Advanced Applications of the Holonomic Systems Approach

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1. The **Holonomic Systems Approach**: Implementation of the *Mathematica* package HolonomicFunctions
   - Noncommutative Gröbner bases in Ore algebras
   - Rational solutions of coupled linear systems of difference or differential equations
   - Closure properties for $\partial$-finite functions
   - Summation/integration algorithms due to Zeilberger, Takayama, and Chyzak
   - [http://www.risc.uni-linz.ac.at/research/combinat/software/](http://www.risc.uni-linz.ac.at/research/combinat/software/)
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2. Three **Advanced Applications**
   - Proof of Ira Gessel’s lattice path conjecture
   - Relations between basis functions in FEM
   - Computer proof of Stembridge’s TSPP theorem
Introduction

**Basic idea:** describe functions/sequences via
- linear relations (PDEs, multivariate recurrences, mixed difference-differential equations) and
- finitely many initial values.

All possible manipulations (addition, multiplication, substitutions, summation, integration) are performed on this level.

(Informal) Definition: A function $f(x_1,...,x_d)$ is called $\partial$-finite w.r.t. the operators $\partial_1,...,\partial_d$ if all its "derivatives" $\partial_{\alpha_1}^{\alpha_1}...\partial_{\alpha_d}^{\alpha_d}(f)$ span a finite-dimensional $K(x_1,...,x_d)$-vector space.

Remark: The notion $\partial$-finiteness is closely related with that of holonomic systems.

Example: The Legendre polynomials $P_n(x)$ are $\partial$-finite with respect to the operators $S_n$ and $D_x$.
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Standard Application 1

Abramowitz/Stegun (10.1.41)

\[ \frac{\partial j_\nu(x)}{\partial \nu} \bigg|_{\nu=0} = \frac{\text{Ci}(2x) \sin(x) - \text{Si}(2x) \cos(x)}{x} \]
What we need

- Closure properties
- a database of functions whose $\partial$-finite description is known:

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Standard Application 1

\[ \text{lhs} = \text{Annihilator}[\text{Derivative}[1,0][\text{SphericalBesselJ}][0,x], \text{Der}[x]] \]

\[
\{ x^3 D_x^4 + 8 x^2 D_x^3 + (2 x^3 + 14 x) D_x^2 + (8 x^2 + 4) D_x + (x^3 + 6 x) \}
\]
Standard Application 1

\[ \text{lhs} = \text{Annihilator}[\text{Derivative}[1,0][\text{SphericalBesselJ}][0,x], \text{Der}[x]] \]

\[ \{x^3 D_x^4 + 8x^2 D_x^3 + (2x^3 + 14x) D_x^2 + (8x^2 + 4) D_x + (x^3 + 6x)\} \]

\[ \text{rhs} = \text{Annihilator}[ \frac{1}{x}(\text{CosIntegral}[2*x]*\text{Sin}[x] - \text{SinIntegral}[2*x]*\text{Cos}[x]), \text{Der}[x]] \]

\[ \{(12x^5 + 5x^3) D_x^6 + (144x^4 + 70x^2) D_x^5 + (132x^5 + 475x^3 + 260x) D_x^4 + (1056x^4 + 796x^2 + 240) D_x^3 + (228x^5 + 1991x^3 + 1288x) D_x^2 + (912x^4 + 1110x^2 + 560) D_x + (108x^5 + 753x^3 + 516x)\} \]
Standard Application 1

One ODE is a multiple of the other one:

\$
\text{OreReduce}[\text{rhs}, \text{lhs}]
\$

\$
\{0\}
\$

Therefore 6 initial values had to be checked.
One ODE is a multiple of the other one:

\[ \text{OreReduce}[\text{rhs}, \text{lhs}] \]

\[ \{0\} \]

Therefore 6 initial values had to be checked.

Alternatively:

\[ \text{Together}[\text{ApplyOreOperator}[\text{lhs}, \frac{1}{x}*(\text{CosIntegral}[2*x] * \text{Sin}[x] - \text{SinIntegral}[2*x] * \text{Cos}[x])]] \]

\[ \{0\} \]

Hence, only 4 initial values have to be compared.
Standard Application 2

A hypergeometric double sum

\[ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{n+r+s} \binom{n}{r} \binom{n+r}{r} \binom{n}{s} \binom{n+s}{s} \binom{2n - (r+s)}{n} = \sum_{k=0}^{\infty} \binom{n}{k}^4 \]
Standard Application 2

Annihilator[(-1)^(n+r+s)*Binomial[n,r]*Binomial[n,s]*Binomial[n+r,r]*Binomial[n+s,s]*Binomial[2*n-(r+s),n], {S[r], S[s], S[n]}]

\{(n + 1)(n - r + 1)(n - s + 1)(n - r - s + 1)S_n
  + (n + r + 1)(n + s + 1)(2n - r - s + 1)(2n - r - s + 2),
  (s + 1)^2(2n - r - s)S_s + (n - s)(n + s + 1)(n - r - s),
  - (r + 1)^2(2n - r - s)S_r - (n - r)(n + r + 1)(n - r - s)\}

Takayama[%, {r, s}]

\{(n+2)^3S_n^2 - 2(2n+3)(3n^2 + 9n + 7)S_n - 4(n+1)(4n+3)(4n+5)\}

Annihilator[Sum[Binomial[n, k]^4, {k, 0, n}], S[n]]

\{(n+2)^3S_n^2 - 2(2n+3)(3n^2 + 9n + 7)S_n - 4(n+1)(4n+3)(4n+5)\}
For $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$, for $l + m + n$ even and the sum of any two of $l, m, n$ is not less than the third:

\[
\int_{-1}^{1} C_{l}^{(\lambda)}(x) C_{m}^{(\lambda)}(x) C_{n}^{(\lambda)}(x) (1 - x^2)^{\lambda-1/2} \, dx =
\]

\[
\pi 2^{1-2\lambda} \Gamma \left( 2\lambda + \frac{1}{2}(l + m + n) \right) \frac{\Gamma(\lambda)^2 \left( \frac{1}{2}(l + m + n) + \lambda \right)}{\left( \frac{1}{2}(m + n - l) \right)! \left( \frac{1}{2}(l + n - m) \right)! \left( \frac{1}{2}(l + m - n) \right)! (\lambda)^{(l+m+n)/2}}
\]

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Standard Application 3

An annihilating ideal for the integral is obtained via creative telescoping using Chyzak’s algorithm. The same ideal is obtained for the right-hand side:

\[ \text{rhs} = \text{Annihilator}[\text{Pochhammer}[\ldots, \{S[1], S[m], S[n]\}]] \]

\[
\begin{align*}
&\{ (l + m - n + 1)(l - m + n + 2\lambda - 1)S_m \\
&\quad - (l - m + n + 1)(l + m - n + 2\lambda - 1)S_n, \\
&\quad (l + m - n + 1)(l - m - n - 2\lambda + 1)S_l \\
&\quad - (l - m - n - 1)(l + m - n + 2\lambda - 1)S_n, \\
&\quad (l - m - n - 2)(l - m + n + 2)(l + m - n + 2\lambda - 2) \\
&\quad \times (l + m + n + 2\lambda + 2)S_n^2 \\
&\quad - (l + m - n)(l - m - n - 2\lambda)(l - m + n + 2\lambda)(l + m + n + 4\lambda) \}
\end{align*}
\]

There seem to be lots of singularities!

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Standard Application 3

AnihilatorSingularities[rhs, {0, 0, 0},
  Assumptions -> lambda > 1/2 &&
  Element[(l+m+n)/2, Integers] &&
  l+m >= n && l+n >= m && m+n >= l]

\[
\begin{align*}
\{\{l \to 0, m \to n\}, \lambda > \frac{1}{2} \land n \geq 0\}, \{\{l \to n, m \to 0\}, \lambda > \frac{1}{2} \land n \geq 0\}, \\
\{\{l \to 0, m \to 0, n \to 0\}, \lambda > \frac{1}{2}\}, \{\{l \to 0, m \to 1, n \to 1\}, \lambda > \frac{1}{2}\}, \\
\{\{l \to 1, m \to 0, n \to 1\}, \lambda > \frac{1}{2}\}, \{\{l \to 2, m \to 0, n \to 2\}, \lambda > \frac{1}{2}\}\end{align*}
\]

The isolated singular points are checked immediately.
For the first two cases, we apply creative telescoping recursively.
\[ \int_{0}^{\infty} J_m(ax) J_n(bx) \, dx = \]
\[ \frac{a^{-n-1} b^n \Gamma \left( \frac{1}{2} (m + n + 1) \right)}{\Gamma(n + 1) \Gamma \left( \frac{1}{2} (m - n + 1) \right)} \times 2F_1 \left( \frac{1}{2} (m + n + 1), \frac{1}{2} (-m + n + 1), n + 1, \frac{b^2}{a^2} \right) \]
Standard Application 4

CreativeTelescoping[BesselJ[m, a*x]*BesselJ[n, b*x],
Der[x], {Der[a], Der[b], S[m], S[n]}]

\{a D_a + b D_b + 1,
(b^2 m^2 - b^2 n^2 - 2 b^2 n - b^2) S_n^2 + (2 a^2 b n + 2 a^2 b - 2 b^3 n - 2 b^3) D_b
- 2 a^2 n^2 - 2 a^2 n + b^2 m^2 + b^2 n^2 - b^2,
(ab m + ab n + ab) S_m S_n + (a^2 b - b^3) D_b - a^2 n - b^2 m - b^2,
(a^2 m^2 + 2 a^2 m - a^2 n^2 + a^2) S_m^2 + (-2 a^2 b m - 2 a^2 b + 2 b^3 m + 2 b^3) D_b
- a^2 m^2 - 2 a^2 m - a^2 n^2 - a^2 + 2 b^2 m^2 + 4 b^2 m + 2 b^2,
(a^2 b - b^3) D_b S_n + (ab n - ab m) S_m + (a^2 n + a^2 - b^2 m - b^2) S_n,
(a^2 b - b^3) D_b S_m + (b^2 m - a^2 n) S_m + (ab m - ab n) S_n,
(a^2 b^2 - b^4) D_b^2 + (a^2 b - 3 b^3) D_b - a^2 n^2 + b^2 m^2 - b^2\},

\{-x, 2(ab m x + ab x) S_m S_n - 2(b m n + b m + b n^2 + 2 b n + b) S_n
+ 2(b^2 n x + b^2 x), ab x S_m S_n + b^2 x,
- 2(ab m x + ab x) S_m S_n + 2(a m^2 + a m n + 2 a m + a n + a) S_m
- 2(b^2 m x + b^2 x), -ab x S_m + b^2 x S_n, b^2 x S_m - ab x S_n,
ab^2 x^2 S_m - b^3 x^2 S_n - b^2 m x + b^2 n x + b^2 x\}\}
gb = OreGroebnerBasis[
Annihilator[BesselJ[m, a*x]*BesselJ[n, b*x],
{Der[x], Der[a], Der[b], S[m], S[n]}],
OreAlgebra[x, Der[x], Der[a], Der[b], S[m], S[n]],
MonomialOrder -> EliminationOrder[1]
];
LeadingPowerProduct /@ gb

\{D_a S_n, D_a S_m, D_a^2, D_x S_n, D_x S_m, D_x D_b,
D_x D_a, S_m^2 S_n, D_b S_n^2, D_b S_m S_n, D_b S_m^2, x\}
Standard Application 5

Gradshteyn/Ryzhik (4.539)

\[
\int_{0}^{1} \frac{(- \log(t))^{s-1} \arctan(at)}{t} \, dt = a 2^{-s-1} \Gamma(s) \Phi(-a^2, s + 1, \frac{1}{2})
\]

where \( \Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s} \) is the Lerch transcendent.
Standard Application 5

Both sides contain non-holonomic expressions (see the ISSAC’09 paper by Chyzak/Kauers/Salvy).

\[
\text{Annihilator}[
\quad \text{Integrate}[((-\log[t])^{s-1}/t*\arctan[a*t], \{t, 0, 1\}],
\quad \{\text{Der}[a], s[s]\}, \text{Assumptions} \rightarrow s>0]
\]

Annihilator::nondf: The expression \((-\log[t])^{(-1 + s)}\) is not recognized to be \(\partial\)-finite. The result might not generate a zero-dimensional ideal.

\[\{a D_\partial S_s - s\}\]

Annihilator\[2^{(-s-1)}*\text{Gamma}[s]*a*\text{LerchPhi}[-a^2, s+1, 1/2]]

\[\{a D_\partial S_s - s\}\]
Advanced Application 1

Proof of Gessel’s conjecture
(joint work with M. Kauers and D. Zeilberger)

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Advanced Application 2

Relations for speeding up FEM

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Problem setting

Joachim Schöberl (RWTH Aachen): Simulate the propagation of electromagnetic waves using the Maxwell equations

\[
\frac{dH}{dt} = \text{curl } E, \quad \frac{dE}{dt} = -\text{curl } H
\]

where \( H \) and \( E \) are the magnetic and the electric field respectively. Define basis functions (in 2D) in order to approximate the solution:

\[
\varphi_{i,j}(x, y) := (1 - x)^i P_j^{(2i+1,0)} (2x - 1) P_i \left( \frac{2y}{1-x} - 1 \right)
\]

**Problem:** need to represent the partial derivatives of \( \varphi_{i,j}(x, y) \) in the original basis (i.e., as linear combinations of shifts of the \( \varphi_{i,j}(x, y) \) itself)
The Gröbner approach

The numerists need a relation of the form

$$\sum_{(k,l) \in A} a_{k,l}(i, j) \frac{d}{dx} \varphi_{i+k,j+l}(x, y) = \sum_{(m,n) \in B} b_{m,n}(i, j) \varphi_{i+m,j+n}(x, y),$$

that is free of $x$ and $y$ (and similarly for $\frac{d}{dy}$).
The Gröbner approach

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that is free of $x$ and $y$ (and similarly for $\frac{d}{dy}$).

- consider the operators $D_x$, $S_i$, and $S_j$
- basis functions $\varphi_{i,j}(x, y)$ are $\partial$-finite with respect to them
- compute generators of an annihilating left ideal for $\varphi_{i,j}(x, y)$
- represent them in the algebra
  $$\mathbb{Q}(i, j)[x, y][D_x; 1, D_x][S_i; S_i, 0][S_j; S_j, 0]$$
- compute a Gröbner basis in order to eliminate $x$ and $y$
The Gröbner approach

The numerists need a relation of the form

\[
\sum_{(k,l) \in A} a_{k,l}(i,j) \frac{d}{dx} \varphi_{i+k,j+l}(x, y) = \sum_{(m,n) \in B} b_{m,n}(i,j) \varphi_{i+m,j+n}(x, y),
\]

that is free of \(x\) and \(y\) (and similarly for \(\frac{d}{dy}\)).

- consider the operators \(D_x\), \(S_i\), and \(S_j\)
- basis functions \(\varphi_{i,j}(x, y)\) are \(\partial\)-finite with respect to them
- compute generators of an annihilating left ideal for \(\varphi_{i,j}(x, y)\)
- represent them in the algebra \(\mathbb{Q}(i, j)[x, y][D_x; 1, D_x][S_i; S_i, 0][S_j; S_j, 0]\)
- compute a Gröbner basis in order to eliminate \(x\) and \(y\)
- takes very long, interrupt as soon as a desired operator is found
- result is quite big (2 pages of output)
- because of “extension/contraction” we can not be sure that we obtain the smallest operator.
The ansatz approach

The numerists need a relation of the form

$$\sum_{(k, l) \in A} a_{k, l}(i, j) \frac{d}{dx} \varphi_{i+k,j+l}(x, y) = \sum_{(m, n) \in B} b_{m, n}(i, j) \varphi_{i+m,j+n}(x, y),$$

that is free of $x$ and $y$ (and similarly for $\frac{d}{dy}$).

- work in the algebra $\mathbb{Q}(i, j, x, y)[D_x; 1, D_x][S_i; S_i, 0][S_j; S_j, 0]$
- compute a Gröbner basis $G$ of a $\partial$-finite annihilating ideal for $\varphi_{i,j}(x, y)$
- choose index sets $A$ and $B$
- reduce the above ansatz with $G$
- do coefficient comparison with respect to $x$ and $y$
- solve the resulting linear system for $a_{k, l}, b_{m, n} \in \mathbb{Q}(i, j)$
- can find the “smallest” relation
- certain optimizations (e.g., using homomorphic images) reduce the computation time to a few seconds
Of course,

$$\text{nf} \left( \sum_k a_k \partial^{\alpha_k} \right) = \sum_k a_k \text{nf} (\partial^{\alpha_k})$$

- reduce each monomial $\partial^{\alpha_k}$ separately
- use previously computed normal forms
Idea: Can we use homomorphic images for finding a good ansatz?

- surely we can compute in $\mathbb{Z}_p(i, j, x, y)[D_x; 1, D_x][S_i; S_i, 0][S_j; S_j, 0]$
- this does not help much
- better: try to reduce polynomial arithmetic
- have to keep $x$ and $y$ symbolically (coefficient comparison)
- what about $i$ and $j$? If we plug in values for them, we loose noncommutativity!
Recall: normal form computation

Let $\mathcal{O}$ be the operator algebra.

**Input:** $p \in \mathcal{O}$, a Gröbner basis $G = \{g_1, \ldots, g_n\} \subseteq \mathcal{O}$

**Output:** normal form of $p$ modulo $\mathcal{O}\langle G \rangle$

while exists $1 \leq k \leq n$ such that $\text{lm}(g_k) | \text{lm}(p)$

\[
\begin{align*}
g & := (\text{lm}(p)/\text{lm}(g_k)) \cdot g_k \\
p & := p - (\text{lc}(p)/\text{lc}(g)) \cdot g
\end{align*}
\]

end while
Modular normal form computation

Input: $p \in \mathbb{O}$, a Gröbner basis $G = \{g_1, \ldots, g_n\} \subseteq \mathbb{O}$

Output: normal form of $p$ modulo $\mathbb{O}\langle G \rangle$

while exists $1 \leq k \leq n$ such that $\text{lm}(g_k) | \text{lm}(p)$

\[ g := h\left(\frac{\text{lm}(p)}{\text{lm}(g_k)}\right) \cdot g_k \]

\[ p := p - \left(\frac{\text{lc}(p)}{\text{lc}(g)}\right) \cdot g \]

end while

where $h$ is an insertion homomorphism, in our example

\[ h : \mathbb{Q}(i, j, x, y) \rightarrow \mathbb{Q}(x, y) \]

\[ f(i, j, x, y) \mapsto f(i_0, j_0, x, y), \text{ for } i_0, j_0 \in \mathbb{Z} \]
Result

With this method, we find in a few seconds relations like

\[
(2i + j + 5)(2i + 2j + 7) \frac{d}{dx} \varphi_{i,j+1}(x, y) \\
+ 2(2i + 1)(i + j + 3) \frac{d}{dx} \varphi_{i,j+2}(x, y) \\
-(j + 3)(2i + 2j + 7) \frac{d}{dx} \varphi_{i,j+3}(x, y) \\
+(j + 1)(2i + 2j + 5) \frac{d}{dx} \varphi_{i+1,j}(x, y) \\
-2(2i + 3)(i + j + 3) \frac{d}{dx} \varphi_{i+1,j+1}(x, y) \\
+(2i + j + 5)(2i + 2j + 7) \frac{d}{dx} \varphi_{i+1,j+2}(x, y) = \\
2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i,j+2}(x, y) \\
-2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i+1,j+1}(x, y)
\]

→ these formulae already caused a speed-up of 20 percent (!) in the numerical simulations.

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3D case

We would like to do the same thing in 3D.

▶ now the basis functions

$$\varphi(i, j, k, x, y, z) := \begin{align*}
& P_i \left( \frac{2z}{(1-x)(1-y)} - 1 \right) \left(1 - x\right)^i \left(1 - y\right)^i \\
& P_j^{(2i+1,0)} \left( \frac{2y}{1-x} - 1 \right) \left(1 - x\right)^j \\
& P_k^{(2i+2j+2,0)} \left(2x - 1\right)
\end{align*}$$

contain 6 variables

▶ computations become too big and too slow

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A first result for 3D

One of the supports looks as follows:

\[ \{ S_j S_k^4, S_j^2 S_k^3, S_j S_k^3 S_k^2, S_j^4 S_k, D_x S_j S_k^3, D_x S_j^2 S_k^2, D_x S_j^3 S_k, D_x S_j^4, S_j S_k^5, S_j^2 S_k^4, S_j^3 S_k^3, S_j^4 S_k^2, S_j S_k^5, S_i S_j S_k^4, S_i S_j^2 S_k^3, S_i S_j S_k^3 S_k^2, D_x S_j S_k^4, D_x S_j^2 S_k^3, D_x S_j^3 S_k, D_x S_j^4, S_j S_k^5, S_j^2 S_k^4, S_j S_k^3 S_k^2, S_j^3 S_k^3 S_k^2, S_j^4 S_k^2, S_j S_k^3 S_k^2, D_x S_j S_k^3, D_x S_j^2 S_k^2, D_x S_j^3 S_k, D_x S_j^4, S_j S_k^5, S_j^2 S_k^4, S_j S_k^3 S_k^2, S_j^3 S_k^3 S_k^2, S_j^4 S_k^2, S_j S_k^3 S_k^2, D_x S_j S_k^2, D_x S_j^2 S_k^1, D_x S_j^3 S_k, D_x S_j^4, S_j S_k^5, S_j^2 S_k^4, S_j S_k^3 S_k^2, S_j^3 S_k^3 S_k^2, S_j^4 S_k^2, S_j S_k^3 S_k^2, \} \]

Joachim Schöberl was impressed but not too happy about these results...
Idea: Write $\varphi = u \cdot v \cdot w$ with

\[
\begin{align*}
    u &= P_i \left( \frac{2z}{(1-x)(1-y)} - 1 \right) (1 - x)^i (1 - y)^i \\
    v &= P_j^{(2i+1,0)} \left( \frac{2y}{1-x} - 1 \right) (1 - x)^j \\
    w &= P_k^{(2i+2j+2,0)} (2x - 1)
\end{align*}
\]

and use the product rule

\[
\frac{d\varphi}{dx} = \frac{du}{dx}vw + u \frac{dv}{dx}w + uv \frac{dw}{dx}
\]

We now want to find a relation between e.g. $uvw$ and $\frac{du}{dx}vw$. 
Task: find relation between $uvw$ and $\frac{du}{dx}vw$

How does this fit into our framework?
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How does this fit into our framework?

Usually we have something like

$$\text{op} \bullet f = 0.$$
Task: find relation between $uvw$ and $\frac{du}{dx}vw$

How does this fit into our framework?

Usually we have something like

$$\text{op} \cdot f = 0.$$  

Now we search for a relation of the form

$$\text{op}_1 \cdot f = \text{op}_2 \cdot g.$$
**Task:** find relation between $uvw$ and $\frac{du}{dx}vw$

How does this fit into our framework?

Usually we have something like

$$op \cdot f = 0.$$  

Now we search for a relation of the form

$$op_1 \cdot f = op_2 \cdot g.$$  

Trivial solution: $op_1 \in \text{Ann } f$ and $op_2 \in \text{Ann } g$. But since $f$ and $g$ are closely related we expect that there exists something “better”.

Christoph Koutschan
The natural way to express a relation like

$$\text{op}_1 \bullet f = \text{op}_2 \bullet g$$

is by using operator vectors in $M = \mathbb{O} \times \mathbb{O}$ which we let act on $\mathcal{F} \times \mathcal{F}$ by

$$P \bullet F = (P_1, P_2) \bullet (f, g) := P_1 \bullet f + P_2 \bullet g,$$

where $P \in M$, $F \in \mathcal{F} \times \mathcal{F}$.

But how to compute a Gröbner basis for the ideal of relations between $f$ and $g$, i.e. the annihilator $\text{Ann}_M(f, g)$?
Let $f = uvw$ and $g = \frac{du}{dx}vw$.

We start with $u$ and $u' = \frac{du}{dx}$:

$\text{Ann}_M(u, u') =$

\[\emptyset \left\langle \{ (p, 0) | p \in \text{Ann}_\square u \} \cup \{ (0, p) | p \in \text{Ann}_\square u' \} \cup \{ (D_x, -1) \} \right\rangle\]
Closure properties

Let \( f = uvw \) and \( g = \frac{du}{dx}vw \).

We start with \( u \) and \( u' = \frac{du}{dx} \):

\[
\text{Ann}_M(u, u') = \langle \{ (p, 0) \mid p \in \text{Ann}_\emptyset u \} \cup \{ (0, p) \mid p \in \text{Ann}_\emptyset u' \} \cup \{ (D_x, -1) \} \rangle
\]

After computing a Gröbner basis of the above, we can perform the closure property “multiplication by \( vw \)” in a very similar fashion as usual (using an FGLM-like approach).
Result

Finally we can use the ansatz technique as before in order to find an \( \{x, y, z\}\)-free operator:

\[
-2(1 + 2i)(2 + j)(3 + 2i + j)(7 + 2i + 2j)(5 + i + j + k)(7 + i + j + k)(8 + i + j + k)(8 + 2i + 2j + k)(9 + 2i + 2j + k)(11 + 2i + 2j + 2k)(15 + 2i + 2j + 2k)f(i, j + 1, k + 3) + \\
\vdots \\
\langle 31 \text{ similar terms} \rangle \\
\vdots \\
-2(4 + 2i + j)(5 + 2i + j)(5 + 2i + 2j)(5 + i + j + k)(6 + i + j + k)(8 + i + j + k)(10 + 2i + 2j + k)(11 + 2i + 2j + k)(11 + 2i + 2j + 2k)(15 + 2i + 2j + 2k)g(i + 1, j + 2, k + 3) = 0
\]

where \( f =uvw \) and \( g = \frac{du}{dx}vw \).
Advanced Application 3

Stembridge’s TSPP Theorem
(motivated by a $300 prize from D. Zeilberger)
Totally Symmetric Plane Partitions (TSPP)

**Theorem:** (John Stembridge, 1995)
The number of TSPPs whose 3D Ferrers diagram is bounded inside the cube $[0, n]^3$ is given by the product-formula

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}.$$

Soichi Okada proved that the TSPP formula is true if

$$\det (a(i, j))_{1 \leq i, j \leq n} = \prod_{1 \leq i \leq j \leq k \leq n} \left( \frac{i + j + k - 1}{i + j + k - 2} \right)^2,$$

where

$$a(i, j) = \binom{i + j - 2}{i - 1} + \binom{i + j - 1}{i} + 2\delta(i, j) - \delta(i, j + 1).$$
Translation to holonomic framework

Doron Zeilberger proposed a method for proving that
\[
\det(a(i, j))_{1 \leq i, j \leq n} = \text{Nice}(n),
\]
for some explicit expressions \(a(i, j)\) and \(\text{Nice}(n)\), and for all \(n \in \mathbb{N}\):

Find another discrete function \(B(n, j)\) such that the following identities hold:

\[
\sum_{j=1}^{n} B(n, j)a(i, j) = 0, \quad i, n \in \mathbb{N}, i < n
\]

\[
B(n, n) = 1, \quad n \in \mathbb{N},
\]

\[
\sum_{j=1}^{n} B(n, j)a(n, j) = \frac{\text{Nice}(n)}{\text{Nice}(n - 1)}, \quad n \in \mathbb{N}.
\]

Then the determinant evaluation follows as a consequence.
How to find $B(n, j)$

- we do not know a closed form for $B(n, j)$, but
- we can guess recurrences for it.
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**Result of guessing:** 65 recurrences for $B(n, j)$, total size about 5MB (done by Manuel Kauers)
How to find $B(n, j)$

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**$\partial$-finite description:** We succeeded in computing a Gröbner basis of these recurrences.

The Gröbner basis consists of 5 operators; their leading monomials $S_j^4, S_j^3 S_n, S_j^2 S_n^2, S_j S_n^3, S_n^4$ form a staircase of regular shape.
Several summation approaches

\[ \sum_{j=1}^{n} B(n, j)a(n, j) = \frac{\text{Nice}(n)}{\text{Nice}(n - 1)} \]

There are several methods for treating such holonomic sums; we unsuccessfully tried

- elimination (Zeilberger’s slow algorithm),
- Takayama’s algorithm,
- Chyzak’s algorithm

(but could not accomplish the necessary computations).
The right ansatz

Chyzak’s algorithm makes an ansatz of the form

$$\sum_{i=0}^{d} p_i(n) S_n^i + (S_j - 1) \sum_{S_j^l S_n^m \in U} q_{l,m}(n,j) S_j^l S_n^m$$

for $p_i \in \mathbb{Q}(n)$ and $q_{l,m} \in \mathbb{Q}(n,j)$. Uncoupling is needed!

Finally, we succeeded by using a “polynomial ansatz” for a creative telescoping operator:

$$\sum_{i=0}^{d} p_i(n) S_n^i + (S_j - 1) \sum_{k,l,m} q_{k,l,m}(n) j^k S_j^l S_n^m$$

Nevertheless, the computations were very much involved; some of the output relations consume up to 700 MB of memory.
Outlook

This technique can be extended in a straight-forward manner to the $q$-case (which is an open problem for more than 25 years!).

From the computational point of view, this is still a big challenge!

→ START project
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Thank you for your attention!