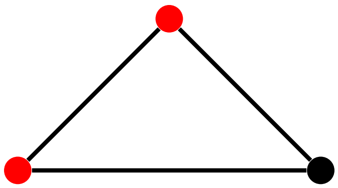

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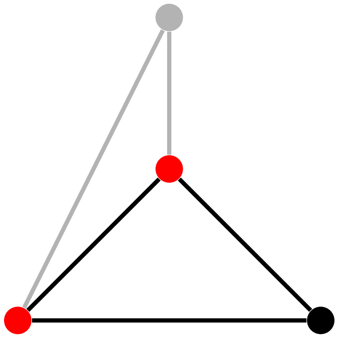

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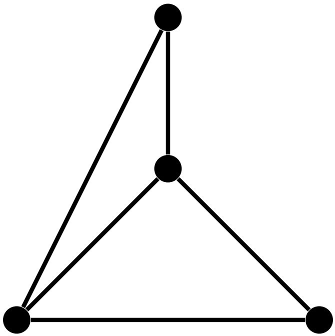

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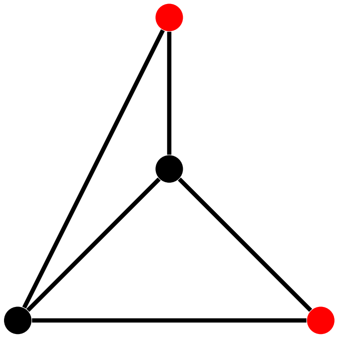

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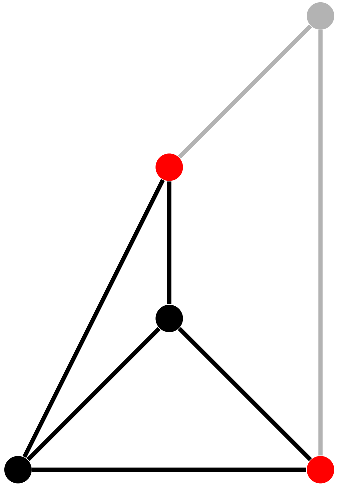

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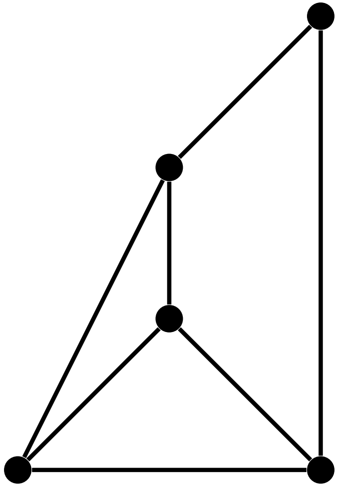

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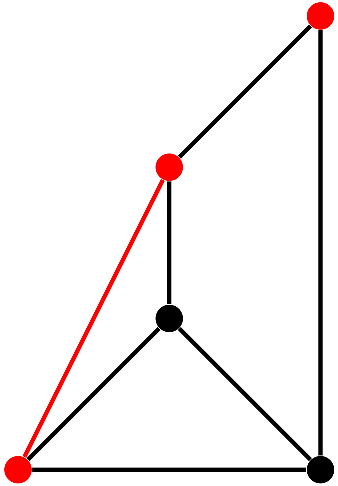

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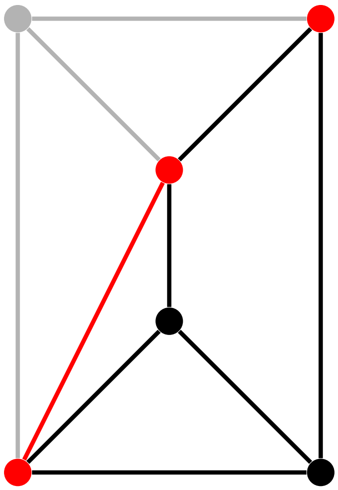

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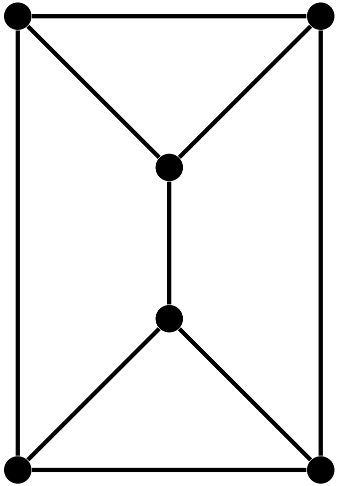
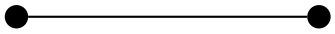
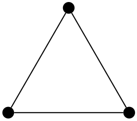
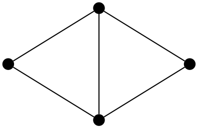


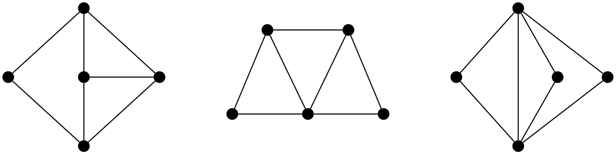
Some Laman Graphs

All Laman graphs with $2 \leqslant n \leqslant 5$ vertices:

$n = 2$: 

$n = 3$: 

$n = 4$: 

$n = 5$: 

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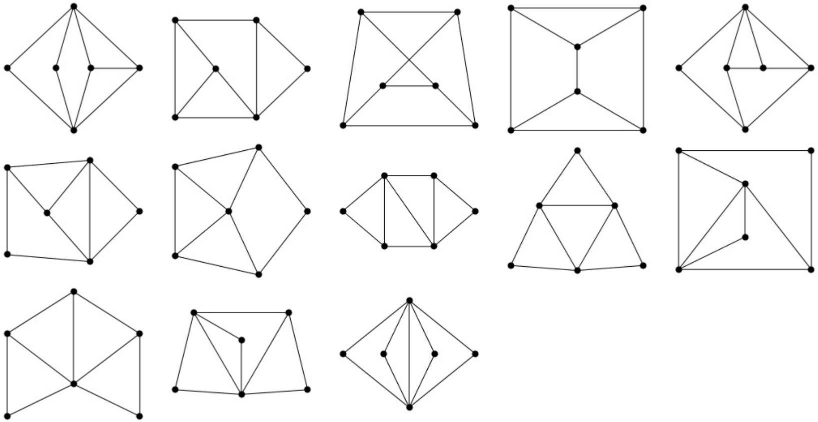
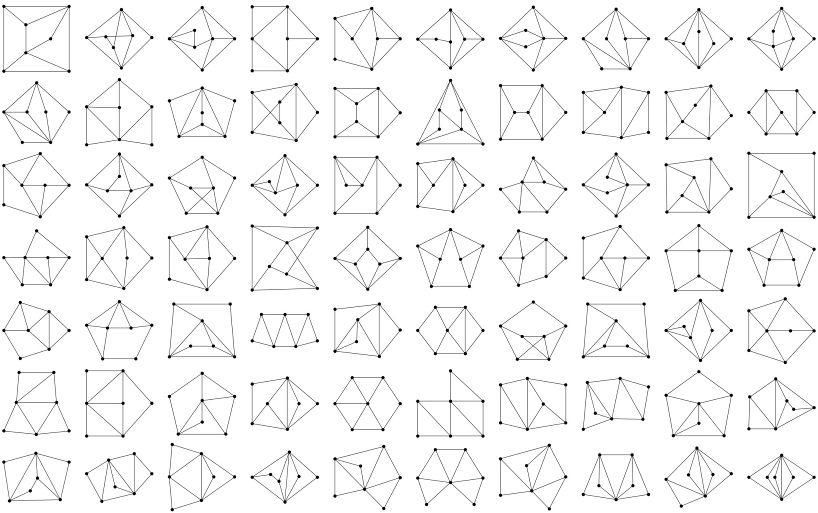
All Laman graphs with 6 vertices:



Some Laman Graphs

There are 70 Laman graphs with 7 vertices:



Enumeration of Laman graphs

Number of Laman graphs with n vertices:

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- Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- Remove duplicates (graphs that are **isomorphic**).

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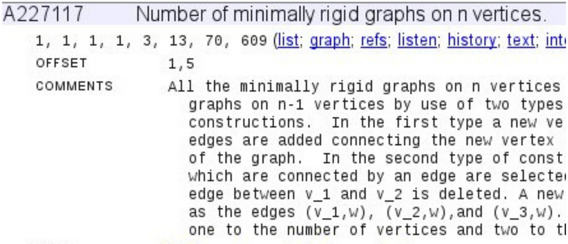
A227117 Number of minimally rigid graphs on n vertices.

1, 1, 1, 1, 3, 13, 70, 609 (list; graph; refs; listen; history; text; int

OFFSET 1,5

COMMENTS All the minimally rigid graphs on n vertices
graphs on n-1 vertices by use of two types
constructions. In the first type a new ve
edges are added connecting the new vertex
of the graph. In the second type of const
which are connected by an edge are selecte
edge between v_1 and v_2 is deleted. A new
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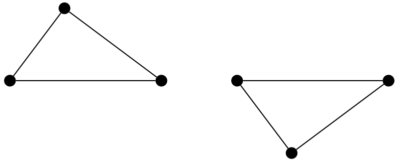

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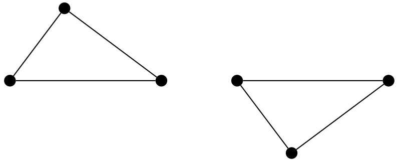
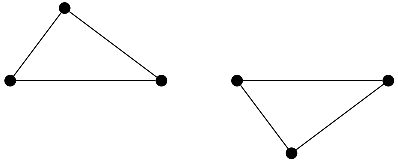


Laman graph with 4 vertices: ?

Number of Realizations

Laman graph with 3 vertices: 2 realizations



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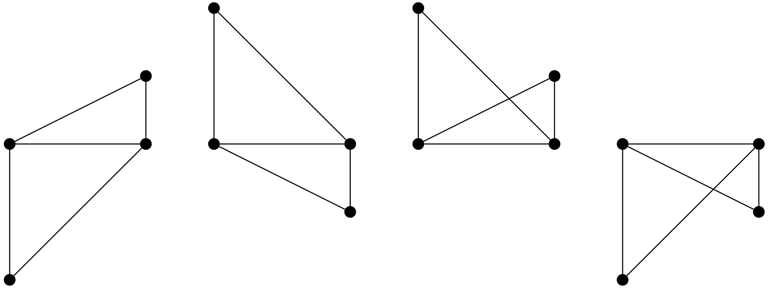
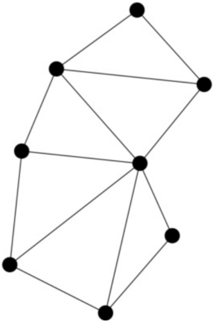


Realizations of H1 Laman Graphs



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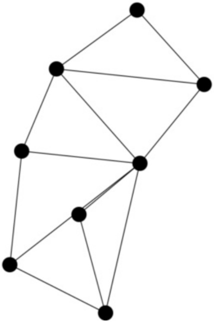

Realizations of H1 Laman Graphs

Definition: An **H1 Laman graph** is a Laman graph that can be obtained by applying a sequence of Henneberg steps of type **I**, starting with the graph $(\{1,2\},\{\{1,2\}\})$.

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- Let $G=(V,E)$ be an H1 Laman graph.
- Fix a realizable labeling $\lambda\colon E\to\mathbb{R}_{>0}$.
- Fix the positions of the first two vertices, respecting $\lambda(1,2)$.
- Each vertex that is added can be put at two different positions.

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Definition: For a Laman graph G the **Laman number** $\mathrm{Lam}(G)$ is the number of realizations of G, for a generic realizable labeling λ.

More Complex Laman Graphs

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Previous work:

- Borcea, Streinu (2004): $\mathrm{Lam}(G) \leqslant \binom{2n-4}{n-2}$ where $n = |V|$.
- Steffens, Theobald (2010): $\mathrm{Lam}(G) < 4^{n-2}$.
- Emiris, Despotakis, Psarros, Tsigaridas, Varvitsiotis (2009, 2012, 2013, 2014):

n	3	4	5	6	7	8	9	10
lower	2	4	8	24	48	96	288	576
upper	2	4	8	24	64	128	512	2048

More Complex Laman Graphs

Question: What about Laman graphs that are not H1?

Set up a system of equations:

- Let (x_v, y_v) be the coordinates of vertex v.
- For $\{u, v\} \in E$:

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

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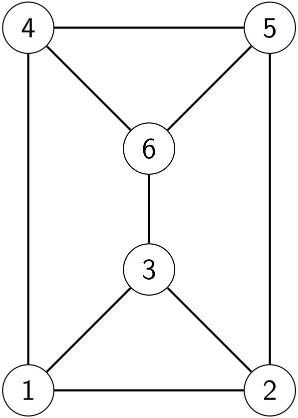
Convention: From now on we work over the complex numbers:

- $\lambda \colon E \to \mathbb{C}$
- $(x_v, y_v) \in \mathbb{C}^2$

Example: Three-Prism Graph



$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = \lambda(1,2)^2$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 = \lambda(1,3)^2$$
$$(x_1 - x_4)^2 + (y_1 - y_4)^2 = \lambda(1,4)^2$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = \lambda(2,3)^2$$
$$(x_2 - x_5)^2 + (y_2 - y_5)^2 = \lambda(2,5)^2$$
$$(x_3 - x_6)^2 + (y_3 - y_6)^2 = \lambda(3,6)^2$$
$$(x_4 - x_5)^2 + (y_4 - y_5)^2 = \lambda(4,5)^2$$
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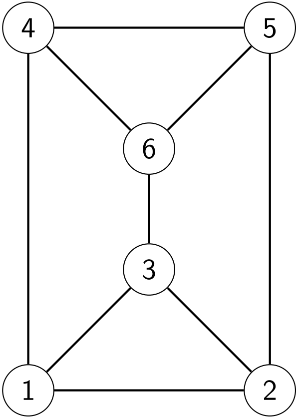

$$
\begin{aligned}
(\phantom{x_1 -{}} x_2)^2 + (\phantom{y_1 -{}} y_2)^2 &= \lambda(1,2)^2 \\
(\phantom{x_1 -{}} x_3)^2 + (\phantom{y_1 -{}} y_3)^2 &= \lambda(1,3)^2 \\
(\phantom{x_1 -{}} x_4)^2 + (\phantom{y_1 -{}} y_4)^2 &= \lambda(1,4)^2 \\
(x_2 - x_3)^2 + (y_2 - y_3)^2 &= \lambda(2,3)^2 \\
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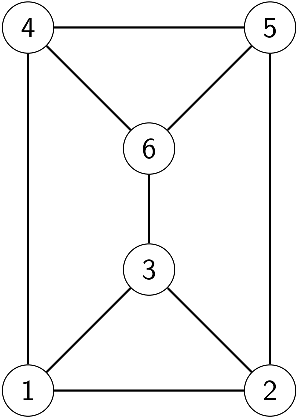

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y_2 &= \lambda(1,2) \\
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(\quad x_5)^2 + (y_2 - y_5)^2 &= \lambda(2,5)^2 \\
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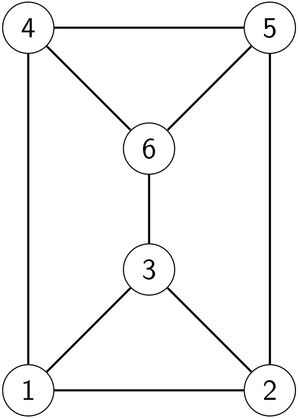

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\end{aligned}
$$

- Take care of translations: $(x_1, y_1) = (0, 0)$
- Take care of rotations: $x_2 = 0$ and $y_2 > 0$

Question: How many solutions does this system have?

- Compute a Gröbner basis!

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Do the computation modulo $p = 2^{31} - 1$:

$\{y_3 + 1727076644,\ x_5 x_6 + 1073741823 x_6^2 + y_5 y_6 + 1073741823 y_6^2 + 2147483458 y_5 + 1073746572,\ x_4 x_6 + 1073741823 x_6^2 + y_4 y_6 + 1073741823 y_6^2 + 2147472199,$

$x_3 x_6 + 1073741823 x_6^2 + 1073741823 y_6^2 + 420407003 y_6 + 2147476519,\ x_3 y_5 + 1449935236 x_4 y_5 + 87139559 x_5 y_5 + 821582392 y_4 x_6 +$

$534432936 y_5 x_6 + 2127003394 x_3 y_6 + 393122455 x_4 y_6 + 739525427 x_5 y_6 + 1428199694 x_6 y_6 + 1318362776 x_3 + 45332622 x_4 + 1666067743 x_5 + 1402190174 x_6,$

$x_5^2 + y_5^2 + 2147483269 y_5 + 2147482119,\ y_4 x_5 + 1431835485 x_4 y_5 + 1585512332 x_5 y_5 + 2099455504 y_4 x_6 + 1274481640 y_5 x_6 + 1926461619 x_3 y_6 + 1819204411 x_4 y_6 +$

$2064309228 x_5 y_6 + 1860755017 x_6 y_6 + 758303990 x_3 + 504327305 x_4 + 513732789 x_5 + 1018326077 x_6,\ x_4 x_5 + y_4 y_5 + 2147483458 y_5 + 2147472715,$

$y_4^2 + 544418756 y_4 y_5 + 47332294 y_5^2 + 1603064889 y_4 y_6 + 1508400303 y_5 y_6 + 591751051 y_6^2 + 1072510925,\ x_4 y_4 + 1252848948 x_4 y_5 + 1309508129 x_5 y_5 + 2016071435 y_4 x_6 +$

$1654953235 y_5 x_6 + 1839606594 x_3 y_6 + 577627465 x_4 y_6 + 876148120 x_5 y_6 + 335588542 x_6 y_6 + 2136682920 x_3 + 1038483051 x_4 + 157778557 x_5 + 540431639 x_6,$

$x_3 y_4 + 204011627 x_4 y_5 + 839002279 x_5 y_5 + 368180718 y_4 x_6 + 1641249205 y_5 x_6 + 430135887 x_3 y_6 + 486556477 x_4 y_6 + 1706891994 x_5 y_6 + 83415671 x_6 y_6 + 123469149 x_3 +$

$554422930 x_4 + 1257780688 x_5 + 1936702634 x_6,\ x_4^2 + 1603064891 y_4 y_5 + 2100151353 y_5^2 + 544418758 y_4 y_6 + 639083344 y_5 y_6 + 1555732596 y_6^2 + 1074934697,$

$x_3^2 + 1527353090,\ y_6^3 + 72446234 x_3 x_4 + 191839850 x_3 x_5 + 1293615843 y_4 y_5 + 2115905836 y_5^2 + 158590087 x_6^2 + 808924606 y_4 y_6 + 945043470 y_5 y_6 + 57464572 y_6^2 +$

$1051760435 y_4 + 458639039 y_5 + 890226333 y_6 + 306458357,\ x_6 y_6^2 + 1202942319 x_4 y_5 + 891621123 x_5 y_5 + 694981073 y_4 x_6 + 1268149853 y_5 x_6 +$

$566843284 x_3 y_6 + 1579449712 x_4 y_6 + 2096672325 x_5 y_6 + 217935702 x_6 y_6 + 1838771945 x_3 + 1574100689 x_4 + 890711649 x_5 + 527754025 x_6,$

$y_5 y_6^2 + 1397298562 x_3 x_4 + 1093626759 x_3 x_5 + 1874498615 y_4 y_5 + 410806791 y_5^2 + 34715881 x_6^2 + 1602680419 y_4 y_6 + 1365806073 y_5 y_6 + 1574368257 y_6^2 +$

$1986672592 y_4 + 1454700418 y_5 + 207782012 y_6 + 817238271,\ x_5 y_6^2 + 906551028 x_4 y_5 + 2088326233 x_5 y_5 + 983660499 y_4 x_6 + 2020744231 y_5 x_6 + 438982960 x_3 y_6 +$

$105460105 x_4 y_6 + 1791795415 x_5 y_6 + 752681903 x_6 y_6 + 1243232341 x_3 + 236567207 x_4 + 2039336095 x_5 + 204724127 x_6,\ y_4 y_6^2 + 1798033564 x_3 x_4 + 1368970181 x_3 x_5 +$

$2111288438 y_4 y_5 + 2116525889 y_5^2 + 631579871 x_6^2 + 2098374939 y_4 y_6 + 14559548 y_5 y_6 + 265925976 y_6^2 + 768097244 y_4 + 197849421 y_5 + 1272087803 y_6 + 1950925264,$

$x_4 y_6^2 + 2000108329 x_4 y_5 + 138882411 x_5 y_5 + 1964621882 y_4 x_6 + 1562649152 y_5 x_6 + 274800980 x_3 y_6 + 381168929 x_4 y_6 + 1561080504 x_5 y_6 + 646135501 x_6 y_6 +$

$1252024999 x_3 + 1828948462 x_4 + 1907059409 x_5 + 1062878925 x_6,\ x_3 y_6^2 + 1940064434 x_4 y_5 + 1699323466 x_5 y_5 + 2767389 y_4 x_6 + 309430653 y_5 x_6 +$

$1746152111 x_3 y_6 + 1486922955 x_4 y_6 + 1042873400 x_5 y_6 + 1877302158 x_6 y_6 + 898857598 x_3 + 2023749908 x_4 + 1369459334 x_5 + 1937240806 x_6,$

$x_6^2 y_6 + 1859350309 x_3 x_4 + 828165967 x_3 x_5 + 1319416915 y_4 y_5 + 1281531769 y_5^2 + 416445396 x_6^2 + 555896977 y_4 y_6 + 838162654 y_5 y_6 + 1094699319 y_6^2 +$

$1025635396 y_4 + 758820774 y_5 + 1932663106 y_6 + 902372666,\ y_5 x_6 y_6 + 1776737250 x_4 y_5 + 1335234339 x_5 y_5 + 197659465 y_4 x_6 + 388691694 y_5 x_6 +$

$1214819713 x_3 y_6 + 1236939013 x_4 y_6 + 1895585096 x_5 y_6 + 1457663787 x_6 y_6 + 1908824636 x_3 + 1937866443 x_4 + 906541898 x_5 + 1779256072 x_6,$

$y_4 x_6 y_6 + 392800087 x_4 y_5 + 43314235 x_5 y_5 + 1752015765 y_4 x_6 + 697637736 y_5 x_6 + 1174862040 x_3 y_6 + 1726470482 x_4 y_6 + 524280549 x_5 y_6 + 1783594194 x_6 y_6 +$

$777027038 x_3 + 1196924612 x_4 + 669351278 x_5 + 136564514 x_6,\ y_5^2 y_6 + 769157270 x_3 x_4 + 30129177 x_3 x_5 + 147541859 y_4 y_5 + 696342885 y_5^2 + 953052903 x_6^2 + 63094058 y_4 y_6 +$

$1607776536 y_5 y_6 + 2003959420 y_6^2 + 1657122998 y_4 + 1041341194 y_5 + 643382090 y_6 + 298205040,\ x_5 y_5 y_6 + 1361368571 x_4 y_5 + 443005480 x_5 y_5 + 749246637 y_4 x_6 +$

$556781711 y_5 x_6 + 268588588 x_3 y_6 + 179323388 x_4 y_6 + 260672145 x_5 y_6 + 542764427 x_6 y_6 + 2031844241 x_3 + 112806238 x_4 + 2024966158 x_5 + 1634398898 x_6,$

$y_4 y_5 y_6 + 1495723262 x_3 x_4 + 1150552515 x_3 x_5 + 647627904 y_4 y_5 + 834052394 y_5^2 + 680400990 x_6^2 + 703082161 y_4 y_6 + 1261907640 y_5 y_6 + 1146980666 y_6^2 +$

$339024153 y_4 + 1829077048 y_5 + 1120614065 y_6 + 420646718,\ x_4 y_5 y_6 + 391789202 x_4 y_5 + 1778622432 x_5 y_5 + 32574434 y_4 x_6 + 638884222 y_5 x_6 +$

$2008976092 x_3 y_6 + 1158838637 x_4 y_6 + 298082231 x_5 y_6 + 579017100 x_6 y_6 + 541015847 x_3 + 1347513279 x_4 + 1774560872 x_5 + 1614705109 x_6,$

$x_3 x_5 y_6 + 1581681716 x_3 x_4 + 486946881 x_3 x_5 + 421622009 y_4 y_5 + 1075313850 y_5^2 + 1564800523 x_6^2 + 198951616 y_4 y_6 + 1466002977 y_5 y_6 + 932669036 y_6^2 + 248319512 y_4 +$

$862020011 y_5 + 649537600 y_6 + 815933435,\ x_3 x_4 y_6 + 1343545648 x_3 x_4 + 1023324514 x_3 x_5 + 40371239 y_4 y_5 + 1905289341 y_5^2 + 1639954889 x_6^2 + 786545101 y_4 y_6 +$

$1219192433 y_5 y_6 + 321512152 y_6^2 + 1631897898 y_4 + 850776521 y_5 + 530499711 y_6 + 2036743747,\ x_6^3 + 1359379754 x_4 y_5 + 4699239 x_5 y_5 + 1446967796 y_4 x_6 +$

$260472488 y_5 x_6 + 701675423 x_3 y_6 + 1889155319 x_4 y_6 + 112548169 x_5 y_6 + 1629096917 x_6 y_6 + 658508665 x_3 + 820850436 x_4 + 665336977 x_5 + 1707190979 x_6,$

$y_5 x_6^2 + 1013046601 x_3 x_4 + 969596453 x_3 x_5 + 1553889292 y_4 y_5 + 1185309841 y_5^2 + 1987921573 x_6^2 + 1033458441 y_4 y_6 + 1320068753 y_5 y_6 + 1102491211 y_6^2 +$

$1104911459 y_4 + 1375116864 y_5 + 672833739 y_6 + 626376074,\ y_4 x_6^2 + 2009134087 x_3 x_4 + 1611713763 x_3 x_5 + 168461479 y_4 y_5 + 1706153267 y_5^2 +$

$1769015690 x_6^2 + 1182579576 y_4 y_6 + 557864255 y_5 y_6 + 503714053 y_6^2 + 176291393 y_4 + 1354065871 y_5 + 954347026 y_6 + 734410570,$

$y_5^2 x_6 + 619010252 x_4 y_5 + 916121455 x_5 y_5 + 1431371638 y_4 x_6 + 969212309 y_5 x_6 + 1949990023 x_3 y_6 + 414782496 x_4 y_6 + 1907745509 x_5 y_6 + 970368126 x_6 y_6 +$

$1740320236 x_3 + 1975330810 x_4 + 2143293978 x_5 + 252311982 x_6,\ y_4 y_5 x_6 + 558167487 x_4 y_5 + 433016430 x_5 y_5 + 2075138717 y_4 x_6 + 1434835475 y_5 x_6 +$

$531264210 x_3 y_6 + 427467244 x_4 y_6 + 1374860777 x_5 y_6 + 149117380 x_6 y_6 + 1826680361 x_3 + 969629736 x_4 + 766694650 x_5 + 1666548268 x_6,$

$y_5^3 + 649714439 x_3 x_4 + 1076476457 x_3 x_5 + 1435812662 y_4 y_5 + 2053151093 y_5^2 + 280374149 x_6^2 + 1469939973 y_4 y_6 + 1400337770 y_5 y_6 + 1634063342 y_6^2 +$

$354162717 y_4 + 737861553 y_5 + 816931778 y_6 + 1428529698,\ x_5 y_5^2 + 1136639110 x_4 y_5 + 121108532 x_5 y_5 + 2127022098 y_4 x_6 + 701800649 y_5 x_6 +$

$1281723728 x_3 y_6 + 2092528324 x_4 y_6 + 1816317333 x_5 y_6 + 1524717023 x_6 y_6 + 737384683 x_3 + 261085830 x_4 + 712596842 x_5 + 1219275979 x_6,$

$y_4 y_5^2 + 1382099903 x_3 x_4 + 1674451197 x_3 x_5 + 1964164303 y_4 y_5 + 610824582 y_5^2 + 1726175807 x_6^2 + 1045412838 y_4 y_6 + 1328732288 y_5 y_6 + 1416893499 y_6^2 +$

$509989107 y_4 + 356562705 y_5 + 701591991 y_6 + 90791056,\ x_4 y_5^2 + 1125381690 x_4 y_5 + 343309511 x_5 y_5 + 412315532 y_4 x_6 + 392837310 y_5 x_6 +$

$1859774430 x_3 y_6 + 1289634195 x_4 y_6 + 511405427 x_5 y_6 + 2104680646 x_6 y_6 + 1304660656 x_3 + 1431387822 x_4 + 2142663821 x_5 + 395031648 x_6\}$

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$y_4^2 + 544418756 y_4 y_5 + 47332294 y_5^2 + 1603064889 y_4 y_6 + 1508400303 y_5 y_6 + 591751051 y_6^2 + 1072510925,\ x_4 y_4 + 1252848948 x_4 y_5 + 1309508129 x_5 y_5 + 2016071435 y_4 x_6 +$
$1654953235 y_5 x_6 + 1839606594 x_3 y_6 + 577627465 x_4 y_6 + 876148120 x_5 y_6 + 335588542 x_6 y_6 + 2136682920 x_3 + 1038483051 x_4 + 157778557 x_5 + 540431639 x_6,$

$x_3 y_4 + 204011627 x_4 y_5 + 839002279 x_5 y_5 + 368180718 y_4 x_6 + 1641249205 y_5 x_6 + 430135887 x_3 y_6 + 486556477 x_4 y_6 + 1706891994 x_5 y_6 + 83415671 x_6 y_6 + 123469149 x_3 +$
$554422930 x_4 + 1257780688 x_5 + 1936702634 x_6,\ x_4^2 + 1603064891 y_4 y_5 + 2100151353 y_5^2 + 544418758 y_4 y_6 + 639083344 y_5 y_6 + 1555732596 y_6^2 + 1074934697,$

$x_3^2 + 1527353090,\ y_6^3 + 72446234 x_3 x_4 + 191839850 x_3 x_5 + 1293615843 y_4 y_5 + 2115905836 y_5^2 + 158590087 x_6^2 + 808924606 y_4 y_6 + 945043470 y_5 y_6 + 57464572 y_6^2 +$
$1051760435 y_4 + 458639039 y_5 + 890226333 y_6 + 306458357,\ x_6 y_6^2 + 1202942319 x_4 y_5 + 891621123 x_5 y_5 + 694981073 y_4 x_6 + 1268149853 y_5 x_6 +$
$566843284 x_3 y_6 + 1579449712 x_4 y_6 + 2096672325 x_5 y_6 + 217935702 x_6 y_6 + 1838771945 x_3 + 1574100689 x_4 + 890711649 x_5 + 527754025 x_6,$

$y_5 y_6^2 + 1397298562 x_3 x_4 + 1093626759 x_3 x_5 + 1874498615 y_4 y_5 + 410806791 y_5^2 + 34715881 x_6^2 + 1602680419 y_4 y_6 + 1365806073 y_5 y_6 + 1574368257 y_6^2 +$
$1986672592 y_4 + 1454700418 y_5 + 207782012 y_6 + 817238271,\ x_5 y_6^2 + 906551028 x_4 y_5 + 2088326233 x_5 y_5 + 983660499 y_4 x_6 + 2020744231 y_5 x_6 + 438982960 x_3 y_6 +$
$105460105 x_4 y_6 + 1791795415 x_5 y_6 + 752681903 x_6 y_6 + 1243232341 x_3 + 236567207 x_4 + 2039336095 x_5 + 204724127 x_6,\ y_4 y_6^2 + 1798033564 x_3 x_4 + 1368970181 x_3 x_5 +$
$2111288438 y_4 y_5 + 2116525889 y_5^2 + 631579871 x_6^2 + 2098374939 y_4 y_6 + 14559548 y_5 y_6 + 265925976 y_6^2 + 768097244 y_4 + 197849421 y_5 + 1272087803 y_6 + 1950925264,$

$x_4 y_6^2 + 2000108329 x_4 y_5 + 138882411 x_5 y_5 + 1964621882 y_4 x_6 + 1562649152 y_5 x_6 + 274800980 x_3 y_6 + 381168929 x_4 y_6 + 1561080504 x_5 y_6 + 646135501 x_6 y_6 +$
$1252024999 x_3 + 1828948462 x_4 + 1907059409 x_5 + 1062878925 x_6,\ x_3 y_6^2 + 1940064434 x_4 y_5 + 1699323466 x_5 y_5 + 2767389 y_4 x_6 + 309430653 y_5 x_6 +$
$1746152111 x_3 y_6 + 1486922955 x_4 y_6 + 1042873400 x_5 y_6 + 1877302158 x_6 y_6 + 898857598 x_3 + 2023749908 x_4 + 1369459334 x_5 + 1937240806 x_6,$

$x_6^2 y_6 + 1859350309 x_3 x_4 + 828165967 x_3 x_5 + 1319416915 y_4 y_5 + 1281531769 y_5^2 + 416445396 x_6^2 + 555896977 y_4 y_6 + 838162654 y_5 y_6 + 1094699319 y_6^2 +$
$1025635396 y_4 + 758820774 y_5 + 1932663106 y_6 + 902372666,\ y_5 x_6 y_6 + 1776737250 x_4 y_5 + 1335234339 x_5 y_5 + 197659465 y_4 x_6 + 388691694 y_5 x_6 +$
$1214819713 x_3 y_6 + 1236939013 x_4 y_6 + 1895585096 x_5 y_6 + 1457663787 x_6 y_6 + 1908824636 x_3 + 1937866443 x_4 + 906541898 x_5 + 1779256072 x_6,$

$y_4 x_6 y_6 + 392800087 x_4 y_5 + 43314235 x_5 y_5 + 1752015765 y_4 x_6 + 697637736 y_5 x_6 + 1174862040 x_3 y_6 + 1726470482 x_4 y_6 + 524280549 x_5 y_6 + 1783594194 x_6 y_6 +$
$777027038 x_3 + 1196924612 x_4 + 669351278 x_5 + 136564514 x_6,\ y_5^2 y_6 + 769157270 x_3 x_4 + 30129177 x_3 x_5 + 147541859 y_4 y_5 + 696342885 y_5^2 + 953052903 x_6^2 + 63094058 y_4 y_6 +$
$1607776536 y_5 y_6 + 2003959420 y_6^2 + 1657122998 y_4 + 1041341194 y_5 + 643382090 y_6 + 298205040,\ x_5 y_5 y_6 + 1361368571 x_4 y_5 + 443005480 x_5 y_5 + 749246637 y_4 x_6 +$
$556781711 y_5 x_6 + 268588588 x_3 y_6 + 179323388 x_4 y_6 + 260672145 x_5 y_6 + 542764427 x_6 y_6 + 2031844241 x_3 + 112806238 x_4 + 2024966158 x_5 + 1634398898 x_6,$

$y_4 y_5 y_6 + 1495723262 x_3 x_4 + 1150552515 x_3 x_5 + 647627904 y_4 y_5 + 834052394 y_5^2 + 680400990 x_6^2 + 703082161 y_4 y_6 + 1261907640 y_5 y_6 + 1146980666 y_6^2 +$
$339024153 y_4 + 1829077048 y_5 + 1120614065 y_6 + 420646718,\ x_4 y_5 y_6 + 391789202 x_4 y_5 + 1778622432 x_5 y_5 + 32574434 y_4 x_6 + 638884222 y_5 x_6 +$
$2008976092 x_3 y_6 + 1158838637 x_4 y_6 + 298082231 x_5 y_6 + 579017100 x_6 y_6 + 541015847 x_3 + 1347513279 x_4 + 1774560872 x_5 + 1614705109 x_6,$

$x_3 x_5 y_6 + 1581681716 x_3 x_4 + 486946881 x_3 x_5 + 421622009 y_4 y_5 + 1075313850 y_5^2 + 1564800523 x_6^2 + 198951616 y_4 y_6 + 1466002977 y_5 y_6 + 932669036 y_6^2 + 248319512 y_4 +$
$862020011 y_5 + 649537600 y_6 + 815933435,\ x_3 x_4 y_6 + 1343545648 x_3 x_4 + 1023324514 x_3 x_5 + 40371239 y_4 y_5 + 1905289341 y_5^2 + 1639954889 x_6^2 + 786545101 y_4 y_6 +$
$1219192433 y_5 y_6 + 321512152 y_6^2 + 1631897898 y_4 + 850776521 y_5 + 530499711 y_6 + 2036743747,\ x_6^3 + 1359379754 x_4 y_5 + 4699239 x_5 y_5 + 1446967796 y_4 x_6 +$
$260472488 y_5 x_6 + 701675423 x_3 y_6 + 1889155319 x_4 y_6 + 112548169 x_5 y_6 + 1629096917 x_6 y_6 + 658508665 x_3 + 820850436 x_4 + 665336977 x_5 + 1707190979 x_6,$

$y_5 x_6^2 + 1013046601 x_3 x_4 + 969596453 x_3 x_5 + 1553889292 y_4 y_5 + 1185309841 y_5^2 + 1987921573 x_6^2 + 1033458441 y_4 y_6 + 1320068753 y_5 y_6 + 1102491211 y_6^2 +$
$1104911459 y_4 + 1375116864 y_5 + 672833739 y_6 + 626376074,\ y_4 x_6^2 + 2009134087 x_3 x_4 + 1611713763 x_3 x_5 + 168461479 y_4 y_5 + 1706153267 y_5^2 +$
$1769015690 x_6^2 + 1182579576 y_4 y_6 + 557864255 y_5 y_6 + 503714053 y_6^2 + 176291393 y_4 + 1354065871 y_5 + 954347026 y_6 + 734410570,$

$y_5^2 x_6 + 619010252 x_4 y_5 + 916121455 x_5 y_5 + 1431371638 y_4 x_6 + 969212309 y_5 x_6 + 1949990023 x_3 y_6 + 414782496 x_4 y_6 + 1907745509 x_5 y_6 + 970368126 x_6 y_6 +$
$1740320236 x_3 + 1975330810 x_4 + 2143293978 x_5 + 252311982 x_6,\ y_4 y_5 x_6 + 558167487 x_4 y_5 + 433016430 x_5 y_5 + 2075138717 y_4 x_6 + 1434835475 y_5 x_6 +$
$531264210 x_3 y_6 + 427467244 x_4 y_6 + 1374860777 x_5 y_6 + 149117380 x_6 y_6 + 1826680361 x_3 + 969629736 x_4 + 766694650 x_5 + 1666548268 x_6,$

$y_5^3 + 649714439 x_3 x_4 + 1076476457 x_3 x_5 + 1435812662 y_4 y_5 + 2053151093 y_5^2 + 280374149 x_6^2 + 1469939973 y_4 y_6 + 1400337770 y_5 y_6 + 1634063342 y_6^2 +$
$354162717 y_4 + 737861553 y_5 + 816931778 y_6 + 1428529698,\ x_5 y_5^2 + 1136639110 x_4 y_5 + 121108532 x_5 y_5 + 2127022098 y_4 x_6 + 701800649 y_5 x_6 +$
$1281723728 x_3 y_6 + 2092528324 x_4 y_6 + 1816317333 x_5 y_6 + 1524717023 x_6 y_6 + 737384683 x_3 + 261085830 x_4 + 712596842 x_5 + 1219275979 x_6,$

$y_4 y_5^2 + 1382099903 x_3 x_4 + 1674451197 x_3 x_5 + 1964164303 y_4 y_5 + 610824582 y_5^2 + 1726175807 x_6^2 + 1045412838 y_4 y_6 + 1328732288 y_5 y_6 + 1416893499 y_6^2 +$
$509989107 y_4 + 356562705 y_5 + 701591991 y_6 + 90791056,\ x_4 y_5^2 + 1125381690 x_4 y_5 + 343309511 x_5 y_5 + 412315532 y_4 x_6 + 392837310 y_5 x_6 +$
$1859774430 x_3 y_6 + 1289634195 x_4 y_6 + 511405427 x_5 y_6 + 2104680646 x_6 y_6 + 1304660656 x_3 + 1431387822 x_4 + 2142663821 x_5 + 395031648 x_6\}$

Determine the Number of Solutions

Leading monomials:

y_3	x_5x_6	x_4x_6	x_3x_6	x_3y_5	x_5^2	y_4x_5	x_4x_5
y_4^2	x_4y_4	x_3y_4	x_4^2	x_3^2	y_6^3	$x_6y_6^2$	$y_5y_6^2$
$x_5y_6^2$	$y_4y_6^2$	$x_4y_6^2$	$x_3y_6^2$	$x_6^2y_6$	$y_5x_6y_6$	$y_4x_6y_6$	$y_5^2y_6$
$x_5y_5y_6$	$y_4y_5y_6$	$x_4y_5y_6$	$x_3x_5y_6$	$x_3x_4y_6$	x_6^3	$y_5x_6^2$	$y_4x_6^2$
$y_5^2x_6$	$y_4y_5x_6$	y_5^3	$x_5y_5^2$	$y_4y_5^2$	$x_4y_5^2$		

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$x_5y_6^2$	$y_4y_6^2$	$x_4y_6^2$	$x_3y_6^2$	$x_6^2y_6$	$y_5x_6y_6$	$y_4x_6y_6$	$y_5^2y_6$
$x_5y_5y_6$	$y_4y_5y_6$	$x_4y_5y_6$	$x_3x_5y_6$	$x_3x_4y_6$	x_6^3	$y_5x_6^2$	$y_4x_6^2$
$y_5^2x_6$	$y_4y_5x_6$	y_5^3	$x_5y_5^2$	$y_4y_5^2$	$x_4y_5^2$		

Monomials under the staircase:

1	y_6	x_6	y_5	x_5	y_4	x_4	x_3
y_6^2	x_6y_6	y_5y_6	x_5y_6	y_4y_6	x_4y_6	x_3y_6	x_6^2
y_5x_6	y_4x_6	y_5^2	x_5y_5	y_4y_5	x_4y_5	x_3x_5	x_3x_4

$\longrightarrow$ 24 complex solutions.

Gröbner Basis Approach

Compute a Gröbner basis of the ideal generated by

$$\{(x_u - x_v)^2 + (y_u - y_v)^2 - \lambda(u,v)^2 \mid \{u,v\} \in E\}$$
$$\cup \{x_1,\, y_1,\, x_2,\, y_2 - \lambda(1,2)\}.$$

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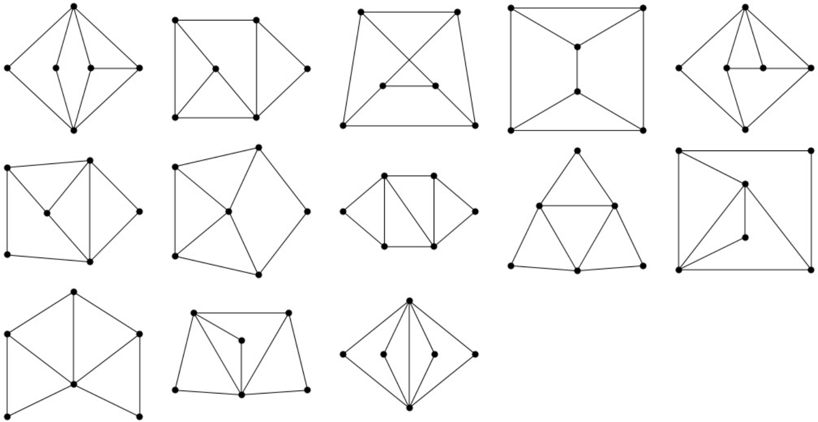
Some tricks for speed-up:

- Preprocessing: undo Henneberg steps of type I.
- Assign random values to $\lambda(u,v)$.
- Discard vertices that are "fixed" by a triangle.
- Compute modulo a prime number.

$\longrightarrow$ This way, we were able to compute the Laman numbers of all (approximately 118,000) Laman graphs with $\leqslant 10$ vertices (using Maple's fgb).

Laman Numbers

All but one Laman graphs with 6 vertices have Laman number 16.



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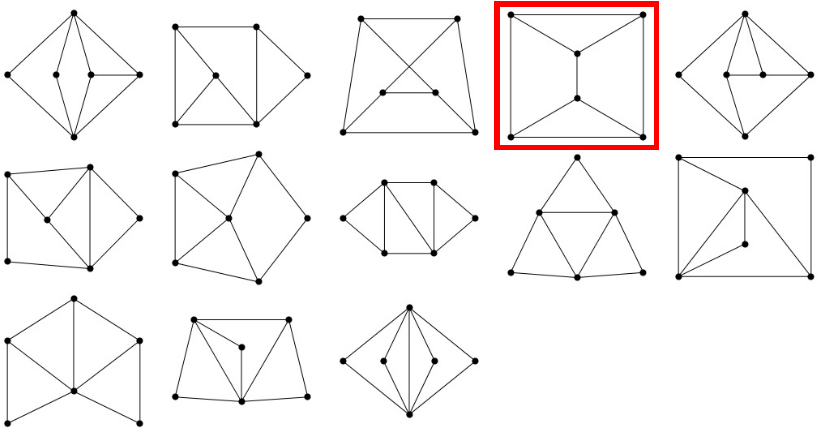

The only exception is the three-prism graph with $\mathrm{Lam}(\blacksquare) = 24$.

Laman Number as Degree

Recall: For each edge $\{u,v\} \in E$ we get an equation

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u,v)^2.$$

Idea: $\operatorname{Lam}(G)$ is given by the **degree** of the map

$$f_G\colon\ \mathbb{C}^V \times \mathbb{C}^V \to \mathbb{C}^E,$$
$$(x_1,\ldots,x_n,y_1,\ldots,y_n) \mapsto \Big((x_u - x_v)^2 + (y_u - y_v)^2\Big)_{\{u,v\}\in E}$$

i.e., by the number how often a generic $\big(\lambda(u,v)\big)_{\{u,v\}\in E} \in \mathbb{C}^E$ is hit by the map f_G (modulo translations and rotations).

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In order to apply methods from algebraic geometry, we want:

- to work in projective space;
- f_G then should be a homogeneous map.

Laman Number as Degree

"Homogenize" the map f_G by a change of coordinates:

$$(x_u - x_v)^2 + (y_u - y_v)^2 =$$

$$\big((x_u - x_v) + \mathrm{i}(y_u - y_v)\big) \cdot \big((x_u - x_v) - \mathrm{i}(y_u - y_v)\big) =$$

$$\big(\underbrace{(x_u + \mathrm{i}y_u)}_{\substack{\downarrow \\ x_u}} - \underbrace{(x_v + \mathrm{i}y_v)}_{\substack{\downarrow \\ x_v}}\big) \cdot \big(\underbrace{(x_u - \mathrm{i}y_u)}_{\substack{\downarrow \\ y_u}} - \underbrace{(x_v - \mathrm{i}y_v)}_{\substack{\downarrow \\ y_v}}\big)$$

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Hence our map becomes

$$\begin{aligned} f_G\colon\ & \mathbb{C}^V \times \mathbb{C}^V \to \mathbb{C}^E, \\ & (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \big((x_u - x_v) \cdot (y_u - y_v)\big)_{\{u,v\} \in E} \end{aligned}$$

Laman Number as Degree

In order to handle translations and rotations, we

- move one vertex to the origin (for each connected component),
- fix the position of another vertex (using projective space $\mathbb{P}$),
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$$L_G := \left\langle \left(\chi_C(v)\right)_{v \in V} \,\middle|\, C \in \mathrm{Comp}(G) \right\rangle \subseteq \mathbb{C}^V,$$

where $\chi_C(v)$ is 1 if $v \in C$ and 0 otherwise.

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where $\chi_C(v)$ is 1 if $v \in C$ and 0 otherwise.

Proposition: The Laman number $\mathrm{Lam}(G)$ of $G = (V, E)$ is given by the degree of the map

$$f_G\colon\ \mathbb{P}(\mathbb{C}^V / L_G) \times \mathbb{P}(\mathbb{C}^V / L_G) \to \mathbb{P}^{|E|-1},$$
$$\left[(x_v)_{v\in V}\right], \left[(y_v)_{v\in V}\right] \mapsto \left((x_u - x_v)\cdot(y_u - y_v)\right)_{\{u,v\}\in E}$$

Bigraphs

Definition: A **bigraph** $B = (G, H)$ is a pair of graphs $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$, allowing several components, multiple edges and self-loops. The set $\mathcal{E}$ is called the set of **biedges**.

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We define the corresponding map f_B for a bigraph:

$$
\begin{aligned}
f_B \colon \mathbb{P}(\mathbb{C}^V / L_G) \times \mathbb{P}(\mathbb{C}^W / L_H) &\to \mathbb{P}^{|\mathcal{E}|-1}, \\
\left[(x_v)_{v \in V}\right], \left[(y_w)_{w \in W}\right] &\mapsto \big((x_u - x_v) \cdot (y_t - y_w)\big)_{e \in \mathcal{E}}
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where $\{u, v\} \subseteq V$ are the vertices to which e is connected in G, and analogously for $\{t, w\} \subseteq W$.

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Definition: The **Laman number** $\mathrm{Lam}(B)$ of a bigraph B is defined to be $\deg(f_B)$.

Proposition: For $B = (G, G)$ we have $\mathrm{Lam}(B) = \mathrm{Lam}(G)$.

Puiseux Series

- $\mathbb{K} = \mathbb{C}\{\{s\}\}$: field of Puiseux series with coefficients in $\mathbb{C}$
- This field comes with a valuation $\nu\colon \mathbb{K} \setminus \{0\} \longrightarrow \mathbb{Q}$:

$$\nu\left(\sum_{i=k}^{+\infty} c_i \, s^{i/n}\right) = \frac{k}{n} \qquad \text{if } c_k \neq 0,$$

  i.e., the order of a Puiseux series.
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For the map $f_{B,\mathbb{K}}\colon \mathbb{P}_{\mathbb{K}}^{(\dots)} \times \mathbb{P}_{\mathbb{K}}^{(\dots)} \longrightarrow \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$, obtained as the extension of scalars from f_B, we have $\deg(f_{B,\mathbb{K}}) = \deg(f_B)$.

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- $\mathbb{K} = \mathbb{C}\{\{s\}\}$: field of Puiseux series with coefficients in $\mathbb{C}$
- This field comes with a valuation $\nu\colon \mathbb{K} \setminus \{0\} \longrightarrow \mathbb{Q}$:

$$\nu\left(\sum_{i=k}^{+\infty} c_i\, s^{i/n}\right) = \frac{k}{n} \qquad \text{if } c_k \neq 0,$$

  i.e., the order of a Puiseux series.
- $\nu(a \cdot b) = \nu(a) + \nu(b)$ and $\nu(a+b) \geqslant \min\{\nu(a), \nu(b)\}$

For the map $f_{B,\mathbb{K}}\colon \mathbb{P}_{\mathbb{K}}^{(\dots)} \times \mathbb{P}_{\mathbb{K}}^{(\dots)} \longrightarrow \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$, obtained as the extension of scalars from f_B, we have $\deg(f_{B,\mathbb{K}}) = \deg(f_B)$.

Study the preimage of a “perturbed” point in $\mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$:

$$f_{B,\mathbb{K}}^{-1}\Big(\big(\lambda_e s^{\mathrm{wt}(e)}\big)_{e\in\mathcal{E}}\Big) \quad \text{for some } \mathrm{wt} \in \mathbb{Q}^{\mathcal{E}} \text{ and } \lambda \in \mathbb{C}^{\mathcal{E}},$$

instead of studying the preimage $f_B^{-1}(p)$ for some $p \in \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}|-1}$.

New Coordinates, New Equations

Introduce new coordinates

- x_{uv} for all $u, v \in V$ that are connected by an edge in G
- y_{tw} for all $t, w \in W$ that are connected by an edge in H

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$\longrightarrow$ They correspond to the factors $(x_u - x_v)$ resp. $(y_t - y_w)$.

Select a distinguished biedge $\bar{e} \in \mathcal{E}$. Then these coordinates satisfy the system of equations:

$$
\begin{aligned}
x_{\bar{u}\bar{v}} = y_{\bar{t}\bar{w}} &= 1 && \\
x_{uv}\, y_{tw} &= \lambda_e s^{\mathrm{wt}(e)} && \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\} \\
\textstyle\sum_{\mathscr{C}} x_{uv} &= 0 && \text{for all cycles } \mathscr{C} \text{ in } G \\
\textstyle\sum_{\mathscr{D}} y_{tw} &= 0 && \text{for all cycles } \mathscr{D} \text{ in } H
\end{aligned}
$$

In particular, $x_{uv} = -x_{vu}$.

Tropicalization

Goal: For a fixed point $p = \left(\lambda_e\, s^{\mathrm{wt}(e)}\right)_{e\in\mathcal{E}} \in \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$ we want to determine its preimages $f_{B,\mathbb{K}}^{-1}(p)$.

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Idea:

- Apply **tropicalization**: look only at the valuations!
- An algebraic relation between Puiseux series implies a piecewise linear relation between their orders.
- For $q \in f_{B,\mathbb{K}}^{-1}(p)$ let $d_V(u,v) = \nu(q_{x_{uv}})$, $d_W(t,w) = \nu(q_{y_{tw}})$.
- This way we obtain a discrete object, a pair of functions (d_V, d_W), that we call **bidistance**.

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Gain: We can then partition the set $f_{B,\mathbb{K}}^{-1}(p)$ according to the bidistances that are determined by its elements.

Bidistances

The functions d_V and d_W satisfy

- $d_V(u,v) = d_V(v,u)$ for all (u,v), and similarly for d_W
- $d_V(u,v) + d_W(t,w) = \mathrm{wt}(e)$ for all $e \in \mathcal{E} \setminus \{\bar{e}\}$
- $d_V(\bar{u},\bar{v}) = d_W(\bar{t},\bar{w}) = 0$
- for every cycle $\mathscr{C}$ in G, the minimum of the values of d_V on the pairs of vertices (u,v) appearing in $\mathscr{C}$ is attained at least twice, and similarly for d_W.

Definition: Every pair of functions (d_V, d_W) satisfying the above conditions is called a **bidistance** compatible with $\mathrm{wt} \in \mathbb{Q}^{|\mathcal{E}|-1}$.

Recursion for the Laman number

Idea: We partition the set $f^{-1}_{B,\mathbb{K}}(p)$ according to the bidistances.

Lemma: The number of preimages sharing the same bidistance d can be obtained as the Laman number of a “simpler” Graph B_d.

Hence we obtain the following recursion:
Theorem:

$$\mathrm{Lam}(B) = \sum_{d} \mathrm{Lam}(B_d).$$

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Unfortunately, it is not very useful for practical purposes:

1. Enumeration of bidistances d: **difficult**
2. Computation of $\mathrm{Lam}(B_d)$: **difficult**

Two specializations in order to get more explicit formulas. . .

First Strategy

By choosing a general weight vector $\mathrm{wt} \in \mathbb{Q}^{|\mathcal{E}|-1}$, one can show that $\mathrm{Lam}(B_d) = 1$ for every bidistance d compatible with wt.

Hence $\mathrm{Lam}(B)$ equals the number of such bidistances.

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The computation of $\mathrm{Lam}(B)$ is therefore reduced to a piecewise linear problem:

1. Enumeration of bidistances d: **difficult**
2. Computation of $\mathrm{Lam}(B_d)$: **trivial**

Second Strategy

Idea: We choose the special weight vector $(1, \ldots, 1) \in \mathbb{Q}^{|\mathcal{E}|-1}$.

We can show that in this case the values of d_V and d_W are

- integers
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Hence, each bidistance can be characterized by a single vector in $\{0,1\}^{|\mathcal{E}|-1}$ (since $d_V + d_W = 1$ for all $e \in \mathcal{E} \setminus \{\bar{e}\}$).

1. Enumeration of bidistances d: **easy**
2. Computation of $\mathrm{Lam}(B_d)$: **feasible**

Operations on Graphs

For constructing the graph B_d, we need to introduce two operations on graphs:

- complement
- quotient

Graph Complement

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph complement** $G \setminus E'$ is defined as

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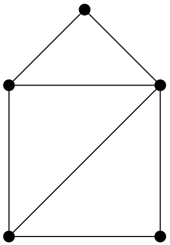

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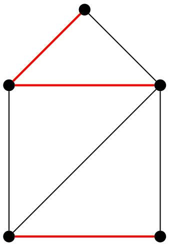

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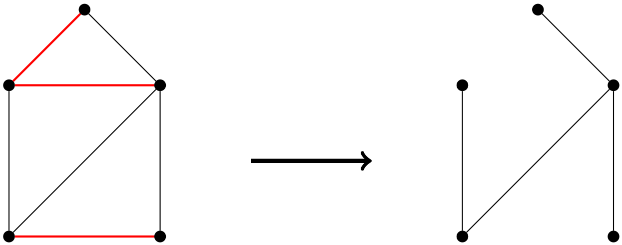

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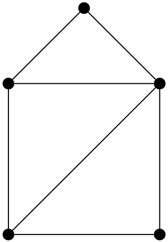

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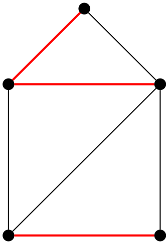

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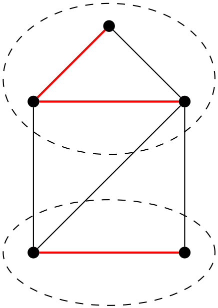

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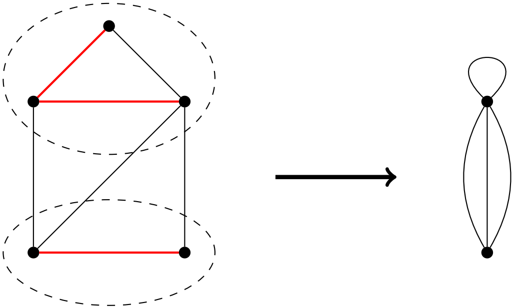

Operations on Bigraphs

We define the following two operations on a bigraph $B = (G, H)$:
For a subset $\mathcal{M} \subseteq \mathcal{E}$ of the biedges $\mathcal{E}$ let

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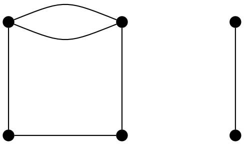
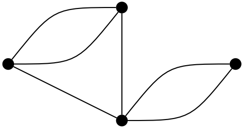


$H =$



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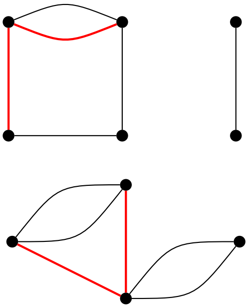

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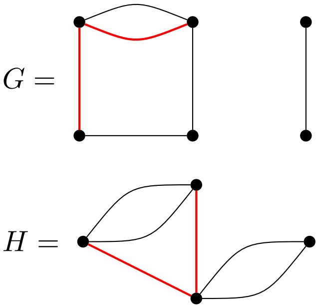

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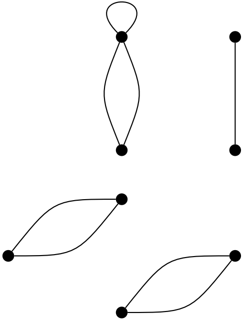

The bigraph ${}^{\mathcal{M}}B$

The Combinatorial Algorithm

Theorem. Let $B = (G, H)$ be a bigraph with $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Choose $\bar{e} \in \mathcal{E}$. Then

$$\mathrm{Lam}(B) = \mathrm{Lam}({}^{\{\bar{e}\}}B) + \mathrm{Lam}(B^{\{\bar{e}\}}) + \sum_{\substack{\mathcal{M} \cup \mathcal{N} = \mathcal{E} \\ \mathcal{M} \cap \mathcal{N} = \{\bar{e}\}}} \mathrm{Lam}({}^{\mathcal{M}}B) \cdot \mathrm{Lam}(B^{\mathcal{N}}).$$

Initial conditions:

- $\mathrm{Lam}(G) = \mathrm{Lam}(G, G)$
- $\mathrm{Lam}(B) = 0$ if G or H contains a loop.
- $\mathrm{Lam}(B) = 0$ if $|V| - |\operatorname{Comp}(G)| + |W| - |\operatorname{Comp}(H)| \neq |\mathcal{E}| + 1$.
- $\mathrm{Lam}(B) = 1$ if $|\mathcal{E}| = 1$ and if there are no loops.

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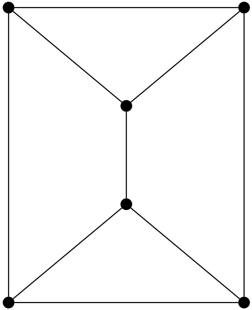
n	6
$\#$	24



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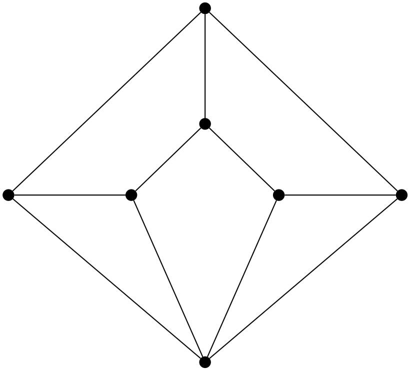
n	6	7
#	24	56



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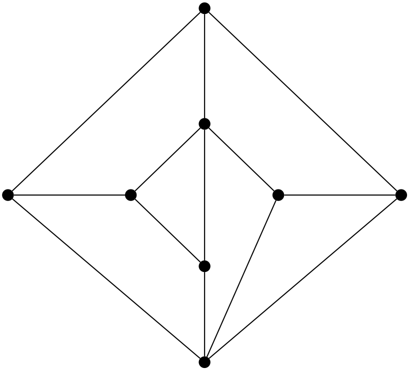
n	6	7	8
#	24	56	136



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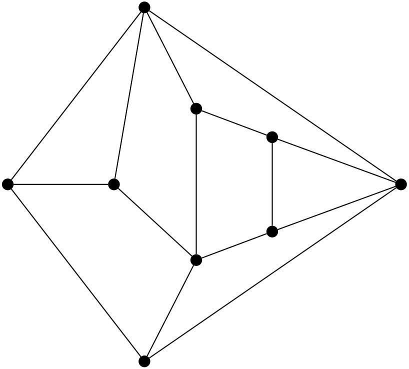
n	6	7	8	9
#	24	56	136	344



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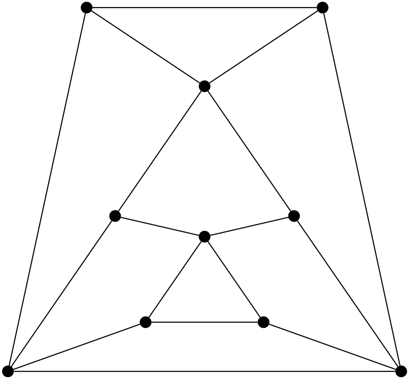
Question: Among all Laman graphs with n vertices, which one has the largest number of realizations?

n	6	7	8	9	10
#	24	56	136	344	880



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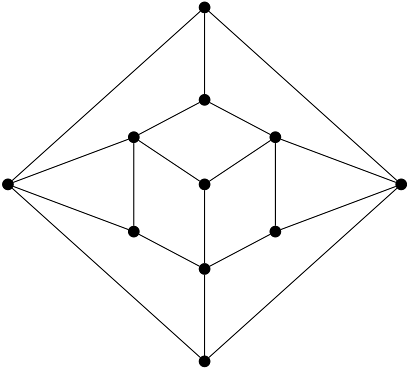
Question: Among all Laman graphs with n vertices, which one has the largest number of realizations?

n	6	7	8	9	10	11
#	24	56	136	344	880	2288



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n	6	7	8	9	10	11
$\#$	24	56	136	344	880	2288
$\frac{1}{8}$	3	7	17	43	110	286

A114589 Number of hill-free Dyck paths of semilength n+3 and having no peaks at even levels (a hill in a Dyck path is a peak at level 1). 1

1, 1, 3, 7, 17, 43, 110, 286, 753, 2003, 5376, 14540, 39589, 108427, 298512, 825664, 2293271, 6393539, 17885835, 50191175, 141247519, 398537101, 1127203038, 3195229662, 9076078057, 25830193513, 73643406563, 210312889095 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS Column 0 of A114588. The number of hill-free Dyck paths having no peaks at odd level are given by the Riordan numbers (A005043).
Contribution from Paul Barry, Jul 05 2009: (Start)
The sequence 1,0,0,1,1,3,7,... has g.f. ((1+x)(1+2x)-sqrt((1+x)(1-3x))/(2x(2+2x+x^2)). It is
the inverse binomial transform of A035929(n+1). (End)

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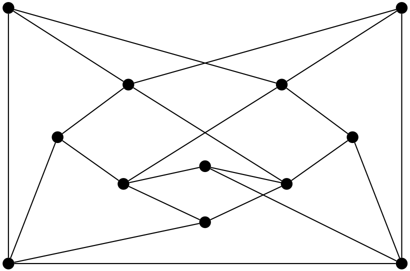
Question: Among all Laman graphs with n vertices, which one has the largest number of realizations?

n	6	7	8	9	10	11	12
#	24	56	136	344	880	2288	6180
$\frac{1}{8}$	3	7	17	43	110	286	772.5



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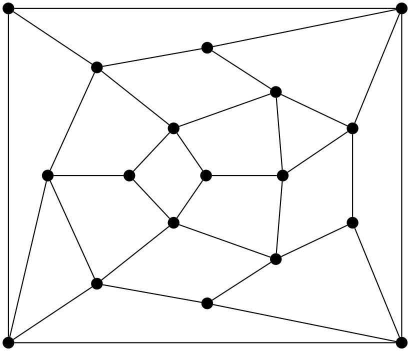
n	6	7	8	9	10	11	12		18
#	24	56	136	344	880	2288	6180	...	$\geqslant 1953816$
$\frac{1}{8}$	3	7	17	43	110	286	772.5		$\geqslant 244227$



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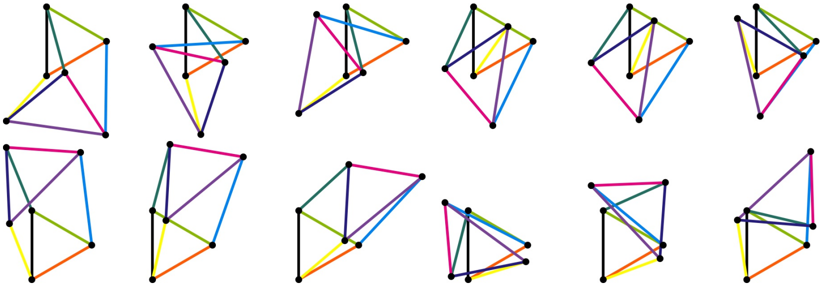

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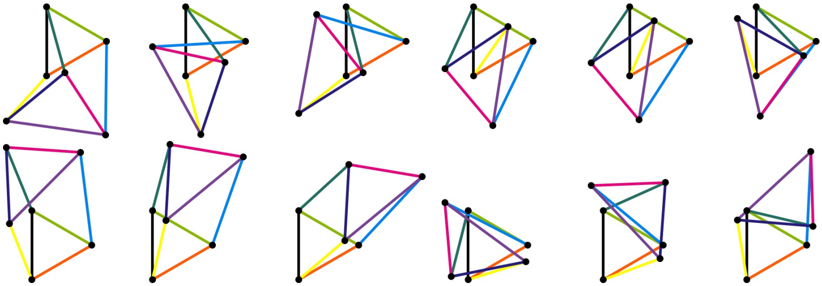

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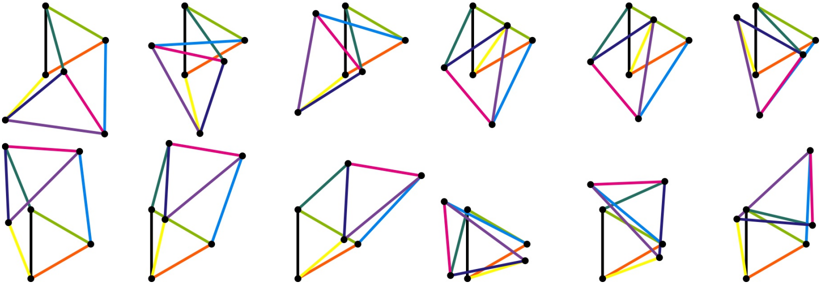

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Answer: Show movie.