Automated Proofs of Mathematical Identities

Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

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RISC Hagenberg

ÖAW RICAM

Special Functions

- arise in mathematical analysis and in real-world phenomena

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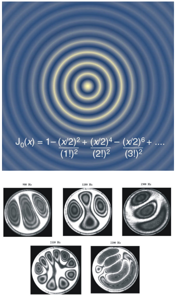

Bessel function

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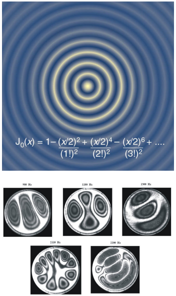

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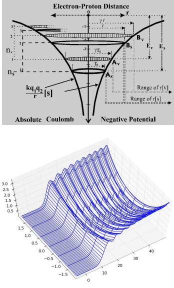

Coulomb function

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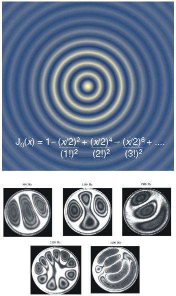

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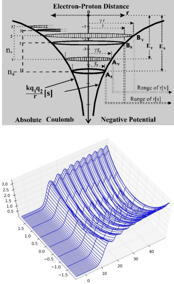

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Special Functions

- arise in mathematical analysis and in real-world phenomena
- are solutions to certain differential equations
- cannot be expressed in terms of the usual elementary functions ($\sqrt{\ }$, exp, log, sin, cos, ...)

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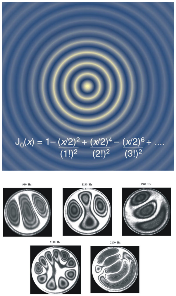

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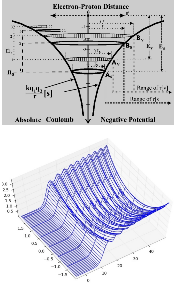

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Differential Equations and Recurrences

Example: The Bessel function $J_\nu(x)$ describes the vibrations of a circular membrane, as well as many other phenomena with cylindrical symmetry.

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- Bessel differential equation:

$$x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} J_\nu(x) + x \frac{\mathrm{d}}{\mathrm{d}x} J_\nu(x) + \left(x^2 - \nu^2\right) J_\nu(x) = 0$$

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$$J_\nu(x) = \frac{2(\nu - 1)}{x} J_{\nu-1}(x) - J_{\nu-2}(x)$$

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Many special functions can be characterized as solutions to systems of linear differential equations and recurrences, and in fact are holonomic.

Holonomic Functions

Definition: A function $f_{\ell,m,\dots,n}(x, y, \dots, z)$ is called holonomic, if it is the solution of a system

- of **linear** differential equations or recurrences,
- whose coefficients are **polynomials**,
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Theorem (Closure Properties): If $f_n(x)$ and $g_n(x)$ are two holonomic functions, then also the following expressions are holonomic:

- $f_n(x) \pm g_n(x)$
- $f_n(x) \cdot g_n(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x} f_n(x)$
- $f_{an+b}(x)$, where $a, b \in \mathbb{Z}$,
- $f_n(h(x))$, where $h(x)$ is an algebraic function.

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$$\begin{aligned}\sum_{k=a}^{b} f(k) &= \sum_{k=a}^{b} \big(g(k) - g(k+1)\big) = \\ &= \big(g(a) - g(a+1)\big) + \big(g(a+1) - g(a+2)\big) + \big(g(a+2) - \dots \\ &= g(a) - g(b+1).\end{aligned}$$

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Result: holonomic description of the sum: $P\big(\sum_{k=a}^{b} f(k)\big) = \dots$
This method works similarly for integrals.

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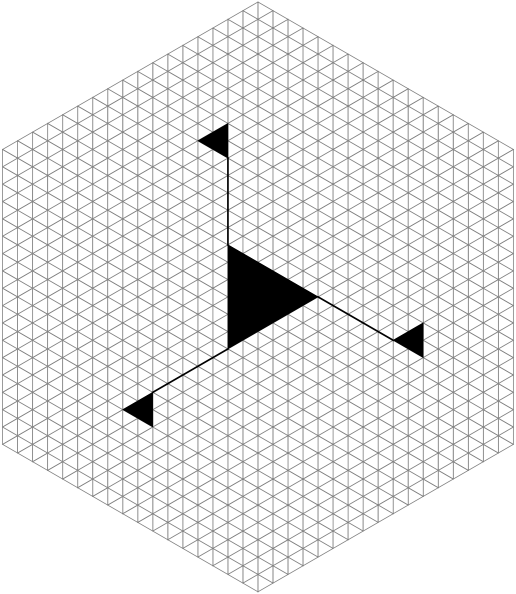
- calculate integrals and summation formulas
- prove special function identities
- computations in q-calculus (e.g., quantum knot invariants)
- fast numerical evaluation of mathematical functions
- number theory (e.g., irrationality proofs)
- evaluate symbolic determinants (e.g., in combinatorics)

A Problem from Enumerative Combinatorics



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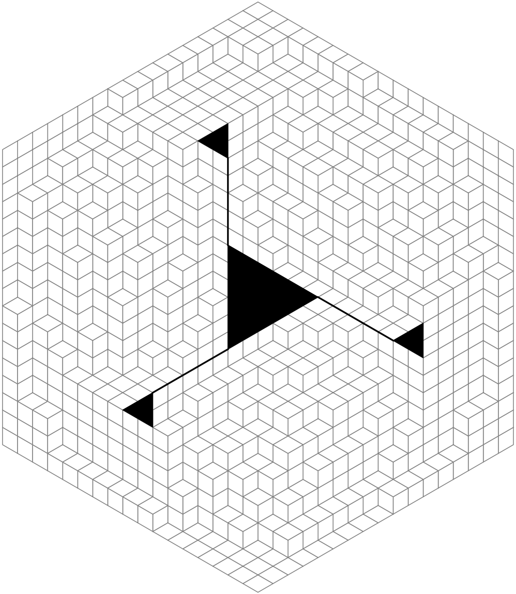

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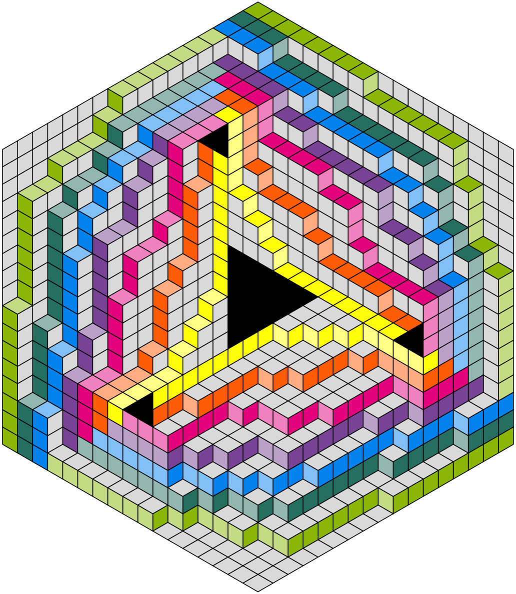
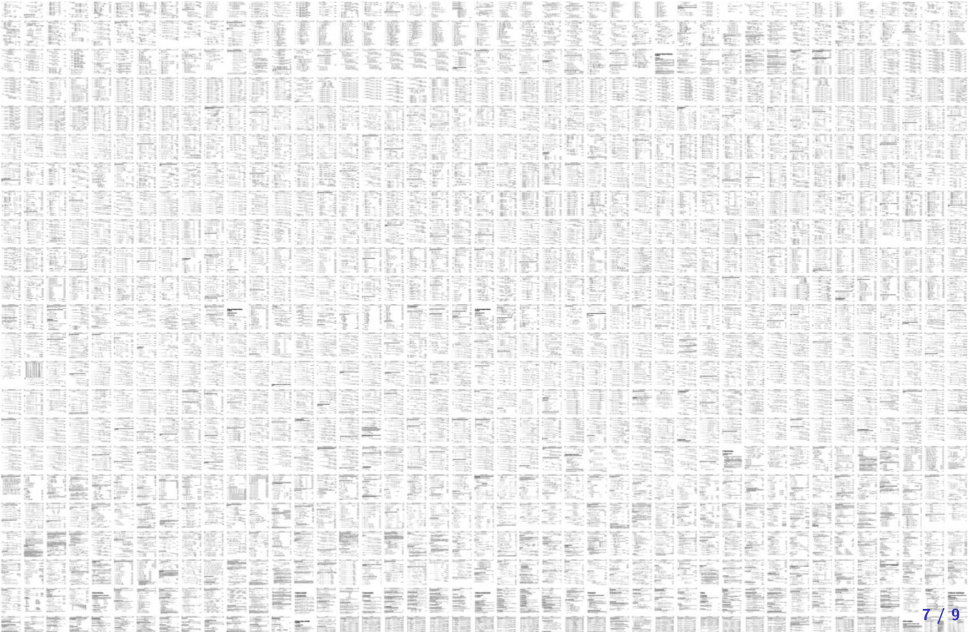


Table of Integrals by Gradshteyn and Ryzhik



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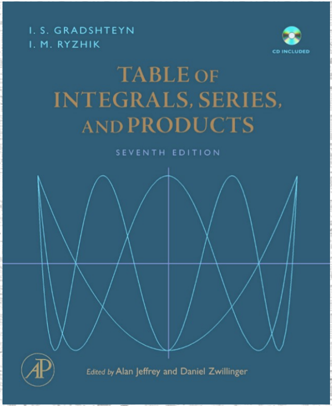

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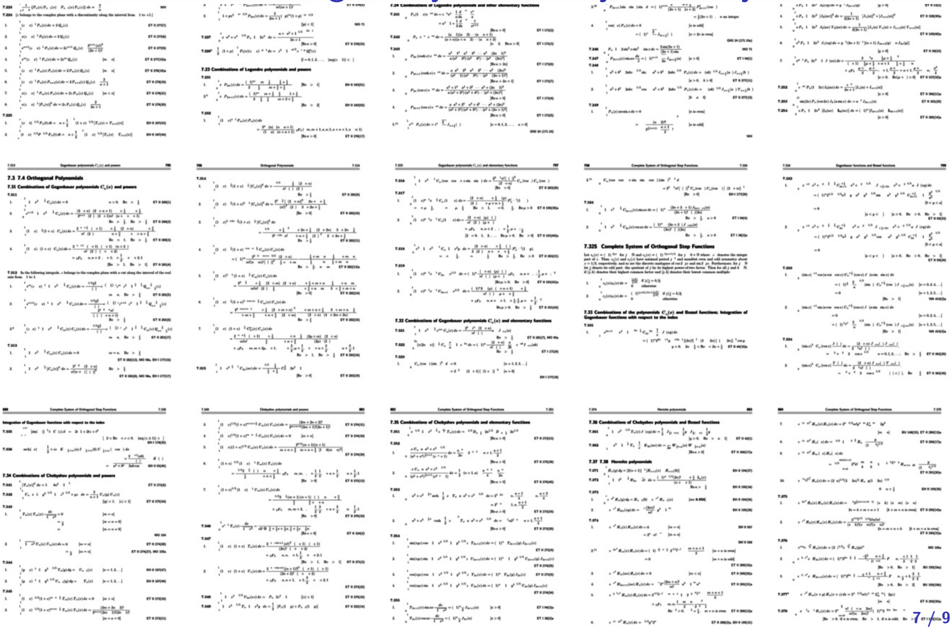

7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n}^{\lambda}\left(\gamma x^{1/2}\right) dx = (-1)^n \frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\; {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n+1}^{\lambda}\left(\gamma x^{1/2}\right) dx = \frac{(-1)^n 2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times {}_3F_2\left(-n, n+\lambda+1, \nu+\tfrac{1}{2}; \tfrac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}]$ ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^{\nu}(x)$ and elementary functions

7.321 $$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

$[\operatorname{Re}\nu > -\tfrac{1}{2}]$ ET II 281(7), MO 99a

7.322 $$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}}\, C_n^{\nu}\left(\frac{x}{a}-1\right) e^{-bx}\, dx = (-1)^n \frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab}\, I_{\nu+n}(ab)$$

$[\operatorname{Re}\nu > -\tfrac{1}{2}]$ ET I 171(9)

7.323

1. $$\int_0^{\pi} C_n^{\nu}(\cos\varphi)\,(\sin\varphi)^{2\nu}\, d\varphi = 0$$ $[n = 1, 2, 3, \ldots]$

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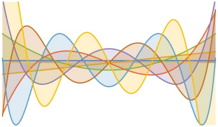

Gegenbauer polynomials $C_n^{(\alpha)}(x)$

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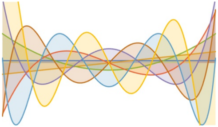

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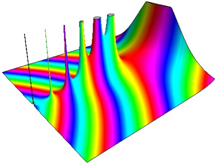

Gamma function $\Gamma(x)$

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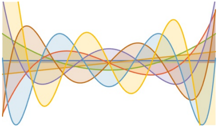

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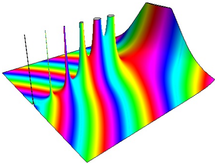

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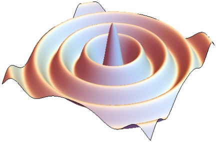

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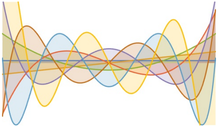

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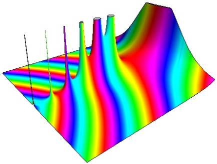

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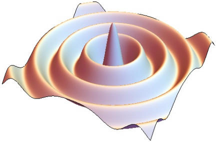

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- A large portion of such identities can be proven via the holonomic systems approach.

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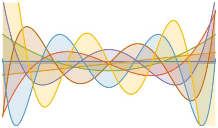

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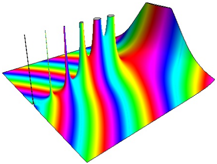

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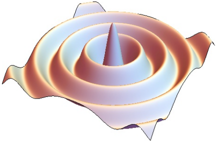

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- A large portion of such identities can be proven via the holonomic systems approach.
- Algorithms are implemented in the HolonomicFunctions package.

The HolonomicFunctions Package

Example: Holonomic system, satisfied by both sides of the identity:

$$ia(n+2\nu)f_n'(a) + a(n+1)f_{n+1}(a) - in(n+2\nu)f_n(a) = 0,$$

$$a(n+1)(n+2)f_{n+2}(a) - 2i(n+1)(n+\nu+1)(n+2\nu+1)f_{n+1}(a)$$
$$- a(n+2\nu)(n+2\nu+1)f_n(a) = 0.$$

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&\quad - a(n+2\nu)(n+2\nu+1)f_n(a) = 0.
\end{aligned}
$$

```
In[42]:= Annihilator[Pi * 2 ^ (1 - ν) * I ^ n * Gamma[2 ν + n] / n ! / Gamma[ν] * a ^ (-ν) *
           BesselJ[ν + n, a], {Der[a], S[n]}] // Factor
Out[42]=
```

$\{i\, a\,(n+2\nu)\, D_a + a\,(1+n)\, S_n - i\, n\,(n+2\nu),$
$\; a\,(1+n)\,(2+n)\, S_n^2 - 2\, i\,(1+n)\,(1+n+\nu)\,(1+n+2\nu)\, S_n - a\,(n+2\nu)\,(1+n+2\nu)\}$

```
In[43]:= CreativeTelescoping[(1 - x ^ 2) ^ (ν - 1 / 2) * Exp[I * a * x] * GegenbauerC[n, ν, x],
          Der[x], {Der[a], S[n]}] // Factor
Out[43]=
```

$\{\{a\,(n+2\nu)\, D_a - i\, a\,(1+n)\, S_n - n\,(n+2\nu),$
$\;\; a\,(1+n)\,(2+n)\, S_n^2 - 2\, i\,(1+n)\,(1+n+\nu)\,(1+n+2\nu)\, S_n - a\,(n+2\nu)\,(1+n+2\nu)\},$
$\; \{(1+n)\, S_n - x\,(n+2\nu),\; 2\, i\,(1+n)\, x\,(1+n+\nu)\, S_n - 2\, i\,(1+n+\nu)\,(n+2\nu)\}\}$

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Goal: Fully automate computer proofs of such identities:

- poles in the certificates $g(k) - g(k+1) = f(k)$
- natural boundaries $\sum_{k=a}^{b} \big(g(k) - g(k+1)\big) = g(a) - g(b+1)$

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- natural boundaries $\sum_{k=a}^{b}\big(g(k) - g(k+1)\big) = g(a) - g(b+1)$
- multiple sums and integrals

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However, in general some human interaction is required.

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Application: automatically verify large collections of mathematical identities, such as Gradshteyn/Ryzhik.