What is a holonomic function?

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Notation: Let $\mathbb{K}$ be a computable field of characteristic zero.

Preliminaries: A finitely generated left module over the Weyl algebra $\mathcal{A}_s(\mathbb{K})$ is **holonomic** if it is zero or if it has Bernstein dimension s, i.e., the dimension of the graded associated module with respect to the Bernstein filtration (which filters according to total degree).

Definition 1: A function $f(x_1, \ldots, x_s)$ is holonomic if the left $\mathcal{A}_s(\mathbb{K})$-module $\mathcal{A}_s(\mathbb{K}) / \operatorname{Ann}_{\mathcal{A}_s(\mathbb{K})}(f)$ is holonomic. In other words, if it is the solution of a maximally overdetermined system of differential equations.

Definition 2: A function f is holonomic if the following $\mathbb{K}(x_1, \ldots, x_s)$-vector space is finite-dimensional: $\dim_{\mathbb{K}(x_1,\ldots,x_s)}(\mathbb{O} / \operatorname{Ann}_{\mathbb{O}}(f)) < \infty$, where $\mathbb{O}$ denotes the Ore algebra $\mathbb{K}(x_1, \ldots, x_s)[D_{x_1}; 1, \frac{\mathrm{d}}{\mathrm{d}x_1}] \cdots [D_{x_s}; 1, \frac{\mathrm{d}}{\mathrm{d}x_s}]$.

D-finite and P-recursive

A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[x]$ not all zero.

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A sequence $f(n)$ is called **P-recursive** (or **P-finite**) if it satisfies a linear recurrence equation with polynomial coefficients:

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$\longrightarrow$ In both cases, only finitely many initial conditions!
$\longrightarrow$ Also called **holonomic function** resp. **holonomic sequence**.

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geq 2)$$

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&= \tfrac{2(2n+5)(n^2+5n+5)}{(n+2)(n+3)(n+4)} H_{n+1} - \tfrac{(n+1)(3n^2+18n+26)}{(n+2)(n+3)(n+4)} H_n
\end{aligned}$$

Closure Properties

If $f(x)$ and $g(x)$ are D-finite then also the following are D-finite

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A sequence is P-recursive iff its generating function is D-finite.

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.

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All coefficients $r_{i,j}$ must vanish: this yields $d_1 d_2$ equations for the unknowns $c_0, \ldots, c_d$. The choice $d = d_1 d_2$ ensures a solution.

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Try to generalize the finiteness property to

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- mixed setting: functions in several continuous and discrete variables $f(x_1, \dots, x_s, n_1, \dots, n_r)$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2 - 1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

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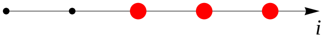

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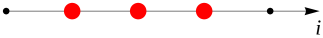

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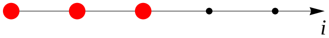

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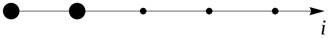

$\longrightarrow$ $P_n(x)$ is D-finite w.r.t. x.

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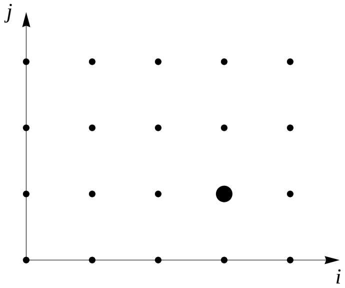

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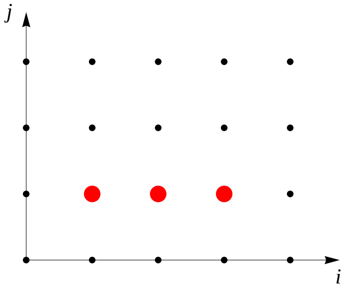

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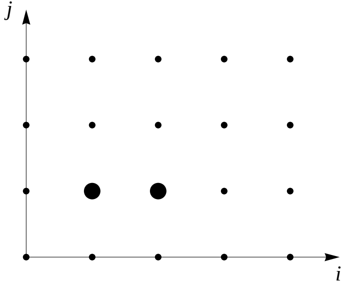

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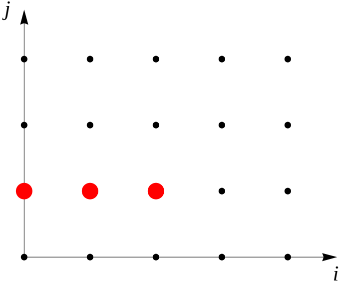

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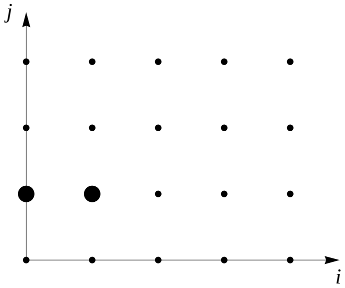

$$P_{n+1}^{(3)}(x) = \frac{(n^2x^2-n^2+3nx^2-3n+8x^2)}{(x^2-1)^2}P_{n+1}'(x) - \frac{4(n^2x+3nx+2x)}{(x^2-1)^2}P_{n+1}(x)$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

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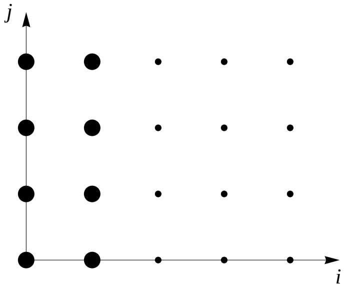

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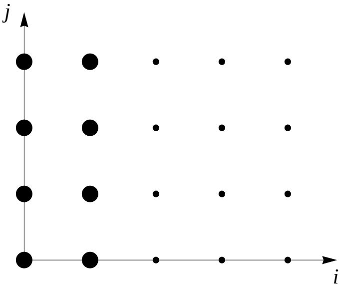

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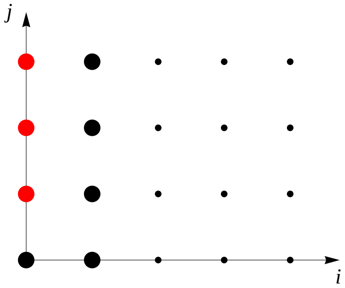

$$P_{n+3}(x) =$$
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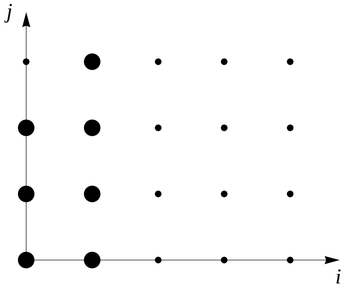

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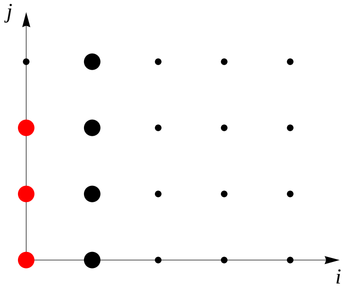

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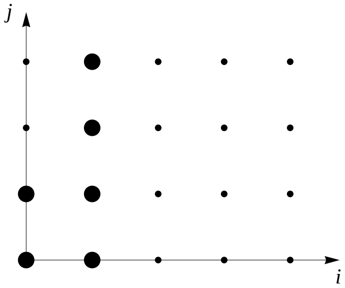

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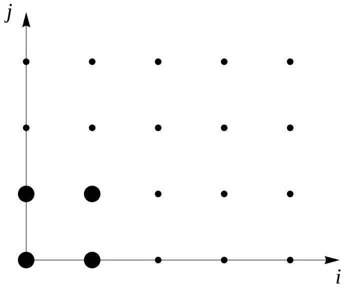

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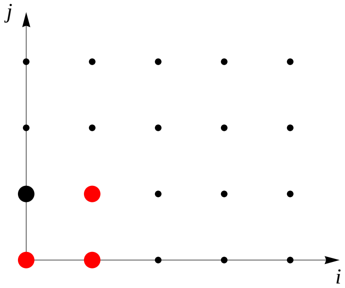

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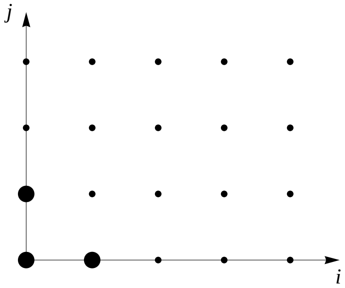

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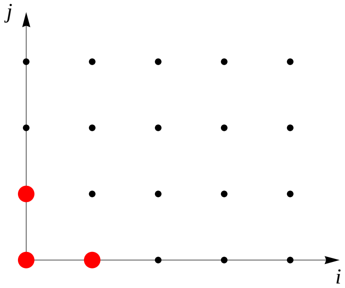

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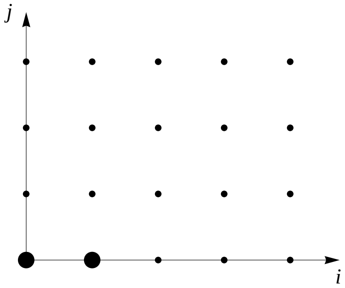

$\longrightarrow$ $P_n(x)$ is ∂-finite w.r.t. n and x (of rank 2).

∂-Finiteness

First attempt at a definition:

Let $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$ be a function in the continuous variables $x_1, \ldots, x_s$ and in the discrete variables $n_1, \ldots, n_r$.
If there is a finite set of basis functions of the form

$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \ldots, x_s, n_1 + j_1, \ldots, n_r + j_r)$$

with $i_1, \ldots, i_s, j_1, \ldots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \ldots, x_s, n_1, \ldots, n_r)$-linear combination of the basis functions.

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Again, finitely many (modulo details) initial conditions suffice to fix f.

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
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$$(n+2)S_n^2 - (2n+3)xS_n + (n+1).$$

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Differential equations and recurrences are translated to skew polynomials.

Noncommutative multiplication:

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Even more general:

$$\partial_v a = \sigma(a)\partial_v + \delta(a)$$

where σ is an automorphism and δ a σ-derivation, i.e.,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

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Such operators form an **Ore algebra**

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i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $v, w, \ldots$, where $\mathbb{K}$ is a computable field of characteristic 0 (i.e., containing $\mathbb{Q}$).

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Example: The operators that we encountered with the Legendre polynomials live in the Ore algebra

$$\mathbb{K}(x, n)\langle D_x, S_n \rangle = \mathbb{K}(x, n)[D_x; 1, \tfrac{\mathrm{d}}{\mathrm{d}x}][S_n; \sigma_n, 0].$$

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

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are also valid equations for f. More generally,

$$P, Q \in \operatorname{Ann}_{\mathbb{O}} f \implies P + Q \in \operatorname{Ann}_{\mathbb{O}} f$$
$$L \in \mathbb{O}, P \in \operatorname{Ann}_{\mathbb{O}} f \implies L\,P \in \operatorname{Ann}_{\mathbb{O}} f$$

which states that $\operatorname{Ann}_{\mathbb{O}} f$ is a **left ideal** in $\mathbb{O}$.

Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function $f(v, w, \dots)$ is ∂-finite w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space:

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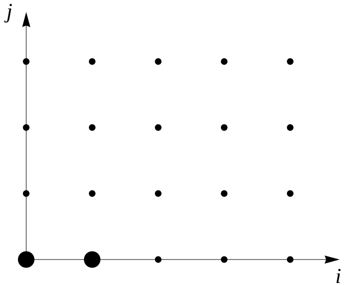

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In other words, if the left ideal of annihilating operators of f

$$\operatorname{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} : P \cdot f = 0\}$$

is a zero-dimensional ideal.

Gröbner Bases

Why ∂-Finite Functions?

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   - annihilating ideal of operators (Gröbner basis is finite)
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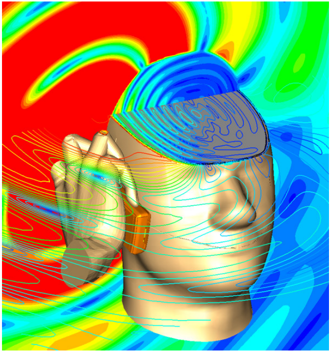
3. These operations (closure properties) can be executed algorithmically.
4. Many elementary and special functions are covered.

(Incomplete) List of ∂-Finite Functions

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, BesselI, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, BesselJ, StruveL, ArcSec, Factorial2, KelvinBer, BesselK, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin,

Finite Elements



Joint work with Joachim Schöberl and Peter Paule

Problem Setting

Simulate the propagation of electromagnetic waves using the Maxwell equations

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively.

Define basis functions (this is the 2D case):

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\tfrac{2y}{1-x} - 1\right)$$

using the Legendre and Jacobi polynomials.

Problem: Represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself).

Make an Ansatz!

More precisely, we need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

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that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

Sketch of the algorithm:

1. Work in the Ore algebra $\mathbb{O} = \mathbb{Q}(i,j,x,y)\langle S_i, S_j, D_x\rangle$.
2. Compute a Gröbner basis G of $\operatorname{Ann}_{\mathbb{O}} \varphi_{i,j}(x,y)$.
3. Choose index sets A and B.
4. Reduce the above ansatz with G and obtain a normal form.
5. Do coefficient comparison with respect to x and y.
6. Solve the resulting linear system for $a_{k,l}, b_{m,n} \in \mathbb{Q}(i,j)$.
7. If there is no solution, go back to step 3.

Result

With this method, we find the relation

$$
\begin{aligned}
&(2i+j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)+\\
&2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)-\\
&(j+3)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)+\\
&(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)-\\
&2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)-\\
&(2i+j+5)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)=0
\end{aligned}
$$

and a similar one for $\frac{\mathrm{d}}{\mathrm{d}y}\varphi_{i,j}(x,y)$.

$\longrightarrow$ The use of these previously unknown formulae caused a considerable speed-up in the numerical simulations.

That was nice, but we want (and can) do more...

What about integrals

$$\int_a^b f(x, \dots)\,\mathrm{d}x$$

and sums

$$\sum_{n=a}^{b} f(n, \dots)$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

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Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

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Creative Telescoping: write

$$c_d(n)f(n+d,k) + \cdots + c_0(n)f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_d(n)F(n+d) + \cdots + c_0(n)F(n) = g(n,b+1) - g(n,a).$$

Creative Telescoping

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Consider the following integration problem: $F(x) = \int_a^b f(x, y)\,\mathrm{d}y$

Telescoping: write $f(x, y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x, y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x, y)\right) \mathrm{d}y \qquad = g(x, b) - g(x, a)$.

Creative Telescoping: write

$$c_d(x) \frac{\mathrm{d}^d}{\mathrm{d}x^d} f(x, y) + \cdots + c_0(x) f(x, y) \;\; = \frac{\mathrm{d}}{\mathrm{d}y} g(x, y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_d(x) \frac{\mathrm{d}^d}{\mathrm{d}x^d} F(x) + \cdots + c_0(x) F(x) \;\; = g(x, b) - g(x, a)$$

Creative Telescoping, $\mathbb{O} = \mathbb{K}(n,k)\langle S_n, S_k\rangle$

$$\begin{aligned} c_d(n)f(n+d,k) + \cdots + c_0(n)f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1)\cdot g(n,k). \end{aligned}$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \bigl(c_d(n)f(n+d,i) + \cdots + c_0(n)f(n,i)\bigr)$$

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A reasonable choice for where to look for g is $\mathbb{O}\cdot f$.

Then the task is to find $P(n,S_n) = c_d(n)S_n^d + \cdots + c_0(n)$ and $Q \in \mathbb{O}$ such that

$$\big(P - (S_k - 1)Q\big)\cdot f = 0 \quad \iff \quad P - (S_k - 1)Q \in \operatorname{Ann}_{\mathbb{O}}(f).$$

Creative Telescoping (Example 1)

Let $F(n)$ denote the double sum over the trinomial coefficients

$$F(n) = \sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i, j, n-i-j} = \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{n!}{i!j!(n-i-j)!}.$$

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Then the creative telescoping operator

$$CT = S_n - 3 + (S_i - 1)\frac{i}{n-i-j+1} + (S_j - 1)\frac{j}{n-i-j+1}$$

with $CT\left(\binom{n}{i,j,n-i-j}\right) = 0$ implies that

$$F(n+1) = 3F(n).$$

Creative Telescoping (Example 2)

The lattice Green's function of the square lattice is given by

$$P(z) = \int_0^1 \int_0^1 \frac{1}{(1 - xyz)\sqrt{1 - x^2}\sqrt{1 - y^2}} \, \mathrm{d}x \, \mathrm{d}y.$$

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The creative telescoping operator

$$(z^3-z)D_z^2 + (3z^2-1)D_z + z + D_x\frac{y(1-x^2)}{xyz-1} + D_y\frac{yz(1-y^2)}{xyz-1}$$

that annihilates the integrand, certifies that $P(z)$ satisfies the differential equation

$$(z^3-z)P''(z) + (3z^2-1)P'(z) + zP(z) = 0.$$

Creative Telescoping in Full Generality

In general, a creative telescoping operator has the form

$$P(\boldsymbol{v}, \boldsymbol{\partial_v}) + \Delta_1 Q_1(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w}) + \cdots + \Delta_m Q_m(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$$

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where $\Delta_i = S_{w_i} - 1$ or $\Delta_i = D_{w_i}$ (depending on the problem).

- Corresponds to an m-fold summation/integration problem.
- $\boldsymbol{w} = w_1, \dots, w_m$ are the summation/integration variables.
- $\boldsymbol{v} = v_1, v_2, \dots$ are the surviving parameters.
- $P(\boldsymbol{v}, \boldsymbol{\partial_v})$ is called the **telescoper**.
- The $Q_i(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$ are called the **delta parts**.
- The delta parts can be viewed as certificates for the correctness of the telescoper.
- Research topic: develop fast algorithms to compute it!

Holonomy

Question: Does there always exist such a telescoping operator?

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Combine the two notions:

- Use ∂-finiteness for computations.
- Use holonomy for justifications.

Holonomic Functions

Assume that $f(x_1, \ldots, x_s)$ depends only on continuous variables.
Consider the **Weyl algebra**

$$\mathbb{W} = \mathbb{K}[x_1, \ldots, x_s]\langle D_{x_1}, \ldots, D_{x_s}\rangle.$$

Then f is holonomic if the left ideal $\mathrm{Ann}_{\mathbb{W}}(f)$ has dimension s (which, by Bernstein's inequality, is the minimum possible).

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Differently stated: f is holonomic if for any $(s-1)$-subset

$$E \subset \left\{x_1, \ldots, x_s, D_{x_1}, \ldots, D_{x_s}\right\}, \quad |E| = s-1,$$

there exists a nonzero element in $\operatorname{Ann}_{\mathbb{W}}(f)$ that is free of all generators in E.

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there exists a nonzero element in $\operatorname{Ann}_{\mathbb{W}}(f)$ that is free of all generators in E.

$\longrightarrow$ This is why a creative telescoping operator always exists.

∂-Finite and Holonomic Functions

Theorem: The function $f(x_1, \ldots, x_s)$ is holonomic if and only if it is ∂-finite.

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Example: The sequence $\frac{1}{n^2+k^2}$ is ∂-finite but not holonomic.

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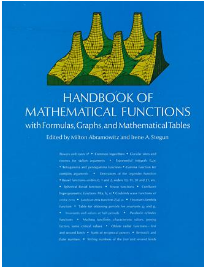
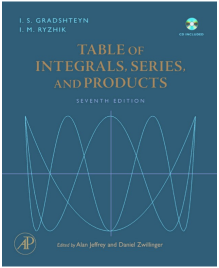
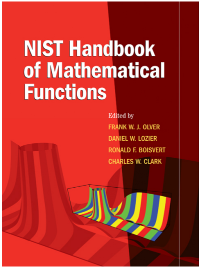
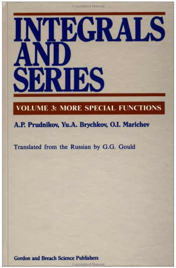
3. Integrals and sums are treated by the method of creative telescoping.
4. The output is always given as an annihilating ideal, not as a closed form.

Special Function Identities









Some Special Function Identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{k+n}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{k+n}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \qquad (1)$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x=\frac{\pi P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \qquad (2)$$

$$e^{-x}x^{a/2}n!L_n^a(x)=\int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\mathrm{d}t \qquad (3)$$

$$\int_{-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!n!}\,\mathrm{d}x=\sqrt{\pi}e^{2rs} \qquad (4)$$

$$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}}e^{iax}C_n^{(\nu)}(x)\,\mathrm{d}x=\frac{\pi i^n\Gamma(n+2\nu)J_{n+\nu}(a)}{2^{\nu-1}a^\nu n!\Gamma(\nu)} \qquad (5)$$

$$\frac{\sin\left(\sqrt{z^2+2tz}\right)}{z}=\sum_{n=0}^{\infty}\frac{(-t)^n y_{n-1}(z)}{n!} \qquad (6)$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

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```
<< HolonomicFunctions.m
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

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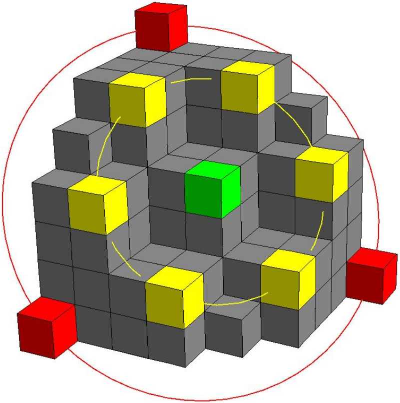
```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*BesselJ[a,2*Sqrt[t*x]
                    Der[t], {S[a], S[n], Der[x]}]
```

$$\{\{-2S_n + 2xD_x + (a+2n+2),\\ 4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),\\ 2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},\\ \{-2t, -4tx, -2tx\}\}$$

$\longrightarrow$ The annihilating ideals agree; check a few initial values.

Enumerative Combinatorics



Joint work with Manuel Kauers and Doron Zeilberger

The q-TSPP Conjecture

"The Holy Grail of Enumerative Combinatorics"

Last surviving open problem of the classic (Stanley, 1986)
A baker's dozen of conjectures concerning plane partitions

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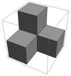
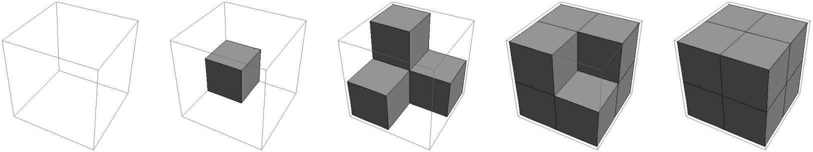
Some notions from partition theory:

- partition of an integer, e.g., $4 = 2 + 1 + 1$
- plane partition: two-dimensional arrangement of summands, e.g., $\begin{matrix} 2 & 1 \\ 1 & \end{matrix}$
- Ferrer’s diagram: three-dimensional representation of PPs, e.g., $\begin{matrix} 2 & 1 \\ 1 & \end{matrix} =$ 
- totally symmetric plane partition (TSPP): Ferrer’s diagram is symmetric w.r.t. rotation and reflection
- orbit: see the picture

Let $T(n)$ denote set of TSPPs with largest part $\leq n$, e.g. $n = 2$:



$1 \cdot q^0$ $\quad$ $1 \cdot q^1$ $\quad$ $1 \cdot q^2$ $\quad$ $1 \cdot q^3$ $\quad$ $1 \cdot q^4$

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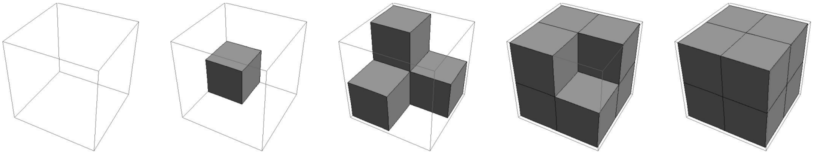

$1 \cdot q^0$ $1 \cdot q^1$ $1 \cdot q^2$ $1 \cdot q^3$ $1 \cdot q^4$

Andrews-Robbins q-TSPP conjecture (1983):

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

Counting formula is obtained by $q \to 1$ (Stembridge 1995):

$$|T(n)| = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}$$

The Determinant

Soichi Okada: the q-TSPP conjecture is true if

$$\det_{1\le i,j\le n}(a_{i,j}) = \prod_{1\le i\le j\le k\le n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 =: b_n$$

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where

$$a_{i,j} := q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q\right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

is the q-binomial coefficient.

The Holonomic Ansatz (by Doron Zeilberger)

"Pull out of the hat" a holonomic function $c_{n,j}$ and prove

$$c_{n,n} = 1 \qquad (n \geq 1),$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0 \qquad (1 \leq i < n),$$

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1).$$

Then $\det(a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds for all n.

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$$\begin{aligned} c_{n,n} &= 1 && (n \geq 1), \\ \sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leq i < n), \\ \sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \geq 1). \end{aligned}$$

Then $\det(a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds for all n.

- $c_{n,j}$ not given explicitly
- perfect application for the holonomic systems approach
- very large computations
- feasible only due to a new algorithm