Selected Applications of Symbolic Summation and Integration

Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

November 5, 2012
Radon Seminar, Linz

Short Curriculum Vitae

1999 – 2005	Undergraduate studies in computer science and mathematics, University of Erlangen-Nürnberg, Germany.
2005 – 2009	Ph.D. studies in symbolic computation at RISC, Johannes Kepler University Linz, Austria.
2009 – 2010	Postdoc at Tulane University, New Orleans, USA.
2010 – 2011	Postdoc at the Research Institute for Symbolic Computation (RISC), Linz.
2011 – 2012	Postdoc at "Institut National de Recherche en Informatique et en Automatique" (INRIA), Paris, France.
2012 –	Postdoc at the Radon Institute for Computational and Applied Mathematics (RICAM), Linz.

Research Interests and Scientific Results

- Algorithmic manipulation of holonomic/∂-finite functions
  - Data structure: o.d.e.'s and recurrences plus initial values
  - Linear (homogeneous) equations with polynomial coefficients
  - Closure properties
  - Symbolic summation and integration
  - Finding and proving identities

Research Interests and Scientific Results

- Algorithmic manipulation of holonomic/∂-finite functions
  - Data structure: o.d.e.'s and recurrences plus initial values
  - Linear (homogeneous) equations with polynomial coefficients
  - Closure properties
  - Symbolic summation and integration
  - Finding and proving identities
- Mathematica package `HolonomicFunctions`
  - Noncommutative arithmetic and Gröbner bases
  - Rational solutions of systems of differential/difference eqns
  - Holonomic closure properties
  - Creative telescoping algorithms
  - $\sim$12,000 lines of code

Research Interests and Scientific Results

- Algorithmic manipulation of holonomic/∂-finite functions
  - Data structure: o.d.e.'s and recurrences plus initial values
  - Linear (homogeneous) equations with polynomial coefficients
  - Closure properties
  - Symbolic summation and integration
  - Finding and proving identities
- Mathematica package `HolonomicFunctions`
  - Noncommutative arithmetic and Gröbner bases
  - Rational solutions of systems of differential/difference eqns
  - Holonomic closure properties
  - Creative telescoping algorithms
  - $\sim$12,000 lines of code
- Applications of the holonomic systems approach
  - Combinatorics
  - Numerical analysis (FEM)
  - Quantum Topology
  - Physics

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

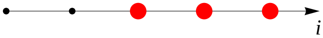
This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$

$$-\frac{6x}{x^2-1}P_n^{(3)}(x) + \frac{(n-2)(n+3)}{x^2-1}P_n''(x)$$



$$(x^2-1)P_n^{(4)}(x) + 6xP_n^{(3)}(x) - (n-2)(n+3)P_n''(x) = 0$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$

$$-\frac{6x}{x^2-1}P_n^{(3)}(x) + \frac{(n-2)(n+3)}{x^2-1}P_n''(x)$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$
$$-\frac{6x}{x^2-1}P_n^{(3)}(x) + \frac{(n-2)(n+3)}{x^2-1}P_n''(x)$$

$$(x^2-1)P_n^{(3)}(x) + 4xP_n''(x) - (n-1)(n+2)P_n'(x) = 0$$

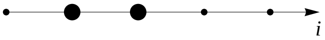
Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) = \frac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x) - \frac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$



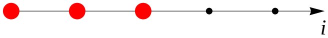
Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) = \frac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x) - \frac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$



$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0$$

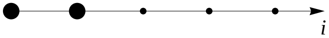
Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) = -\frac{8x(n^2x^2-n^2+nx^2-n+3x^2+3)}{(x^2-1)^3}P_n'(x) + \frac{n(n+1)(n^2x^2-n^2+nx^2-n+18x^2+6)}{(x^2-1)^3}P_n(x)$$



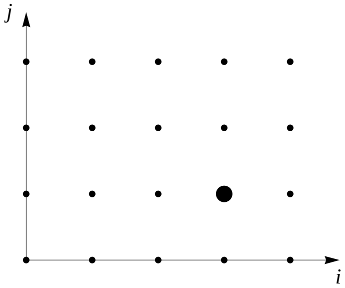
$\longrightarrow$ $P_n(x)$ is ∂-finite w.r.t. x.

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2 - 1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



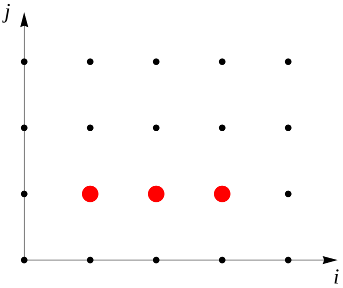
$$P_{n+1}^{(3)}(x) =$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+1}^{(3)}(x) =$$

$$-\tfrac{4x}{x^2-1}P_{n+1}''(x) + \tfrac{n(n+3)}{x^2-1}P_{n+1}'(x)$$

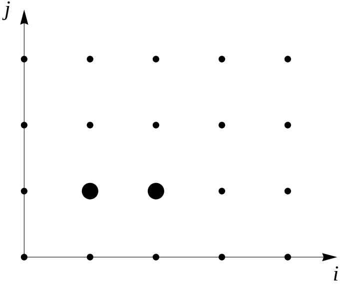
$$(x^2-1)P_{n+1}^{(3)}(x) + 4xP_{n+1}''(x) - n(n+3)P_{n+1}'(x) = 0$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+1}^{(3)}(x) =$$

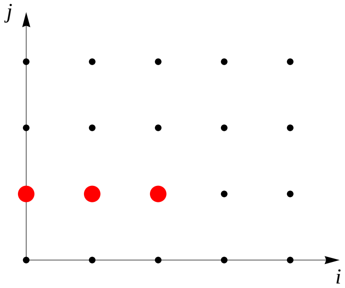
$$-\frac{4x}{x^2-1}P_{n+1}''(x) + \frac{n(n+3)}{x^2-1}P_{n+1}'(x)$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i,j \geq 0\right\}$.



$$P_{n+1}^{(3)}(x) =$$

$$-\tfrac{4x}{x^2-1}P_{n+1}''(x) + \tfrac{n(n+3)}{x^2-1}P_{n+1}'(x)$$

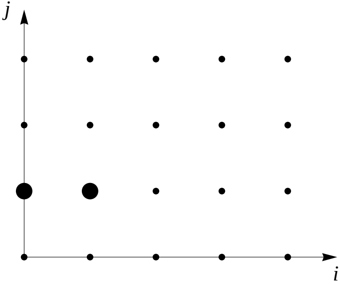
$$(x^2-1)P_{n+1}''(x) + 2xP_{n+1}'(x) - (n+1)(n+2)P_{n+1}(x) = 0$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



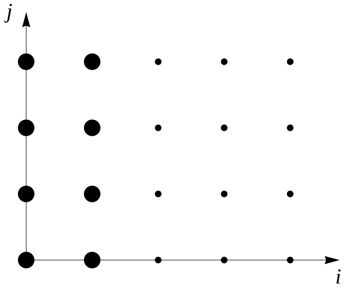
$$P_{n+1}^{(3)}(x) = \frac{(n^2x^2-n^2+3nx^2-3n+8x^2)}{(x^2-1)^2}P_{n+1}'(x) - \frac{4(n^2x+3nx+2x)}{(x^2-1)^2}P_{n+1}(x)$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

This family of (orthogonal) polynomials is a solution of the differential equation

$$(x^2-1)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i,j \geq 0\right\}$.

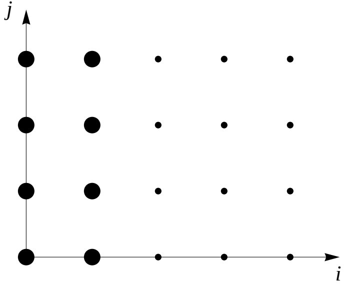


Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P^{(i)}_{n+j}(x) : i, j \geq 0\right\}$.

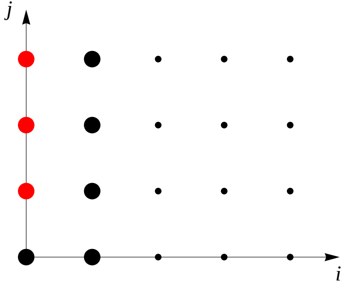


Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+3}(x) = \frac{(2n+5)x}{n+3}P_{n+2}(x) - \frac{n+2}{n+3}P_{n+1}(x)$$

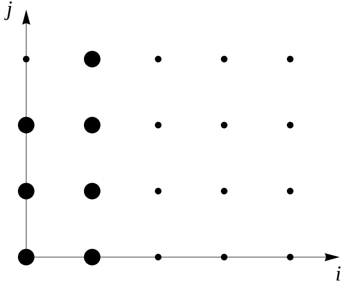
$$(n+3)P_{n+3}(x) - (2n+5)xP_{n+2}(x) + (n+2)P_{n+1}(x) = 0$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+3}(x) =$$

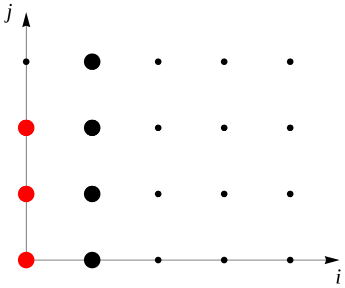
$$\frac{(2n+5)x}{n+3} P_{n+2}(x) - \frac{n+2}{n+3} P_{n+1}(x)$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+3}(x) =$$
$$\frac{4n^2x^2-n^2+16nx^2-4n+15x^2-4}{(n+2)(n+3)}P_{n+1}(x)$$
$$-\frac{2n^2x+7nx+5x}{(n+2)(n+3)}P_n(x)$$

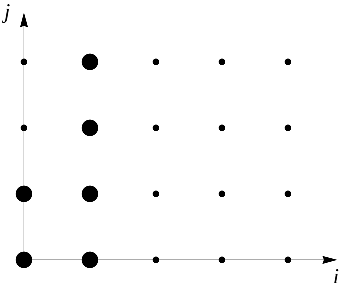
$$(n+2)P_{n+2}(x) - (2n+3)xP_{n+1}(x) + (n+1)P_n(x) = 0$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



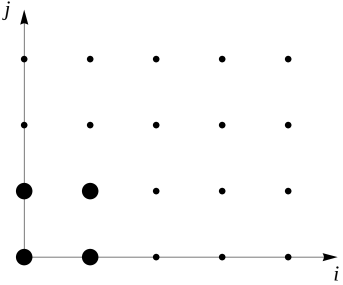
$$P_{n+3}(x) = \frac{4n^2x^2 - n^2 + 16nx^2 - 4n + 15x^2 - 4}{(n+2)(n+3)} P_{n+1}(x) - \frac{2n^2x + 7nx + 5x}{(n+2)(n+3)} P_n(x)$$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.

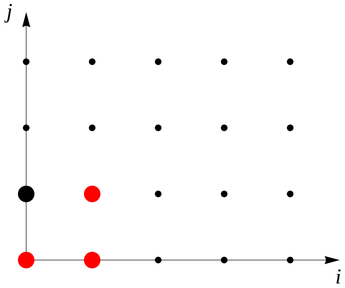


Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



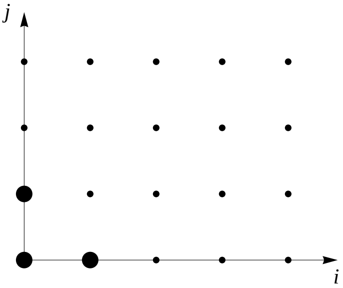
$P'_{n+1}(x) - xP'_n(x) - (n+1)P_n(x) = 0$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.

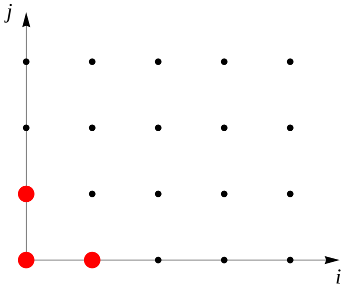


Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



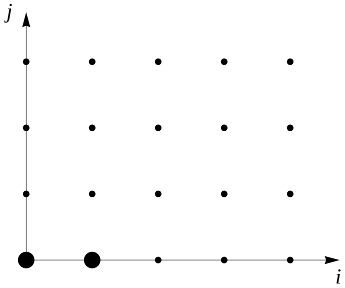
$(n+1)P_{n+1}(x) + (1-x^2)P_n'(x) - (n+1)xP_n(x) = 0$

Example: the Legendre Polynomials $P_n(x)$ are ∂-Finite

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P^{(i)}_{n+j}(x) : i, j \geq 0\right\}$.



$\longrightarrow$ $P_n(x)$ is ∂-finite w.r.t. n and x (of rank 2).

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
- q-shift operator $S_{v,q}$: $S_{v,q} f(v) = f(qv)$
- arbitrary operator: ∂_v any of the above

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
- q-shift operator $S_{v,q}$: $S_{v,q} f(v) = f(qv)$
- arbitrary operator: ∂_v any of the above

Example 1: The Legendre differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0$$

is written as

$$(x^2-1)D_x^2 + 2xD_x - n(n+1).$$

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
- q-shift operator $S_{v,q}$: $S_{v,q} f(v) = f(qv)$
- arbitrary operator: ∂_v any of the above

Example 1: The Legendre differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0$$

is written as

$$(x^2-1)D_x^2 + 2xD_x - n(n+1).$$

Example 2: The three-term recurrence

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

is written as

$$(n+2)S_n^2 - (2n+3)xS_n + (n+1).$$

Operator Algebra

Noncommutative multiplication

$$D_x x = x D_x + 1 \quad \text{and} \quad S_n n = n S_n + S_n$$

or more general

$$D_x \cdot a(x) = a(x) D_x + a'(x) \quad \text{and} \quad S_n \cdot a(n) = a(n+1) S_n$$

Operator Algebra

Noncommutative multiplication

$$D_x x = x D_x + 1 \quad \text{and} \quad S_n n = n S_n + S_n$$

or more general

$$D_x \cdot a(x) = a(x) D_x + a'(x) \quad \text{and} \quad S_n \cdot a(n) = a(n+1) S_n$$

Such operators form an **Ore algebra**

$$\mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle,$$

i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $v, w, \dots$, where $\mathbb{K}$ is a computable field of characteristic 0 (i.e., containing $\mathbb{Q}$).

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P(f) = 0\}$$

is called the **annihilator** of the function f.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P(f) = 0\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two equations for a function f, given by $P(f) = Q(f) = 0$.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P(f) = 0\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two equations for a function f, given by $P(f) = Q(f) = 0$. Then

- $\mathrm{Eq}_1 + \mathrm{Eq}_2$
- $r(x) \cdot \mathrm{Eq}_1$
- $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Eq}_1$
- $\mathrm{Eq}_1|_{n \to n+1}$

are also valid equations for f.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P(f) = 0\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two equations for a function f, given by $P(f) = Q(f) = 0$. Then

- $\mathrm{Eq}_1 + \mathrm{Eq}_2$ corresponds to $P + Q$
- $r(x) \cdot \mathrm{Eq}_1$ corresponds to $r(x) \cdot P$
- $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Eq}_1$ corresponds to $D_x \cdot P$
- $\mathrm{Eq}_1|_{n \to n+1}$ corresponds to $S_n \cdot P$

are also valid equations for f.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P(f) = 0\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two equations for a function f, given by $P(f) = Q(f) = 0$. Then

- $\mathrm{Eq}_1 + \mathrm{Eq}_2$ corresponds to $P + Q$
- $r(x) \cdot \mathrm{Eq}_1$ corresponds to $r(x) \cdot P$
- $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Eq}_1$ corresponds to $D_x \cdot P$
- $\mathrm{Eq}_1|_{n \to n+1}$ corresponds to $S_n \cdot P$

are also valid equations for f. More generally,

$$P, Q \in \operatorname{Ann}_{\mathbb{O}} f \implies P + Q \in \operatorname{Ann}_{\mathbb{O}} f$$
$$L \in \mathbb{O}, P \in \operatorname{Ann}_{\mathbb{O}} f \implies L \cdot P \in \operatorname{Ann}_{\mathbb{O}} f$$

which states that $\operatorname{Ann}_{\mathbb{O}} f$ is a **left ideal** in $\mathbb{O}$.

Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function f is ∂-finite w.r.t. $\mathbb{O}$ if all its shifts and derivatives

$$\mathbb{O}(f) = \{P(f) : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space.

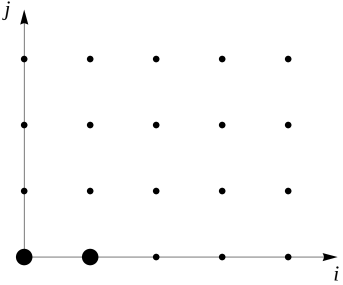
Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function f is ∂-finite w.r.t. $\mathbb{O}$ if all its shifts and derivatives

$$\mathbb{O}(f) = \{P(f) : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space.



Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function f is ∂-finite w.r.t. $\mathbb{O}$ if all its shifts and derivatives

$$\mathbb{O}(f) = \{P(f) : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space.

In other words, if the left ideal of annihilating operators of f

$$\mathrm{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} : P(f) = 0\}$$

is a zero-dimensional ideal.

Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function f is ∂-finite w.r.t. $\mathbb{O}$ if all its shifts and derivatives

$$\mathbb{O}(f) = \{P(f) : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space.

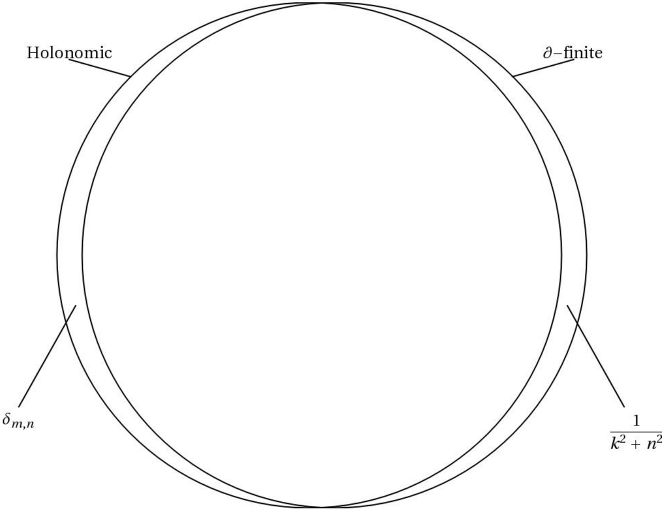
In other words, if the left ideal of annihilating operators of f

$$\mathrm{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} : P(f) = 0\}$$

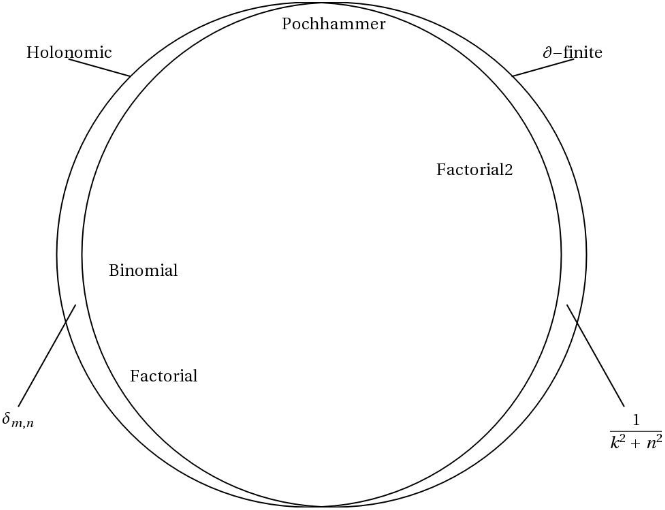
is a zero-dimensional ideal.

The definition of "holonomic" is a bit technical, so we skip it here.

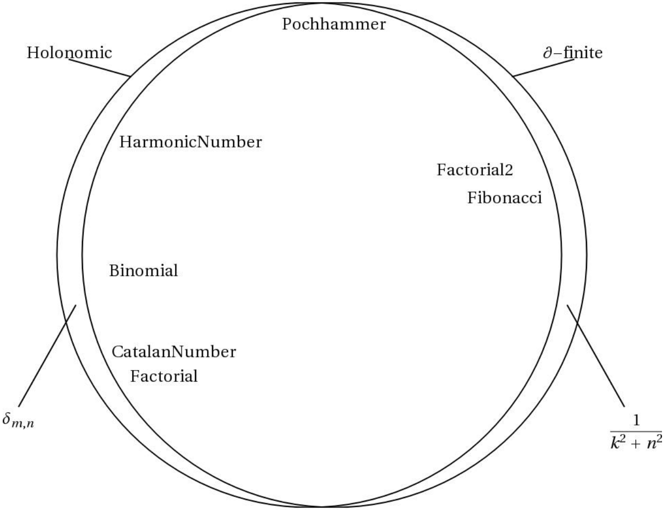
Holonomic Versus ∂-Finite Functions



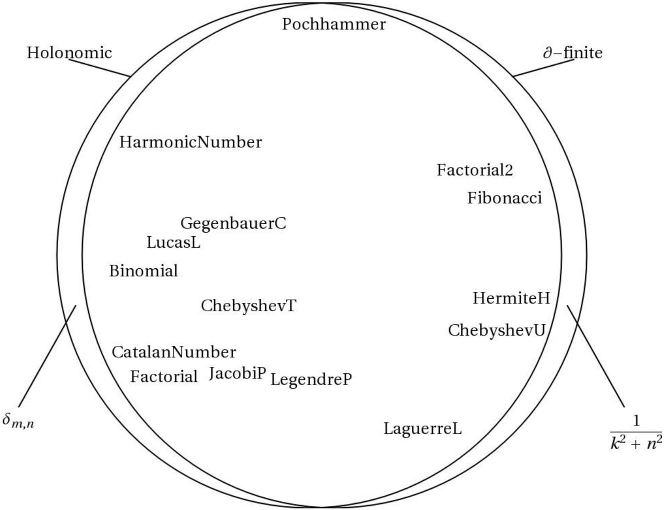
Holonomic Versus ∂-Finite Functions



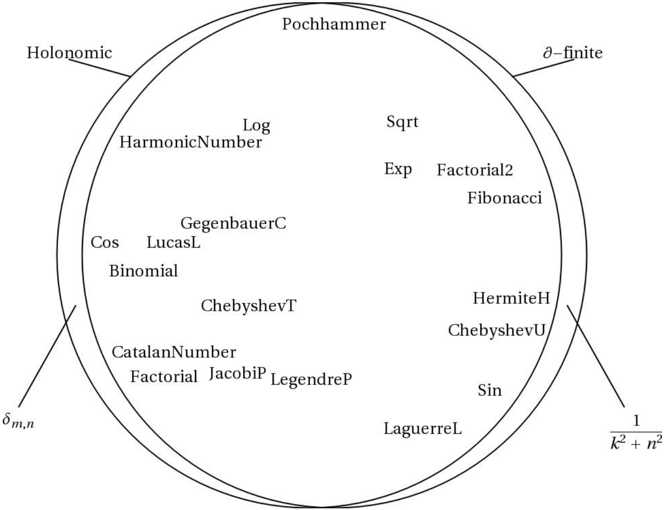
Holonomic Versus ∂-Finite Functions



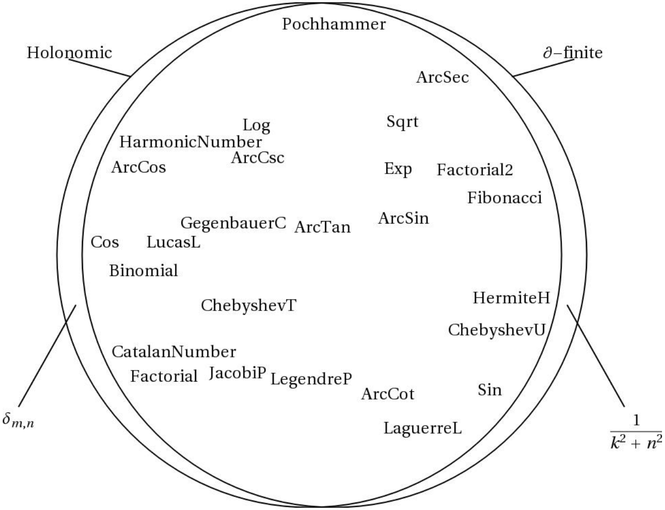
Holonomic Versus ∂-Finite Functions



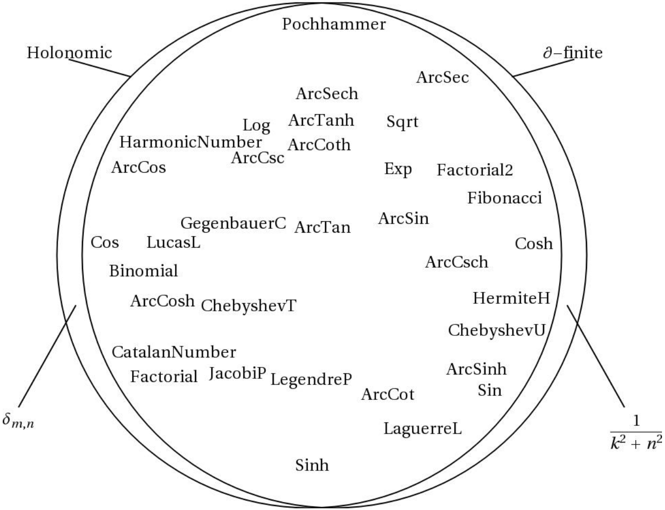
Holonomic Versus ∂-Finite Functions



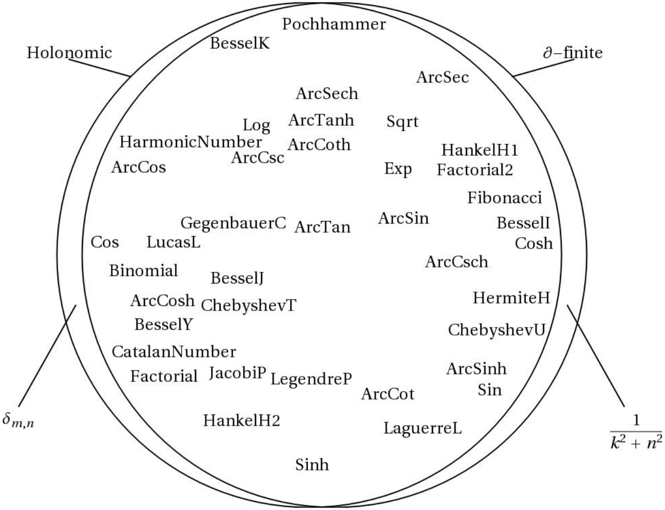
Holonomic Versus ∂-Finite Functions



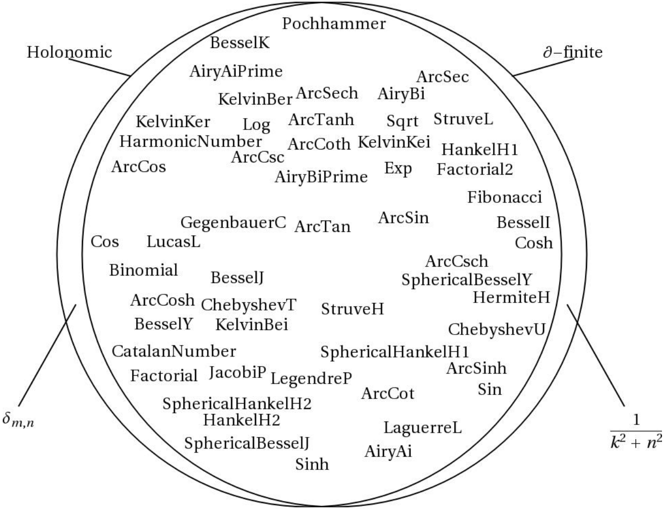
Holonomic Versus ∂-Finite Functions



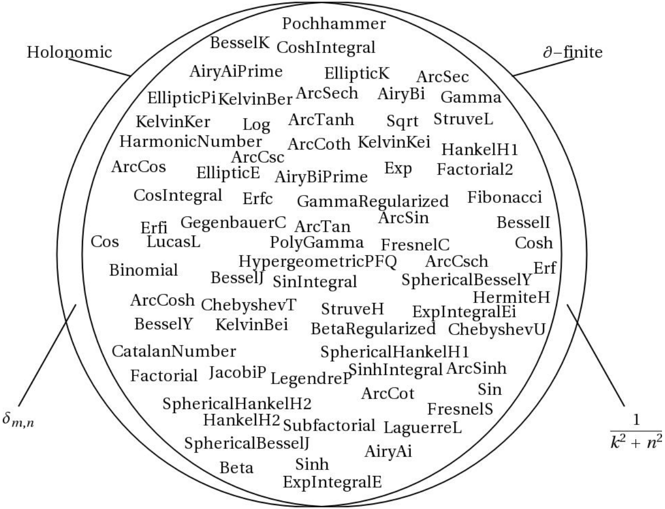
Holonomic Versus ∂-Finite Functions



Holonomic Versus ∂-Finite Functions



Holonomic Versus ∂-Finite Functions



Why ∂-Finite/Holonomic Functions?

1. Definition gives rise to a finite data structure.
    - annihilating ideal of operators (Gröbner basis is finite)
    - finitely many initial values

Why ∂-Finite/Holonomic Functions?

1. Definition gives rise to a finite data structure.
   - annihilating ideal of operators (Gröbner basis is finite)
   - finitely many initial values
2. This set of functions is closed under many operations.
   - addition, e.g., $x^n + P_n(x)$
   - multiplication, e.g., $P_n^2(x)$
   - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
   - operator application, e.g., $P_n'(x)$
   - definite summation, e.g., $\sum_{n=0}^{\infty} P_n(x)t^n$
   - definite integration, e.g., $\int_0^1 P_n(x)\,\mathrm{d}x$

Why ∂-Finite/Holonomic Functions?

1. Definition gives rise to a finite data structure.
   - annihilating ideal of operators (Gröbner basis is finite)
   - finitely many initial values
2. This set of functions is closed under many operations.
   - addition, e.g., $x^n + P_n(x)$
   - multiplication, e.g., $P_n^2(x)$
   - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
   - operator application, e.g., $P_n'(x)$
   - definite summation, e.g., $\sum_{n=0}^{\infty} P_n(x)t^n$
   - definite integration, e.g., $\int_0^1 P_n(x)\,\mathrm{d}x$
3. These operations (closure properties) can be executed algorithmically.

Why ∂-Finite/Holonomic Functions?

1. Definition gives rise to a finite data structure.
    - annihilating ideal of operators (Gröbner basis is finite)
    - finitely many initial values
2. This set of functions is closed under many operations.
    - addition, e.g., $x^n + P_n(x)$
    - multiplication, e.g., $P_n^2(x)$
    - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
    - operator application, e.g., $P_n'(x)$
    - definite summation, e.g., $\sum_{n=0}^{\infty} P_n(x) t^n$
    - definite integration, e.g., $\int_0^1 P_n(x)\,\mathrm{d}x$
3. These operations (closure properties) can be executed algorithmically.
4. Many elementary and special functions are covered.

Principia Holonomica

1. Functions and sequences are represented by their annihilating left ideals (and initial values).

Principia Holonomica

1. Functions and sequences are represented by their annihilating left ideals (and initial values).
2. An annihilating ideal is given by its Gröbner basis (i.e. a finite set of generators that allows to decide ideal membership and equality of ideals).

Principia Holonomica

1. Functions and sequences are represented by their annihilating left ideals (and initial values).
2. An annihilating ideal is given by its Gröbner basis (i.e. a finite set of generators that allows to decide ideal membership and equality of ideals).
3. Integrals and sums are treated by the method of creative telescoping.

Principia Holonomica

1. Functions and sequences are represented by their annihilating left ideals (and initial values).
2. An annihilating ideal is given by its Gröbner basis (i.e. a finite set of generators that allows to decide ideal membership and equality of ideals).
3. Integrals and sums are treated by the method of creative telescoping.
4. The output is always given as an annihilating ideal, not as a closed form.

Finite Elements

Joint work with Joachim Schöberl and Peter Paule

Problem Setting

Simulate the propagation of electromagnetic waves using the Maxwell equations

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively.

Define basis functions (this is the 2D case):

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\big(\tfrac{2y}{1-x} - 1\big)$$

using the Legendre and Jacobi polynomials.

Problem: Represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself).

Make an Ansatz!

More precisely, we need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

Make an Ansatz!

More precisely, we need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

Sketch of the algorithm:

1. Work in the Ore algebra $\mathbb{O} = \mathbb{Q}(i,j,x,y)\langle S_i, S_j, D_x\rangle$.
2. Compute a Gröbner basis G of $\operatorname{Ann}_{\mathbb{O}} \varphi_{i,j}(x,y)$.
3. Choose index sets A and B.
4. Reduce the above ansatz with G.
5. Do coefficient comparison with respect to x and y.
6. Solve the resulting linear system for $a_{k,l}, b_{m,n} \in \mathbb{Q}(i,j)$.
7. If there is no solution, go back to step 3.

Result

With this method, we find the relation

$$\begin{aligned}
&(2i+j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)+\\
&2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)-\\
&(j+3)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)+\\
&(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)-\\
&2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)-\\
&(2i+j+5)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)=0
\end{aligned}$$

and a similar one for $\frac{\mathrm{d}}{\mathrm{d}y}\varphi_{i,j}(x,y)$.

$\longrightarrow$ These formulae caused a speed-up of 20% in the numerical simulations.

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$. Then

$$F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a).$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$. Then

$$F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a).$$

Creative Telescoping: write

$$c_d(n) f(n+d,k) + \cdots + c_0(n) f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_d(n) F(n+d) + \cdots + c_0(n) F(n) = g(n,b+1) - g(n,a).$$

Creative Telescoping (Example 1)

Let $F(n)$ denote the double sum over the trinomial coefficients

$$F(n) = \sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i, j, n-i-j} = \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{n!}{i!j!(n-i-j)!}.$$

Creative Telescoping (Example 1)

Let $F(n)$ denote the double sum over the trinomial coefficients

$$F(n) = \sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i, j, n-i-j} = \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{n!}{i!j!(n-i-j)!}.$$

Then the creative telescoping operator

$$CT = S_n - 3 + (S_i - 1)\frac{i}{n-i-j+1} + (S_j - 1)\frac{j}{n-i-j+1}$$

with $CT\left(\binom{n}{i,j,n-i-j}\right) = 0$ implies that

$$F(n+1) = 3F(n).$$

Creative Telescoping (Example 2)

The lattice Green's function of the square lattice is given by

$$P(z) = \int_0^1 \int_0^1 \frac{1}{(1 - xyz)\sqrt{1 - x^2}\sqrt{1 - y^2}} \, \mathrm{d}x \, \mathrm{d}y.$$

Creative Telescoping (Example 2)

The lattice Green's function of the square lattice is given by

$$P(z) = \int_0^1 \int_0^1 \frac{1}{(1-xyz)\sqrt{1-x^2}\sqrt{1-y^2}}\,\mathrm{d}x\,\mathrm{d}y.$$

The creative telescoping operator

$$(z^3-z)D_z^2 + (3z^2-1)D_z + z + D_x\frac{y(1-x^2)}{xyz-1} + D_y\frac{yz(1-y^2)}{xyz-1}$$

that annihilates the integrand, certifies that $P(z)$ satisfies the differential equation

$$(z^3-z)P''(z) + (3z^2-1)P'(z) + zP(z) = 0.$$

Creative Telescoping

In general, a creative telescoping operator has the form

$$P(\boldsymbol{v}, \boldsymbol{\partial_v}) + \Delta_1 Q_1(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w}) + \cdots + \Delta_m Q_m(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$$

where $\Delta_i = S_{w_i} - 1$ or $\Delta_i = D_{w_i}$ (depending on the problem).

Creative Telescoping

In general, a creative telescoping operator has the form

$$P(\boldsymbol{v}, \boldsymbol{\partial_v}) + \Delta_1 Q_1(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w}) + \cdots + \Delta_m Q_m(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$$

where $\Delta_i = S_{w_i} - 1$ or $\Delta_i = D_{w_i}$ (depending on the problem).

- Corresponds to an m-fold summation/integration problem.
- $\boldsymbol{w} = w_1, \ldots, w_m$ are the summation/integration variables.
- $\boldsymbol{v} = v_1, v_2, \ldots$ are the surviving parameters.
- $P(\boldsymbol{v}, \boldsymbol{\partial_v})$ is called the **telescoper**.
- The $Q_i(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$ are called the **delta parts**.
- The delta parts can be viewed as certificates for the correctness of the telescoper.
- Research topic: develop fast algorithms to compute it!

Application 2

Special Function Identities

Some Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{1}$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x = \frac{\pi P_m^{(m+\frac{1}{2},-m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \tag{2}$$

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\mathrm{d}t \tag{3}$$

$$\int_{-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!n!}\,\mathrm{d}x = \sqrt{\pi}e^{2rs} \tag{4}$$

$$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu)J_{n+\nu}(a)}{2^{\nu-1}a^\nu n!\Gamma(\nu)} \tag{5}$$

$$\frac{\sin\left(\sqrt{z^2+2tz}\right)}{z} = \sum_{n=0}^{\infty}\frac{(-t)^n y_{n-1}(z)}{n!} \tag{6}$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

```
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

```
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

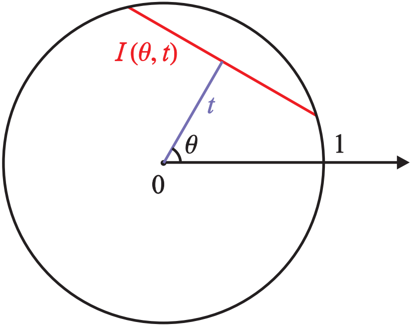
```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*BesselJ[a,2*Sqrt[t*x]
                    Der[t], {S[a], S[n], Der[x]}]
```

$$\{\{-2S_n + 2xD_x + (a+2n+2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},$$
$$\{-2t, -4tx, -2tx\}\}$$

$\longrightarrow$ The annihilating ideals agree; complete the proof by comparing a few initial values.

Application 3

Radon Projections



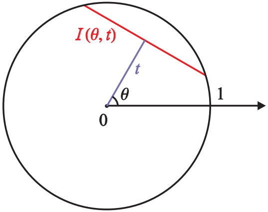
Joint work with Irina Georgieva, Clemens Hofreither, Veronika Pillwein, and Thotsaporn Thanatipanonda

Radon Projection

Consider real bivariate functions $f(x,y)$ in the unit disk in $\mathbb{R}^2$.

Let $I(\theta,t)$ denote a chord of the unit circle at angle $\theta \in [0,2\pi)$ and distance $t \in (-1,1)$ from the origin.

The **Radon projection** $\mathcal{R}_\theta(f;t)$ of f in direction θ is defined by the line integral



$$\mathcal{R}_\theta(f;t) := \int_{I(\theta,t)} f(\mathbf{x})\,d\mathbf{x}$$

$$= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds.$$

Interpolation Using Radon Projections

Find $p \in \Pi_n^2$ (bivariate polynomials of degree $\leq n$) such that

$$\int_I p \, dx = \gamma_I \qquad \forall\, I \in \mathcal{I},$$

where

- $\mathcal{I} = \left\{ I_1, I_2, \ldots, I_{\binom{n+2}{2}} \right\}$ are chords of the unit circle,
- $\gamma_I \in \mathbb{R}$ are given values.

Question: For which configurations of chords $\mathcal{I}$ does this problem have a unique solution?

Radon Projections of the Harmonic Basis

Show that the matrix with the following entries is non-singular:

$$\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \binom{k}{2\ell} (-1)^\ell (t\cos\theta - s\sin\theta)^{k-2\ell} (t\sin\theta + s\cos\theta)^{2\ell}\,\mathrm{d}s =$$

$$\sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{2\ell} (-1)^\ell \sum_{p=0}^{k-2\ell} \sum_{q=0}^{2\ell} \binom{k-2\ell}{p} \binom{2\ell}{q} t^{p+q} \times$$

$$\times (\cos\theta)^{2\ell+p-q} (\sin\theta)^{k-2\ell-(p-q)} \frac{(-1)^{k-2\ell-p}}{k-p-q+1} \times$$

$$\times (1-t^2)^{\frac{1}{2}(k-p-q+1)} \left(1-(-1)^{k-p-q+1}\right)$$

$\longrightarrow$ Symbolic computation succeeded in simplifying these expressions drastically.

Application 4

Quantum Topology

Joint work with Stavros Garoufalidis

A Multiple Sum

While studying invariants of such knots, Stavros Garoufalidis asked me to look at the following sum:

$$\sum_{k=0}^{2n}(2k+1)\left((-1)^k a(k,n)\right)^s$$

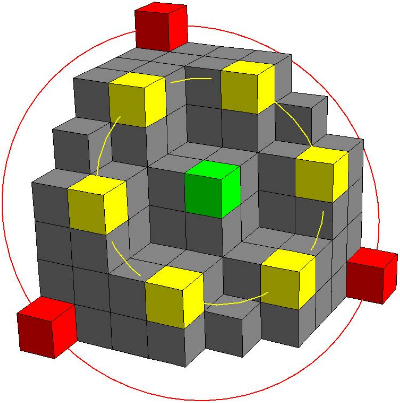
with

$$a(k,n)=\sum_{j=\max(3n,k+2n)}^{\min(4n,k+3n)}(-1)^j\binom{j+1}{k+2n+1}\binom{k}{j-3n}^2\binom{2n-k}{4n-j}.$$

The package `HolonomicFunctions` can compute the recurrence for the sum up to $s=6$. This recurrence has order 12, coefficient degrees about 500 and integer coefficients with several hundred decimal digits.

Application 5

Enumerative Combinatorics



Joint work with Manuel Kauers and Doron Zeilberger

The q-TSPP Conjecture

"The Holy Grail of Enumerative Combinatorics"

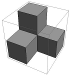
Last surviving open problem of the classic (Stanley, 1986)
A baker's dozen of conjectures concerning plane partitions

The q-TSPP Conjecture

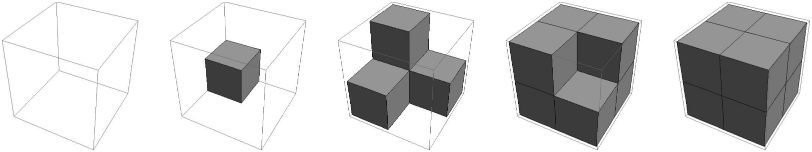
"The Holy Grail of Enumerative Combinatorics"

Last surviving open problem of the classic (Stanley, 1986)
A baker's dozen of conjectures concerning plane partitions

Some notions from partition theory:

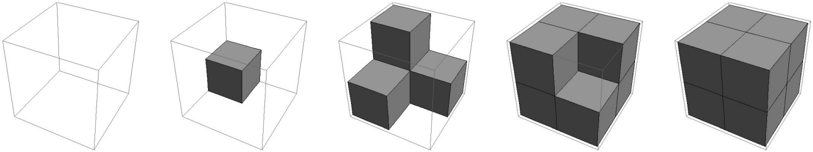
- partition of an integer, e.g., $4 = 2 + 1 + 1$
- plane partition: two-dimensional arrangement of summands, e.g., $\begin{matrix} 2 & 1 \\ 1 & \end{matrix}$
- Ferrer's diagram: three-dimensional representation of PPs, e.g., $\begin{matrix} 2 & 1 \\ 1 & \end{matrix}$ = 
- totally symmetric plane partition (TSPP): Ferrer's diagram is symmetric w.r.t. rotation and reflection
- orbit: see the picture

Let $T(n)$ denote set of TSPPs with largest part $\leq n$, e.g. $n = 2$:



$1 \cdot q^0$ $\quad$ $1 \cdot q^1$ $\quad$ $1 \cdot q^2$ $\quad$ $1 \cdot q^3$ $\quad$ $1 \cdot q^4$

Let $T(n)$ denote set of TSPPs with largest part $\leq n$, e.g. $n = 2$:



$1 \cdot q^0$ $\quad$ $1 \cdot q^1$ $\quad$ $1 \cdot q^2$ $\quad$ $1 \cdot q^3$ $\quad$ $1 \cdot q^4$

Andrews-Robbins q-TSPP conjecture (1983):

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

Counting formula is obtained by $q \to 1$ (Stembridge 1995):

$$|T(n)| = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$$

The Determinant

First reduction by Soichi Okada:

The q-TSPP conjecture is true if

$$\det_{1\leq i,j\leq n}(a_{i,j}) = \prod_{1\leq i\leq j\leq k\leq n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 =: b_n$$

The Determinant

First reduction by Soichi Okada:

The q-TSPP conjecture is true if

$$\det_{1\le i,j\le n}(a_{i,j}) = \prod_{1\le i\le j\le k\le n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 =: b_n$$

where

$$a_{i,j} := q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q\right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

is the q-binomial coefficient.

The Holonomic Ansatz

Second reduction by Doron Zeilberger:

"Pull out of the hat" a holonomic function $c_{n,j}$ and prove

$$
\begin{aligned}
c_{n,n} &= 1 && (n \geq 1),\\
\sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leq i < n),\\
\sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \geq 1).
\end{aligned}
$$

Then $\det(a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds for all n.

The Holonomic Ansatz

Second reduction by Doron Zeilberger:

"Pull out of the hat" a holonomic function $c_{n,j}$ and prove

$$\begin{aligned} c_{n,n} &= 1 && (n \geq 1), \\ \sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leq i < n), \\ \sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \geq 1). \end{aligned}$$

Then $\det(a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds for all n.

- $c_{n,j}$ not given explicitly
- perfect application for the holonomic systems approach
- very large computations
- feasible only due to a new algorithm

The Result

. . . is about 7GB large (corresponding to some million printed pages).

A short version of this appeared in PNAS (Proceedings of the National Academy of Sciences of the USA):

Christoph Koutschan, Manuel Kauers, Doron Zeilberger:
A proof of George Andrews' and David Robbins' q-TSPP conjecture