Minimally Rigid Graphs

Christoph Koutschan
and the Symbolic Computation Group at RICAM

Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences

July 16, 2016
Milestones in Computer Algebra (MICA 2016)
University of Waterloo, Canada

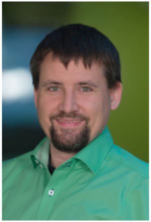
ÖAW ÖSTERREICHISCHE AKADEMIE DER WISSENSCHAFTEN

RICAM JOHANN · RADON · INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

The Symbolic Computation Group at RICAM

Matteo Gallet

Georg Grasegger



C. Koutschan

Niels Lubbes

Clemens Raab

G. Regensburger

Josef Schicho

Nelly Villamizar

Erich

Rigid and Non-Rigid Graphs

Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Rigid and Non-Rigid Graphs

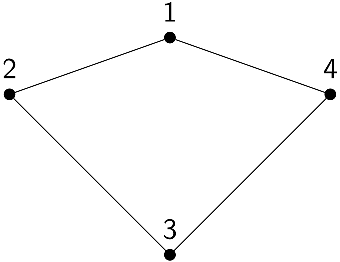
Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1,2), (2,3), (3,4), (1,4)\}$$

and

$$\lambda(1,2) = \lambda(1,4) = 0.75$$
$$\lambda(2,3) = \lambda(3,4) = 1$$

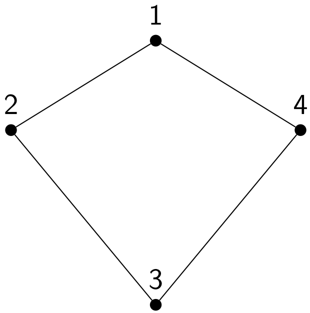


Rigid and Non-Rigid Graphs

Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1,2), (2,3), (3,4), (1,4)\}$$



and

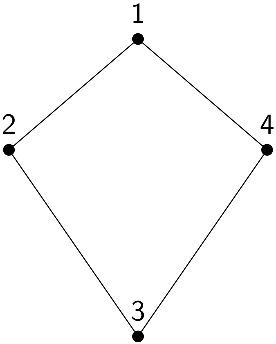
Rigid and Non-Rigid Graphs

Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1, 2), (2, 3), (3, 4), (1, 4)\}$$

and



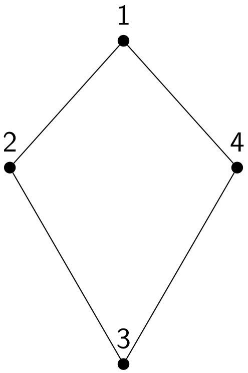
Rigid and Non-Rigid Graphs

Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1, 2), (2, 3), (3, 4), (1, 4)\}$$

and



Rigid and Non-Rigid Graphs

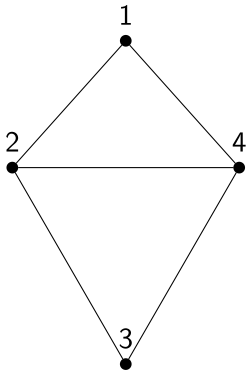
Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1,2), (2,3), (3,4), (1,4), (2,4)\}$$

and

$$\lambda(1,2) = \lambda(1,4) = 0.75$$
$$\lambda(2,3) = \lambda(3,4) = 1$$
$$\lambda(2,4) = 1$$



Rigid and Non-Rigid Graphs

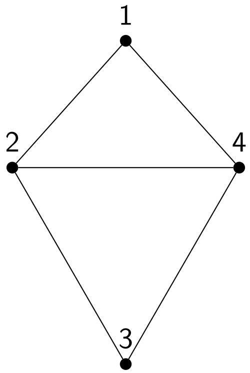
Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1,2), (2,3), (3,4), (1,4), (2,4)\}$$

and

$$\lambda(1,2) = \lambda(1,4) = 0.75$$
$$\lambda(2,3) = \lambda(3,4) = 1$$
$$\lambda(2,4) = 1$$



Definition: G is called **rigid**, if there is exactly one way, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Rigid and Non-Rigid Graphs

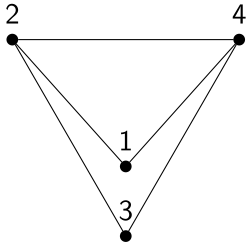
Notation: Let $G = (V, E)$ be a graph, and let $\lambda \colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1,2), (2,3), (3,4), (1,4), (2,4)\}$$

and

$$\lambda(1,2) = \lambda(1,4) = 0.75$$
$$\lambda(2,3) = \lambda(3,4) = 1$$
$$\lambda(2,4) = 1$$



Definition: G is called **rigid**, if there is exactly one way, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Rigid and Non-Rigid Graphs

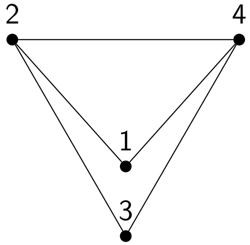
Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{(1,2), (2,3), (3,4), (1,4), (2,4)\}$$

and

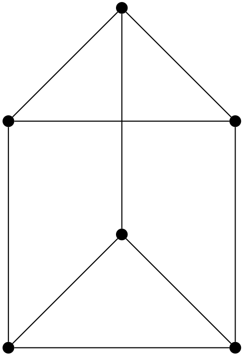
$$\lambda(1,2) = \lambda(1,4) = 0.75$$
$$\lambda(2,3) = \lambda(3,4) = 1$$
$$\lambda(2,4) = 1$$



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

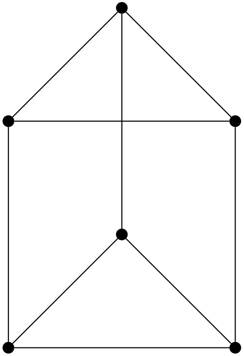
Is this graph rigid?



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

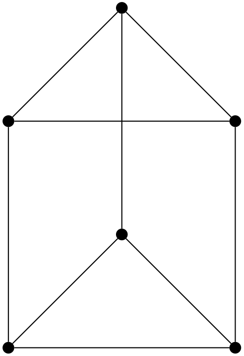
Is this graph rigid? No!



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

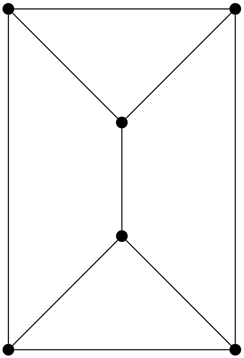
Is this graph rigid? No!



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

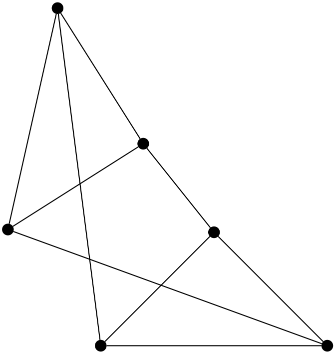
Is this graph rigid? Yes!



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

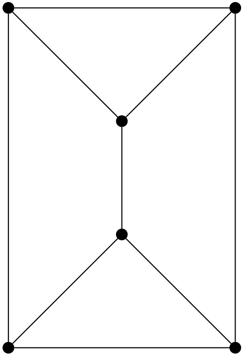
Is this graph rigid? Yes!



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

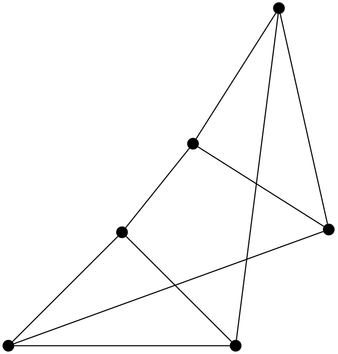
Is this graph rigid? Yes!



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

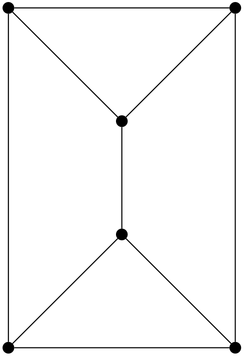
Is this graph rigid? Yes!



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given.

Three-Prism Graph

Is this graph rigid? Yes!



Definition: G is called **rigid**, if there are only finitely many ways, modulo rotations and translations, how it can be embedded in the plane, when the edge lengths λ are given generically.

Laman Graphs

Definition: A rigid graph G is called **minimally rigid** (or **Laman**) if removing any single edge makes G non-rigid.

Laman Graphs

Definition: A rigid graph G is called **minimally rigid** (or **Laman**) if removing any single edge makes G non-rigid.

Question: When can we expect rigidity?

- # unknowns (coordinates of the vertices): $2 \cdot |V|$
- # equations: $|E|$
- dim(direct isometries): 3

$\longrightarrow$ Hence, $|E| \geqslant 2|V| - 3$ is a necessary condition for rigidity.

Laman Graphs

Definition: A rigid graph G is called **minimally rigid** (or **Laman**) if removing any single edge makes G non-rigid.

Question: When can we expect rigidity?

- $\#$ unknowns (coordinates of the vertices): $2 \cdot |V|$
- $\#$ equations: $|E|$
- dim(direct isometries): 3

$\longrightarrow$ Hence, $|E| \geqslant 2|V| - 3$ is a necessary condition for rigidity.

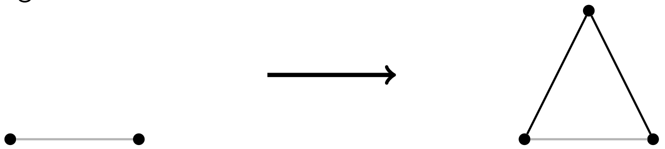
Theorem. (Laman, 1970)
A graph $G = (V, E)$ is minimally rigid if and only if

1. $|E| = 2|V| - 3$,
2. $|E'| \leqslant 2|V'| - 3$ for each subgraph $G' = (V', E')$ of G.

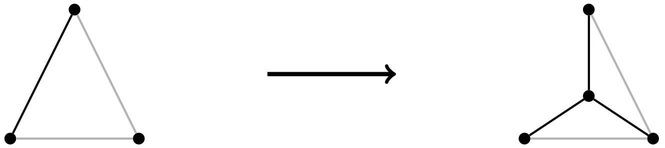
Henneberg Steps

Theorem. (Henneberg, 1911) The following two simple rules allow to construct any Laman graph, starting from a single edge.

Henneberg step (type I): add a new vertex and connect it to two existing vertices.



Henneberg step (type II): select three vertices of the graph, at least two of which are connected by an edge e; delete the edge e; add a new vertex and connect it to the three chosen ones.



Henneberg Steps

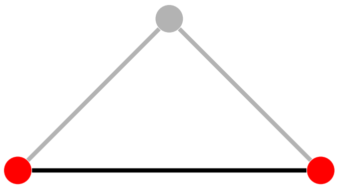
Construct the three-prism graph by Henneberg steps:

Henneberg Steps

Construct the three-prism graph by Henneberg steps:

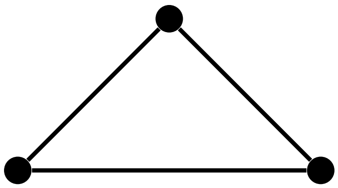
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



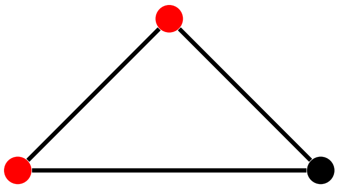
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



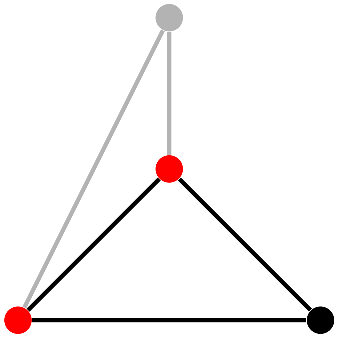
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



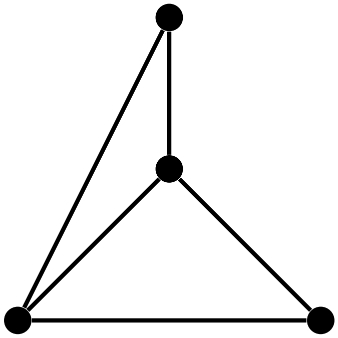
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



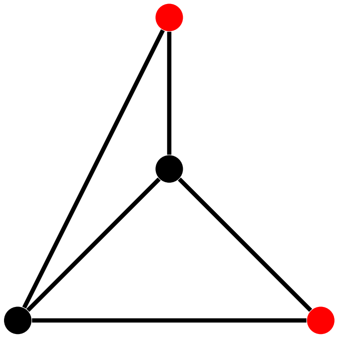
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



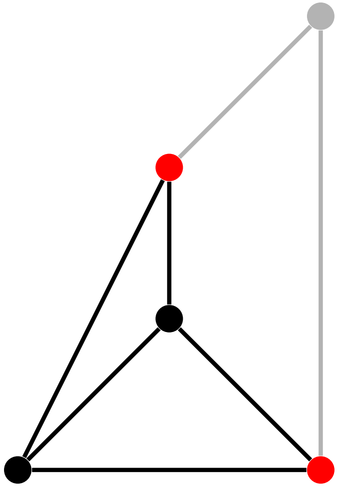
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



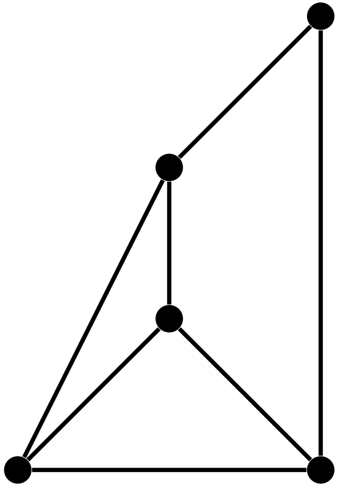
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



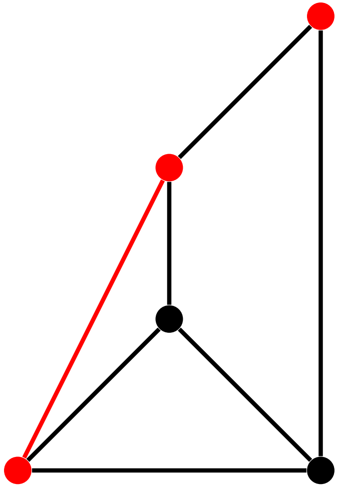
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



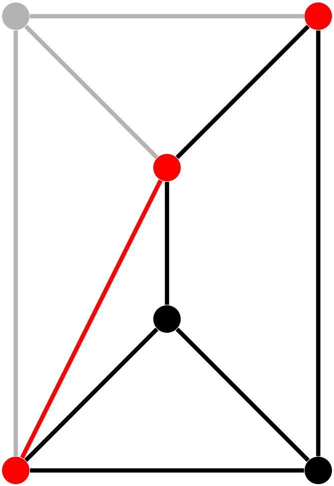
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



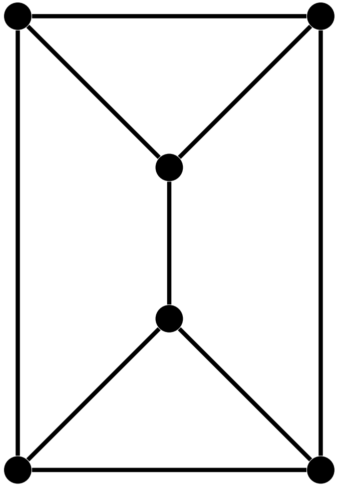
Henneberg Steps

Construct the three-prism graph by Henneberg steps:



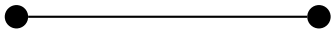
Henneberg Steps

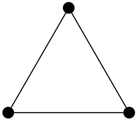
Construct the three-prism graph by Henneberg steps:

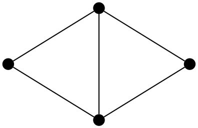


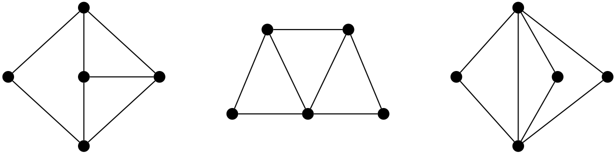
Some Laman Graphs

All Laman graphs with $2 \leqslant n \leqslant 5$ vertices:

$n = 2$: 

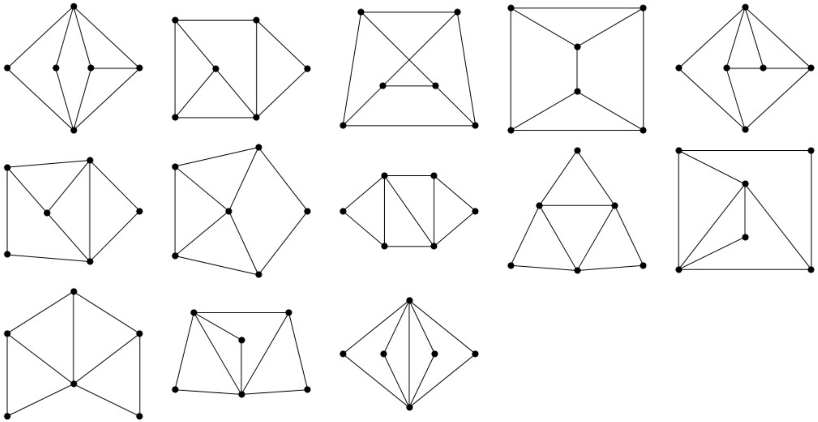
$n = 3$: 

$n = 4$: 

$n = 5$: 

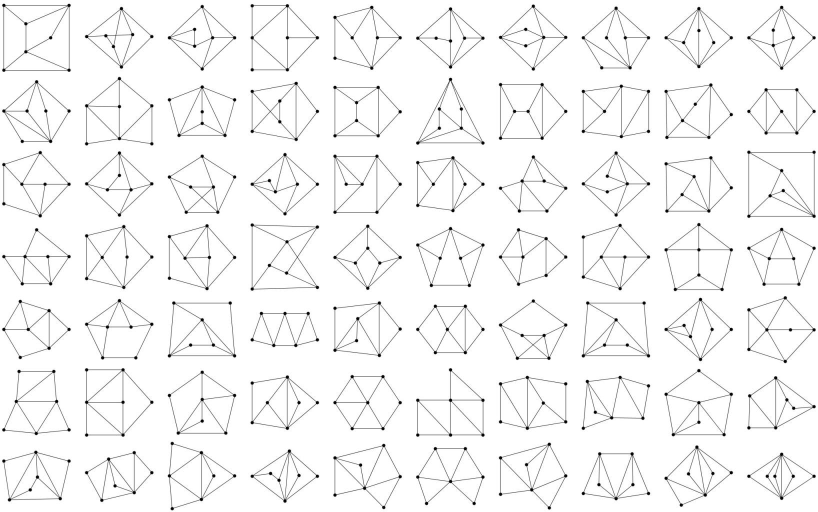
Some Laman Graphs

All Laman graphs with 6 vertices:



Some Laman Graphs

There are 70 Laman graphs with 7 vertices:



Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70

Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70

Question: How to compute these numbers?

Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70

Question: How to compute these numbers?

- Start with all Laman graphs on n vertices.
- Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- Remove duplicates (graphs that are **isomorphic**).

Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70
8	608

Question: How to compute these numbers?

- Start with all Laman graphs on n vertices.
- Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- Remove duplicates (graphs that are **isomorphic**).

Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70
8	608

Question: How to compute these numbers?

- Start with all Laman graphs on n vertices.
- Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- Remove duplicates (graphs that are **isomorphic**).

A227117 Number of minimally rigid graphs on n vertices.

1, 1, 1, 1, 3, 13, 70, 609 (list; graph; refs; listen; history; text; int

OFFSET 1,5

COMMENTS All the minimally rigid graphs on n vertices
graphs on n-1 vertices by use of two types
constructions. In the first type a new ve
edges are added connecting the new vertex
of the graph. In the second type of const
which are connected by an edge are selecte
edge between v_1 and v_2 is deleted. A new
as the edges (v_1,w), (v_2,w),and (v_3,w).
one to the number of vertices and two to tl

Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70
8	608

Question: How to compute these numbers?

- Start with all Laman graphs on n vertices.
- Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- Remove duplicates (graphs that are **isomorphic**).

Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70
8	608

Question: How to compute these numbers?

- Start with all Laman graphs on n vertices.
- Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- Remove duplicates (graphs that are **isomorphic**).
- Define a **normal form** for graphs.
- Encode graphs as integers.

Enumeration of Laman graphs

Number of Laman graphs with n vertices:

n	#
2	1
3	1
4	1
5	3
6	13
7	70
8	608
9	7222
10	110132
11	2039273
12	44176717

Question: How to compute these numbers?

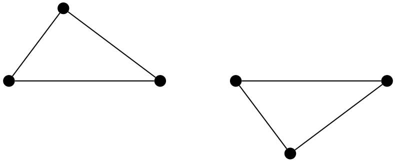
- Start with all Laman graphs on n vertices.
- Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- Remove duplicates (graphs that are **isomorphic**).
- Define a **normal form** for graphs.
- Encode graphs as integers.

Number of Embeddings

Laman graph with 3 vertices: ?

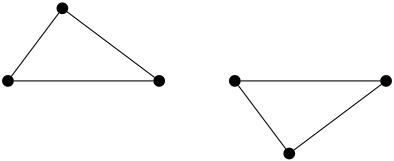
Number of Embeddings

Laman graph with 3 vertices: 2 embeddings



Number of Embeddings

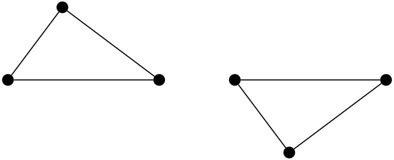
Laman graph with 3 vertices: 2 embeddings



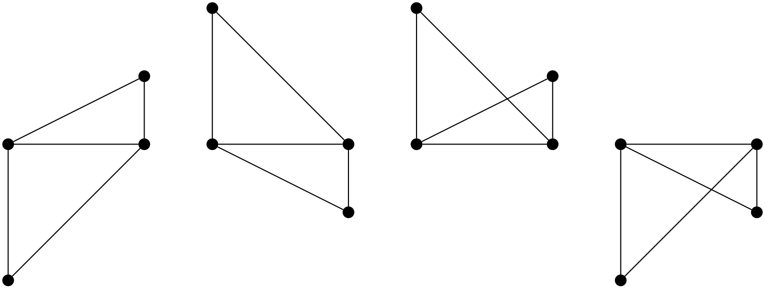
Laman graph with 4 vertices: ?

Number of Embeddings

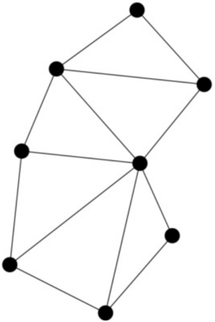
Laman graph with 3 vertices: 2 embeddings



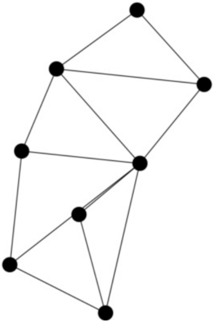
Laman graph with 4 vertices: 4 embeddings



Embeddings of H1 Laman Graphs



Embeddings of H1 Laman Graphs



Embeddings of H1 Laman Graphs

Definition: An **H1 Laman graph** is a Laman graph that can be obtained by applying a sequence of Henneberg steps of type **I**, starting with the graph $(\{1,2\},\{\{1,2\}\})$.

Embeddings of H1 Laman Graphs

Definition: An **H1 Laman graph** is a Laman graph that can be obtained by applying a sequence of Henneberg steps of type I, starting with the graph $(\{1,2\},\{\{1,2\}\})$.

Number of embeddings:

- Let $G=(V,E)$ be an H1 Laman graph.
- Fix a realizable labeling $\lambda\colon E\to\mathbb{R}_{>0}$.
- Fix the positions of the first two vertices, respecting $\lambda(1,2)$.
- Each vertex that is added can be put at two different positions.

$\longrightarrow$ There are $2^{|V|-2}$ embeddings.

Embeddings of H1 Laman Graphs

Definition: An **H1 Laman graph** is a Laman graph that can be obtained by applying a sequence of Henneberg steps of type I, starting with the graph $(\{1,2\},\{\{1,2\}\})$.

Number of embeddings:

- Let $G = (V, E)$ be an H1 Laman graph.
- Fix a realizable labeling $\lambda\colon E \to \mathbb{R}_{>0}$.
- Fix the positions of the first two vertices, respecting $\lambda(1,2)$.
- Each vertex that is added can be put at two different positions.

$\longrightarrow$ There are $2^{|V|-2}$ embeddings.

Definition: Let G be a Laman graph; the **Laman number** $L(G)$ is the number of embeddings of G, for a generic labeling λ.

More Complex Laman Graphs

Question: What about Laman graphs that are not H1?

More Complex Laman Graphs

Question: What about Laman graphs that are not H1?

Previous work:

- Borcea, Streinu (2004): $L(G) \leqslant \binom{2n-4}{n-2}$ where $n = |V|$.
- Steffens, Theobald (2010): $L(G) < 4^{n-2}$.
- Emiris, Despotakis, Psarros, Tsigaridas, Varvitsiotis (2009, 2012, 2013, 2014):

n	3	4	5	6	7	8	9	10
lower	2	4	8	24	48	96	288	576
upper	2	4	8	24	64	128	512	2048

More Complex Laman Graphs

Question: What about Laman graphs that are not H1?

Set up a system of equations:

- Let (x_v, y_v) be the coordinates of vertex v.
- For $(u, v) \in E$:

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

More Complex Laman Graphs

Question: What about Laman graphs that are not H1?

Set up a system of equations:

- Let (x_v, y_v) be the coordinates of vertex v.
- For $(u, v) \in E$:

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

Convention: From now on we work over the complex numbers:

- $\lambda \colon E \to \mathbb{C}$
- $(x_v, y_v) \in \mathbb{C}^2$

Gröbner Basis Approach

Compute a Gröbner basis of

$$\{(x_u - x_v)^2 + (y_u - y_v)^2 - \lambda(u,v)^2 \mid (u,v) \in E\}$$
$$\cup \{x_1, y_1, x_2\}.$$

Gröbner Basis Approach

Compute a Gröbner basis of

$$\{(x_u - x_v)^2 + (y_u - y_v)^2 - \lambda(u,v)^2 \mid (u,v) \in E\} \\ \cup \{x_1, y_1, x_2\}.$$

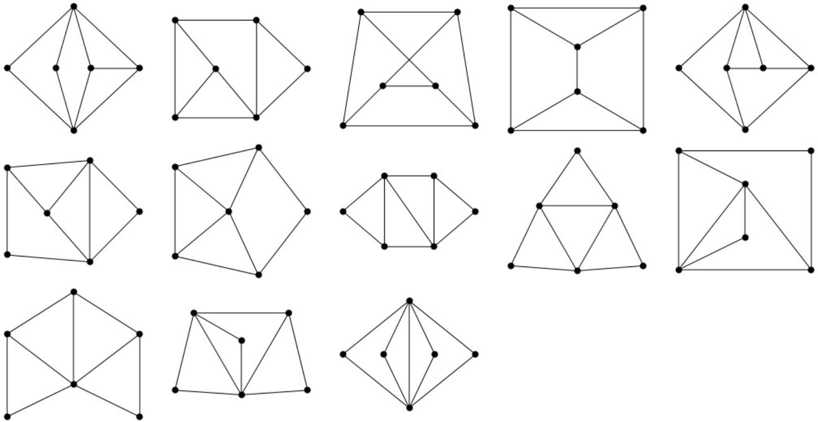
Some tricks for speed-up:

- Preprocessing: undo Henneberg steps of type I.
- Assign random values to $\lambda(u,v)$.
- Discard vertices that are "fixed" by a triangle.
- Compute modulo a prime number.

$\longrightarrow$ This way, we were able to compute the Laman numbers of all Laman graphs with $\leqslant 9$ vertices.

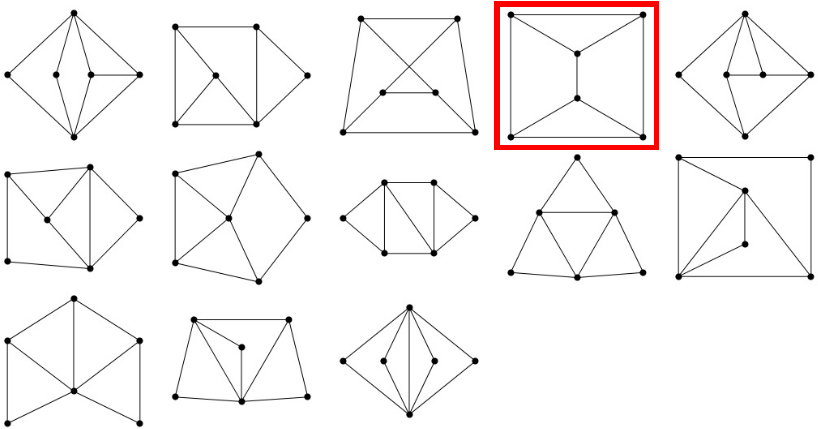
Laman Numbers

All but one Laman graphs with 6 vertices have Laman number 16.



Laman Numbers

All but one Laman graphs with 6 vertices have Laman number 16.



The only exception is the three-prism graph with $L(\blacksquare) = 24$.

Laman Number as Degree

Recall: For each edge $(u,v) \in E$ we get an equation

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u,v)^2.$$

Idea: $L(G)$ is given by the "degree" of the map

$$f_G \colon \mathbb{C}^V \times \mathbb{C}^V \to \mathbb{C}^E,$$
$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \Big((x_u - x_v)^2 + (y_u - y_v)^2 \Big)_{(u,v)\in E}$$

i.e., by the number how often a generic $\big(\lambda(u,v)\big)_{(u,v)\in E}$ is hit by the map f_G (modulo translations and rotations).

Laman Number as Degree

Recall: For each edge $(u, v) \in E$ we get an equation

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

Idea: $L(G)$ is given by the "degree" of the map

$$f_G \colon \mathbb{C}^V \times \mathbb{C}^V \to \mathbb{C}^E,$$
$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \Big((x_u - x_v)^2 + (y_u - y_v)^2\Big)_{(u,v)\in E}$$

i.e., by the number how often a generic $\big(\lambda(u, v)\big)_{(u,v)\in E}$ is hit by the map f_G (modulo translations and rotations).

In order to apply methods from **algebraic geometry**, we want:

- to work in projective space;
- f_G then should be a homogeneous map.

Laman Number as Degree

"Homogenize" the map f_G by a change of coordinates:

$$(x_u - x_v)^2 + (y_u - y_v)^2 =$$

$$\big((x_u - x_v) + \mathrm{i}(y_u - y_v)\big) \cdot \big((x_u - x_v) - \mathrm{i}(y_u - y_v)\big) =$$

$$\big(\underbrace{(x_u + \mathrm{i}y_u)}_{\substack{\downarrow \\ x_u}} - \underbrace{(x_v + \mathrm{i}y_v)}_{\substack{\downarrow \\ x_v}}\big) \cdot \big(\underbrace{(x_u - \mathrm{i}y_u)}_{\substack{\downarrow \\ y_u}} - \underbrace{(x_v - \mathrm{i}y_v)}_{\substack{\downarrow \\ y_v}}\big)$$

Laman Number as Degree

"Homogenize" the map f_G by a change of coordinates:

$$(x_u - x_v)^2 + (y_u - y_v)^2 =$$

$$\big((x_u - x_v) + \mathrm{i}(y_u - y_v)\big) \cdot \big((x_u - x_v) - \mathrm{i}(y_u - y_v)\big) =$$

$$\big(\underbrace{(x_u + \mathrm{i}y_u)}_{\substack{\downarrow \\ x_u}} - \underbrace{(x_v + \mathrm{i}y_v)}_{\substack{\downarrow \\ x_v}}\big) \cdot \big(\underbrace{(x_u - \mathrm{i}y_u)}_{\substack{\downarrow \\ y_u}} - \underbrace{(x_v - \mathrm{i}y_v)}_{\substack{\downarrow \\ y_v}}\big)$$

Hence our map becomes

$$f_G \colon \mathbb{C}^V \times \mathbb{C}^V \to \mathbb{C}^E,$$

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \big((x_u - x_v) \cdot (y_u - y_v)\big)_{(u,v) \in E}$$

Laman Number as Degree

In order to mod out translations and rotations, we

- move one vertex to the origin (for each connected component),
- fix the position of another vertex (using projective space $\mathbb{P}$),
- and fix the length of one edge (again, by projectivization).

Laman Number as Degree

In order to mod out translations and rotations, we

- move one vertex to the origin (for each connected component),
- fix the position of another vertex (using projective space $\mathbb{P}$),
- and fix the length of one edge (again, by projectivization).

Define $\mathrm{Comp}(G)$, the set of connected components of a graph G,

$$\mathrm{Comp}(G) := \{C \subseteq V \mid C \text{ is a connected component of } G\}.$$

Laman Number as Degree

In order to mod out translations and rotations, we

- move one vertex to the origin (for each connected component),
- fix the position of another vertex (using projective space $\mathbb{P}$),
- and fix the length of one edge (again, by projectivization).

Define $\mathrm{Comp}(G)$, the set of connected components of a graph G,

$$\mathrm{Comp}(G) := \{C \subseteq V \mid C \text{ is a connected component of } G\}.$$

Let

$$U(G) := \left\langle \left(\chi_C(v)\right)_{v \in V} \,\middle|\, C \in \mathrm{Comp}(G) \right\rangle \subseteq \mathbb{C}^V,$$

where $\chi_C(v)$ is 1 if $v \in C$ and 0 otherwise.

Laman Number as Degree

In order to mod out translations and rotations, we

- move one vertex to the origin (for each connected component),
- fix the position of another vertex (using projective space $\mathbb{P}$),
- and fix the length of one edge (again, by projectivization).

Define $\mathrm{Comp}(G)$, the set of connected components of a graph G,

$$\mathrm{Comp}(G) := \{C \subseteq V \mid C \text{ is a connected component of } G\}.$$

Let

$$U(G) := \left\langle \left(\chi_C(v)\right)_{v \in V} \,\middle|\, C \in \mathrm{Comp}(G) \right\rangle \subseteq \mathbb{C}^V,$$

where $\chi_C(v)$ is 1 if $v \in C$ and 0 otherwise.

Proposition: The Laman number $L(G)$ is given by the degree of

$$f_G \colon \mathbb{P}\left(\mathbb{C}^V / U(G)\right) \times \mathbb{P}\left(\mathbb{C}^V / U(G)\right) \to \mathbb{P}^{|E|-1},$$
$$(x_1 : \ldots : x_n), (y_1 : \ldots : y_n) \mapsto \left((x_u - x_v) \cdot (y_u - y_v)\right)_{(u,v) \in E}$$

Laman Number as Degree

Our strategy:

1. Algebraic geometry methods allow to compute the degree of the map f_G.
2. Underlying proofs require a lot of work (under construction).
3. In the end we obtain a "combinatorial" algorithm to compute the Laman number $L(G)$.

We first need to introduce two operations on graphs:

- complement
- quotient

Graph Complement

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph complement** $G \setminus E'$ is defined as

$$G \setminus E' := (V, E \setminus E').$$

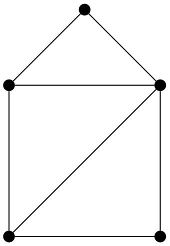
Graph Complement

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph complement** $G \setminus E'$ is defined as

$$G \setminus E' := (V, E \setminus E').$$

Example:



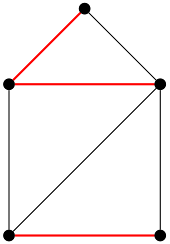
Graph Complement

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph complement** $G \setminus E'$ is defined as

$$G \setminus E' := (V, E \setminus E').$$

Example:



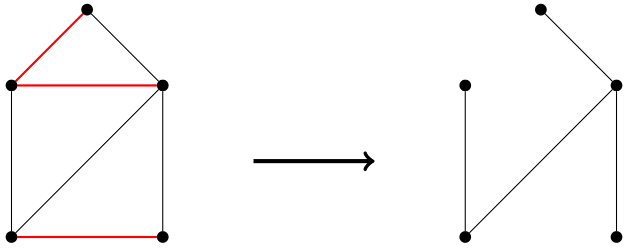
Graph Complement

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph complement** $G \setminus E'$ is defined as

$$G \setminus E' := (V, E \setminus E').$$

Example:



Graph Quotient

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph quotient** $G \,/\, E'$ is constructed as follows:

- Connected components of (V, E') become vertices of $G \,/\, E'$.
- Each edge in $E \setminus E'$ induces an edge of $G \,/\, E'$.

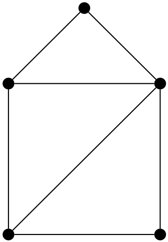
Graph Quotient

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph quotient** $G \,/\, E'$ is constructed as follows:

- Connected components of (V, E') become vertices of $G \,/\, E'$.
- Each edge in $E \setminus E'$ induces an edge of $G \,/\, E'$.

Example:



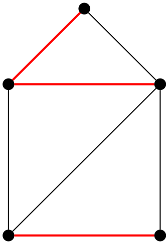
Graph Quotient

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph quotient** $G \,/\, E'$ is constructed as follows:

- Connected components of (V, E') become vertices of $G \,/\, E'$.
- Each edge in $E \setminus E'$ induces an edge of $G \,/\, E'$.

Example:



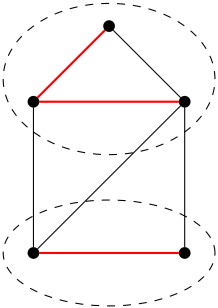
Graph Quotient

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph quotient** $G \,/\, E'$ is constructed as follows:

- Connected components of (V, E') become vertices of $G \,/\, E'$.
- Each edge in $E \setminus E'$ induces an edge of $G \,/\, E'$.

Example:



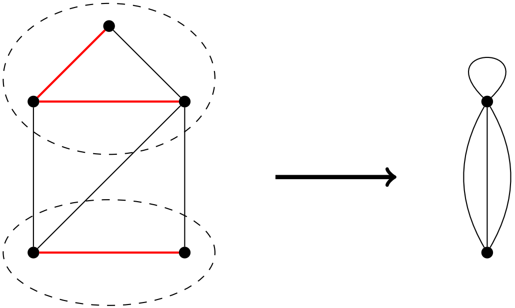
Graph Quotient

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph quotient** $G \,/\, E'$ is constructed as follows:

- Connected components of (V, E') become vertices of $G \,/\, E'$.
- Each edge in $E \setminus E'$ induces an edge of $G \,/\, E'$.

Example:



Bigraphs

Definition: A **bigraph** $B = (G, H, \varphi)$ is a pair of graphs $G = (V, E)$ and $H = (W, F)$, together with a bijective map

$$\varphi\colon E \to F.$$

Bigraphs

Definition: A **bigraph** $B = (G, H, \varphi)$ is a pair of graphs $G = (V, E)$ and $H = (W, F)$, together with a bijective map

$$\varphi\colon E \to F.$$

We define the following two operations on bigraphs:
For $E' \subseteq E$ let

- $B \setminus E' := \big(G \setminus E', H \,/\, \varphi(E')\big)$
- $B \,/\, E' := \big(G \,/\, E', H \setminus \varphi(E')\big)$

Bigraphs

Definition: A **bigraph** $B = (G, H, \varphi)$ is a pair of graphs $G = (V, E)$ and $H = (W, F)$, together with a bijective map

$$\varphi \colon E \to F.$$

We define the following two operations on bigraphs:
For $E' \subseteq E$ let

- $B \setminus E' := \big(G \setminus E', H \,/\, \varphi(E')\big)$
- $B \,/\, E' := \big(G \,/\, E', H \setminus \varphi(E')\big)$

We define the rational map f_B for a bigraph:

$$f_B \colon \mathbb{P}\big(\mathbb{C}^V / U(G)\big) \times \mathbb{P}\big(\mathbb{C}^W / U(H)\big) \to \mathbb{P}^{|E|-1},$$
$$(x_1 : \ldots : x_n), (y_1 : \ldots : y_n) \mapsto \big((x_u - x_v) \cdot (y_t - y_w)\big)_{(u,v) \in E}$$

where $(t, w) \in F$ with $(t, w) = \varphi(u, v)$.

The Algorithm

Theorem. Let $B = (G, H, \varphi)$ be a bigraph with $G = (V, E)$ and $H = (W, F)$. Choose $e \in E$. Then

$$L(B) = L(B \setminus \{e\}) + L(B \,/\{e\}) + \sum_{\substack{M \cup N = E \\ M \cap N = \{e\}}} L(B \setminus M) \cdot L(B \,/\, N).$$

Initial conditions:

- $L(G) = L(G, G, \mathrm{id})$
- $L(B) = 0$ if G or H contains a loop.
- $L(B) = 0$ if $|V| - |\operatorname{Comp}(G)| + |W| - |\operatorname{Comp}(H)| \neq |E| + 1$.
- $L(B) = 1$ if $|E| = |F| = 1$ (no loops).

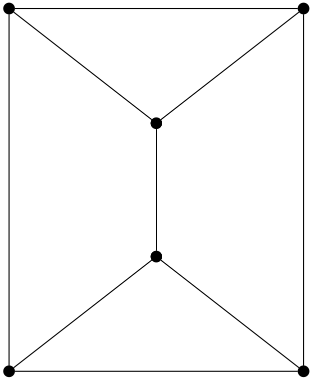
Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

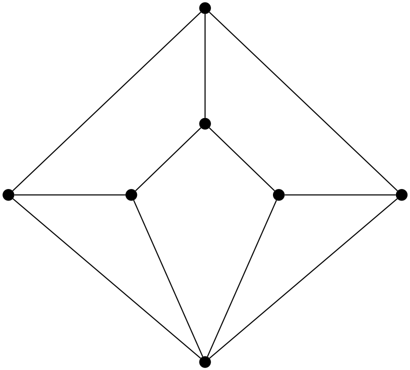
n	6
$\#$	24



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

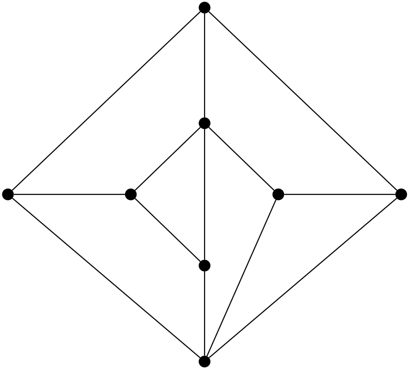
n	6	7
$\#$	24	56



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

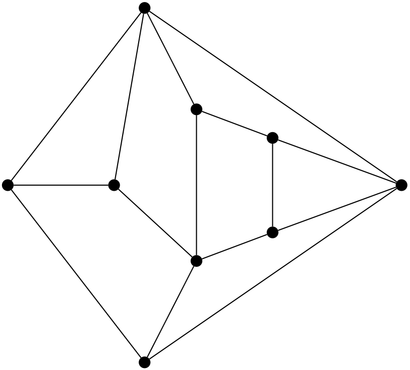
n	6	7	8
#	24	56	136



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

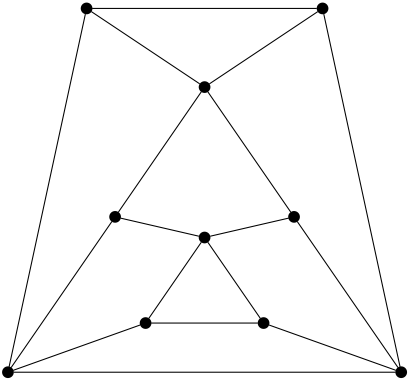
n	6	7	8	9
#	24	56	136	344



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

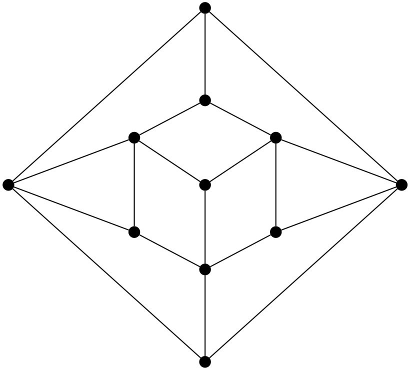
n	6	7	8	9	10
#	24	56	136	344	880



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

n	6	7	8	9	10	11
#	24	56	136	344	880	2288



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

n	6	7	8	9	10	11
#	24	56	136	344	880	2288
$\frac{1}{8}$	3	7	17	43	110	286

A114589 Number of hill-free Dyck paths of semilength n+3 and having no peaks at even levels (a hill in a Dyck path is a peak at level 1). 1

1, 1, 3, 7, 17, 43, 110, 286, 753, 2003, 5376, 14540, 39589, 108427, 298512, 825664, 2293271, 6393539, 17885835, 50191175, 141247519, 398537101, 1127203038, 3195229662, 9076078057, 25830193513, 73643406563, 210312889095 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS Column 0 of A114588. The number of hill-free Dyck paths having no peaks at odd level are given by the Riordan numbers (A005043).
Contribution from Paul Barry, Jul 05 2009: (Start)
The sequence 1,0,0,1,1,3,7,... has g.f. ((1+x)(1+2x)-sqrt((1+x)(1-3x))/(2x(2+2x+x^2)). It is
the inverse binomial transform of A035929(n+1). (End)

Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

n	6	7	8	9	10	11
$\#$	24	56	136	344	880	2288
$\frac{1}{8}$	3	7	17	43	110	286

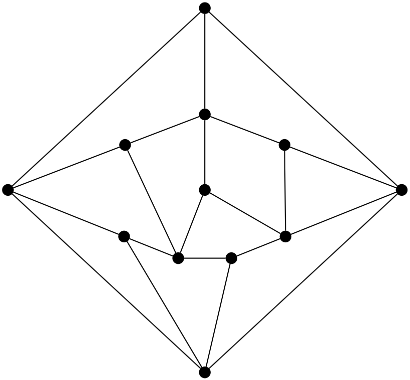
Conjecture: For each $n \geqslant 6$ there is a unique Laman graph $G_n^{\max}$ with n vertices that has the maximal number of embeddings; moreover $G_n^{\max}$ has the following properties:

- $G_n^{\max}$ is a planar graph.
- $G_n^{\max}$ has exactly 6 vertices with valency 3.
- $G_n^{\max}$ has exactly $n-6$ vertices with valency 4.
- $G_n^{\max}$ has exactly 2 triangles and $n-3$ quadrilaterals.
- The two triangles do not share an edge.

Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

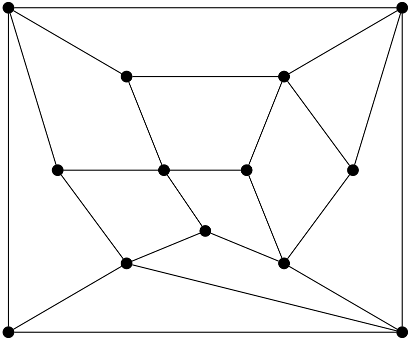
n	6	7	8	9	10	11	12
$\#$	24	56	136	344	880	2288	5952
$\frac{1}{8}$	3	7	17	43	110	286	744



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

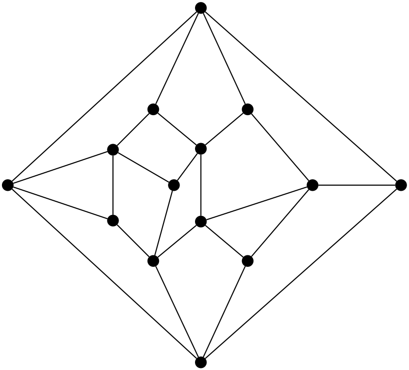
n	6	7	8	9	10	11	12	13
$\#$	24	56	136	344	880	2288	5952	15056
$\frac{1}{8}$	3	7	17	43	110	286	744	1882



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

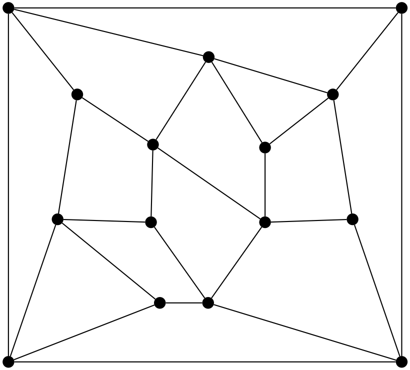
n	6	7	8	9	10	11	12	13	14
$\#$	24	56	136	344	880	2288	5952	15056	39696
$\frac{1}{8}$	3	7	17	43	110	286	744	1882	4962



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

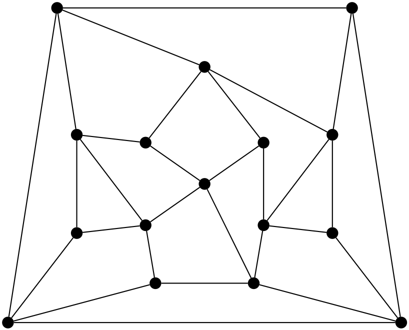
n	6	7	8	9	10	11	12	13	14	15
$\#$	24	56	136	344	880	2288	5952	15056	39696	105384
$\frac{1}{8}$	3	7	17	43	110	286	744	1882	4962	13173



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

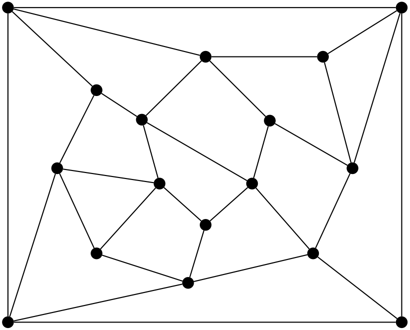
n	6	7	8	9	10	11	12	13	14	15	16
#	24	56	136	344	880	2288	5952	15056	39696	105384	277864
$\frac{1}{8}$	3	7	17	43	110	286	744	1882	4962	13173	34733



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

n	6	7	8	9	10	11	12	13	14	15	16	17
#	24	56	136	344	880	2288	5952	15056	39696	105384	277864	731336
$\frac{1}{8}$	3	7	17	43	110	286	744	1882	4962	13173	34733	91417



Laman Graphs with Maximal Embeddings

Question: Among all Laman graphs with n vertices, which one has the largest number of embeddings?

n	6	7	8	9	10	11	12	13	14	15	16	17
#	24	56	136	344	880	2288	5952	15056	39696	105384	277864	731336
$\frac{1}{8}$	3	7	17	43	110	286	744	1882	4962	13173	34733	91417

A114589 Number of hill-free Dyck paths of semilength n+3 and having no peaks at even levels (a hill in a Dyck path is a peak at level 1). 1

1, 1, 3, 7, 17, 43, 110, 286, 753, 2003, 5376, 14540, 39589, 108427, 298512, 825664, 2293271, 6393539, 17885835, 50191175, 141247519, 398537101, 1127203038, 3195229662, 9076078057, 25830193513, 73643406563, 210312889095 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS Column 0 of A114588. The number of hill-free Dyck paths having no peaks at odd level are given by the Riordan numbers (A005043).
Contribution from Paul Barry, Jul 05 2009: (Start)
The sequence 1,0,0,1,1,3,7,... has g.f. ((1+x)(1+2x)-sqrt((1+x)(1-3x))/(2x(2+2x+x^2)). It is
the inverse binomial transform of A035929(n+1). (End)

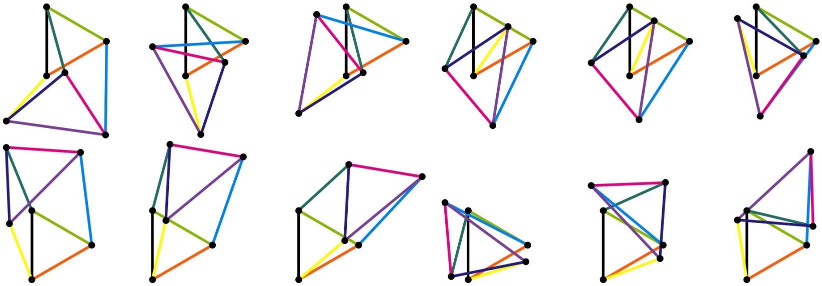
Real embeddings

Question: Given a Laman graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $L(G)$ **real** embeddings?

Real embeddings

Question: Given a Laman graph G, can we find a **real** labeling $\lambda \colon E \to \mathbb{R}_{>0}$ such that there exist $L(G)$ **real** embeddings?

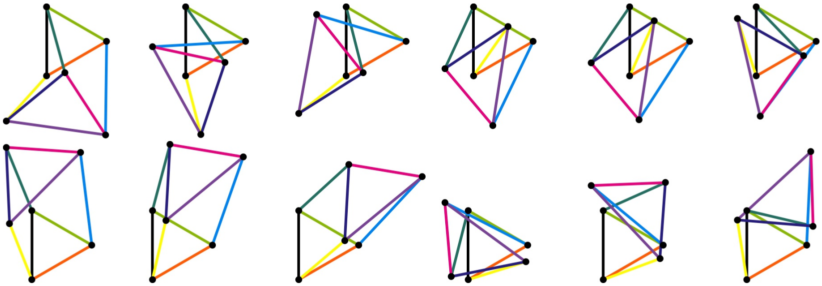
Answer: Maybe. At least sometimes:



Real embeddings

Question: Given a Laman graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $L(G)$ **real** embeddings?

Answer: Maybe. At least sometimes:

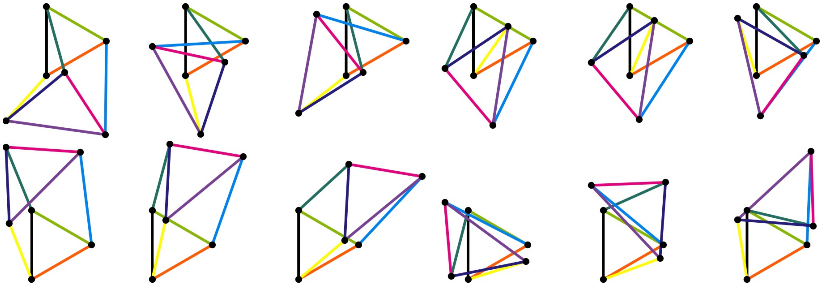


Question: Can we move (in 3D) from any one position to any other, without “disassembling” the graph?

Real embeddings

Question: Given a Laman graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $L(G)$ **real** embeddings?

Answer: Maybe. At least sometimes:



Question: Can we move (in 3D) from any one position to any other, without "disassembling" the graph?

Answer: Show movie.