On Christol's Conjecture

Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

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ÖAW RICAM

Diagonals in Theoretical Physics

Joint work with

Youssef Abdelaziz, Salah Boukraa, Jean-Marie Maillard

- Diagonals of rational functions, pullbacked ${}_2F_1$ hypergeometric functions and modular forms (JPA 51(45), 455201, 2018)
- Heun functions and diagonals of rational functions (JPA 53(7), 075206, 2020)
- On Christol's conjecture (JPA 53(20), 205201, 2020)

Diagonals of Rational Functions

Given a rational function in n variables

$$R(x_1, \ldots, x_n) = \frac{A(x_1, \ldots, x_n)}{B(x_1, \ldots, x_n)},$$

where $A, B \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $B(0, \ldots, 0) \neq 0$.

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Definition: The diagonal of R is defined through its multi-Taylor expansion around $(0, \ldots, 0)$

$$R(x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} r_{m_1, \ldots, m_n} \cdot x_1^{m_1} \cdots x_n^{m_n}$$

as the power series in one variable:

$$\mathrm{Diag}\big(R(x_1, \ldots, x_n)\big) := \sum_{m=0}^{\infty} r_{m,m,\ldots,m} \cdot x^m$$

Example of a Diagonal

Consider the Taylor expansion of the bivariate rational function

$$\begin{aligned} f(x,y) &= \frac{1}{1-x-y-2xy} \\ &= 1+x+y+x^2+4xy+y^2+x^3+7x^2y+7xy^2+\dots \end{aligned}$$

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$$\begin{array}{cccccccccccccc} = & 1 & + & y & + & y^2 & + & y^3 & + & y^4 & + & y^5 & + \dots \\ + & x & + & 4xy & + & 7xy^2 & + & 10xy^3 & + & 13xy^4 & + & 16xy^5 & + \dots \\ + & x^2 & + & 7x^2y & + & 22x^2y^2 & + & 46x^2y^3 & + & 79x^2y^4 & + & 121x^2y^5 & + \dots \\ + & x^3 & + & 10x^3y & + & 46x^3y^2 & + & 136x^3y^3 & + & 307x^3y^4 & + & 586x^3y^5 & + \dots \\ + & x^4 & + & 13x^4y & + & 79x^4y^2 & + & 307x^4y^3 & + & 886x^4y^4 & + & 2086x^4y^5 & + \dots \\ + & x^5 & + & 16x^5y & + & 121x^5y^2 & + & 586x^5y^3 & + & 2086x^5y^4 & + & 5944x^5y^5 & + \dots \end{array}$$

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$$

$$
\begin{array}{llllllll}
= \color{red}{1} & + \; y & + \; y^2 & + \; y^3 & + \; y^4 & + \; y^5 & + \dots \\
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Then the diagonal of f is

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\mathrm{Diag}(f) = 1 + 4x + 22x^2 + 136x^3 + 886x^4 + 5944x^5 + \dots
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Properties of Diagonals

Theorem: The diagonal $f(x)$ of every rational function is

- **globally bounded:** there exist integers $c, d \in \mathbb{N}^*$, such that $d \cdot f(cx) \in \mathbb{Z}[[x]]$ and $f(x)$ has nonzero radius of convergence.

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- This conjecture was first formulated in a paper in 1986 and it is still widely open.
- It doesn't say anything about the number of variables in the rational function.
- One needs at least three variables, but no explicit example requiring more than three variables is known.

Christol's Conjecture

PROPOSITION : *Toute diagonale de fraction rationnelle f satisfait les propriétés suivantes :*

a) Elle est solution d'une équation différentielle linéaire L à coefficients dans $\mathbb{Q}[\lambda]$.

a') Cette équation différentielle est une équation de Picard Fuchs.

b) Pour toute place p (finie ou non) de $\mathbb{Q}$ *, le rayon de convergence $r_p(f)$ de la série f dans le corps* $\mathbb{C}_p$ *est non nul.*

c) Pour presque toute place p de $\mathbb{Q}$ *, on a* $r_p(f) = 1$.

c') Pour presque toute place p de $\mathbb{Q}$ *, la fonction f est bornée dans le disque* $D_p(0,1) = \{x \in \mathbb{C}_p ; |x|<1\}$.

c") Pour presque toute place p de $\mathbb{Q}$ *, on a :*

$$\|f\|_p = \sup_{x \in D_p(0,1)} |f(x)| = 1.$$

Seules les propriétés *a)* et *a')* ne sont pas immédiates. On en trouvera une démonstration dans [1] .

Dans cet article nous nous proposons de tester la conjecture suivante sur les fonctions hypergéométriques ${}_sF_{s-1}$:

CONJECTURE : *Une série entière f qui vérifie les propriétés a), b), c), c') et c") est la diagonale d'une fraction rationnelle.*

Hypergeometric Functions

Definition: Let $(a)_k := a \cdot (a+1) \cdots (a+k-1)$. Then

$$ {}_pF_q\big([a_1, \ldots, a_p], [b_1, \ldots, b_q]\,, x\big) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \cdot \frac{x^k}{k!} $$

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Note: Any such hypergeometric function is D-finite, for example: the classical Gauß hypergeometric $_2F_1([a,b],[c],x)$ function satisfies Euler's differential equation:

$$x(x-1)\,y''(x) + ((a+b+1)x - c)\,y'(x) + ab\,y(x) = 0.$$

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Hypergeometric functions of the form ${}_pF_{p-1}$ provide a natural testing ground for Christol's conjecture:

- If $q < p-1$ then the ${}_pF_q$ series has zero radius of convergence.

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- If $q < p-1$ then the ${}_pF_q$ series has zero radius of convergence.
- If $q > p-1$ then the ${}_pF_q$ series cannot be globally bounded.

Main Result

The hypergeometric function

$$ {}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big). $$

is the diagonal of a rational function.

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The following six-variable rational function witnesses this fact:

$$ 1 + \frac{au^3v\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-v)(v-w)} $$
$$ - \frac{av^4\,(1-vx-vy-vz)\,(1+v)^{a-1}(1-vx-vy-vz)^{a-1}}{(1+v)^a(1-vx-vy-vz)^a-(1-vx-vy)^b(u-v)(v-w)} $$
$$ - \frac{au^3w\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-w)(v-w)} $$
$$ - \frac{aw^4\,(1-wx-wy-wz)\,(1+w)^{a-1}(1-wx-wy-wz)^{a-1}}{(1+w)^a(1-wx-wy-wz)^a-(1-wx-wy)^b(u-w)(v-w)} $$

Hadamard Product

Definition: The Hadamard product of two series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n$$

is given by

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Example:

Let $f(x) = \mathrm{Diag}\big(F(x_1, \ldots, x_r)\big)$ and $g(x) = \mathrm{Diag}\big(G(x_1, \ldots, x_s)\big)$.
Then:

$$f(x) \star g(x) = \mathrm{Diag}\Big(F(x_1, \ldots, x_r) \cdot G(x_{r+1}, \ldots, x_{r+s})\Big).$$

Height

Definition: Let $f(x)$ be a hypergeometric function of the form

$$f(x) = {}_pF_{p-1}\big([a_1, \ldots, a_p], [b_1 \ldots, b_{p-1}], x\big).$$

Setting $b_p = 1$, the height of f is

$$h = \big|\{1 \leqslant j \leqslant p \mid b_j \in \mathbb{Z}\}\big| - \big|\{1 \leqslant j \leqslant p \mid a_j \in \mathbb{Z}\}\big|.$$

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Theorem (Christol): A ${}_pF_{p-1}$ hypergeometric function of height 1 is globally bounded if and only if it is algebraic.

Theorem (Christol): If $f(x)$ can be written as the Hadamard product of h globally bounded series of height 1, then $f(x)$ is the diagonal of a rational function.

Rational vs. Algebraic

Example: The globally bounded hypergeometric function

$$f(x) = {}_3F_2\big(\big[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\big], [1, 1], x\big) = 1 + \tfrac{1}{27}x + \tfrac{8}{729}x^2 + \dots$$

has height 3

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has height 3, and it can be written as the **Hadamard product** of three hypergeometric ${}_1F_0$ functions of height 1:

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By noting that ${}_1F_0([\frac{1}{3}], [\,], x) = (1-x)^{-1/3}$, we see that $f(x)$ is the diagonal of an **algebraic function** in three variables:

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Theorem (Denef, Lipshitz): Any power series in $\mathbb{Q}\,[[x_1, \dots, x_n]]$, algebraic over $\mathbb{Q}(x_1, \dots, x_n)$, is the diagonal of a rational function in $2n$ variables.

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- $c \in \mathbb{Q} \setminus \mathbb{Z}$: the ${}_2F_1$ function has height 1 and therefore is algebraic by Christol's theorem.

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- $c \in \mathbb{Q} \setminus \mathbb{Z}$: the ${}_2F_1$ function has height 1 and therefore is algebraic by Christol's theorem.

Hence, this situation is not particularly interesting for our purposes!

Situation for ${}_3F_2$ Functions

We now look at hypergeometric functions ${}_3F_2([a,b,c],[d,e],x)$ that are globally bounded, and consider the case $a,b,c \in \mathbb{Q} \setminus \mathbb{Z}$.

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But even in this case, a lot of the ${}_3F_2$ functions are easily seen to be diagonals of rational functions. . .

Situation for ${}_3F_2$ Functions

Suppose now that $f(x) = {}_3F_2([a,b,c],[d,1],x)$ is globally bounded, with the parameters $a,b,c,d \in \mathbb{Q} \setminus \mathbb{Z}$.

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Then there are six ways to write this function as a Hadamard product of hypergeometric functions:

$$f(x) = {}_2F_1([a,b],[d],x) \star {}_1F_0([c],[\,],x) \qquad = \dots$$

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Thus if one of ${}_2F_1([a,b],[d],x)$, ${}_2F_1([b,c],[d],x)$, ${}_2F_1([a,c],[d],x)$ is algebraic, then $f(x)$ is the diagonal of a rational function.

Potential Counterexamples

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- G. Christol, *Fonctions hypergéométriques bornées*, Groupe d'Etude d'Analyse ultramétrique, vol. 14 (1986–1987), Exposé N$^\circ$ 8, p. 1–16.

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A longer list was generated by Christol and his co-authors in 2012.

- A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard *Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity.* Journal of Physics A: Mathematical and Theoretical **46**(18)

Potential Counterexamples

For example, these two hypergeometric functions are globally bounded, as they can be recast into series with integer coefficients:

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], 3^6 x\right) = 1 + 120x + 47124x^2 + 23483460x^3 + \dots $$

$$ {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}, 1\right], 3^6 x\right) = 1 + 84x + 32760x^2 + 16302000x^3 + \dots $$

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But they cannot be obtained as diagonals through Hadamard products, since the following series are not globally bounded:

$$ {}_2F_1\left(\left[\tfrac{2}{9}, \tfrac{5}{9}\right], \left[\tfrac{2}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{2}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}\right], x\right), $$

$$ {}_2F_1\left(\left[\tfrac{1}{9}, \tfrac{4}{9}\right], \left[\tfrac{1}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{1}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}\right], x\right). $$

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$$ \dots + \frac{2 \cdot 11 \cdot 20 \cdots (9k-7) \cdot 5 \cdot 14 \cdot 23 \cdots (9k-4)}{2 \cdot 5 \cdot 8 \cdots (3k-1) \cdot 1 \cdot 2 \cdot 3 \cdots k} \cdot \left(\frac{x}{27}\right)^k + \dots $$

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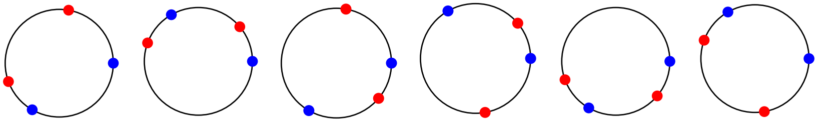
There are infinitely many prime factors in the Taylor expansion, and therefore the function is not globally bounded.

Criterion for Global Boundedness

Theorem (Christol): Assume that the parameters $\{a_1, \ldots, a_p\}$ and $\{b_1, \ldots, b_{p-1}, b_p = 1\}$ are rational and disjoint modulo $\mathbb{Z}$, and let N be their common denominator.
Then ${}_pF_{p-1}([a_1, \ldots, a_p], [b_1, \ldots, b_{p-1}], x)$ is globally bounded if and only if for all $1 \leqslant r < N$ with $\gcd(r, N) = 1$, one encounters more numbers in $\{\exp(2\pi i\, r\, b_j)\}$ than in $\{\exp(2\pi i\, r\, a_j)\}$, when running through the unit circle counter-clockwise.

Example: For our ${}_2F_1\left(\left[\frac{2}{9}, \frac{5}{9}\right], \left[\frac{2}{3}\right], x\right)$ we obtain:



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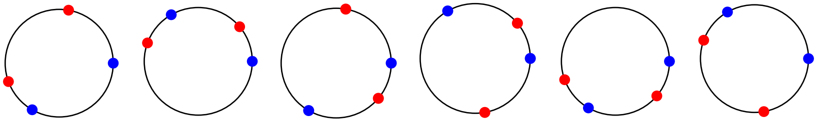

Theorem (Beukers-Heckman): Such ${}_pF_{p-1}$ function is algebraic if and only if the red and blue dots interlace.

Towards Christol

Theorem: The hypergeometric functions

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], 27x\right) \quad \text{and} \quad {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}, 1\right], 27x\right) $$

are diagonals of rational functions.

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More precisely, we have:

$$ {}_3F_2\big(\big[\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}\big],\big[\tfrac{2}{3},1\big],27x\big) = \mathrm{Diag}\left(\frac{(1-x-y)^{1/3}}{1-x-y-z}\right), $$

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More generally, $\mathrm{Diag}\left(\frac{(1-x-y)^{a/b}}{1-x-y-z}\right)$ is shown to evaluate to

$${}_3F_2\Big(\Big[\frac{3a-b}{3a},\frac{2a-b}{3a},\frac{a-b}{3a}\Big],\Big[\frac{a-b}{a},1\Big],27x\Big).$$

Proof

The denominator of the algebraic function $\dfrac{(1-x-y)^{a/b}}{(1-x-y-z)}$ is expanded as a geometric series:

$$(1-x-y-z)^{-1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\binom{n}{m}\binom{m}{l}\cdot x^l y^{m-l} z^{n-m},$$

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Multiplying these two sums and re-indexing, we obtain:

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}x^s y^t z^u\sum_{j=0}^{s}\sum_{k=0}^{\infty}\frac{(-a/b)_k}{k!}\binom{k}{j}\binom{s+t+u-k}{s+t-k}\binom{s+t-k}{s-j}.$$

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Hence the diagonal coefficient of $x^n y^n z^n$ is given by

$$\sum_{j=0}^{n} \sum_{k=0}^{\infty} \frac{(-a/b)_k}{k!} \cdot \binom{k}{j} \binom{3n-k}{2n-k} \binom{2n-k}{n-j},$$

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Now use a computer algebra tool like Mathematica or Maple to simplify this sum further into a closed form. . .

Proof

More precisely, we employ creative telescoping to find that

$$\binom{2n}{n} \cdot \sum_{k=0}^{2n} \frac{(-a/b)_k}{k!} \cdot \binom{3n-k}{2n-k} \quad =: S(n)$$

satisfies the first-order recurrence

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satisfies the first-order recurrence

$$\begin{aligned}&(a-3b-3bn)\cdot(a-2b-3bn)\cdot(a-b-3bn)\cdot S(n)\\&= b^2\cdot(n+1)^2\cdot(a-b-bn)\cdot S(n+1).\end{aligned}$$

Together with the initial value $S(0)=1$, we get the closed form

$$S(n) = \frac{3^{3n}\cdot\left(\frac{b-a}{3b}\right)_n\cdot\left(\frac{2b-a}{3b}\right)_n\cdot\left(\frac{3b-a}{3b}\right)_n}{\left(\frac{b-a}{b}\right)_n\cdot\left(n!\right)^2},$$

yielding the hypergeom. function representation of the diagonal.

Diagonals as Integrals

Note that a diagonal $\operatorname{Diag}\big(R(x,y,z)\big)$ can also be expressed as

$$\langle y^0 z^0 \rangle \, R\Big(\frac{x}{y}, \frac{y}{z}, z\Big) = \operatorname{res}_{y,z} \frac{1}{yz} R\Big(\frac{x}{y}, \frac{y}{z}, z\Big) = \oint \frac{1}{yz} R\Big(\frac{x}{y}, \frac{y}{z}, z\Big) \mathrm{d}y \, \mathrm{d}z.$$

where $\langle y^0 z^0 \rangle$ denotes the constant coefficient w.r.t. y and z.

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Indeed, writing

$$R(x,y,z) = \sum_{l \geqslant 0} \sum_{m \geqslant 0} \sum_{n \geqslant 0} r_{l,m,n}\, x^l y^m z^n$$

one obtains

$$R\Big(\frac{x}{y}, \frac{y}{z}, z\Big) = \sum_{l \geqslant 0} \sum_{m \geqslant 0} \sum_{n \geqslant 0} a_{l,m,n}\, x^l y^{m-l} z^{n-m}.$$

Proof by Creative Telescoping

Compute a linear differential operator that annihilates the diagonal of our algebraic function, by applying creative telescoping to

$$\oint \frac{1}{yz} R\Big(\frac{x}{y}, \frac{y}{z}, z\Big)\,\mathrm{d}y\,\mathrm{d}z = \oint \frac{(1 - x/y - y/z)^{a/b}}{yz - xz - y^2 - yz^2}\,\mathrm{d}y\,\mathrm{d}z$$

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We obtain the following telescoper of order three:

$$\begin{aligned} & b^3x^2(1-27x)\cdot D_x^3 + b^2x((27a-135b)\cdot x - a + 3b)\cdot D_x^2 \\ & - b\cdot((9a^2 - 63ab + 114b^2)\cdot x + ab - b^2)\cdot D_x \\ & + (a-3b)\cdot(a-2b)\cdot(a-b). \end{aligned}$$

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One of its solutions is the claimed ${}_3F_2$ hypergeometric function

$${}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big).$$

Software Demo

In[1]:=
```
<< RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

In[2]:=
```
alg = (1 - x - y) ^ (1 / 3) / (1 - x - y - z);
intg = ExpandAll[(alg /. {x → x / y, y → y / z}) / (y z)]
```

Out[3]= $$\frac{\left(1-\frac{x}{y}-\frac{y}{z}\right)^{1/3}}{-y^2-x\,z+y\,z-y\,z^2}$$

In[4]:=
```
CreativeTelescoping[intg, Der[y], {Der[x], Der[z]}][[1]]
```

Out[4]= $\{(144 x^2 z^2 - 72 x z^3 + 9 z^4 + 72 x z^4 - 18 z^5 - 36 x z^5 + 9 z^6) D_z^2 + (-6 x^2 z - 972 x^3 z - 3 x z^2 + 324 x^2 z^2 - 12 x z^3 - 3 x z^4) D_x +$
$(264 x^2 z - 180 x z^2 - 324 x^2 z^2 + 24 z^3 + 366 x z^3 - 66 z^4 - 174 x z^4 + 42 z^5) D_z + (16 x^2 - 46 x z - 540 x^2 z + 6 z^2 + 308 x z^2 -$
$(144 x^2 z - 72 x z^2 + 9 z^3 + 72 x z^3 - 18 z^4 - 36 x z^4 + 9 z^5) D_x D_z + (24 x^2 - 24 x z + 324 x^2 z + 9 z^2 - 6 x z^2 - 27 z^3 - 60 x z^3 + 1$
$(48 x z + 6 z^2 + 108 x z^2 - 48 z^3 + 6 z^4) D_z + (8 x + 16 z + 180 x z - 74 z^2 + 10 z^3), (-144 x^3 + 72 x^2 z - 9 x z^2 - 72 x^2 z^2 + 18$
$(-336 x^2 + 138 x z + 108 x^2 z - 9 z^2 - 132 x z^2 + 18 z^3 + 48 x z^3 - 9 z^4) D_x + (-24 x z + 24 z^2 + 36 x z^2 - 30 z^3 + 6 z^4) D_z + (-64$

In[5]:=
```
CreativeTelescoping[%, Der[z]][[1]]
```

Out[5]= $\{(-27 x^2 + 729 x^3) D_x^3 + (-72 x + 3402 x^2) D_x^2 + (-18 + 2538 x) D_x + 80\}$

In[6]:=
```
Annihilator[HypergeometricPFQ[{2 / 9, 5 / 9, 8 / 9}, {2 / 3, 1}, 27 x], Der[x]]
```

Out[6]= $\{(-27 x^2 + 729 x^3) D_x^3 + (-72 x + 3402 x^2) D_x^2 + (-18 + 2538 x) D_x + 80\}$

From Algebraic to Rational

Denef and Lipshitz: For a given algebraic power series $f(x_1, \ldots, x_n)$ in n variables, construct a rational function $R(x_1, \ldots, x_{2n})$ in $2n$ variables such that

$$\mathrm{Diag}\big(R(x_1, \ldots, x_{2n})\big) = \mathrm{Diag}\big(f(x_1, \ldots, x_n)\big).$$

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Moreover, the "partial diagonal" of R, w.r.t. the pairs of variables

$$(x_1, x_{n+1}), \ldots, (x_{n-1}, x_{2n}),$$

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Example: We use the three-variable algebraic function

$$\begin{aligned} f(x, y, z) &= \frac{(1 - x - y)^{1/3}}{1 - x - y - z} \\ &= 1 + \tfrac{2}{3}x + \tfrac{2}{3}y + z + \tfrac{10}{9}xy + \tfrac{5}{3}xz + \tfrac{5}{3}yz + \tfrac{40}{9}xyz + \ldots \end{aligned}$$

Etale Extensions

The minimal polynomial of $f = \frac{(1-x-y)^{1/3}}{1-x-y-z}$ is given by

$$p(x, y, z, f) = \big((x + y + z - 1) \cdot f\big)^3 + 1 - x - y.$$

Denef and Lipshitz's theorem is formulated for étale extensions, which basically means that $\frac{\partial p}{\partial f}$ has a nonzero constant coefficient.

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By considering $\tilde{f} = f - 1$, i.e. by removing the constant term of f, we can achieve an étale extension. The minimal polynomial then reads

$$\tilde{p}(x,y,z,f) = \big((x+y+z-1)\cdot(f+1)\big)^3 + 1 - x - y.$$

and indeed, $\frac{\partial \tilde{p}}{\partial f}(0,0,0,0) = -3 \neq 0$.

Special Diagonal

Now, the rational function

$$\tilde{r}(x, y, z, f) = f^2 \cdot \frac{\frac{\partial \tilde{p}}{\partial f}(xf, yf, zf, f)}{\tilde{p}(xf, yf, zf, f)}$$

has the property that $\mathcal{D}\big(\tilde{r}(x, y, z, f)\big) = \tilde{f}(x, y, z)$, where the operator $\mathcal{D}$ denotes a special kind of "diagonalization" with respect to the last variable:

$$\mathcal{D}\Big(\sum a_{i_1,\ldots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n} y^j\Big) = \sum_{j=i_1+\cdots+i_n} a_{i_1,\ldots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n}.$$

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Hence $\mathcal{D}\big(r(x, y, z, f)\big) = f(x, y, z)$ for $r(x, y, z, f) = \tilde{r}(x, y, z, f) + 1$.

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In our example we obtain:

$$r(x,y,z,f) = \frac{3f^2 \cdot (f+1)^2 \cdot (xf+yf+zf-1)^3}{(f+1)^3 \cdot (xf+yf+zf-1)^3 - xf - yf + 1} + 1.$$

Rational Function

Transform the rational function r (that has $n+1$ variables) into another rational function (having $2n$ variables) such that its “true” partial diagonal gives the n-variable algebraic series f.

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This process consists of a sequence of $n-1$ elementary steps, each of which is adding one more variable:

$$r_1(x,y,z,u_1,v_1) = \frac{u_1 \cdot r(x,y,z,u_1) - v_1 \cdot r(x,y,z,v_1)}{u_1 - v_1}$$

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Then r_2 is the desired rational function in six variables.

Final Result

The hypergeometric function

$$ {}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big). $$

is the diagonal of the following rational function in the six variables x, y, z, u, v, w:

$$ \begin{aligned} 1 &+ \frac{au^3v\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-v)(v-w)} \\ &- \frac{av^4\,(1-vx-vy-vz)\,(1+v)^{a-1}(1-vx-vy-vz)^{a-1}}{(1+v)^a(1-vx-vy-vz)^a-(1-vx-vy)^b(u-v)(v-w)} \\ &- \frac{au^3w\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-w)(v-w)} \\ &- \frac{aw^4\,(1-wx-wy-wz)\,(1+w)^{a-1}(1-wx-wy-wz)^{a-1}}{(1+w)^a(1-wx-wy-wz)^a-(1-wx-wy)^b(u-w)(v-w)} \end{aligned} $$

Other Potential Counterexamples

Christol's original example:

$$ {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{5}{9}\right], \left[\tfrac{1}{3}, 1\right], 27x\right) $$

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It seems that this example cannot be treated in a similar way.

Note that our examples,

$$ {}_3F_2\big(\big[\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}\big],\big[\tfrac{2}{3},1\big],x\big) \quad \text{and} \quad {}_3F_2\big(\big[\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9}\big],\big[\tfrac{1}{3},1\big],x\big), $$

have an arithmetic progression in the top parameters.

Integral Representation

Recalling the integral representation of the hypergeometric function

$$ {}_3F_2([a,b,c],[d,e],x) = \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(a)\,\Gamma(b)\,\Gamma(d-a)\,\Gamma(e-b)} \times $$
$$ \times \int_0^1 \int_0^1 y^{a-1} z^{b-1} (1-y)^{-a+d-1} (1-z)^{-b+e-1} (1-xyz)^{-c} \,\mathrm{d}y\,\mathrm{d}z $$

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$$

one can try to find suitable algebraic functions...

For example, let

$$
A(x,y,z) = (1-y)^{d-b-1}\, y^b\, (1-xy^2)^{-a}\, (1-z)^{-c}
$$

then the telescoper of

$$
\frac{1}{yz} A\Big(\frac{x}{y}, \frac{y}{z}, z\Big)
$$

gives precisely the differential equation of ${}_3F_2([a,b,c],[d,1],x)$.

The End

Taking the parameter values $a = \frac{1}{9}, b = \frac{4}{9}, c = \frac{5}{9}, d = \frac{1}{3}$, one could hope that the diagonal of the algebraic function

$$\frac{y^{4/9}}{(1-y)^{10/9}\,(1-xy^2)^{1/9}\,(1-z)^{5/9}}$$

gives rise to the notorious ${}_3F_2\big(\big[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\big], \big[\frac{1}{3}, 1\big], 27x\big)$.

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Note: The diagonal of a rational function and a solution of the corresponding telescoper are close, yet distinct notions: the telescoper annihilates the n-fold integral over all integration cycles.

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Outlook: Meanwhile, Alin Bostan and Sergey Yurkevich came up with a generalization of our result, but Christol's original example still **resists**!