# Apéry limits and irrationality proofs

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# Apéry

**Theorem (Apéry, 1978):** $\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

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- The story and the proof were written up in the classic paper *A Proof that Euler Missed* by Alfred van der Poorten

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Apéry's proof uses the recurrence equation:

$$n^3 f_n = (34n^3 - 51n^2 + 27n - 5) f_{n-1} - (n-1)^3 f_{n-2} \qquad (1)$$

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be defined by (1) as follows:

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$$\begin{aligned} u_0 = 1,\, u_1 = 5 \implies u_2 &= 73, & u_3 &= 1445, & u_4 &= 33001 \\ v_0 = 0,\; v_1 = 6 \implies v_2 &= \frac{351}{4}, & v_3 &= \frac{62531}{36}, & v_4 &= \frac{11424695}{288} \end{aligned}$$

- The numbers $u_n$ are all integers.
- $d_n^3 v_n \in \mathbb{Z}$ where $d_n := \operatorname{lcm}(1, 2, \ldots, n)$.

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$$\int_0^1 \int_0^1 \frac{1}{1 - xy} \, \mathrm{d}x \, \mathrm{d}y = \frac{\pi^2}{6} = \zeta(2)$$

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$$I(n) := \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y.$$

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$$I(0) = \frac{\pi^2}{6}, \quad I(1) = 5 - \frac{\pi^2}{2}, \quad I(2) = -\frac{125}{4} + \frac{19\pi^2}{6}, \ \ldots$$

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$$I(n) = u_n \frac{\pi^2}{6} - v_n$$

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$$\begin{aligned}
I(0) &= \tfrac{\pi^2}{6} &&= 1.6449340668482264365\\
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We see

$$I(n) = u_n \frac{\pi^2}{6} - v_n \quad \text{and} \quad \lim_{n\to\infty} \frac{v_n}{u_n} = \frac{\pi^2}{6}$$

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Hence, the family of integrals $I(n)$ yields a sequence of rational approximations to $\zeta(2)$:

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Since $u_n, v_n \in \mathbb{Q}$, let's clear denominators and write

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Since $u_n, v_n \in \mathbb{Q}$, let's clear denominators and write

$$\frac{v_n}{u_n} = \frac{v'_n}{u'_n}, \quad u'_n, v'_n \in \mathbb{Z}.$$

By estimating the size of the integral $I(n)$, one can show, by denoting $I'(n) = u'_n \frac{\pi^2}{6} - v'_n$:

$$\lim_{n\to\infty} |I'(n)| = 0 \quad \text{and} \quad I'(n) \neq 0.$$

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Starting from the integral

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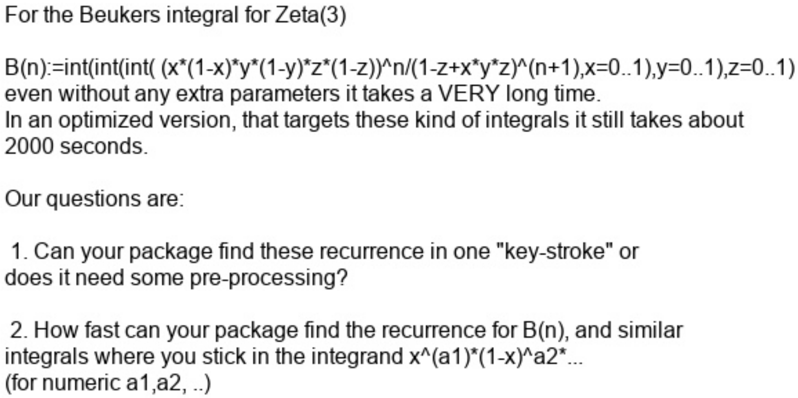
and more generally: $I(n) = u_n\zeta(3) - v_n$. In fact, $I(n)$ satisfies

$$(n+2)^3 I(n+2) = (2n+3)(17n^2+51n+39)I(n+1) - (n+1)^3 I(n).$$

# Zeilberger

An e-mail from Doron Zeilberger (dated 28.08.2020):



For the Beukers integral for Zeta(3)

B(n):=int(int(int( (x*(1-x)*y*(1-y)*z*(1-z))^n/(1-z+x*y*z)^(n+1),x=0..1),y=0..1),z=0..1)
even without any extra parameters it takes a VERY long time.
In an optimized version, that targets these kind of integrals it still takes about
2000 seconds.

Our questions are:

1. Can your package find these recurrence in one "key-stroke" or
does it need some pre-processing?

2. How fast can your package find the recurrence for B(n), and similar
integrals where you stick in the integrand x^(a1)*(1-x)^a2*...
(for numeric a1,a2, ..)

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**Task:** Show that the Beukers integral for $\zeta(3)$ satisfies Apéry's second-order recurrence:

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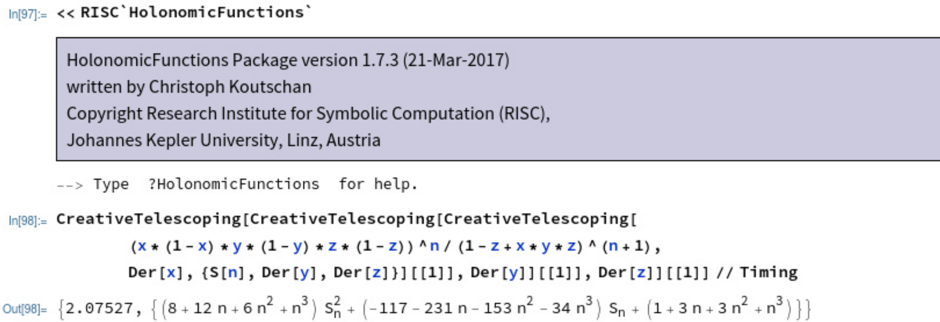

```
In[97]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.

In[98]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
           (x * (1 - x) * y * (1 - y) * z * (1 - z)) ^n / (1 - z + x * y * z) ^ (n + 1),
           Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing
```

Out[98]= $\{2.07527, \{(8+12n+6n^2+n^3)S_n^2+(-117-231n-153n^2-34n^3)S_n+(1+3n+3n^2+n^3)\}\}$

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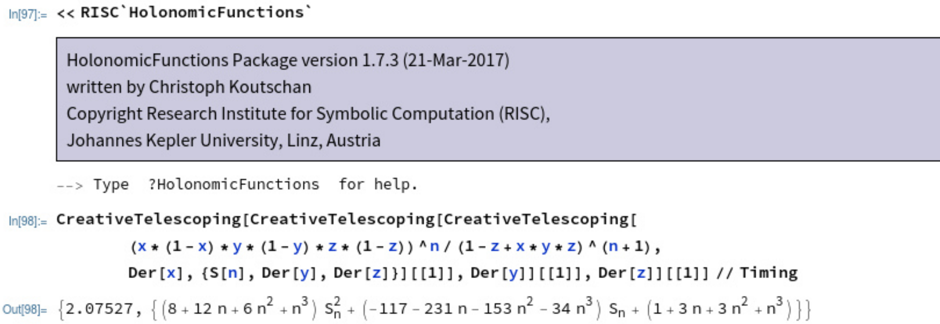

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$\longrightarrow$ *Wow, we are really impressed!*
*We will rave about your package in our forthcoming paper...*

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for another function $S(x)$ (and their multidimensional analogs).

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for another function $S(x)$ (and their multidimensional analogs). Of course $I(0) = C$.

# Generalization of the Beukers Integral

- Look at the following generalization of Beukers' integral for certain values of the parameters $a_1, a_2, b_1, b_2, c_1, c_2, d$:

$$\int_0^1 \int_0^1 \int_0^1 x^{a_1}(1-x)^{a_2} y^{b_1}(1-y)^{b_2} z^{c_1}(1-z)^{c_2} \\ \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+d+1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

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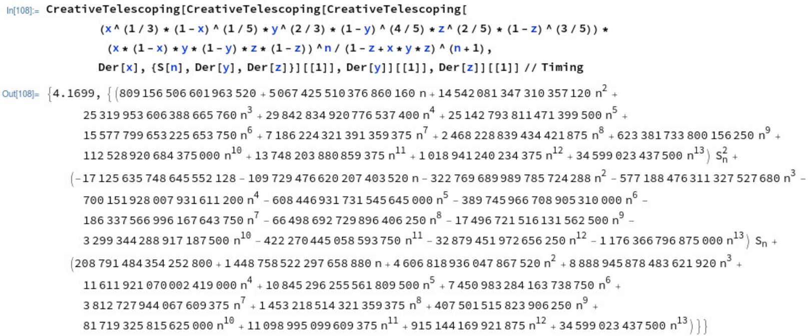
- Hope that this gives irrationality proos of some interesting constants...

# Generalized Integral with Numeric Parameters

$$\int_0^1 \int_0^1 \int_0^1 x^{1/3}(1-x)^{1/5} y^{2/3}(1-y)^{4/5} z^{2/5}(1-z)^{3/5} \\ \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

# Generalized Integral with Numeric Parameters

$$\int_0^1\int_0^1\int_0^1 x^{1/3}(1-x)^{1/5}y^{2/3}(1-y)^{4/5}z^{2/5}(1-z)^{3/5} \\ \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$



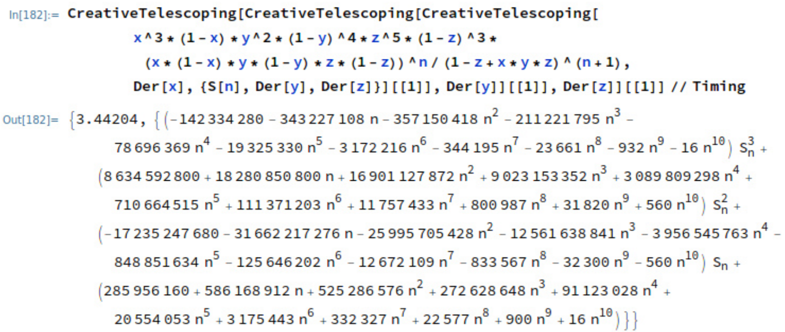
```
In[108]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
            (x ^ (1 / 3) * (1 - x) ^ (1 / 5) * y ^ (2 / 3) * (1 - y) ^ (4 / 5) * z ^ (2 / 5) * (1 - z) ^ (3 / 5)) *
             (x * (1 - x) * y * (1 - y) * z * (1 - z)) ^ n / (1 - z + x * y * z) ^ (n + 1),
            Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing

Out[108]= {4.1699, {(809 156 506 601 963 520 + 5 067 425 510 376 860 160 n + 14 542 081 347 310 357 120 n^2 +
           25 319 953 606 388 665 760 n^3 + 29 842 834 920 776 537 400 n^4 + 25 142 793 811 471 399 500 n^5 +
           15 577 799 653 225 653 750 n^6 + 7 186 224 321 391 359 375 n^7 + 2 468 228 839 434 421 875 n^8 + 623 381 733 800 156 250 n^9 +
           112 528 920 684 375 000 n^10 + 13 748 203 880 859 375 n^11 + 1 018 941 240 234 375 n^12 + 34 599 023 437 500 n^13) S_n^2 +
          (-17 125 635 748 645 552 128 - 109 729 476 620 207 403 520 n - 322 769 689 989 785 724 288 n^2 - 577 188 476 311 327 527 680 n^3 -
           700 151 928 007 931 611 200 n^4 - 608 446 931 731 545 645 000 n^5 - 389 745 966 708 905 310 000 n^6 -
           186 337 566 996 167 643 750 n^7 - 66 498 692 729 896 406 250 n^8 - 17 496 721 516 131 562 500 n^9 -
           3 299 344 288 917 187 500 n^10 - 422 270 445 058 593 750 n^11 - 32 879 451 972 656 250 n^12 - 1 176 366 796 875 000 n^13) S_n +
          (208 791 484 354 252 800 + 1 448 758 522 297 658 880 n + 4 606 818 936 047 867 520 n^2 + 8 888 945 878 483 621 920 n^3 +
           11 611 921 070 002 419 000 n^4 + 10 845 296 255 561 809 500 n^5 + 7 450 983 284 163 738 750 n^6 +
           3 812 727 944 067 609 375 n^7 + 1 453 218 514 321 359 375 n^8 + 407 501 515 823 906 250 n^9 +
           81 719 325 815 625 000 n^10 + 11 098 995 099 609 375 n^11 + 915 144 169 921 875 n^12 + 34 599 023 437 500 n^13)}}
```

# Generalized Integral with Numeric Parameters

$$\int_0^1 \int_0^1 \int_0^1 x^3(1-x)y^2(1-y)^4 z^5(1-z)^3 \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

# Generalized Integral with Numeric Parameters

$$\int_0^1\int_0^1\int_0^1 x^3(1-x)y^2(1-y)^4z^5(1-z)^3 \times \frac{\bigl(x(1-x)y(1-y)z(1-z)\bigr)^n}{(1-z+xyz)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$



```
In[182]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
             x^3*(1-x)*y^2*(1-y)^4*z^5*(1-z)^3*
              (x*(1-x)*y*(1-y)*z*(1-z))^n/(1-z+x*y*z)^(n+1),
             Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing

Out[182]= {3.44204, {(-142334280 - 343227108 n - 357150418 n^2 - 211221795 n^3 -
              78696369 n^4 - 19325330 n^5 - 3172216 n^6 - 344195 n^7 - 23661 n^8 - 932 n^9 - 16 n^10) S_n^3 +
            (8634592800 + 18280850800 n + 16901127872 n^2 + 9023153352 n^3 + 3089809298 n^4 +
              710664515 n^5 + 111371203 n^6 + 11757433 n^7 + 800987 n^8 + 31820 n^9 + 560 n^10) S_n^2 +
            (-17235247680 - 31662217276 n - 25995705428 n^2 - 12561638841 n^3 - 3956545763 n^4 -
              848851634 n^5 - 125646202 n^6 - 12672109 n^7 - 833567 n^8 - 32300 n^9 - 560 n^10) S_n +
            (285956160 + 586168912 n + 525286576 n^2 + 272628648 n^3 + 91123028 n^4 +
              20554053 n^5 + 3175443 n^6 + 332327 n^7 + 22577 n^8 + 900 n^9 + 16 n^10)}}
```

# Generalized Integral with Six Symbolic Parameters

$$\int_0^1\int_0^1\int_0^1 x^b(1-x)^{c-f}y^e(1-y)^{a+f}z^a(1-z)^c \times \frac{\bigl(x(1-x)y(1-y)z(1-z)\bigr)^n}{(1-z+xyz)^{n+d+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\begin{aligned}
&(-2+a-b-c+f-n)\,(2+a+c-d+n)\,(2+d+n)\,(2+b+c-d-f+n)\,(2+b+c-e-f+n)\,(2+e+f+n)\,(2+a-b+e+f+n)\,(2+a-d+e+f+n)\\
&\bigl(12+15 a+5 a^2+15 b+18 a b+6 a^2 b+5 b^2+6 a b^2+2 a^2 b^2+30 c+32 a c+9 a^2 c+28 b c+27 a b c+7 a^2 b c+6 b^2 c+5 a b^2 c+a^2 b^2 c+28 c^2+24 a c^2+5 a^2 c^2+18 b c^2+13 a b c^2+2 a^2 b c^2+2 b^2 c^2+a b^2 c^2+12 c^3+8 a c^3+a^2 c^3+4 b c^3+2 a b c^3+2 c^4+a c^4-15 d-14 a d-3 a^2 d-\\
&14 b d-12 a b d-2 a^2 b d-3 b^2 d-2 a b^2 d-28 c d-21 a c d-3 a^2 c d-18 b c d-12 a b c d-a^2 b c d-2 b^2 c d-a b^2 c d-18 c^2 d-11 a c^2 d-a^2 c^2 d-6 b c^2 d-3 a b c^2 d-4 c^3 d-2 a c^3 d+5 d^2+3 a d^2+3 b d^2+2 a b d^2+6 c d^2+3 a c d^2+2 b c d^2+a b c d^2+2 c^2 d^2+a c^2 d^2+15 e+\\
&14 a e+3 a^2 e+18 b e+15 a b e+3 a^2 b e+6 b^2 e+5 a b^2 e+a^2 b^2 e+32 c e+24 a c e+4 a^2 c e+27 b c e+16 a b c e+2 a^2 b c e+5 b^2 c e+2 a b^2 c e+24 c^2 e+12 a c^2 e+a^2 c^2 e+13 b c^2 e+4 a b c^2 e+b^2 c^2 e+8 c^3 e+2 a c^3 e+2 b c^3 e+c^4 e-14 d e-9 a d e-a^2 d e-12 b d e-\\
&8 a b d e-a^2 b d e-2 b^2 d e-a b^2 d e-21 c d e-8 a c d e-12 b c d e-4 a b c d e-b^2 c d e-11 c^2 d e-2 a c^2 d e-3 b c^2 d e-2 c^3 d e+3 d^2 e+a d^2 e+2 b d^2 e+a b d^2 e+3 c d^2 e+b c d^2 e+c^2 d^2 e+5 e^2+3 a e^2+6 b e^2+3 a b e^2+2 b^2 e^2+a b^2 e^2+9 c e^2+4 a c e^2+7 b c e^2+\\
&2 a b c e^2+b^2 c e^2+5 c^2 e^2+a c^2 e^2+2 b c^2 e^2+c^3 e^2-3 d e^2-a d e^2-2 b d e^2-a b d e^2-3 c d e^2-b c d e^2-c^2 d e^2-4 a f-3 a^2 f+4 b f-2 a^2 b f+3 b^2 f+2 a b^2 f+4 c f-3 a c f-3 a^2 c f+9 b c f+2 a b c f-a^2 b c f+3 b^2 c f+a b^2 c f+6 c^2 f-a c^2 f-a^2 c^2 f+7 b c^2 f+\\
&a b c^2 f+b^2 c^2 f+4 c^3 f+2 b c^3 f+c^4 f+3 a d f+a^2 d f-3 b d f-b^2 d f-3 c d f+4 a c d f+a^2 c d f-6 b c d f-b^2 c d f-5 c^2 d f+a c^2 d f-3 b c^2 d f-2 c^3 d f-a d^2 f+b d^2 f+c d^2 f-a c d^2 f+b c d^2 f+c^2 d^2 f-4 e f-6 a e f-2 a^2 e f-2 a b e f-a^2 b e f+2 b^2 e f+a b^2 e f-\\
&3 c e f-4 a c e f-a^2 c e f+2 b c e f+b^2 c e f-c^2 e f-a c^2 e f+b c^2 e f+3 d e f+2 a d e f+4 c d e f+2 a c d e f+c^2 d e f-d^2 e f-c d^2 e f-3 e^2 f-2 a e^2 f-2 b e^2 f-a b e^2 f-3 c e^2 f-a c e^2 f-b c e^2 f-c^2 e^2 f+d e^2 f+c d e^2 f-4 f^2-3 a f^2-3 b f^2-2 a b f^2-6 c f^2-\\
&3 a c f^2-3 b c f^2-a b c f^2-4 c^2 f^2-a c^2 f^2-b c^2 f^2-c^3 f^2+3 d f^2+a d f^2+b d f^2+5 c d f^2+a c d f^2+b c d f^2+2 c^2 d f^2-d^2 f^2-c d^2 f^2-3 e f^2-2 a e f^2-2 b e f^2-a b e f^2-3 c e f^2-a c e f^2-b c e f^2-c^2 e f^2+d e f^2+c d e f^2+60 n+60 a n+15 a^2 n+60 b n+54 a b n+\\
&12 a^2 b n+15 b^2 n+12 a b^2 n+2 a^2 b^2 n+120 c n+96 a c n+18 a^2 c n+84 b c n+54 a b c n+7 a^2 b c n+12 b^2 c n+5 a b^2 c n+84 c^2 n+48 a c^2 n+5 a^2 c^2 n+36 b c^2 n+13 a b c^2 n+2 b^2 c^2 n+24 c^3 n+8 a c^3 n+4 b c^3 n+2 c^4 n-60 d n-42 a d n-6 a^2 d n-42 b d n-24 a b d n-\\
&2 a^2 b d n-6 b^2 d n-2 a b^2 d n-84 c d n-42 a c d n-3 a^2 c d n-36 b c d n-12 a b c d n-2 b^2 c d n-36 c^2 d n-11 a c^2 d n-6 b c^2 d n-4 c^3 d n+15 d^2 n+6 a d^2 n+6 b d^2 n+2 a b d^2 n+12 c d^2 n+3 a c d^2 n+2 b c d^2 n+2 c^2 d^2 n+60 e n+42 a e n+6 a^2 e n+54 b e n+30 a b e n+\\
&3 a^2 b e n+12 b^2 e n+5 a b^2 e n+96 c e n+48 a c e n+4 a^2 c e n+54 b c e n+16 a b c e n+5 b^2 c e n+48 c^2 e n+12 a c^2 e n+13 b c^2 e n+8 c^3 e n-42 d e n-18 a d e n-a^2 d e n-24 b d e n-8 a b d e n-2 b^2 d e n-42 c d e n-8 a c d e n-12 b c d e n-11 c^2 d e n+6 d^2 e n+a d^2 e n+\\
&2 b d^2 e n+3 c d^2 e n+15 e^2 n+6 a e^2 n+12 b e^2 n+3 a b e^2 n+2 b^2 e^2 n+18 c e^2 n+4 a c e^2 n+7 b c e^2 n+5 c^2 e^2 n-6 d e^2 n-a d e^2 n-2 b d e^2 n-3 c d e^2 n-12 a f n-6 a^2 f n+12 b f n-2 a^2 b f n+6 b^2 f n+2 a b^2 f n+12 c f n-6 a c f n-3 a^2 c f n+18 b c f n+2 a b c f n+\\
&3 b^2 c f n+12 c^2 f n-a c^2 f n+7 b c^2 f n+4 c^3 f n+6 a d f n+a^2 d f n-6 b d f n-b^2 d f n-6 c d f n+4 a c d f n-6 b c d f n-5 c^2 d f n-a d^2 f n+b d^2 f n+c d^2 f n-12 e f n-12 a e f n-2 a^2 e f n-2 a b e f n+2 b^2 e f n-6 c e f n-4 a c e f n+2 b c e f n-c^2 e f n+6 d e f n+\\
&2 a d e f n+4 c d e f n-d^2 e f n-6 e^2 f n-2 a e^2 f n-2 b e^2 f n-3 c e^2 f n+d e^2 f n-12 f^2 n-6 a f^2 n-6 b f^2 n-2 a b f^2 n-12 c f^2 n-3 a c f^2 n-3 b c f^2 n-4 c^2 f^2 n+6 d f^2 n+a d f^2 n+b d f^2 n+5 c d f^2 n-d^2 f^2 n-6 e f^2 n-2 a e f^2 n-2 b e f^2 n-3 c e f^2 n+d e f^2 n+120 n^2+\\
&90 a n^2+15 a^2 n^2+90 b n^2+54 a b n^2+6 a^2 b n^2+15 b^2 n^2+6 a b^2 n^2+180 c n^2+96 a c n^2+9 a^2 c n^2+84 b c n^2+27 a b c n^2+6 b^2 c n^2+84 c^2 n^2+24 a c^2 n^2+18 b c^2 n^2+12 c^3 n^2-90 d n^2-42 a d n^2-3 a^2 d n^2-42 b d n^2-12 a b d n^2-3 b^2 d n^2-84 c d n^2-21 a c d n^2-18 b c d n^2-\\
&18 c^2 d n^2+15 d^2 n^2+3 a d^2 n^2+3 b d^2 n^2+6 c d^2 n^2+90 e n^2+42 a e n^2+3 a^2 e n^2+54 b e n^2+15 a b e n^2+6 b^2 e n^2+96 c e n^2+24 a c e n^2+27 b c e n^2+24 c^2 e n^2-42 d e n^2-9 a d e n^2-12 b d e n^2-21 c d e n^2+3 d^2 e n^2+15 e^2 n^2+3 a e^2 n^2+6 b e^2 n^2+9 c e^2 n^2-3 d e^2 n^2-\\
&12 a f n^2-3 a^2 f n^2+12 b f n^2+3 b^2 f n^2+12 c f n^2-3 a c f n^2+9 b c f n^2+6 c^2 f n^2+3 a d f n^2-3 b d f n^2-3 c d f n^2-12 e f n^2-6 a e f n^2-3 c e f n^2+3 d e f n^2-3 e^2 f n^2-12 f^2 n^2-3 a f^2 n^2-3 b f^2 n^2-6 c f^2 n^2+3 d f^2 n^2-3 e f^2 n^2+120 n^3+60 a n^3+5 a^2 n^3+60 b n^3+\\
&18 a b n^3+5 b^2 n^3+120 c n^3+32 a c n^3+28 b c n^3+28 c^2 n^3-60 d n^3-14 a d n^3-14 b d n^3-28 c d n^3+5 d^2 n^3+60 e n^3+14 a e n^3+18 b e n^3+32 c e n^3-14 d e n^3+5 e^2 n^3-4 a f n^3+4 b f n^3+4 c f n^3-4 e f n^3-4 f^2 n^3+60 n^4+15 a n^4+15 b n^4+30 c n^4-15 d n^4+15 e n^4+12 n^5\bigr)\,S_n^2+\\
&\bigl(44\,928+106\,416 a+100\,632 a^2+47\,144 a^3+10\,920 a^4+1000 a^5+106\,416 b+248\,688 a b+233\,648 a^2 b+109\,224 a^3 b+25\,328 a^4 b+2328 a^5 b+100\,632 b^2+233\,648 a b^2+219\,252 a^2 b^2+102\,692 a^3 b^2+23\,916 a^4 b^2+2212 a^5 b^2+47\,144 b^3+109\,224 a b^3+102\,692 a^2 b^3+\\
&48\,312 a^3 b^3+11\,324 a^4 b^3+1056 a^5 b^3+10\,920 b^4+25\,328 a b^4+23\,916 a^2 b^4+11\,324 a^3 b^4+2676 a^4 b^4+252 a^5 b^4+1000 b^5+2328 a b^5+2212 a^2 b^5+1056 a^3 b^5+252 a^4 b^5+24 a^5 b^5+212\,832 c+473\,664 a c+421\,188 a^2 c+185\,676 a^3 c+40\,492 a^4 c+3492 a^5 c+449\,952 b c+\\
&980\,700 a b c+859\,542 a^2 b c+374\,942 a^3 b c+81\,138 a^4 b c+6958 a^5 b c+375\,080 b^2 c+804\,126 a b^2 c+696\,505 a^2 b^2 c+300\,993 a^3 b^2 c+64\,631 a^4 b^2 c+5505 a^5 b^2 c+152\,904 b^3 c+322\,906 a b^3 c+276\,357 a^2 b^3 c+118\,146 a^3 b^3 c+25\,107 a^4 b^3 c+2116 a^5 b^3 c+30\,328 b^4 c+\\
&63\,042 a b^4 c+53\,175 a^2 b^4 c+22\,395 a^3 b^4 c+4681 a^4 b^4 c+387 a^5 b^4 c+2328 b^5 c+4746 a b^5 c+3921 a^2 b^5 c+1612 a^3 b^5 c+327 a^4 b^5 c+26 a^5 b^5 c+449\,952 c^2+934\,592 a c^2+774\,426 a^2 c^2+317\,646 a^3 c^2+64\,326 a^4 c^2+5138 a^5 c^2+842\,376 b c^2+1\,701\,618 a b c^2+\\
&1\,378\,298 a^2 b c^2+554\,076 a^3 b c^2+110\,118 a^4 b c^2+8634 a^5 b c^2+612\,444 b^2 c^2+1\,205\,179 a b^2 c^2+953\,829 a^2 b^2 c^2+374\,865 a^3 b^2 c^2+72\,771 a^4 b^2 c^2+5560 a^5 b^2 c^2+214\,004 b^3 c^2+409\,581 a b^3 c^2+315\,452 a^2 b^3 c^2+120\,405 a^3 b^3 c^2+22\,607 a^4 b^3 c^2+1659 a^5 b^3 c^2+\\
&35\,556 b^4 c^2+65\,845 a b^4 c^2+48\,927 a^2 b^4 c^2+17\,905 a^3 b^4 c^2+3189 a^4 b^4 c^2+218 a^5 b^4 c^2+2212 b^5 c^2+3921 a b^5 c^2+2764 a^2 b^5 c^2+945 a^3 b^5 c^2+153 a^4 b^5 c^2+9 a^5 b^5 c^2+561\,584 c^3+1\,083\,168 a c^3+829\,881 a^2 c^3+313\,371 a^3 c^3+58\,111 a^4 c^3+4221 a^5 c^3+919\,080 b c^3+\\
&1\,712\,510 a b c^3+1\,271\,277 a^2 b c^3+465\,300 a^3 b c^3+83\,499 a^4 b c^3+5846 a^5 b c^3+572\,736 b^2 c^3+1\,029\,828 a b^2 c^3+737\,797 a^2 b^2 c^3+259\,869 a^3 b^2 c^3+44\,616 a^4 b^2 c^3+2958 a^5 b^2 c^3+167\,256 b^3 c^3+288\,770 a b^3 c^3+197\,883 a^2 b^3 c^3+66\,156 a^3 b^3 c^3+\cdots 82\,236 \cdots+\\
&15\,510 d f^2 n^9+2474 a d f^2 n^9+2474 b d f^2 n^9+5302 c d f^2 n^9-1192 d^2 f^2 n^9-15\,510 e f^2 n^9-2384 a e f^2 n^9-3008 b e f^2 n^9-5482 c e f^2 n^9+2474 d e f^2 n^9-970 e^2 f^2 n^9+208 a f^3 n^9-208 b f^3 n^9-208 c f^3 n^9+208 e f^3 n^9+104 f^4 n^9+385\,704 n^{10}+195\,558 a n^{10}+\\
&28\,941 a^2 n^{10}+1177 a^3 n^{10}+195\,558 b n^{10}+70\,290 a b n^{10}+5599 a^2 b n^{10}+28\,941 b^2 n^{10}+5599 a b^2 n^{10}+1177 b^3 n^{10}+391\,116 c n^{10}+134\,376 a c n^{10}+10\,164 a^2 c n^{10}+128\,172 b c n^{10}+23\,430 a b c n^{10}+9130 b^2 c n^{10}+128\,172 c^2 n^{10}+22\,396 a c^2 n^{10}+20\,328 b c^2 n^{10}+\\
&13\,552 c^3 n^{10}-195\,558 d n^{10}-64\,086 a d n^{10}-4565 a^2 d n^{10}-64\,086 b d n^{10}-11\,198 a b d n^{10}-4565 b^2 d n^{10}-128\,172 c d n^{10}-21\,362 a c d n^{10}-20\,328 b c d n^{10}-20\,328 c^2 d n^{10}+28\,941 d^2 n^{10}+4565 a d^2 n^{10}+4565 b d^2 n^{10}+9130 c d^2 n^{10}-1177 d^3 n^{10}+195\,558 e n^{10}+\\
&64\,086 a e n^{10}+4565 a^2 e n^{10}+70\,290 b e n^{10}+12\,232 a b e n^{10}+5599 b^2 e n^{10}+134\,376 c e n^{10}+22\,396 a c e n^{10}+23\,430 b c e n^{10}+22\,396 c^2 e n^{10}-64\,086 d e n^{10}-10\,164 a d e n^{10}-11\,198 b d e n^{10}-21\,362 c d e n^{10}+4565 d^2 e n^{10}+28\,941 e^2 n^{10}+4565 a e^2 n^{10}+5599 b e^2 n^{10}+\\
&10\,164 c e^2 n^{10}-4565 d e^2 n^{10}+1177 e^3 n^{10}-6204 a f n^{10}-1034 a^2 f n^{10}+6204 b f n^{10}+1034 b^2 f n^{10}+6204 c f n^{10}-1034 a c f n^{10}+3102 b c f n^{10}+2068 c^2 f n^{10}+1034 a d f n^{10}-1034 b d f n^{10}-1034 c d f n^{10}-6204 e f n^{10}-2068 a e f n^{10}-1034 c e f n^{10}+1034 d e f n^{10}-\\
&1034 e^2 f n^{10}-6204 f^2 n^{10}-1034 a f^2 n^{10}-1034 b f^2 n^{10}-2068 c f^2 n^{10}+1034 d f^2 n^{10}-1034 e f^2 n^{10}+71\,112 n^{11}+23\,868 a n^{11}+1754 a^2 n^{11}+23\,868 b n^{11}+4260 a b n^{11}+1754 b^2 n^{11}+47\,736 c n^{11}+8144 a c n^{11}+7768 b c n^{11}+7768 c^2 n^{11}-23\,868 d n^{11}-3884 a d n^{11}-\\
&3884 b d n^{11}-7768 c d n^{11}+1754 d^2 n^{11}+23\,868 e n^{11}+3884 a e n^{11}+4260 b e n^{11}+8144 c e n^{11}-3884 d e n^{11}+1754 e^2 n^{11}-376 a f n^{11}+376 b f n^{11}+376 c f n^{11}-376 e f n^{11}-376 f^2 n^{11}+7956 n^{12}+1326 a n^{12}+1326 b n^{12}+2652 c n^{12}-1326 d n^{12}+1326 e n^{12}+408 n^{13}\bigr)\,S_n-\\
&(1+a+n)\,(1+b+n)\,(1+c+n)\,(1+b+c-d+n)\,(1+e+n)\,(1+c-d+e+n)\,(1+c-f+n)\,(1+a+f+n)\,\bigl(384+240 a+40 a^2+240 b+144 a b+24 a^2 b+40 b^2+24 a b^2+4 a^2 b^2+480 c+256 a c+36 a^2 c+224 b c+108 a b c+14 a^2 b c+24 b^2 c+10 a b^2 c+a^2 b^2 c+\\
&224 c^2+96 a c^2+10 a^2 c^2+72 b c^2+26 a b c^2+2 a^2 b c^2+4 b^2 c^2+a b^2 c^2+48 c^3+16 a c^3+a^2 c^3+8 b c^3+2 a b c^3+4 c^4+a c^4-240 d-112 a d-12 a^2 d-112 b d-48 a b d-4 a^2 b d-12 b^2 d-4 a b^2 d-224 c d-84 a c d-6 a^2 c d-72 b c d-24 a b c d-a^2 b c d-\\
&4 b^2 c d-a b^2 c d-72 c^2 d-22 a c^2 d-a^2 c^2 d-12 b c^2 d-3 a b c^2 d-8 c^3 d-2 a c^3 d+40 d^2+12 a d^2+12 b d^2+4 a b d^2+24 c d^2+6 a c d^2+4 b c d^2+a b c d^2+4 c^2 d^2+a c^2 d^2+240 e+112 a e+12 a^2 e+144 b e+60 a b e+6 a^2 b e+24 b^2 e+10 a b^2 e+a^2 b^2 e+\\
&256 c e+96 a c e+8 a^2 c e+108 b c e+32 a b c e+2 a^2 b c e+10 b^2 c e+2 a b^2 c e+96 c^2 e+24 a c^2 e+a^2 c^2 e+26 b c^2 e+4 a b c^2 e+b^2 c^2 e+16 c^3 e+2 a c^3 e+2 b c^3 e+c^4 e-112 d e-36 a d e-2 a^2 d e-48 b d e-16 a b d e-a^2 b d e-4 b^2 d e-a b^2 d e-84 c d e-\\
&16 a c d e-24 b c d e-4 a b c d e-b^2 c d e-22 c^2 d e-2 a c^2 d e-3 b c^2 d e-2 c^3 d e+12 d^2 e+2 a d^2 e+4 b d^2 e+a b d^2 e+6 c d^2 e+b c d^2 e+c^2 d^2 e+40 e^2+12 a e^2+24 b e^2+6 a b e^2+4 b^2 e^2+a b^2 e^2+36 c e^2+8 a c e^2+14 b c e^2+2 a b c e^2+b^2 c e^2+10 c^2 e^2+\\
&a c^2 e^2+2 b c^2 e^2+c^3 e^2-12 d e^2-2 a d e^2-4 b d e^2-a b d e^2-6 c d e^2-b c d e^2-c^2 d e^2-32 a f-12 a^2 f+32 b f-4 a^2 b f+12 b^2 f+4 a b^2 f+32 c f-12 a c f-6 a^2 c f+36 b c f+4 a b c f-a^2 b c f+6 b^2 c f+a b^2 c f+24 c^2 f-2 a c^2 f-a^2 c^2 f+14 b c^2 f+\\
&a b c^2 f+b^2 c^2 f+8 c^3 f+2 b c^3 f+c^4 f+12 a d f+2 a^2 d f-12 b d f-2 b^2 d f-12 c d f+8 a c d f+a^2 c d f-12 b c d f-b^2 c d f-10 c^2 d f+a c^2 d f-3 b c^2 d f-2 c^3 d f-2 a d^2 f+2 b d^2 f+2 c d^2 f-a c d^2 f+b c d^2 f+c^2 d^2 f-32 e f-24 a e f-4 a^2 e f-4 a b e f-\\
&a^2 b e f+4 b^2 e f+a b^2 e f-12 c e f-8 a c e f-a^2 c e f+4 b c e f+b^2 c e f-2 c^2 e f-a c^2 e f+b c^2 e f+12 d e f+4 a d e f+8 c d e f+2 a c d e f+c^2 d e f-2 d^2 e f-c d^2 e f-12 e^2 f-4 a e^2 f-4 b e^2 f-a b e^2 f-6 c e^2 f-a c e^2 f-b c e^2 f-c^2 e^2 f+2 d e^2 f+\\
&c d e^2 f-32 f^2-12 a f^2-12 b f^2-4 a b f^2-24 c f^2-6 a c f^2-6 b c f^2-a b c f^2-8 c^2 f^2-a c^2 f^2-b c^2 f^2-c^3 f^2+12 d f^2+2 a d f^2+2 b d f^2+10 c d f^2+a c d f^2+b c d f^2+2 c^2 d f^2-2 d^2 f^2-c d^2 f^2-12 e f^2-4 a e f^2-4 b e f^2-a b e f^2-6 c e f^2-a c e f^2-\\
&b c e f^2-c^2 e f^2+2 d e f^2+c d e f^2+960 n+480 a n+60 a^2 n+480 b n+216 a b n+24 a^2 b n+60 b^2 n+24 a b^2 n+2 a^2 b^2 n+960 c n+384 a c n+36 a^2 c n+336 b c n+108 a b c n+7 a^2 b c n+24 b^2 c n+5 a b^2 c n+336 c^2 n+96 a c^2 n+5 a^2 c^2 n+72 b c^2 n+13 a b c^2 n+\\
&2 b^2 c^2 n+48 c^3 n+8 a c^3 n+4 b c^3 n+2 c^4 n-480 d n-168 a d n-12 a^2 d n-168 b d n-48 a b d n-2 a^2 b d n-12 b^2 d n-2 a b^2 d n-336 c d n-84 a c d n-3 a^2 c d n-72 b c d n-12 a b c d n-2 b^2 c d n-72 c^2 d n-11 a c^2 d n-6 b c^2 d n-4 c^3 d n+60 d^2 n+12 a d^2 n+\\
&12 b d^2 n+2 a b d^2 n+24 c d^2 n+3 a c d^2 n+2 b c d^2 n+2 c^2 d^2 n+480 e n+168 a e n+12 a^2 e n+216 b e n+60 a b e n+3 a^2 b e n+24 b^2 e n+5 a b^2 e n+384 c e n+96 a c e n+4 a^2 c e n+108 b c e n+16 a b c e n+5 b^2 c e n+96 c^2 e n+12 a c^2 e n+13 b c^2 e n+\\
&8 c^3 e n-168 d e n-36 a d e n-a^2 d e n-48 b d e n-8 a b d e n-2 b^2 d e n-84 c d e n-8 a c d e n-12 b c d e n-11 c^2 d e n+12 d^2 e n+a d^2 e n+2 b d^2 e n+3 c d^2 e n+60 e^2 n+12 a e^2 n+24 b e^2 n+3 a b e^2 n+2 b^2 e^2 n+36 c e^2 n+4 a c e^2 n+7 b c e^2 n+5 c^2 e^2 n-\\
&12 d e^2 n-a d e^2 n-2 b d e^2 n-3 c d e^2 n-48 a f n-12 a^2 f n+48 b f n-2 a^2 b f n+12 b^2 f n+2 a b^2 f n+48 c f n-12 a c f n-3 a^2 c f n+36 b c f n+2 a b c f n+3 b^2 c f n+24 c^2 f n-a c^2 f n+7 b c^2 f n+4 c^3 f n+12 a d f n+a^2 d f n-12 b d f n-b^2 d f n-12 c d f n+\\
&4 a c d f n-6 b c d f n-5 c^2 d f n-a d^2 f n+b d^2 f n+c d^2 f n-48 e f n-24 a e f n-2 a^2 e f n-2 a b e f n+2 b^2 e f n-12 c e f n-4 a c e f n+2 b c e f n-c^2 e f n+12 d e f n+2 a d e f n+4 c d e f n-d^2 e f n-12 e^2 f n-2 a e^2 f n-2 b e^2 f n-3 c e^2 f n+d e^2 f n-\\
&48 f^2 n-12 a f^2 n-12 b f^2 n-2 a b f^2 n-24 c f^2 n-3 a c f^2 n-3 b c f^2 n-4 c^2 f^2 n+12 d f^2 n+a d f^2 n+b d f^2 n+5 c d f^2 n-d^2 f^2 n-12 e f^2 n-2 a e f^2 n-2 b e f^2 n-3 c e f^2 n+d e f^2 n+960 n^2+360 a n^2+30 a^2 n^2+360 b n^2+108 a b n^2+6 a^2 b n^2+30 b^2 n^2+\\
&6 a b^2 n^2+720 c n^2+192 a c n^2+9 a^2 c n^2+168 b c n^2+27 a b c n^2+6 b^2 c n^2+168 c^2 n^2+24 a c^2 n^2+18 b c^2 n^2+12 c^3 n^2-360 d n^2-84 a d n^2-3 a^2 d n^2-84 b d n^2-12 a b d n^2-3 b^2 d n^2-168 c d n^2-21 a c d n^2-18 b c d n^2-18 c^2 d n^2+30 d^2 n^2+3 a d^2 n^2+3 b d^2 n^2+
\end{aligned}$$

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- Hence one can write $I(n) = u_n C - v_n$ for two sequences of rational numbers $(u_n)$ and $(v_n)$ that both satisfy the same recurrence as $I(n)$.
- Let $E(n)$ be such that $u'_n := u_n\, E(n)$ and $v'_n := v_n\, E(n)$ are always integers and $\gcd(u'_n, v'_n) = 1$. We call $E(n)$ the **integer-ating factor**.

# Irrationality Measure

**Irrationality Criterion:** If there is a $\delta > 0$ and a sequence $(v_n/u_n)_{n\in\mathbb{N}}$ of rational numbers such that $v_n/u_n \neq C$ with

$$\left|C - \frac{v_n}{u_n}\right| < \frac{1}{u_n^{1+\delta}},$$

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- For rational numbers, the maximum is $\delta = 0$.
- For (non-rational) algebraic numbers, the maximum is $\delta = 1$.

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We write $x(n) = \Omega(\alpha^n)$ to mean that $\lim_{n\to\infty} \frac{\log x(n)}{n} = \alpha$.

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$$\left|C - \frac{v_n}{u_n}\right| = \Omega\left(\frac{1}{(\alpha\beta)^n}\right) = \left|C - \frac{v'_n}{u'_n}\right|.$$

- Let $\nu := \lim_{n\to\infty} \frac{\log E(n)}{n}$, then we need a positive $\delta$ such that $(e^{n\nu}\alpha^n)^{1+\delta} = (\alpha\beta)^n$. This translates into

$$\delta = \frac{\log\beta - \nu}{\log\alpha + \nu}.$$

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- The constants $\alpha$ and $\beta$
- A conjectured integer-ating factor $E(n)$, or else conjecture that one exists, and find, or estimate (respectively), $\nu := \lim_{n\to\infty} \frac{\log E(n)}{n}$.

# Some Results

Generalizing the Alladi-Robinson family of integrals

$$I(n) := \int_0^1 \frac{1}{1+cx} \left( \frac{x(1-x)}{1+cx} \right)^n \mathrm{d}x,$$

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- Many of these constants are expressible terms of algebraic numbers and/or logarithms of rational numbers.
- Hence proving them irrational is not that exciting...
- However, there are also some unidentified cases.

# Some Results

Generalizing the Beukers Integral for $\zeta(2)$, we define

$$I_2(a_1,a_2,b_1,b_2)(n) := \frac{1}{B(1-a_1,1-a_2)B(1-b_1,1-b_2)}$$
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It allows us to realize the following constants as weak Apéry limits:

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Generalizing the Beukers Integral for $\zeta(2)$, we define

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- Several hundred cases with positive $\delta$, but many of them are equivalent via transformations $C \mapsto \frac{a+bC}{c+dC}$ with integer coeffs.
- Again, there are some cases that could not be identified.

# Examples

$$\begin{aligned}
C_2(0,0,\tfrac{1}{2},0) &= {}_3F_2(1,1,\tfrac{1}{2};2,\tfrac{3}{2};1) &&= 2\log 2\\
C_2(0,0,\tfrac{1}{3},-\tfrac{2}{3}) &= {}_3F_2(1,1,\tfrac{2}{3};2,\tfrac{7}{3};1) &&= -6+4\pi\sqrt{3}/3\\
C_2(-\tfrac{3}{4},-\tfrac{3}{4},-\tfrac{1}{4},-\tfrac{3}{4}) &= {}_3F_2(1,\tfrac{7}{4},\tfrac{5}{4};\tfrac{7}{2},3;1) &&= -240\;+\;\frac{512}{3}\sqrt{2}\\
C_2(-\tfrac{4}{5},-\tfrac{4}{5},-\tfrac{2}{5},-\tfrac{3}{5}) &= {}_3F_2(1,\tfrac{9}{5},\tfrac{7}{5};\tfrac{18}{5},3;1) &&= -\frac{845}{2}\;+\;\frac{2275}{12}\sqrt{5}\\
C_2(-\tfrac{5}{6},-\tfrac{5}{6},-\tfrac{1}{2},-\tfrac{1}{2}) &= {}_3F_2(1,\tfrac{11}{6},\tfrac{3}{2};\tfrac{11}{3},3;1) &&= -\frac{1344}{5}+\frac{16384\sqrt{3}}{105}\\
C_2(-\tfrac{5}{6},-\tfrac{5}{6},-\tfrac{1}{3},-\tfrac{2}{3}) &= {}_3F_2(1,\tfrac{11}{6},\tfrac{4}{3};\tfrac{11}{3},3;1) &&= \frac{972\,2^{2/3}}{5}-\frac{1536}{5}
\end{aligned}$$

# Some Results

Using the generalized Beukers integral for $\zeta(3)$,

$$J_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(n) := \int_0^1\int_0^1\int_0^1 \left(\frac{x(1-x)y(1-y)z(1-z)}{1-z+xyz}\right)^n \times \frac{x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2}}{(1-z+xyz)^e}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

we define

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$$K(a,b,c,d,e)(n) := I_3(b,c,e,a,a,c,d)(n)$$

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*The output file contains many such conjectured evaluations and we challenge the birthday boy [WZ], or anyone else, to prove them.*

# The Birthday Boy Problem

In their recent preprint arXiv:2101.08308, Robert Dougherty-Bliss, Christoph Koutschan and Doron Zeilberger come up with a powerful strategy to prove the irrationality, in a quantitative form, of some numbers that are given as multiple integrals or quotients of such. What is really missing there, for many examples given, is an explicit identification of those irrational numbers. Without an identification, the numbers are hardly appealing to human (number theorists). The goal of this note is to outline a strategy to do the job and illustrate it on several promising entries discussed in the preprint above.

# Zudilin

$$K(0,0,0,\tfrac{2}{3},\tfrac{1}{3}) = -\frac{K_1-2}{2(K_1-3)}, \qquad \text{where } K_1 = \log 3 + \frac{\pi}{\sqrt{3}}$$

$$K(0,0,0,\tfrac{1}{3},\tfrac{2}{3}) = -\frac{2(K_2+1)}{K_2+1/2}, \qquad \text{where } K_2 = \log 3 + \frac{\pi}{\sqrt{3}}$$

$$K(0,\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{3},\tfrac{2}{3}) = -\frac{20(7-54K_3)}{52-405K_3}, \text{ where } K_3 = \frac{\Gamma(2/3)^3}{\Gamma(1/3)^3}$$

$$K(0,\tfrac{1}{5},0,\tfrac{3}{5},\tfrac{2}{5}) = -\frac{4(1-4K_4)}{5-24K_4}, \qquad \text{where } K_4 = \frac{1}{\sqrt{5}} \log \frac{\sqrt{5}+1}{2}$$

$$K(\tfrac{1}{7},0,\tfrac{2}{7},\tfrac{3}{7},\tfrac{4}{7}) = -\frac{189(8-5K_5)}{832-525K_5}, \text{ where } K_5 = \frac{2^{2/7}\sqrt{\pi}\,\Gamma(9/14)}{\cos(3\pi/14)\,\Gamma(4/7)^2}$$

*Perhaps, a real pearl in this collection of "quantitatively" irrational numbers is the number* $K_3$.

# References

- Robert Dougherty-Bliss, Christoph Koutschan, Doron Zeilberger: Tweaking the Beukers integrals in search of more miraculous irrationality proofs á la Apéry. To appear in The Ramanujan Journal, arXiv:2101.08308.
- Wadim Zudilin: The birthday boy problem. arXiv:2108.06586.
- Christoph Koutschan, Wadim Zudilin: Apéry limits for elliptic $L$-values. To appear in the Bulletin of the Australian Mathematical Society, arXiv:2111.08796.