A Generalized Apagodu-Zeilberger Algorithm

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(joint work with Shaoshi Chen and Manuel Kauers)

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Different approaches to creative telescoping:

**Elimination approach:**
Zeilberger’s slow algorithm (1990), Takayama’s algorithm (1990)
→ works for general $\partial$-finite holonomic functions
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\[\rightarrow\] generalization to \( \partial \)-finite functions (Chyzak 1998)
Background

Different approaches to creative telescoping:

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Zeilberger’s slow algorithm (1990), Takayama’s algorithm (1990) → works for general $\partial$-finite holonomic functions

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**Prediction approach:**
Apagodu-Zeilberger algorithms (2005, 2006) → generalization to $\partial$-finite functions (NEW!)
Ore Algebras

Definition.

1. Field: $K \supseteq \mathbb{Q}$
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2. Rational Functions: $K(x, y)$
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2. Rational Functions: $K(x, y)$
3. Automorphisms: $\sigma_x, \sigma_y : K(x, y) \to K(x, y)$ s.t. $\sigma_x \sigma_y = \sigma_y \sigma_x$
4. Derivations: $K$-linear maps $\delta_x, \delta_y : K(x, y) \to K(x, y)$ s.t.
   $$\delta_x(ab) = \delta_x(a)b + \sigma_x(a)\delta_x(b), \quad \delta_y(ab) = \delta_y(a)b + \sigma_y(a)\delta_y(b)$$
5. Algebra: $A = K(x, y)[\partial_x, \partial_y]$, noncommutative multiplication:
   $$\partial_x a = \sigma_x(a)\partial_x + \delta_x(a), \quad \partial_y a = \sigma_y(a)\partial_y + \delta_y(a), \quad a \in K(x, y)$$

Additional assumptions:
For all $p \in K[x, y]$: $\sigma_x(p), \sigma_y(p), \delta_x(p), \delta_y(p) \in K[x, y]$, $\deg_x(\sigma_x(p)) = \deg_x(p)$,
$\deg_y(\sigma_x(p)) = \deg_y(p)$, $\deg_x(\delta_x(p)) \leq \deg_x(p) - 1$, $\deg_y(\delta_x(p)) \leq \deg_y(p)$,
and likewise for $\sigma_y, \delta_y$. 
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\deg_x(\sigma_x(p)) = \deg_x(p), \quad \deg_y(\sigma_x(p)) = \deg_y(p),
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\deg_x(\delta_x(p)) \leq \deg_x(p) - 1, \quad \deg_y(\delta_x(p)) \leq \deg_y(p),
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and likewise for $\sigma_y, \delta_y$. 
\(\partial\)-Finite Functions

**Hypergeometric term:**

\(f(x, y)\) is hg. if \(f(x + 1, y)/f(x, y), f(x, y + 1)/f(x, y) \in K(x, y)\).

\(\rightarrow\) \(f(x, y)\) satisfies first-order recurrence equations in \(x\) and \(y\).
\( \partial \)-Finite Functions

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**Hyperexponential function:**
\( f(x, y) \) is hyperexponential if \( f_x/f \) and \( f_y/f \) are rational.
\( \rightarrow \) \( f(x, y) \) satisfies first-order differential equations in \( x \) and \( y \).
Partial Differential Functions

**Hypergeometric term:**

$f(x, y)$ is hypergeometric if $f(x+1, y)/f(x, y), f(x, y+1)/f(x, y) \in K(x, y)$.  
$\rightarrow f(x, y)$ satisfies first-order recurrence equations in $x$ and $y$.

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**∂-finite function:**

$f(x, y)$ is ∂-finite if the annihilator $\text{ann}_A(f) := \{ P \in A \mid P \cdot f = 0 \}$ is a zero-dimensional left ideal, i.e., $\dim_{K(x,y)}(A/\text{ann}_A(f)) < \infty$.  
$\rightarrow f(x, y)$ satisfies a higher-order system of linear equations.
The Apagodu-Zeilberger Algorithm

**Setting:** Work in the Ore algebra $\mathbb{A} = K(x, y)[\partial_x, \partial_y]$ where

- $\partial_x$ denotes the $x$-shift operator ($\sigma_x(x) = x + 1$, $\delta_x = 0$)
- $\partial_y$ denotes the forward $y$-difference ($\sigma_y(y) = y + 1$, $\delta_y(y) = 1$)

**Problem:** Find $T = t_0 + t_1 \partial_x + \cdots + t_r \partial_x^r$ where $t_i \in K(x)$ such that $T \cdot h(x, y) = \partial_y C \cdot h(x, y)$.

**Idea:**
1. Bound the shape (numerator degree, denominator) of the left-hand side in dependence of $r$.
2. Choose $C$ such that the right-hand side matches these data.
3. The condition $\#\text{unknowns} > \#\text{equations}$ yields an upper bound for $r$, the order of the telescoper.
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The Apagodu-Zeilberger Algorithm (Example)

Consider the simple example \( h(x, y) := \frac{1}{\Gamma(ax + by)}, \quad a, b \in \mathbb{N}. \)
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1. Investigate telescoper part, \( T = t_0 + t_1 \partial_x + \cdots + t_r \partial_x^r \):

\[
T \cdot h(x, y) = \frac{u}{(ax + by) \cdots (ax + by + ra - 1)} h(x, y)
\]

for some polynomial \( u \) of \( y \)-degree \( ra \).
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2. Choose certificate part such that rhs matches lhs:

$$C = \frac{c_0 + c_1 y + \cdots + c_s y^s}{(ax + by)(ax + by + 1) \cdots (ax + by + ra - b - 1)}$$

$$\partial_y C \cdot h(x, y) = \frac{v}{(ax + by) \cdots (ax + by + ra - 1)} h(x, y)$$

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\[ \longrightarrow \text{For } r \geq b \text{ we get a nontrivial solution.} \]
Technical Definitions

Factorials:

\[ (a; i)_y := \prod_{j=0}^{i-1} \sigma_y^j(a) \text{ for } a \in K(x, y) \text{ and } i \in \mathbb{N} \]
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Left and Right Borders:

- $p[ y := \prod_{i=1}^{n} p_i$ (left border)
- $p]_y := \prod_{i=1}^{n} \sigma_y^{i-1}(p_i)$ (right border)
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\[ p\mathcal{[}y := \prod_{i=1}^{n} p_i \quad \text{(left border)} \]
\[ p\mathcal{]}_y := \prod_{i=1}^{n} \sigma_y^{i-1}(p_i) \quad \text{(right border)} \]

- if \(\sigma_y = \text{id}:\) \(p\mathcal{[}y = p\mathcal{]}_y = \text{squarefree part of } p.\)
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Left and Right Borders:

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\begin{align*}
  p[\]_y &:= \prod_{i=1}^{n} p_i \quad \text{(left border)} \\
  p[_y &:= \prod_{i=1}^{n} \sigma^{i-1}_y(p_i) \quad \text{(right border)}
\end{align*}
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- if \(\sigma_y = \text{id}\): \(p[\]_y = p[_y = \text{squarefree part of } p\).
- \(p \sigma_y(p[_y) = p[\]_y \sigma_y(p) \) and other similar identities
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p[y] := \prod_{i=1}^{n} p_i \quad \text{(left border)}
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\[\rightarrow\] Analogously: $(a; i)_x$, $p[x, p]_x$. 
Definition.

\[ \mathfrak{a} \subseteq \mathcal{A} = K(x, y)[\partial_x, \partial_y], \text{ a } \partial\text{-finite ideal} \]
Multiplication Matrices

Definition.

- \( a \subseteq A = K(x, y)[\partial_x, \partial_y] \), a \( \partial \)-finite ideal
- \( B = \{b_1, \ldots, b_n\} \), a \( K(x, y) \)-basis of \( A/\mathfrak{a} \)
Multiplication Matrices

Definition.

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- $B = \{b_1, \ldots, b_n\}$, a $K(x, y)$-basis of $\mathbb{A}/\mathfrak{a}$
- every $w \in \mathbb{A}/\mathfrak{a}$ can be written uniquely $wb = \sum_{i=1}^{n} w_i b_i$ for $w = (w_1, \ldots, w_n) \in K(x, y)^n$ and $b = (b_1, \ldots, b_n)^T$. 
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- for all \( b_i \in B \): \( \partial_x b_i = \sum_{j=1}^{n} m_{i,j} b_j \) with \( m_{i,j} \in K(x, y) \).
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- with \( M = (m_{i,j})_{1 \leq i, j \leq n} \in K(x, y)^{n \times n} \) rewrite to \( \partial_x b = Mb \).
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- Every \( w \in \mathbb{A}/\mathfrak{a} \) can be written uniquely \( wb = \sum_{i=1}^{n} w_ib_i \)
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- With \( M = (m_{i,j})_{1 \leq i, j \leq n} \in K(x, y)^{n \times n} \), rewrite to \( \partial_xb = Mb \).
- \( \partial_x(wb) = (\sigma_x(w)\partial_x + \delta_x(w))b = (\sigma_x(w)M + \delta_x(w))b \)
Definition.

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3. every \( w \in \mathbb{A}/\mathfrak{a} \) can be written uniquely \( wb = \sum_{i=1}^{n} w_i b_i \) for \( w = (w_1, \ldots, w_n) \in K(x, y)^n \) and \( b = (b_1, \ldots, b_n)^T \).

4. for all \( b_i \in B \): \( \partial_x b_i = \sum_{j=1}^{n} m_{i,j} b_j \) with \( m_{i,j} \in K(x, y) \).

5. with \( M = (m_{i,j})_{1 \leq i, j \leq n} \in K(x, y)^{n \times n} \) rewrite to \( \partial_x b = Mb \)

6. \( \partial_x (wb) = (\sigma_x(w) \partial_x + \delta_x(w)) b = (\sigma_x(w) M + \delta_x(w)) b \)

7. Similarly, there exists a matrix \( N \in K(x, y)^{n \times n} \) such that \( \partial_y b = Nb \) and \( \partial_y (wb) = (\sigma_y(w) N + \delta_y(w)) b \).
Conventions

The matrices $M$ and $N$ correspond to the

- shift quotients $\partial_x h/h$ and $\partial_y h/h$ in the hypergeometric case,
- logarithmic derivatives in the hyperexponential case.
Conventions

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Notation:
Set $M = \frac{1}{u} U$, $N = \frac{1}{v} V$ with $u, v \in K[x, y]$ and $U, V \in K[x, y]^{n \times n}$. 
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The matrices $M$ and $N$ correspond to the
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Set $M = \frac{1}{u} U$, $N = \frac{1}{v} V$ with $u, v \in K[x, y]$ and $U, V \in K[x, y]^{n \times n}$.

Admissible basis:
1 $\in A/\alpha$ is represented by a polynomial vector $e \in K(x)[y]^n$. 
Telescoper Part

Ansatz: $T = t_0 + t_1 \partial_x + \cdots + t_r \partial_x^r \in K(x)[\partial_x], \quad t_i \in K(x)$.

**Task:** Predict the shape of the vector $Te \in K(x, y)^n$. 
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Task: Predict the shape of the vector \( Te \in K(x, y)^n. \)

Lemma. Let \( e \in K(x)[y]^n \) be some polynomial vector. For every \( i \geq 0 \) we have \( \partial_x^i e = w/(u; i)_x \) for some vector \( w \in K(x)[y]^n \) with

\[
\deg_y(w) \leq \deg_y(e) + i \max\{\deg_y(u), \deg_y(U)\}
\]

where \( \deg_y \) refers to the maximum degree of all components.

Proof. By induction on \( i \).
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Ansatz: \( T = t_0 + t_1 \partial_x + \cdots + t_r \partial^r_x \in K(x)[\partial_x], \quad t_i \in K(x). \)

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**Lemma.** Let \( e \in K(x)[y]^n \) be some polynomial vector. For every \( i \geq 0 \) we have \( \partial^i_x e = w/(u; i)_x \) for some vector \( w \in K(x)[y]^n \) with

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where \( \deg_y \) refers to the maximum degree of all components.

**Proof.** By induction on \( i. \)

\[ \text{Thus we obtain } Te = w/(u; r)_x \text{ for some polynomial vector } w \]

- whose entries are \( K(x)[y] \)-linear combinations of \( t_0, \ldots, t_r, \)
- whose degree is bounded by \( \deg_y(e) + r \max\{\deg_y(u), \deg_y(U)\}. \)
Certificate Part

Task: Characterize those certificates $C \in \mathbb{A}$ for which the vector $\partial_y C e$ matches a prescribed numerator degree and a prescribed denominator $d \in K(x)[y]$ (coming from the telescoper part).
Task: Characterize those certificates $C \in \mathcal{A}$ for which the vector $\partial_y Ce$ matches a prescribed numerator degree and a prescribed denominator $d \in K(x)[y]$ (coming from the telescoper part).

Observation: Common factors of $d$ and $v$ behave slightly different than other factors. This motivates the decomposition

$$d = (f_1; p_1)_y \cdots (f_m; p_m)_y g, \quad p_1, \ldots, p_m > 0,$$

$$v = (f_1; q_1)_y \cdots (f_m; q_m)_y \sigma_y(h), \quad q_1, \ldots, q_m > 0.$$  

(no coprimeness conditions on the $f_i$’s with $g$ and $h$ is imposed.)
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(no coprimeness conditions on the $f_i$’s with $g$ and $h$ is imposed.)

$\longrightarrow$ W.l.o.g. assume $p_i \geq q_i$, otherwise move some factors to $\sigma_y(h)$. 

Certificate Part
Certificate Part (2)

For convenience, set $c := Ce \in K(x, y)^n$.

**Lemma.** Assume that $p_i \geq q_i \geq 1$ for $i = 1, \ldots, m$ and let

$$z = \sigma_y^{-1} \left( \frac{(f_1; p_1)y \cdots (f_m; p_m)y}{(f_1; q_1)y \cdots (f_m; q_m)y} \right) \frac{g}{g \mid y} \in K(x)[y].$$

Let $w \in K(x)[y]^n$ be any polynomial vector and consider $c = \frac{h}{z} w$. Then $\partial_y c = \frac{1}{d} \tilde{w}$ for some polynomial vector $\tilde{w} \in K(x)[y]^n$ with $\deg_y(\tilde{w}) \leq \deg_y(w) + \deg_y(g \mid y) + \max\{\deg_y(v) - 1, \deg_y(V)\}$.

**Proof.** By “straight-forward” calculation, but a bit technical.
Complication

Are we already there? No!

Subtle problem:

- The denominator \((u; r_x)\) coming from the telescoper part is expressed with respect to \(\sigma_x\).
- For the certificate part, it has to be written in terms of \(\sigma_y\).

Solution:

1. Differential case: no problem here since \(\sigma_x = \sigma_y = \text{id}\).
2. Hypergeometric case: admit only proper hypergeometric terms.
3. General case: impose certain conditions on the input ideal \(a\); this leads to our definition of proper \(\partial\)-finite ideals:
   - It generalizes the notion of proper hypergeometric terms.
   - It refines properness by distinguishing the free variable \(x\) from the summation/integration variable \(y\).
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1. Differential case: no problem here since $\sigma_x = \sigma_y = \text{id}$.
2. Hypergeometric case: admit only proper hypergeometric terms.
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   ▶ It generalizes the notion of proper hypergeometric terms.

   ▶ It refines properness by distinguishing the free variable $x$ from the summation/integration variable $y$. 
Proper $\partial$-Finite Ideals

Definition.

1. A polynomial $u \in K[x,y]$ is called $y$-proper (w.r.t. $\sigma_x, \sigma_y$) if
   \[ \deg_y\left( (u; r) x \left[ y \right] \right) = O(1) \] as $r \to \infty$.

Examples.

1. Let $u \in K[x,y]$ and $\sigma_x = \sigma_y = \text{id}$. Then trivially we get
   \[ (u; r) x \left[ y \right] = u r = (u; r) y \] and
   \[ (u; r) x \left\lceil y \right\rceil = \text{sfp}(u) \] for all $r \geq 1$.

2. Let $\sigma_x(x) = x + 1$, $\sigma_y(y) = y + 1$, and take $u = x + 2y$. Then
   \[ (u; r) x \left[ y \right] = r - 1 \prod_{i=0}^{r-1} (x + 2y) = \left( r - 1 \right) / 2 \prod_{i=0}^{r-2} (x + 2(y + i)) \]
   \[ r/2 - 1 \prod_{i=0}^{r-2} (x + 2(y + i + 1)) = (x + 2y; \lfloor r/2 \rfloor) y \] and hence
   \[ (u; r) x \left\lceil y \right\rceil = (x + 2y)(x + 2y + 1) \] for all $r \geq 2$. 

Proper \( \partial \)-Finite Ideals

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2. A \( \partial \)-finite ideal \( \mathfrak{a} \subseteq A \) is called proper (with respect to \( y \)) if there exists a \( y \)-admissible basis \( B \) of \( A/\mathfrak{a} \), for which the denominator \( u \) of the multiplication matrix \( M \) is \( y \)-proper.
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Examples.

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   $$(u; r)_x = u^r = (u; r)_y$$ and $$(u; r)_x[y] = \text{sfp}(u)$$ for all $r \geq 1$. 
Proper $\partial$-Finite Ideals

**Definition.**

1. A polynomial $u \in K[x, y]$ is called $y$-proper (w.r.t. $\sigma_x, \sigma_y$) if $\deg_y((u; r)_x\lceil_y) = O(1)$ as $r \to \infty$.

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1. Let $u \in K[x, y]$ and $\sigma_x = \sigma_y = \text{id}$. Then trivially we get $(u; r)_x = u^r = (u; r)_y$ and $(u; r)_x\lceil_y = \text{sfp}(u)$ for all $r \geq 1$.

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$$
(u; r)_x = \prod_{i=0}^{r-1} (x + 2y + i) = \prod_{i=0}^{(r-1)/2} (x + 2(y+i)) \prod_{i=0}^{r/2-1} (x + 2(y+i) + 1)
$$

$$
= (x + 2y; \left\lfloor \frac{r-1}{2} \right\rfloor)_y (x + 2y + 1; \left\lfloor \frac{r}{2} \right\rfloor - 1)_y
$$

and hence $(u; r)_x\lceil_y = (x + 2y)(x + 2y + 1)$ for all $r \geq 2$. 

Definition.

1. Let $\eta \in \mathbb{N}$ be the smallest number such that for all $r \geq 1$ there exist $f_1, \ldots, f_m, g, h \in K\{x, y\}$, $p_1, \ldots, p_m, q_1, \ldots, q_m \in \mathbb{N}$, $p_i \geq q_i \geq 1$ for $i = 1, \ldots, m$, with

\[ v = \sigma_y(h) \prod_{i=1}^{m} (f_i; q_i)_y \quad \text{and} \quad (u; r)_x = g \prod_{i=1}^{m} (f_i; p_i)_y \]

and $\deg_y(g[y]) \leq \eta$. Then

\[ \eta + \max\{\deg_y(v) - 1, \deg_y(V)\} \]

is called the height of $a$ with respect to the basis $B$. 

Height of $\partial$-Finite Ideals

Definition.

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$$v = \sigma_y(h) \prod_{i=1}^{m} (f_i; q_i)_y \quad \text{and} \quad (u; r)_x = g \prod_{i=1}^{m} (f_i; p_i)_y$$

and $\deg_y(g[y]) \leq \eta$. Then

$$\eta + \max\{\deg_y(v) - 1, \deg_y(V)\}$$

is called the height of $\alpha$ with respect to the basis $B$.

2. Let $\alpha \subseteq A$ be a proper $\partial$-finite ideal. The height of $\alpha$ is defined as the minimum height of $\alpha$ with respect to all admissible bases of $A/\alpha$. 
The Classic Example

Shift case: consider the bivariate sequence \( f = 1/(x^2 + y^2) \).
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- The annihilating ideal \( \mathfrak{a} \) is generated by

\[
\langle ((x + 1)^2 + y^2) \partial_x - x^2 - y^2, (x^2 + (y + 1)^2) \partial_y - x^2 - y^2 \rangle.
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- Choose \( 1 \in \mathbb{A}/\mathfrak{a} \) as the single basis element \( b_1 \).
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- $M = \frac{1}{u} U = \frac{x^2 + y^2}{(x + 1)^2 + y^2}$

- The denominator $u$ is not $y$-proper.
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- \( M = \frac{1}{u} \mathcal{U} = \frac{x^2 + y^2}{(x+1)^2 + y^2} \)

- The denominator \( u \) is not \( y \)-proper.

- Try a basis change to obtain \( \tilde{M} = 1 \).
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- The denominator $u$ is not $y$-proper.

- Try a basis change to obtain $\tilde{M} = 1$.

- However, this basis is not admissible since $1 \in \mathbb{A}/\mathfrak{a}$ is not represented by a polynomial vector.
Example

Again shift case: consider \( f(x, y) = 1/(x + y)! + 1/(x - y)! \), which is not hypergeometric but \( \partial \)-finite with holonomic rank 2.
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- Take standard monomial basis \( B = \{1, \partial_y\} \), where the basis elements \( b_1 \) and \( b_2 \) correspond to \( f(x, y) \) and \( f(x, y + 1) \).
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\[
M = \frac{1}{p} \begin{pmatrix}
\frac{x^2 - 2xy + y^2 + x - y - 1}{y - x + 1} \\
\frac{2(y + 1)}{x + y + 2} & - \frac{2y}{y - x - 1} & - \frac{x^2 + 2xy + y^2 + 3x + 3y + 1}{x + y + 2}
\end{pmatrix}
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where

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p = y^2 - x^2 + y - x + 1.
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- The denominator is not \( y \)-proper.
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- The denominator is not $y$-proper.

- On the other hand, by a basis change we can get

  $\tilde{M} = \begin{pmatrix} \frac{1}{x+y+1} & 0 \\ 0 & \frac{1}{x-y+1} \end{pmatrix}$, \quad $\tilde{N} = \begin{pmatrix} \frac{1}{x+y+1} & 0 \\ 0 & x - y \end{pmatrix}$. 
Example

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\]

- The basis elements \( \tilde{b}_1, \tilde{b}_2 \) are now \( 1/(x + y)! \) and \( 1/(x - y)! \).
Example

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- The basis elements \( \tilde{b}_1, \tilde{b}_2 \) are now \( 1/(x + y)! \) and \( 1/(x - y)! \).

- This example is proper \( \partial \)-finite.
**Theorem.** Assume that \( \mathfrak{a} \subseteq \mathbb{A} = K(x, y)[\partial_x, \partial_y] \) is proper \( \partial \)-finite w.r.t. \( y \). Let \( \varrho \) be the height of \( \mathfrak{a} \), let \( n = \dim_{K(x,y)} \mathbb{A}/\mathfrak{a} \), and let

\[
\phi = \dim_{K(x)} \{ W \in \mathbb{A}/\mathfrak{a} \mid \partial_y W = 0 \}.
\]

Then there exist \( T \in K(x)[\partial_x] \setminus \{0\} \) and \( C \in \mathbb{A} \) such that

\( T - \partial_y C \in \mathfrak{a} \) and \( \ord(T) \leq n\varrho + \phi \).
Main Theorem

**Theorem.** Assume that $\mathfrak{a} \subseteq \mathbb{A} = K(x, y)[\partial_x, \partial_y]$ is proper $\partial$-finite w.r.t. $y$. Let $\varrho$ be the height of $\mathfrak{a}$, let $n = \dim_{K(x,y)} \mathbb{A}/\mathfrak{a}$, and let

$$\phi = \dim_{K(x)} \{W \in \mathbb{A}/\mathfrak{a} \mid \partial_y W = 0\}.$$

Then there exist $T \in K(x)[\partial_x] \setminus \{0\}$ and $C \in \mathbb{A}$ such that $T - \partial_y C \in \mathfrak{a}$ and $\text{ord}(T) \leq n\varrho + \phi$.

**Note:** The quantity $\phi$ ensures solutions with nonzero telescoper. Apagodu and Zeilberger excluded rational functions as input.
Differential Case

\[ \sigma_x = \sigma_y = \text{id}, \delta_x = \frac{\partial}{\partial x}, \delta_y = \frac{\partial}{\partial y} \]
Differential Case

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\sigma_x = \sigma_y = \text{id}, \quad \delta_x = \frac{\partial}{\partial x}, \quad \delta_y = \frac{\partial}{\partial y}
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**Facts:**

1. Every $\partial$-finite ideal is proper $\partial$-finite.
Differential Case

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Facts:

1. Every \( \partial \)-finite ideal is proper \( \partial \)-finite.
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1. Every \( \partial \)-finite ideal is proper \( \partial \)-finite.
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3. In the definition of the height, we have always \( \eta = 0 \).
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2. Our bound reduces exactly to the known bound for the hyperexponential case.
3. In the definition of the height, we have always \( \eta = 0 \).

Proposition. If \( \mathfrak{a} \subseteq \mathbb{A} \) is \( \partial \)-finite, \( B \) is a basis of \( \mathbb{A}/\mathfrak{a} \) and the multiplication matrices are \( \frac{1}{u}U, \frac{1}{v}V \), then the squarefree part of \( u \) in \( K(x)[y] \) divides the squarefree part of \( v \) in \( K(x)[y] \).
Example: Sharp Family

Consider the bivariate function \( f(x, y) = p(x, y)^{-1/3} + p(x, y)^{-1/5} \)
where \( p \) is a random polynomial of \( y \)-degree 2.
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Consider the bivariate function $f(x, y) = p(x, y)^{-1/3} + p(x, y)^{-1/5}$ where $p$ is a random polynomial of $y$-degree 2.

- $n = \dim_{\mathbb{Q}(x,y)} A/a = 2$
Example: Sharp Family
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\[
\begin{align*}
\text{▶ } n &= \dim_{\mathbb{Q}(x,y)} A/\mathfrak{a} = 2 \\
\text{▶ } \phi &= \dim_{\mathbb{Q}(x)} \{ W \in A/\mathfrak{a} | \partial_y W = 0 \} = 0
\end{align*}
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Consider the bivariate function \( f(x, y) = p(x, y)^{-1/3} + p(x, y)^{-1/5} \) where \( p \) is a random polynomial of \( y \)-degree 2.

\[ \begin{align*}
\text{▶ } n &= \dim_{Q(x, y)} \mathbb{A}/\mathfrak{a} = 2 \\
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\text{▶ } B &= \{ p^{-1/3}, p^{-1/5} \}
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5. \( M = \frac{D_x(p)}{p} \begin{pmatrix} -1/3 & 0 \\ 0 & -1/5 \end{pmatrix}, \quad N = \frac{D_y(p)}{p} \begin{pmatrix} -1/3 & 0 \\ 0 & -1/5 \end{pmatrix} \)
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&\quad B = \{ p^{-1/3}, p^{-1/5} \} \\
&\quad 1 \in \mathbb{A}/\mathfrak{a} \text{ is represented by the vector } (1,1) \in \mathbb{K}(x)[y]^2. \\
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- Predicted bound \( 1 \cdot 2 + 0 = 2 \) is exact.
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\[ \text{Predicted bound } 1 \cdot 2 + 0 = 2 \text{ is exact.} \]

More generally, consider \( f = p^{e_1} + \cdots + p^{e_n} \) with random polynomial \( p \) of \( y \)-degree \( d \); our theorem produces the bound \( n(d - 1) \) which is exact for \( d = 2, \ldots, 5 \) and \( n = 1, \ldots, 4. \)
Shift Case

\[ \sigma_x(x) = x + 1, \sigma_y(y) = y + 1, \delta_x = \sigma_x - \text{id}, \delta_y = \sigma_y - \text{id} \]
Shift Case

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\sigma_x(x) = x + 1, \sigma_y(y) = y + 1, \delta_x = \sigma_x - \text{id}, \delta_y = \sigma_y - \text{id}
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Our bound does not exactly reduce to the hypergeometric case:
- It is worse: because of the additional term \( \eta = \deg_y(g_{\overline{y}}) \).
- It is better: because we take \( \partial_y \) to be the forward difference rather than the shift operator (this sometimes improves the bound by 1).

Proposition.
A \( \partial \)-finite ideal \( a \) is proper if and only if there exists an admissible basis \( B \) of \( A/\sqrt{a} \) for which the multiplication matrices \( 1_u \mathbf{U}, 1_v \mathbf{V} \) are such that \( u \) is a product of integer-linear polynomials.

Note: This implies that a function \( f(x,y) \) is proper hypergeometric if and only if its annihilating ideal is proper \( \partial \)-finite with respect to both \( x \) and \( y \).
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\sigma_x(x) = x + 1, \sigma_y(y) = y + 1, \delta_x = \sigma_x - \text{id}, \delta_y = \sigma_y - \text{id}
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\sigma_x(x) &= x + 1, \\
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**Proposition.** A \( \partial \)-finite ideal \( \alpha \) is proper if and only if there exists an admissible basis \( B \) of \( \mathbb{A}/\alpha \) for which the multiplication matrices \( \frac{1}{u} U, \frac{1}{v} V \) are such that \( u \) is a product of integer-linear polynomials.
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Example: Sharp Family

For fixed $n \geq 0$ and $\varrho$, consider the bivariate sequence

$$f(x, y) = \frac{1 + 2^y + 3^y + \cdots + n^y}{\Gamma(x + \varrho y)}$$
Example: Sharp Family

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\[ \dim_{K(x,y)}(\mathbb{A}/\mathfrak{a}) = n \]
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- $\text{dim}_{K(x,y)}(\mathbb{A}/\mathfrak{a}) = n$
- $\text{height } \varrho$
- $\phi = 0$
- Our theorem gives the bound $n\varrho$.
- The minimal telescoper is

$$T = (\partial^o_x - 1)(\partial^o_x - 2) \cdots (\partial^o_x - n).$$
Example (Mixed Case)

\[\sigma_x = \text{id}, \delta_x = \frac{\partial}{\partial x}, \sigma_y(y) = y + 1, \delta_y = \sigma_y - \text{id}.\]
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The family \( f_k(x, y) \) involving the Bessel function of the first kind

\[
f_k(x, y) = (y + 1)^{-k} J_y(x), \quad k \in \mathbb{N},
\]

is \( \partial \)-finite w.r.t. \( A = K(x, y)[\partial_x, \partial_y] \).
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- For any fixed \( k \), the annihilator \( a \) of \( f_k(x, y) \) is generated by two operators: \( a = \mathbb{A}\langle x^2 \partial_x^2 + x \partial_x + x^2 - y^2, \ldots \rangle \).
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\begin{align*}
\sigma_x &= \text{id}, \quad \delta_x = \frac{\partial}{\partial x}, \\
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- \( n = \dim_{K(x,y)}(A/\mathfrak{a}) = 2 \)
- As a basis \( B \) for \( A/\mathfrak{a} \) choose the monomials 1 and \( \partial_x \).
Example (Mixed Case)

- multiplication matrices:

\[
U = \begin{pmatrix}
0 & x^2 \\
y^2 - x^2 & -x
\end{pmatrix}
\]

\[
V = \begin{pmatrix}
xy(y + 1)^k - x^2(y + 2)^k & -x^2(y + 1)^k \\
(y + 1)^k(x^2 - y^2 - y) & x(y + 1)^{k+1} - x^2(y + 2)^k
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with denominators \( u = x^2 \) and \( v = x^2(y + 2)^k \).
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- \( \phi = 0 \)

\[\rightarrow\] Our theorem produces the bound \( 2(k + 2) \) for the order of \( T \).

\[\rightarrow\] The minimal telescoper (conjecturally) has order \( 2k + 1 \).
Conclusion and Outlook

Conclusion:

1. We presented an a priori estimate of the order of telescopers for general \( \partial \)-finite functions, generalizing some ideas of Apagodu and Zeilberger.
2. We propose a definition for “proper \( \partial \)-finite”. 
3. These results may help to speed up creative telescoping algorithms.

Outlook and further work:

▶ We pose a generalization of the Wilf-Zeilberger Conjecture.
▶ How to find “the” right basis (i.e., an admissible one which make the bound as small as possible)?
▶ Use these results in implementations.
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