

# Computing the number of DSASMs (diagonally symmetric alternating sign matrices)

Christoph Koutschan

(joint work with Roger Behrend and Ilse Fischer)

Notebook: [www.koutschan.de/data/asm/Demo\\_DSASM.nb](http://www.koutschan.de/data/asm/Demo_DSASM.nb)

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# Alternating Sign Matrix (ASM)

## Definition:

- ▶ quadratic matrix ( $n \times n$ )
- ▶ entries are 0, 1, or  $-1$
- ▶ 1's and  $-1$ 's alternate along rows and along columns
- ▶ all row sums and all column sums are 1

## Example:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# A bit of history

- ▶ ASMs arose during the study of Dodgson condensation for determinants (Mills, Robbins, Rumsey, 1982), conjectures
- ▶ Proof: Zeilberger (1996), Kuperberg (1996)

$$\text{ASM}(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE <sup>1</sup>

*Doron ZEILBERGER*<sup>2</sup>

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Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquet-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

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$$\text{ASM}(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

- ▶ Various symmetry classes: ASM, VSASM, VHSASM, HTSASM, QTSASM, DSASM, DASASM, TSASM
- ▶ Behrend/Fischer/Konvalinka (2016):

$$\text{DASASM}(2n+1) = \prod_{i=0}^n \frac{(3i)!}{(n+i)!}$$



# David Robbins (2000): Symmetry Classes of ASMs

## Symmetry Type

size	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	2	0	2	2	0	0	2	0
3	7	1	3	5	1	1	3	1
4	42	0	10	16	2	0	8	0
5	429	3	25	67	3	1	15	1
6	7436	0	140	368	0	0	52	0
7	218348	26	588	2630	12	2	126	2
8	*	0	5544		40	0	568	0
9	*	646	39204		100	6	1782	4
10	*	0	622908		0	0	10436	0
11	*	45885	7422987		1225	33	42471	13
12	*	0	*		6460	0	323144	0
13	*	9304650	*		28812	286	1706562	46
14	*	0	*		0	0	*	0
15	*	*	*		1037232	4420	*	248
16	*	0	*		9779616	0	*	0
17	*	*	*	*	*	109820	*	1516

## Number of ASMs

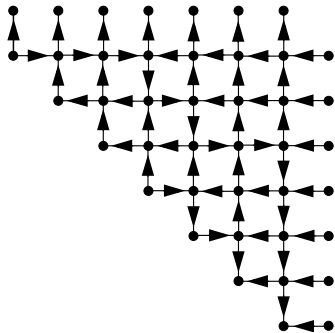
$n = 1$ :	1	=	1
$n = 2$ :	2	=	2
$n = 3$ :	7	=	7
$n = 4$ :	42	=	$2 \cdot 3 \cdot 7$
$n = 5$ :	429	=	$3 \cdot 11 \cdot 13$
$n = 6$ :	7436	=	$2^2 \cdot 11 \cdot 13^2$
$n = 7$ :	218348	=	$2^2 \cdot 13^2 \cdot 17 \cdot 19$
$n = 8$ :	10850216	=	$2^3 \cdot 13 \cdot 17^2 \cdot 19^2$
$n = 9$ :	911835460	=	$2^2 \cdot 5 \cdot 17^2 \cdot 19^3 \cdot 23$
$n = 10$ :	129534272700	=	$2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 19^3 \cdot 23^2$
$n = 11$ :	31095744852375	=	$3^2 \cdot 5^3 \cdot 7 \cdot 19^2 \cdot 23^3 \cdot 29 \cdot 31$
$n = 12$ :	12611311859677500	=	$2^2 \cdot 3^3 \cdot 5^4 \cdot 19 \cdot 23^3 \cdot 29^2 \cdot 31^2$
$n = 13$ :	8639383518297652500	=	$2^2 \cdot 3^5 \cdot 5^4 \cdot 23^2 \cdot 29^3 \cdot 31^3 \cdot 37$
$n = 14$ :	9995541355448167482000	=	$2^4 \cdot 3^5 \cdot 5^3 \cdot 23 \cdot 29^4 \cdot 31^4 \cdot 37^2$
$n = 15$ :	19529076234661277104897200	=	$2^4 \cdot 3^3 \cdot 5^2 \cdot 29^4 \cdot 31^5 \cdot 37^3 \cdot 41 \cdot 43$

## Number of DSASMs

$n = 1$ :	1	=	1
$n = 2$ :	2	=	2
$n = 3$ :	5	=	5
$n = 4$ :	16	=	$2^4$
$n = 5$ :	67	=	67
$n = 6$ :	368	=	$2^4 \cdot 23$
$n = 7$ :	2630	=	$2 \cdot 5 \cdot 263$
$n = 8$ :	24376	=	$2^3 \cdot 11 \cdot 277$
$n = 9$ :	293770	=	$2 \cdot 5 \cdot 29 \cdot 1013$
$n = 10$ :	4610624	=	$2^6 \cdot 61 \cdot 1181$
$n = 11$ :	94080653	=	$4679 \cdot 20107$
$n = 12$ :	2492747656	=	$2^3 \cdot 7 \cdot 2063 \cdot 21577$
$n = 13$ :	85827875506	=	$2 \cdot 29 \cdot 73 \cdot 20271109$
$n = 14$ :	3842929319936	=	$2^{13} \cdot 7 \cdot 67015369$
$n = 15$ :	223624506056156	=	$2^2 \cdot 67 \cdot 7547 \cdot 110563111$
$n = 16$ :	16901839470598576	=	$2^4 \cdot 13 \cdot 12343 \cdot 6583394929$
$n = 17$ :	1659776507866213636	=	$2^2 \cdot 263 \cdot 1577734323066743$
$n = 18$ :	211853506422044996288	=	$2^6 \cdot 13 \cdot 254631618295727159$
$n = 19$ :	35137231473111223912310	=	$2 \cdot 5 \cdot 1601 \cdot 2194705276271781631$
$n = 20$ :	7569998079873075147860464	=	$2^4 \cdot 473124879992067196741279$

## Six-vertex model

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



- ▶ The degree-4 vertices have two incoming and two outgoing edges.
- ▶ The top vertical edges point up.
- ▶ The rightmost horizontal edges point to the left.

$$\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array}, \begin{array}{c} \uparrow \\ \leftarrow \\ \downarrow \end{array} \leftrightarrow 1,$$

$$\begin{array}{c} \uparrow \\ \leftarrow \\ \downarrow \end{array}, \begin{array}{c} \uparrow \\ \rightarrow \\ \downarrow \end{array} \leftrightarrow -1,$$

$$\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array}, \begin{array}{c} \uparrow \\ \leftarrow \\ \downarrow \end{array}, \begin{array}{c} \uparrow \\ \leftarrow \\ \downarrow \end{array}, \begin{array}{c} \uparrow \\ \leftarrow \\ \downarrow \end{array} \leftrightarrow 0,$$

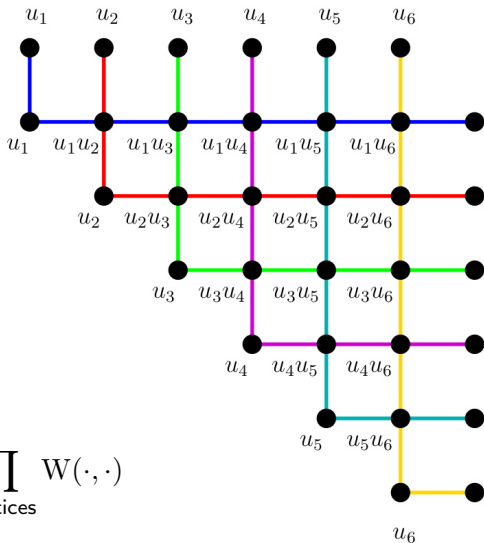
$$\begin{array}{c} \uparrow \\ \leftarrow \\ \downarrow \end{array}, \begin{array}{c} \uparrow \\ \leftarrow \\ \downarrow \end{array} \leftrightarrow 0.$$

## Vertex weights

Bulk weights	Left boundary weights	Matrix entry
$W(\begin{array}{c} \leftarrow \\ \uparrow \\ \rightarrow \\ \downarrow \end{array}, u) = 1$	$W(\begin{array}{c} \uparrow \\ \leftarrow \end{array}, u) = \frac{\beta qu + \gamma \bar{q}\bar{u}}{\sigma(q^2)}$	1
$W(\begin{array}{c} \rightarrow \\ \uparrow \\ \leftarrow \\ \downarrow \end{array}, u) = 1$	$W(\begin{array}{c} \uparrow \\ \rightarrow \end{array}, u) = \frac{\gamma qu + \beta \bar{q}\bar{u}}{\sigma(q^2)}$	-1
$W(\begin{array}{c} \leftarrow \\ \uparrow \\ \rightarrow \\ \downarrow \end{array}, u) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$	$W(\begin{array}{c} \uparrow \\ \leftarrow \end{array}, u) = \alpha \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	0
$W(\begin{array}{c} \rightarrow \\ \uparrow \\ \leftarrow \\ \downarrow \end{array}, u) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$	$W(\begin{array}{c} \uparrow \\ \rightarrow \end{array}, u) = \delta \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	0
$W(\begin{array}{c} \leftarrow \\ \uparrow \\ \rightarrow \\ \downarrow \end{array}, u) = \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		0
$W(\begin{array}{c} \rightarrow \\ \uparrow \\ \leftarrow \\ \downarrow \end{array}, u) = \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		0

**Notation:**  $\bar{x} = x^{-1}$  and  $\sigma(x) = x - \bar{x}$ .

## Vertex labeling



Partition function:

$$Z_n(u_1, \dots, u_n) = \sum_{\text{conf's}} \prod_{\text{vertices}} W(\cdot, \cdot)$$

## Partition function

The even-order partition function is  $Z_{2n}(u_1, \dots, u_{2n}) =$

$$\sigma(q^4)^{-n(2n-1)} \sigma(q^2)^{-2n} \prod_{1 \leq i < j \leq 2n} \frac{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \times$$

$$\text{Pf}_{1 \leq i < j \leq 2n} \left( \frac{\sigma(u_i \bar{u}_j) (\alpha \delta \sigma(q^4) \sigma(q^2 u_i^2) \sigma(q^2 u_j^2) + \sigma(q^2 \bar{u}_i \bar{u}_j) (\beta q u_i + \gamma \bar{q} \bar{u}_i) (\beta q u_j + \gamma \bar{q} \bar{u}_j))}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)} \right)$$

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$$\text{Pf}_{1 \leq i < j \leq 2n} \left( \frac{\sigma(u_i \bar{u}_j) (\alpha \delta \sigma(q^4) \sigma(q^2 u_i^2) \sigma(q^2 u_j^2) + \sigma(q^2 \bar{u}_i \bar{u}_j) (\beta q u_i + \gamma \bar{q} \bar{u}_i) (\beta q u_j + \gamma \bar{q} \bar{u}_j))}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)} \right)$$

The odd-order partition function is  $Z_{2n-1}(u_1, \dots, u_{2n-1}) =$

$$\sigma(q^4)^{-(n-1)(2n-1)} \sigma(q^2)^{-2n+1} \prod_{1 \leq i < j \leq 2n-1} \frac{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \times$$

$$\text{Pf}_{0 \leq i < j \leq 2n-1} \left( \begin{cases} \beta q u_j + \gamma \bar{q} \bar{u}_j, & \text{if } i = 0, \\ \frac{\sigma(u_i \bar{u}_j) (\alpha \delta \sigma(q^4) \sigma(q^2 u_i^2) \sigma(q^2 u_j^2) + \sigma(q^2 \bar{u}_i \bar{u}_j) (\beta q u_i + \gamma \bar{q} \bar{u}_i) (\beta q u_j + \gamma \bar{q} \bar{u}_j))}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)} \end{cases} \right)$$

## Pfaffian

For an array  $A = (a_{i,j})_{1 \leq i < j \leq 2n}$  its Pfaffian is defined as

$$\text{Pf}(A) = \sum_{\pi = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}} \text{sgn}(\pi) \cdot \prod_{k=1}^n a_{i_k, j_k},$$

where  $\pi = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}$  is a perfect matching of  $\{1, 2, \dots, 2n\}$ , assuming  $i_k < j_k$  and

$$\text{sgn}\{\{i_1, j_1\}, \dots, \{i_n, j_n\}\} := \text{sgn}(i_1 j_1 \dots i_n j_n).$$

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- ▶ Setting  $a_{i,j} = -a_{j,i}$  if  $i > j$ , we need not require  $i_k < j_k$ .
- ▶ For a skew-symmetric matrix we have  $\text{Pf}(A)^2 = \det(A)$ .

## Evaluate the partition function

Note that all weights become 1 under the substitution

$$\alpha = \delta = 1,$$

$$\beta = \gamma = \sigma(q),$$

$$q = \exp\left(\frac{2\pi i}{12}\right) = \frac{1}{2}(\sqrt{3} + i),$$

$$u_1 = \cdots = u_n = 1.$$

Hence, after this substitution,  $Z_n$  gives the number DSASM( $n$ ).

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Hence, after this substitution,  $Z_n$  gives the number DSASM( $n$ ).

**Problem:** One first has to evaluate the Pfaffian and simplify before the substitution  $u_1 = \cdots = u_n$  is admissible.

## Example of partition function

The partition function for  $(2 \times 2)$ -DSASMs is

$$\begin{aligned} Z_2(u_1, u_2) = & \frac{1}{(q^4 - 1)^3 (q^4 + 1) u_1^2 u_2^2} \\ & \times \left( \gamma^2 q^8 - \alpha \delta + \alpha \delta q^{16} u_1^4 u_2^4 - \alpha \delta q^{12} u_1^4 - \alpha \delta q^{12} u_2^4 + \beta^2 q^{12} u_1^2 u_2^2 \right. \\ & + \beta \gamma q^{10} u_1^2 + \beta \gamma q^{10} u_2^2 + \alpha \delta q^8 - \alpha \delta q^8 u_1^4 u_2^4 - \beta^2 q^8 u_1^4 u_2^4 \\ & \left. - \beta \gamma q^6 u_1^2 u_2^4 - \beta \gamma q^6 u_1^4 u_2^2 + \alpha \delta q^4 u_1^4 + \alpha \delta q^4 u_2^4 - \gamma^2 q^4 u_1^2 u_2^2 \right) \end{aligned}$$

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Setting  $u_1 = u_2 = 1$  yields

$$\frac{1}{(q^4 - 1)^3 (q^4 + 1)} \cdot \left( -\alpha\delta + \alpha\delta q^{16} - 2\alpha\delta q^{12} + \beta^2 q^{12} + 2\beta\gamma q^{10} \right. \\ \left. - \beta^2 q^8 + \gamma^2 q^8 - 2\beta\gamma q^6 + 2\alpha\delta q^4 - \gamma^2 q^4 \right)$$

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Setting  $\alpha = \delta = 1$  and  $\beta = \gamma = q - q^{-1}$  yields  $\frac{q^4 + q^2 + 1}{q^4 + 1}$ ,  
which is 2 for  $q$  any primitive 12-th root of unity.

# Mathematica Demo

## More numbers of DSASMs

$n = 1 : 1$   
 $n = 2 : 2$   
 $n = 3 : 5$   
 $n = 4 : 16$   
 $n = 5 : 67$   
 $n = 6 : 368$   
 $n = 7 : 2630$   
 $n = 8 : 24376$   
 $n = 9 : 293770$   
 $n = 10 : 4610624$   
 $n = 11 : 94080653$   
 $n = 12 : 2492747656$   
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 $n = 19 : 35137231473111223912310$   
 $n = 20 : 7569998079873075147860464$   
 $n = 21 : 2118828647238536587298828059$   
 $n = 22 : 770673990478689868790208726560$   
 $n = 23 : 364208960426686596537755921369204$   
 $n = 24 : 223584747192421482835961510980545856$   
 $n = 25 : 178316052131585634729418453829335054936$   
 $n = 26 : 184783530298259828417276720806393969283072$   
 $n = 27 : 248781049045288108090429913919027669986240120$   
 $n = 28 : 435100651401570160848422958340223675778192652096$   
 $n = 29 : 988575714030048455539523534323540042730905113789712$   
 $n = 30 : 2918268808268512558661062634728957323873319565816585216$   
 $n = 31 : 11191977556595306787688924514392387401160522562509404752480$   
 $n = 32 : 55758361084253080973582654412549007021500164037613319497525120$   
 $n = 33 : 360873619910724091463085427354561273174898813007073520286270385312$   
 $n = 34 : 3034428094821742605659499518550072906286077860183906562736432838885888$   
 $n = 35 : 33147799463054177459141651964502103162909097171877989104468852980628830640$   
 $n = 36 : 470387500964073698951254185900620768347698608335597100251088754974122927207552$   
 $n = 37 : 8671511724276778346310776146790389784220888235304663578971016972018061943458672488$   
 $n = 38 : 207681592952826862584324117895542918615491954128661522770180883458509427151696694500864$   
 $n = 39 : 6461752399123675883811274927599968774322623175693197510050988605401080962582639480482415880$   
 $n = 40 : 261171997590465540151318762930933288702318505263690005394345900119582121231418206978624887208000$

→ With a new implementation of the Bousquet-Mélou-Habsieger algorithm: DSASM( $n$ ) for  $n$  up to 40.

## Elimination of variables

After lots of manipulations.....

**Theorem.** Let  $g(u, v) = \frac{(u - v) \left( \frac{i}{\sqrt{3}} (uv - 1) - u - v \right)}{1 + uv + u^2v^2}$

Then the number of  $(2n \times 2n)$ -DSASMs is equal to

$$3^{n^2} \text{Pf}_{1 \leq i < j \leq 2n} \left( [u^{i-1}v^{j-1}]g(u + 1, v + 1) \right),$$

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$$3^{n^2} \text{Pf}_{1 \leq i < j \leq 2n} \left( [u^{i-1}v^{j-1}]g(u + 1, v + 1) \right),$$

while the number of  $(2n + 1) \times (2n + 1)$ -DSASMs is equal to

$$3^{(n+1)n} (-1)^n \text{Pf}_{1 \leq i < j \leq 2n} \left( [u^i v^j]g(u + 1, v + 1) \right).$$

## Compute matrix entries

$$[u^k v^n]g(u+1, v+1) \text{ for } g(u, v) = \frac{(u-v) \left( \frac{i}{\sqrt{3}} (uv-1) - u - v \right)}{1 + uv + u^2 v^2}$$

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→ Show Mathematica

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→ Show Mathematica

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$$\text{with } \bar{g}(u, v) := \frac{u-v}{(u+1)^2(v+1)^2 + (u+1)(v+1) + 1}$$

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Apply Cauchy's integral formula:

$$\bar{c}(k, n) := [u^k v^n] \bar{g}(u, v) = \left( \frac{1}{2\pi i} \right)^2 \oint \oint \frac{\bar{g}(u, v)}{u^{k+1} v^{n+1}} du dv$$

$$c(k, n) := [u^k v^n] g(u+1, v+1) = \frac{i}{\sqrt{3}} \bar{c}(k+1, n+1) + \left( \frac{i}{\sqrt{3}} - 1 \right) \left( \bar{c}(k+1, n) + \bar{c}(k, n+1) \right) - 2 \bar{c}(k, n)$$

## Recurrences for matrix entries

Employ Holonomic Functions to derive recurrences for  $\bar{c}(k, n)$ :

- ▶  $\text{int1} = \frac{\bar{g}(u,v)}{u^{k+1}v^{n+1}}$
- ▶  $\text{int2} = \text{CreativeTelescoping}[\text{int1}, \text{Der}[u], \{\text{Der}[v], S[k], S[n]\}]$
- ▶  $\text{CreativeTelescoping}[\text{int2}[[1]], \text{Der}[v]][[1]]$

$$\begin{aligned} & \{(k+1)(n-k+1)S_k + (n+1)(k-n+1)S_n - (k-n-1)(k-n+1), \\ & 3(n+3)(n+4)(k-n-3)(k-n-2)(k-n-1)(k-n)S_n^4 \\ & - 3(n+3)(k-3n-7)(k-n-4)(k-n-2)(k-n-1)(k-n)S_n^3 \\ & + (k-n-4)(k-n-3)(k-n-1)(k-n) \\ & \quad \times (k^2 - 4kn - 7k + 10n^2 + 38n + 36) S_n^2 \\ & - (n+1)(k-5n-7)(k-n-4)(k-n-3)(k-n-2)(k-n)S_n \\ & + n(n+1)(-k+n+1)(-k+n+2)(-k+n+3)(-k+n+4)\} \end{aligned}$$

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Next, we have to evaluate the determinant...

→ Show Mathematica

## Homomorphic images

- ▶ Choose primes  $p$  such that  $x^2 + 3 = 0$  has a solution in  $\mathbb{Z}_p$
- ▶ compute the matrix entries with the recurrences
- ▶ replace  $i\sqrt{3}$  by such a solution
- ▶ evaluate the determinant mod  $p$  and use chinese remaindering

## Homomorphic images

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- ▶ evaluate the determinant mod  $p$  and use chinese remaindering

This way we computed  $\text{DSASM}(n)$  for  $n$  up to 600:

$$\text{DSASM}(600) = 149641 \dots 695744 \approx 1.49641 \cdot 10^{20523}$$

## New Pfaffian formula for DSASMs

After lots of manipulations.....

**Theorem.** The number of  $(n \times n)$ -DSASMs is equal to

$$\text{Pf}_{\epsilon(n) \leq i < j \leq n-1} \left( [u^i v^j] \frac{(v-u)(2+uv)}{(1-uv)(1-u-v)} \right),$$

where  $\epsilon(n) = 0$  for even  $n$  and  $\epsilon(n) = 1$  for odd  $n$ .

## Refined enumeration of DSASMs

For an ASM  $A$  let  $\rho(A)$  denote the column of the 1 in the top row.

$$\begin{array}{cccc}
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
 \end{array}$$

**Example:** for  $n = 4$  one gets  $F_4(y) = 5y + 5y^2 + 4y^3 + 2y^4$ .

## Refined enumeration of DSASMs

For an ASM  $A$  let  $\rho(A)$  denote the column of the 1 in the top row.

**Proposition.** Let  $q = \frac{\sqrt{3+i}}{2}$ . Then

$$\begin{aligned} F_n(y) &= \sum_{A \text{ DSASM of order } 2n} y^{\rho(A)} = \sum_{k=1}^n \text{DSASM}(n, k) \cdot y^k = \frac{1}{1-2y} \\ &\times \left( y(1-y) \text{DSASM}(2n-1) + (-1)^n i 3^{n(n-1)} \sqrt{3} y^{2n+1} (1-y)^{-2n+1} \right. \\ &\quad \times \sum_{l=1}^{2n-1} (-1)^{l-1} [u^{i-1}] g \left( u+1, \frac{1+q^4 y}{q^4+y} \right) \\ &\quad \left. \times \text{Pf}_{1 \leq i < j \leq 2n-1, i, j \neq l} ([u^{i-1} v^{j-1}] g(u+1, v+1)) \right). \end{aligned}$$

**Example:** for  $n = 4$  one gets  $F_4(y) = 5y + 5y^2 + 4y^3 + 2y^4$ .

## Top 1 at first, second, or last position

We have  $\text{DSASM}(n, 1) = \text{DSASM}(n, 2) = \text{DSASM}(n - 1)$   
and  $\text{DSASM}(n, n) = \text{DSASM}(n - 2)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & & & & 0 \\ 0 & & & & 0 \\ 0 & & & & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 1 & & & & \\ 0 & 0 & & & & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{pmatrix}$$

## Top 1 at third position

**Lemma.**  $\text{DSASM}(n, 3) = 2 \cdot \text{DSASM}(n - 1) - 3 \cdot \text{DSASM}(n - 2)$ .

**Proof.** We construct two sets,  $M_1$  and  $M_2$ , of  $(n \times n)$  matrices.

$$M_1: \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 0 & \dots & 0 \\ \hline 0 & & & & & \\ 1 & & -1 & & & \\ 0 & & \text{DSASM} & & & \\ \vdots & & & & & \\ 0 & & & & & \\ \hline \end{array}$$
$$M_2: \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 0 & \dots & 0 \\ \hline 0 & +1 & -1 & & & \\ 1 & -1 & & & & \\ 0 & & \text{DSASM} & & & \\ \vdots & & & & & \\ 0 & & & & & \\ \hline \end{array}$$

Observations:

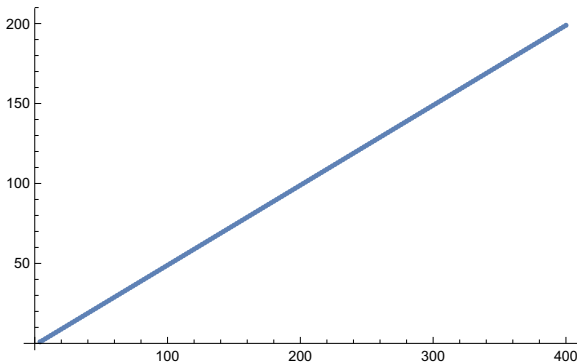
- ▶  $|M_1| = |M_2| = \text{DSASM}(n - 1)$
- ▶  $|M_1 \cap M_2| = \text{DSASM}(n - 2)$
- ▶  $\text{DSASM}(n - 2)$  elements in  $M_1$  are not ASM
- ▶  $\text{DSASM}(n - 2)$  elements in  $M_2$  are not ASM

## What about the fourth position?

Look at the coefficient of  $y^3$  in the transformed generating function

$$\bar{F}_n(y) = \frac{1}{3 \text{ DSASM}(n-2)} \left( \frac{2y-1}{y} F_n(y) - \text{DSASM}(n-1) \cdot (y-1) \right)$$

For  $n = 4, 6, 8, \dots$  plot  $q_3(n)$

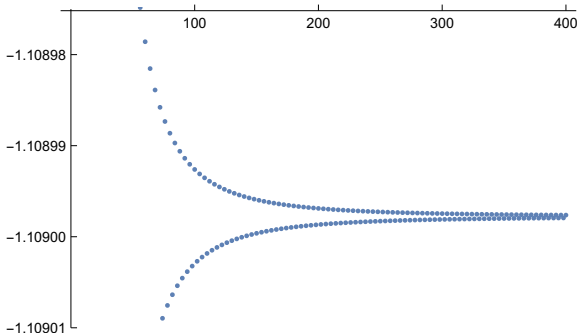


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For  $n = 4, 6, 8, \dots$  plot  $q_3(n) - \frac{n}{2}$

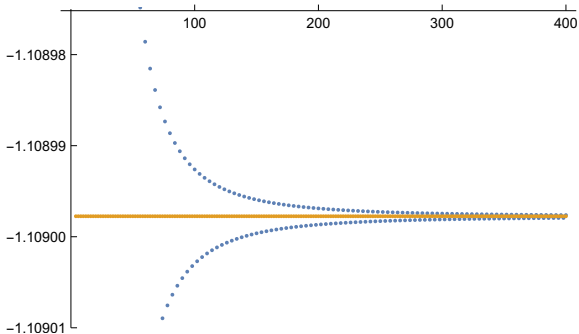


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For  $n = 4, 6, 8, \dots$  plot  $q_3(n) - \frac{n}{2} + 1.10899775$

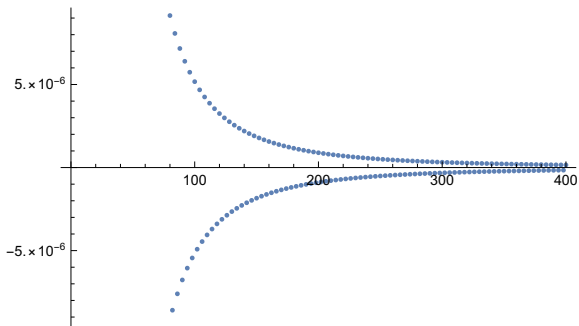


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For  $n = 4, 6, 8, \dots$  plot  $q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi}$

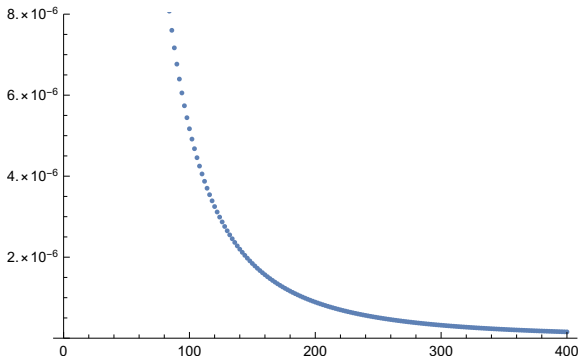


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For  $n = 4, 6, 8, \dots$  plot  $\left( q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi} \right) \cdot (-1)^{n/2}$

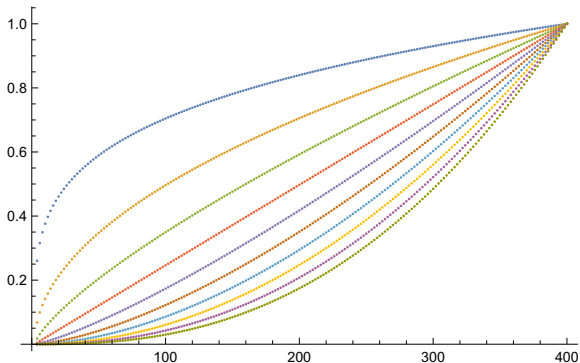


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For  $n = 4, 6, 8, \dots$  plot  $\left( (q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi}) \cdot (-1)^{n/2} \right)^{-j/10}$

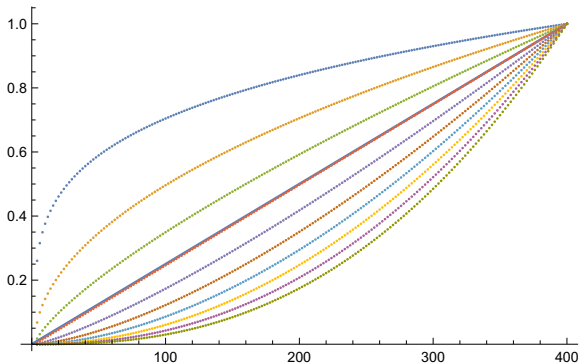


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For  $n = 4, 6, 8, \dots$  plot  $\left( (q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi}) \cdot (-1)^{n/2} \right)^{-j/10}$



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These experiments suggest an asymptotic expansion of the form

$$q_3(n) \approx \frac{n}{2} - \frac{5}{6} - \frac{\sqrt{3}}{2\pi} + c \cdot (-1)^{n/2} \cdot n^{-5/2} + \dots$$

- ▶ Confirmed by Jay Pantone, using differential approximants
- ▶ He estimates  $c \approx 0.086873868\dots$
- ▶ For DASASMs, these coefficients have a closed form as a rational function

# DSASMs in the OEIS

<b>A005163</b>	Number of alternating sign $n \times n$ matrices that are symmetric about a diagonal. (Formerly M1500)	+40 0
	1, 2, 5, 16, 67, 368, 2630, 24376, 293770, 4610624, 94080653, 2492747656, 85827875506, 3842929319936, 223624506056156, 16901839470598576, 1659776507866213636, 211853506422044996288, 35137231473111223912310, 7569998079873075147860464 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ; <a href="#">history</a> ; <a href="#">text</a> ; <a href="#">internal format</a> )	
OFFSET	1,2	
COMMENTS	Robbins's paper does not give a formula for this sequence. On the contrary he states: "Apparently these numbers do not factor into small primes, so a simple product formula seems unlikely. Of course this does not rule out other very simple formulas, but these would be more difficult to discover (let alone prove)." As far as I know no formula is currently known. - Herman Jamke (hermanjamke(AT)fastmail.fm), Feb 23 2008	
REFERENCES	N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence). R. P. Stanley, A baker's dozen of conjectures concerning plane partitions, pp. 285-293 of "Combinatoire Enumerative (Montreal 1985)", Lect. Notes Math. 1234, 1986.	
LINKS	Christoph Koutschan, <a href="#">Table of n, a(n) for n = 1..131</a> Mireille Bousquet-Mélou and Laurent Habsieger, <a href="#">Sur les matrices à signes alternants</a> , [On alternating-sign matrices] in Formal power series and algebraic combinatorics (Montreal, PQ, 1992). Discrete Math. 139 (1995), 57-72. D. P. Robbins, <a href="#">Symmetry classes of alternating sign matrices</a> , arXiv:math/0008045 [math.CO], 2000. R. P. Stanley, <a href="#">A baker's dozen of conjectures concerning plane partitions</a> , pp. 285-293 of "Combinatoire Enumerative (Montreal 1985)", Lect. Notes Math. 1234, 1986. Preprint. [Annotated scanned copy]	
KEYWORD	nonn,easy,nice,changed	