qGeneratingfunctionology

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q-P-finite Sequences

$(f_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is called P-finite if $\exists a_0, \ldots, a_d \in \mathbb{C}[x]$ s.t.

$$a_d(n) f_{n+d} + \cdots + a_0(n) f_n = 0 \quad (n \geq 0)$$

In the univariate case this is equivalent to holonomic.

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Analogously we call these "q-holonomic".

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"P-finite $\cap$ q-P-finite $=$ C-finite"

q-D-finite Power Series

$f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{C}[\![x]\!]$ is called D-finite if $\exists a_0, \ldots, a_d \in \mathbb{C}[x]$ s.t.

$$a_d(x) f^{(d)}(x) + \cdots + a_1(x) f'(x) + a_0(x) f(x) = 0$$

Again this is equivalent to saying f is holonomic (univariate case!).

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$\exists a_0, \ldots, a_d \in \mathbb{C}(q)[x]$ s.t.

$$a_d(x) D_q^d f(x) + \cdots + a_1(x) D_q f(x) + a_0(x) f(x) = 0$$

Again we call these "q-holonomic".

q-Differentiation

Definition for the q-differential operator:

$$D_q f(x) := \frac{f(qx) - f(x)}{qx - x} = \frac{f(x) - f(qx)}{(1 - q)x}$$

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Example: $D_q x^n = \frac{(qx)^n - x^n}{(1-q)x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n] x^{n-1}$

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Product rule:

$$\begin{aligned} D_q(f(x)g(x)) &= f(x)D_q g(x) + g(qx)D_q f(x) \\ &= g(x)D_q f(x) + f(qx)D_q g(x) \end{aligned}$$

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Definition for the q-shift operator:

$$\epsilon f(x) := f(qx)$$

Of course we have $\epsilon f(x) = f(x) + (q-1)xD_q f(x)$.

Basic Facts

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Closure Properties

Holonomic sequences / power series are closed by

- sum
- Cauchy product
- Hadamard product
- taking subsequences
- algebraic substitution

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q-holonomic sequences / power series are closed by

- sum
- Cauchy product
- Hadamard product
- taking subsequences
- substitution of x^k (!)

Proof: Goto blackboard

A Formula

$$f(q^n x) = \sum_{k=0}^{n} (-1)^k (1-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k D_q^k f(x)$$

Commands

RE2L

Commands

RE2L QRE2L

Commands

RE2L	QRE2L
-	QDE2L, QSE2L

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	

Commands

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Commands

RE2L	QRE2L
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DE2RE	QDE2RE, QSE2RE
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Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
-	QDE2SE, QSE2DE
REPlus	

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
-	QDE2SE, QSE2DE
REPlus	QREPlus

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
-	QDE2SE, QSE2DE
REPlus	QREPlus
REHadamard	

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
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REPlus	QREPlus
REHadamard	QREHadamard

Commands

RE2L	QRE2L
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RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
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REPlus	QREPlus
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RECauchy	

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REPlus	QREPlus
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RESubsequence	

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
-	QDE2SE, QSE2DE
REPlus	QREPlus
REHadamard	QREHadamard
RECauchy	QRECauchy
RESubsequence	QRESubstitute

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
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DE2RE	QDE2RE, QSE2RE
-	QDE2SE, QSE2DE
REPlus	QREPlus
REHadamard	QREHadamard
RECauchy	QRECauchy
RESubsequence	QRESubstitute
DEPlus	

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
-	QDE2SE, QSE2DE
REPlus	QREPlus
REHadamard	QREHadamard
RECauchy	QRECauchy
RESubsequence	QRESubstitute
DEPlus	QDEPlus, QSEPlus

Commands

RE2L	QRE2L
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RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
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REPlus	QREPlus
REHadamard	QREHadamard
RECauchy	QRECauchy
RESubsequence	QRESubstitute
DEPlus	QDEPlus, QSEPlus
DEHadamard	

Commands

RE2L	QRE2L
-	QDE2L, QSE2L
RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
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REPlus	QREPlus
REHadamard	QREHadamard
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Commands

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REPlus	QREPlus
REHadamard	QREHadamard
RECauchy	QRECauchy
RESubsequence	QRESubstitute
DEPlus	QDEPlus, QSEPlus
DEHadamard	QDEHadamard, QSEHadamard
DECauchy	

Commands

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REHadamard	QREHadamard
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RESubsequence	QRESubstitute
DEPlus	QDEPlus, QSEPlus
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REHadamard	QREHadamard
RECauchy	QRECauchy
RESubsequence	QRESubstitute
DEPlus	QDEPlus, QSEPlus
DEHadamard	QDEHadamard, QSEHadamard
DECauchy	QDECauchy, QSECauchy
(REInterlace)	QDESubstitute, QSESubstitute

Commands

RE2L	QRE2L
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RE2DE	QRE2DE, QRE2SE
DE2RE	QDE2RE, QSE2RE
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REPlus	QREPlus
REHadamard	QREHadamard
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RESubsequence	QRESubstitute
DEPlus	QDEPlus, QSEPlus
DEHadamard	QDEHadamard, QSEHadamard
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HomogenousRE	

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RE2DE	QRE2DE, QRE2SE
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REPlus	QREPlus
REHadamard	QREHadamard
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DEPlus	QDEPlus, QSEPlus
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GuessRE	

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HomogenousDE	QDEHomogen**e**ous, QSEHomogen**e**ous
GuessRE	QREGuess

Task 1

Image that there is a guest from Vienna who asks you to guess the following sequence

$1,$

$1,$

$q^2 + 1,$

$q^6 + q^4 + q^3 + q^2 + 1,$

$q^{12} + q^{10} + q^9 + 2q^8 + q^7 + 2q^6 + q^5 + 2q^4 + q^3 + q^2 + 1,$

$\ldots$

Task 2

Prove identities involving the q-trigonometric functions

$$\sin_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}}, \quad \mathrm{Sin}_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(2n+1)n} x^{2n+1}}{(q;q)_{2n+1}},$$

$$\cos_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}}, \qquad \mathrm{Cos}_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(2n-1)n} x^{2n}}{(q;q)_{2n}}.$$

1. $\sin_q(x)\,\mathrm{Sin}_q(x) + \cos_q(x)\,\mathrm{Cos}_q(x) = 1$
2. $\sin_q(x)\,\mathrm{Cos}_q(x) - \mathrm{Sin}_q(x)\cos_q(x) = 0$

Task 3

Prove the following identity:

$$\sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{k=-n}^{n} (-1)^k q^{k(5k+1)/2} \begin{bmatrix} 2n \\ n+2k \end{bmatrix}_q$$

Thanks for your attention!

Clap now!