

Proof of George Andrews' and David Robbins' q -TSP Conjecture

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- enumeration formula for certain plane partitions
(TSPP = “totally symmetric plane partition”)



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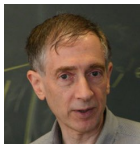
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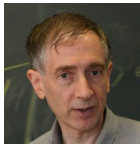
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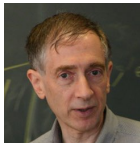
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- \$1000 prize!



Plane Partitions

A plane partition π is:

- two-dimensional array $\pi = (\pi_{i,j})_{1 \leq i,j}$
- $\pi_{i,j} \in \mathbb{N}$ with finite sum $|\pi| = \sum \pi_{i,j}$
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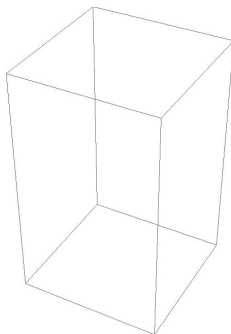
A plane partition π is identified with its 3D Ferrers diagram:

→ stack $\pi_{i,j}$ unit cubes on top of the location (i,j) !



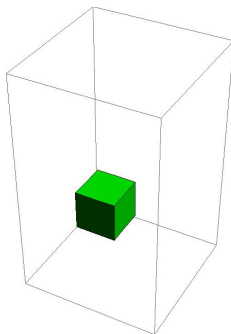
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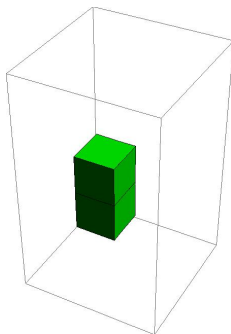
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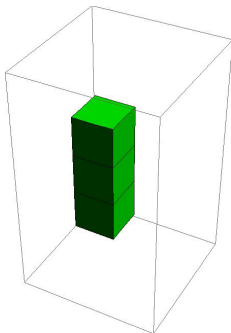
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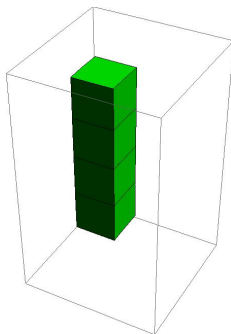
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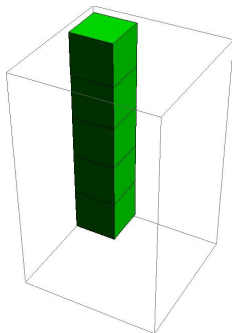
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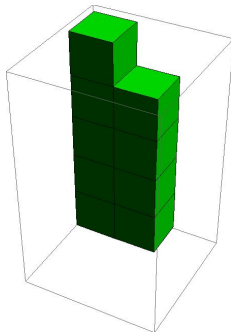
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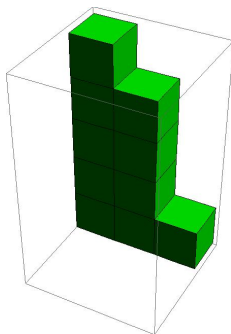
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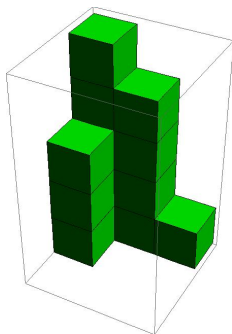
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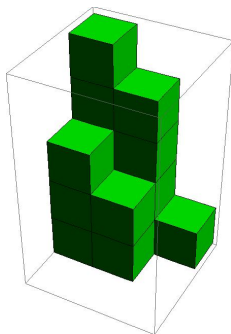
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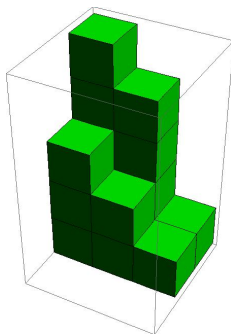
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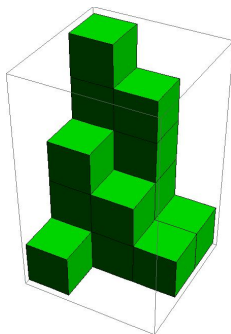
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The enumeration formula for TSPPs (one of Stanley's problems)

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$$

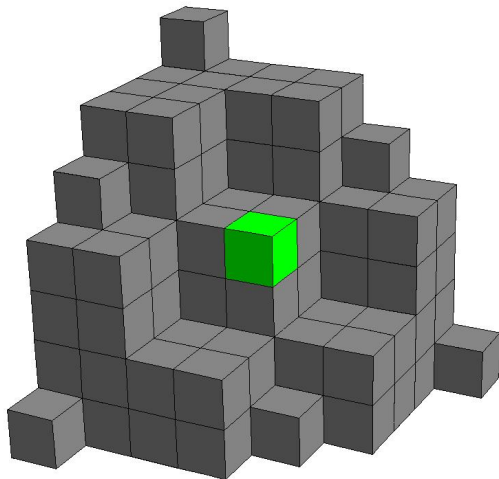
has been proven in 1995 by



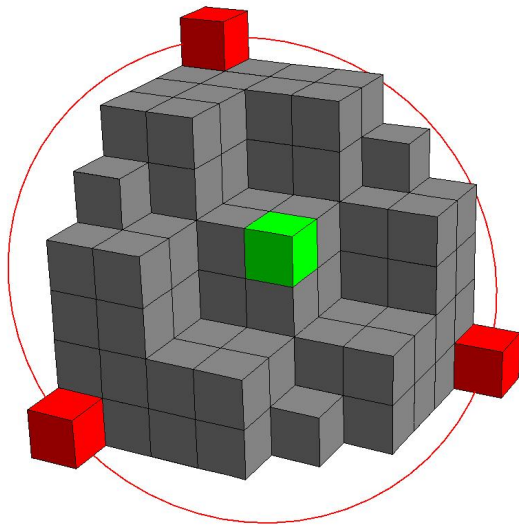
(John Stembridge).



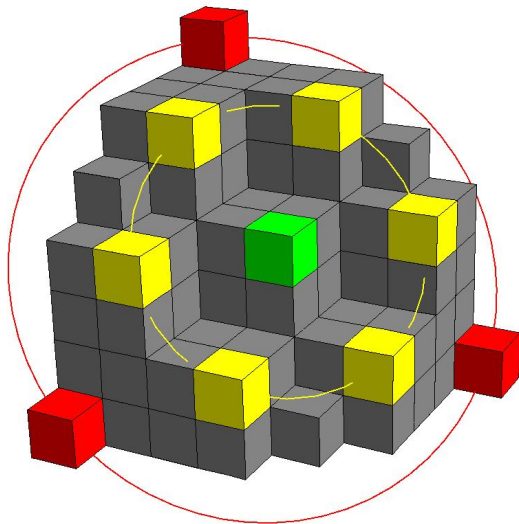
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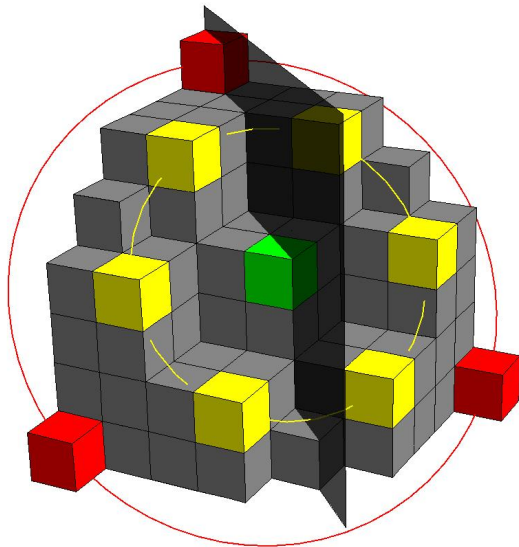
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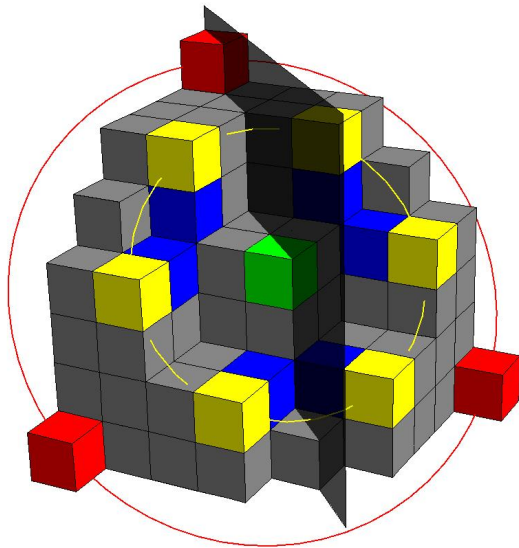
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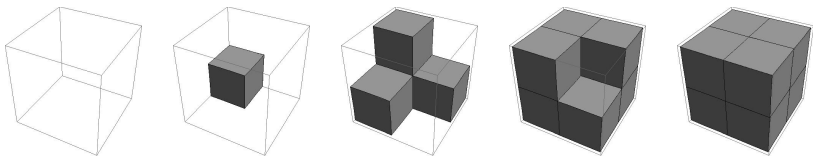
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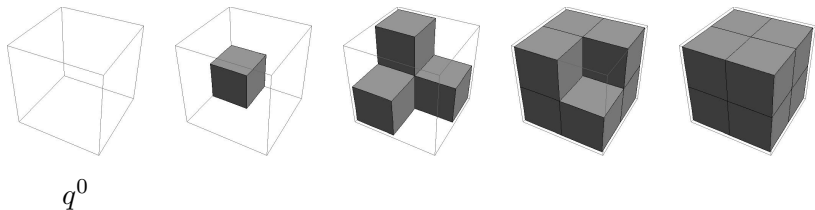
Let $T(n)$ denote set of TSPPs with largest part at most n .



Stembridge's Theorem: $|T(n)| = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$

q -TSPP conjecture: $\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$

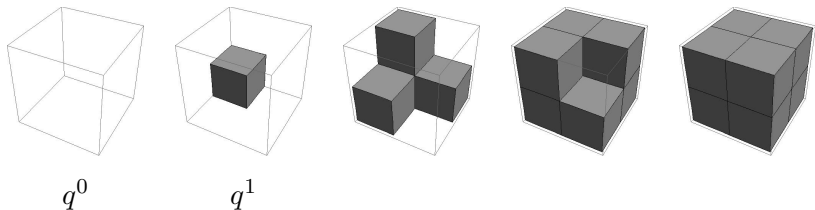
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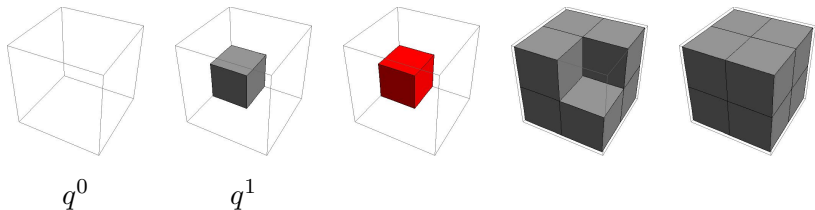
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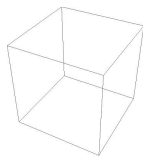
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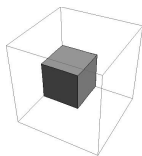
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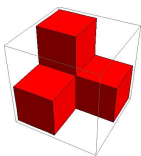
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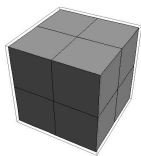
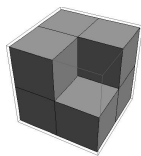
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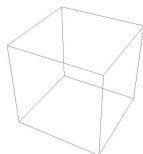
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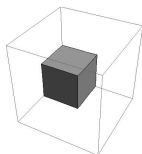
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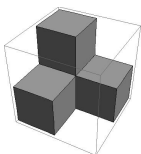
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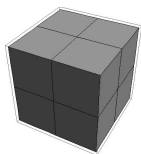
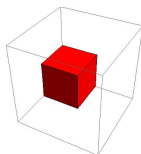
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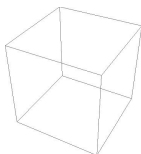
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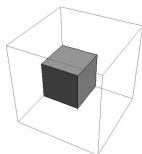
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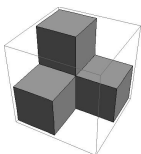
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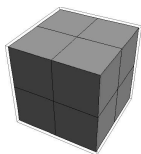
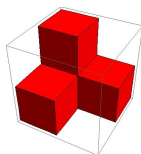
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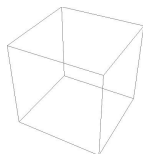
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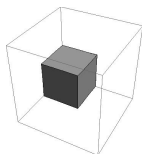
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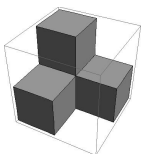
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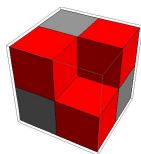
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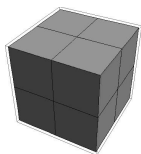
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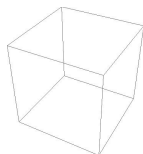
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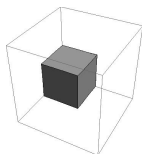
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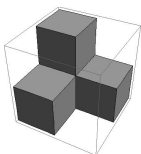
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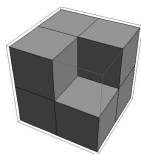
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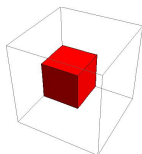
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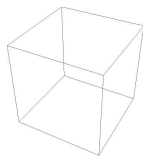
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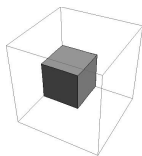
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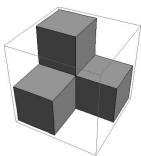
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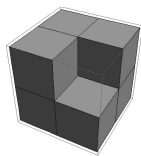
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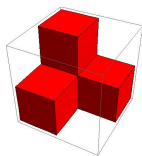
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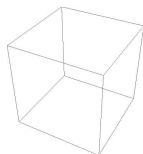
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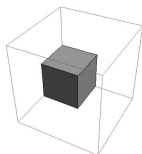
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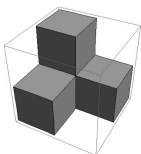
Let $T(n)$ denote set of TSPPs with largest part at most n .



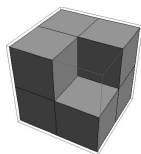
q^0



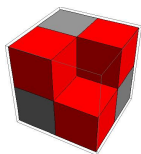
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q^2



q^3



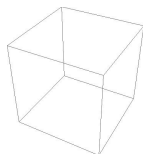
Stembridge's Theorem:

$$|T(n)| = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$$

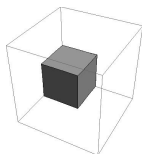
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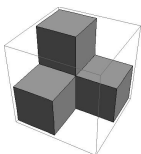
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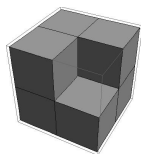
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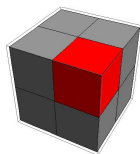
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A parallel universe

q -bracket: $[n]_q := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}$
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q -binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-k+1})}{(1 - q^k)(1 - q^{k-1}) \cdots (1 - q)}$$

(for $q = 1$ we obtain the binomial coefficient $\binom{n}{k}$).



The determinant

We employ an elegant reduction by



(Soichi Okada):

The q -TSPP conjecture is true if

$$\det(a_{i,j})_{1 \leq i, j \leq n} = \prod_{1 \leq i \leq j \leq k \leq n} \left(\frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \right)^2 =: b_n.$$

where

$$a_{i,j} := q^{i+j-1} \left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q \right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}.$$

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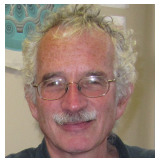
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For $q = 1$: Stembridge's theorem!



The holonomic ansatz

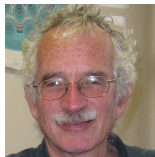
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The holonomic ansatz

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“Pull out of the hat” a discrete function $c_{n,j}$ and prove

$$c_{n,n} = 1 \quad (n \geq 1),$$

$$\sum_{j=1}^n c_{n,j} a_{i,j} = 0 \quad (1 \leq i < n),$$

$$\sum_{j=1}^n c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \quad (n \geq 1).$$

Then $\det(a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds.

Why???



Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!
Instead we aim at some “suitable description”, viz. implicitly via linear recurrences (“holonomic systems”) plus initial values.



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Example: The binomial coefficient $f_{n,k} = \binom{n}{k}$ can be described by

$$\begin{aligned}(n - k + 1)f_{n+1,k} &= (n + 1)f_{n,k} \\ (k + 1)f_{n,k+1} &= (n - k)f_{n,k} \\ f_{0,0} &= 1\end{aligned}$$



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Analogously, we get for the q -binomial coefficient $\bar{f}_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$:

$$\begin{aligned}(q^{n+1} - q^k)\bar{f}_{n+1,k} &= (q^{k+n+1} - q^k)\bar{f}_{n,k} \\ (q^{2k+1} - q^k)\bar{f}_{n,k+1} &= (q^n - q^k)\bar{f}_{n,k} \\ \bar{f}_{0,0} &= 1\end{aligned}$$



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All linear combinations of shifts are again valid recurrences:

$$\begin{array}{l}(n - k)f_{n+1,k+1} - (n + 1)f_{n,k+1} = 0 \\ (k + 1)f_{n+1,k+1} - (n - k + 1)f_{n+1,k} = 0 \\ \hline (n + 1)f_{n+1,k+1} - (n - k + 1)f_{n+1,k} - (n + 1)f_{n,k+1} = 0\end{array}$$

They form a left ideal in some noncommutative operator algebra.



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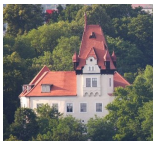
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Guessing



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Their Gröbner basis has the form (without their coefficients which are polynomials in $\mathbb{Q}[q, q^j, q^n]$ with degree ≤ 100):

$$c_{n,j+4} = c_{n,j} + c_{n,j+1} + c_{n,j+2} + c_{n,j+3} + c_{n+2,j} + c_{n+2,j+1}$$

$$c_{n+1,j+3} = c_{n,j} + c_{n,j+1} + c_{n,j+2} + c_{n,j+3} + c_{n+1,j} + c_{n+1,j+1} + c_{n+1,j+2} + c_{n+2,j} + c_{n+2,j+1} + c_{n+3,j}$$

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$$c_{n+3,j+1} = c_{n,j} + c_{n,j+1} + c_{n,j+2} + c_{n,j+3} + c_{n+1,j} + c_{n+1,j+1} + c_{n+1,j+2} + c_{n+2,j} + c_{n+2,j+1} + c_{n+3,j}$$

$$c_{n+4,j} = c_{n,j} + c_{n,j+1} + c_{n,j+2} + c_{n,j+3} + c_{n+2,j} + c_{n+2,j+1}$$

Let's have a closer look at it!



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Let's have a closer look at it! Its total size is 244MB.



The first identity (normalization)

With this implicit description of $c_{n,j}$ we are going to prove the necessary identities (there are algorithms for automatically proving such identities, and they are implemented in our package `HolonomicFunctions`).

How to prove $c_{n,n} = 1$ for all $n \geq 1$?



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How to prove $c_{n,n} = 1$ for all $n \geq 1$?

- We find an element in the annihilating ideal of $c_{n,j}$ of the form

$$p_7 c_{n+7,j+7} = p_6 c_{n+6,j+6} + \cdots + p_1 c_{n+1,j+1} + p_0 c_{n,j}$$

with $p_i \in \mathbb{Q}[q, q^j, q^n]$.

- Substituting $j \rightarrow n$ yields a recurrence for the diagonal sequence $c_{n,n}$.
- Show that the corresponding operator factors into $P_1 P_2$ where P_2 corresponds to $c_{n+1,n+1} = c_{n,n}$.
- Show that $c_{1,1} = \cdots = c_{7,7} = 1$.



The third identity

How to prove $(1 + q^n) - c_{n,n-1} + \sum_{j=1}^n c'_{n,j} = \frac{b_n}{b_{n-1}}$

with $c'_{n,j} = q^{n+j-1} \left(\begin{bmatrix} n+j-2 \\ n-1 \end{bmatrix}_q + q \begin{bmatrix} n+j-1 \\ n \end{bmatrix}_q \right) c_{n,j}?$



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- Compute an annihilating ideal for $c'_{n,j}$ via closure properties.
- Find a relation in this ideal of the form

$$p_7 c'_{n+7,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where the p_7, \dots, p_0 are rational functions in $\mathbb{Q}(q, q^n)$ and $t_{n,j}$ is a $\mathbb{Q}(q, q^j, q^n)$ -linear combination of certain shifts of $c'_{n,j}$.

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- Creative telescoping yields a recurrence for the sum.
- Closure properties yield a recurrence for the left-hand side.
- Recurrence for right-hand side is a right factor.
- Compare finitely many initial values.



A computational challenge

How to find the certificate

$$p_7(q, q^n)c'_{n+7,j} + \cdots + p_0(q, q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(q, q^n, q^j)c'_{n+3,j+2} + \cdots + r_{10}(q, q^n, q^j)c'_{n,j}$?



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Zeilberger's slow algorithm: eliminate (e.g. with Gröbner bases) the variable q^j .

Input recurrences have j -degrees between 24 and 30 (in the $q=1$ case). After 48h, this was reduced to 23.

Estimate: 1677721600 days



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Takayama's algorithm: a faster variant which is also based on elimination.

Estimate: 52428800 days



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Chyzak's algorithm: ansatz with unknown $p_i(q, q^n)$ and $r_k(q, q^n, q^j)$. Leads to a coupled first-order parametrized linear system of q -difference equations.

Estimate: ∞ ?



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CK's polynomial ansatz: refine

$$r_k(q, q^n, q^j) = \sum_{l=0}^L r_{k,l}(q, q^n)(q^j)^l.$$

Leads to a linear system over $\mathbb{Q}(q, q^n)$.

We used this ansatz for proving TSPP (took about 40 days).

Estimate: 4000 days



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CK's rational ansatz: ansatz with

$$r_k(q, q^n, q^j) = \frac{\sum_{l=0}^L r_{k,l}(q, q^n)(q^j)^l}{d_k(q, q^n, q^j)}$$

where the denominators d_k can be “guessed” by looking at the leading coefficients of the Gröbner basis.

Leads to a linear system over $\mathbb{Q}(q, q^n)$.

Estimate: 294 days



A computational challenge

How to solve a linear system that does not even fit into the memory?



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Homomorphic images (modular computations):

- plug in concrete integral values for q and q^n
- do all computations modulo some prime, e.g., 2147483629

Doing the computation for sufficiently many values and primes allows to reconstruct the solution. We need:



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- 1167 interpolation points for q
- 363 interpolation points for q^n
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- estimated computation time: $1167 \cdot 363 \cdot 60s = 294$ days!



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Applying further tricks reduced the actual time to 35 days.



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Further technical problems:

- check singularities of leading coefficients
- initial values
- cleaning ladies

Quod erat demonstrandum.

THEOREM. Let π/S_3 denote the set of orbits of a plane partition π under the action of the symmetric group S_3 . Then the orbit-counting generating function is given by

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

where $T(n)$ denotes the set of totally symmetric plane partitions with largest part at most n .

