Guessing with Little Data

Manuel Kauers and Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

13 October 2022
AriC Seminar, ENS Lyon, France

Motivating Example: OEIS A172671

$a_n := \#\big\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \;\big|\; \text{row sums} = 2,\ \text{column sums} = n\big\}$

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- Recurrence? Efficient method for computing next terms?
- Nature of the generating function?

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1,\ldots,c_{21} \leqslant n \\ +\text{lin. constraints}}} \binom{3n}{c_1, c_2, \ldots, c_{21}}$$

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Answer 2: Interpret arrays as walks in $\mathbb{Z}^6_{\geqslant 0}$ ending at $(n, \dots, n)$:

$$\sum_{n=0}^{\infty} a_n x^n = \operatorname{Diag}\left(\frac{1}{1 - x_1x_2 - x_1x_3 - \cdots - x_5x_6 - x_1^2 - \cdots - x_6^2} \right)$$

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How to construct this recurrence / ODE?

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Ansatz: $c_0 a_n + c_1 a_{n+1} + \cdots + c_r a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

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Ansatz: $\big(c_{00} + c_{01}n\big)a_n + \cdots + \big(c_{r0} + c_{r1}n\big)a_{n+r} = 0$
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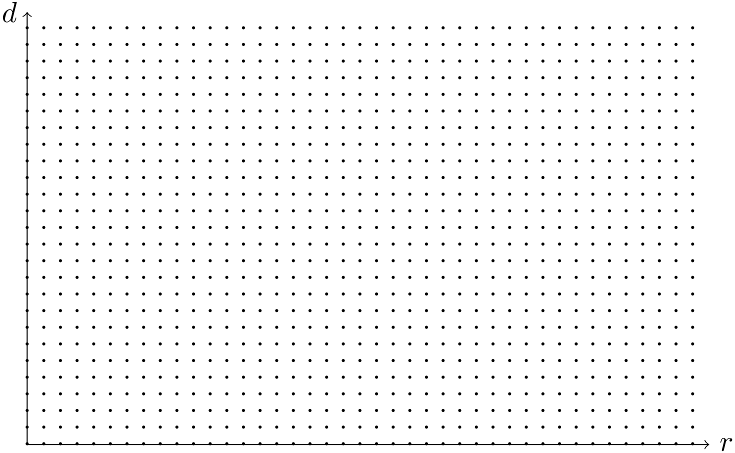
- To trust the result, we need $N-r+1\geqslant(r+1)(d+1)$.
- For sequence A172671 we know only 33 terms, i.e. we can try $(r,d)=(1,15),(2,9),(3,6),(4,5),(6,3),(7,2),(10,1),(16,0)$.

Trading Order vs. Degree



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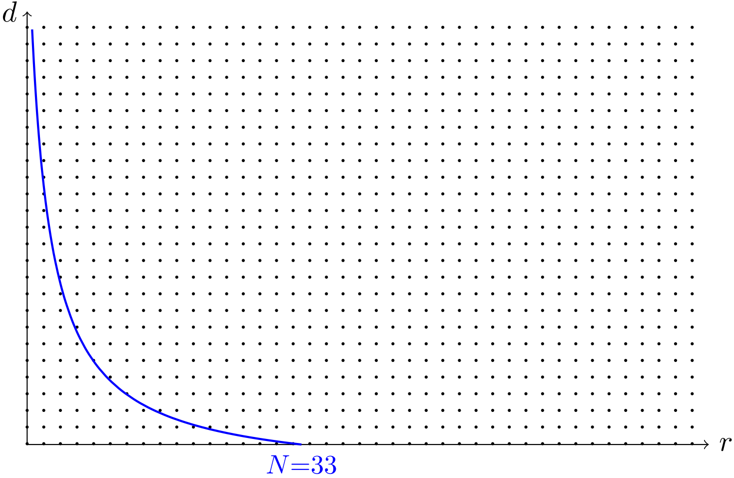

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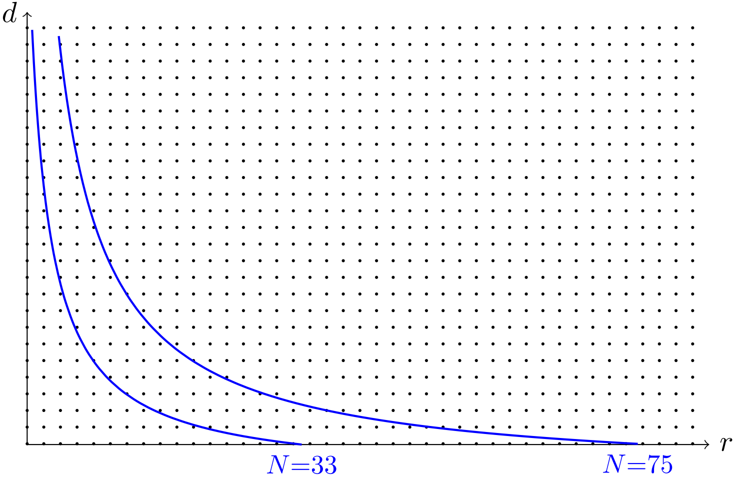

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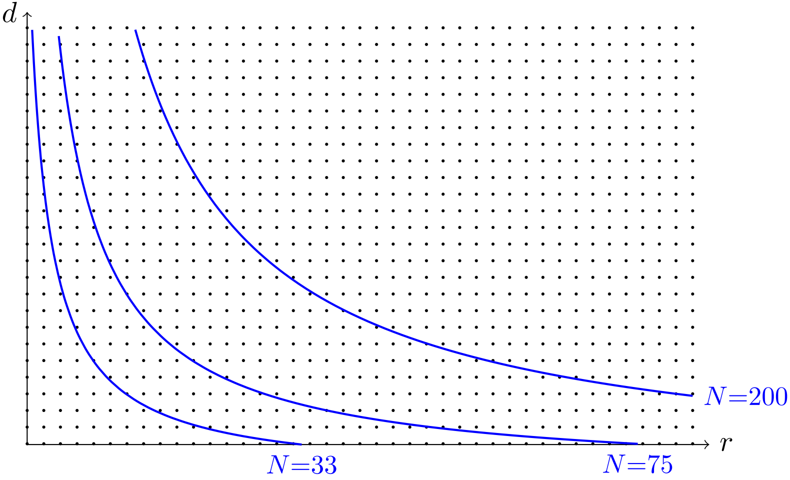

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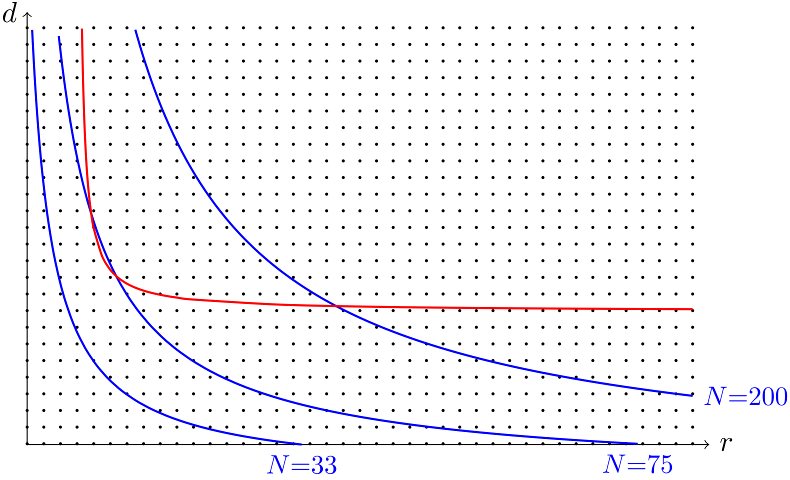

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- The coefficients of the recurrence involve "small" integers.

$\longrightarrow$ Employ a lattice reduction algorithm (LLL, BKZ, ...).

Lattice Basis

Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \ldots, c_\ell \in \mathbb{Z}\}.$$

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Caveat: Before we had computed a $\mathbb{Q}$-vector space basis of $\ker M$.

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Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

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- It can be computed, e.g. using the Hermite normal form.

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Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

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- Incorporate homomorphic images and Chinese remaindering
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The Generic Case

Theorem [Bombieri–Vaaler, 1983] Let $M \in \mathbb{Z}^{k \times m}$ with $k < m$, and let g be the $\gcd$ of all $k \times k$ minors of M.
Then $\ker_{\mathbb{Z}} M$ contains a nonzero element $x \in \mathbb{Z}^m$ with

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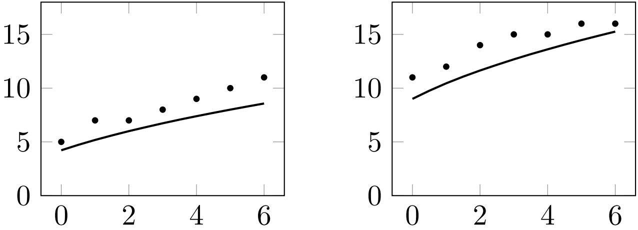

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- If we seek for an order-4 and degree-5 recurrence we obtain

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- Same with order-4 and degree-3, we could get something like

$$\begin{aligned}&(n-4)(2234n^2 + 719n - 6083) \cdot a_n + \\ &(n-4)(5672n^2 - 4375n + 6727) \cdot a_{n+1} - \\ &(n-4)(9959n^2 - 4399n - 7710) \cdot a_{n+2} + \\ &(n-4)(n-3)(1322n + 1174) \cdot a_{n+3} + \\ &(n-4)(n-3)(n-2)1909 \cdot a_{n+4} = 0.\end{aligned}$$

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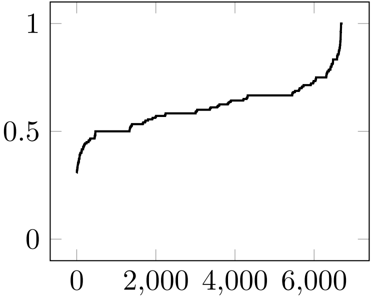

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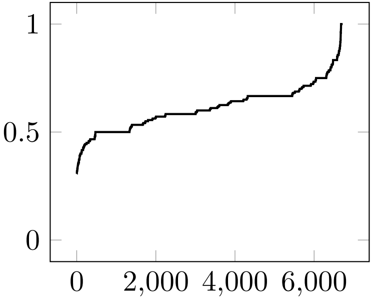

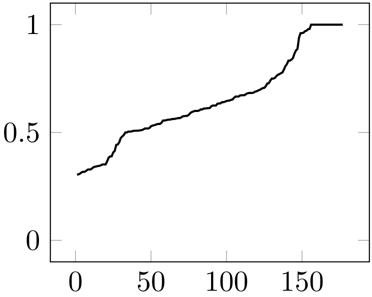

Result for A172671

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737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
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792606936905424716827805609592848631050897983368000
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390229178478432343758493287708395462786699986146463590205462500
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25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
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972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000
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&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}\\
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&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4}\\
&+248660072114197834440n^{3} + 195845152107619591920n^{2}\\
&+92743576895010081600n + 19927056990544704000)a_{n}\\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10}\\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7}\\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4}\\
&+11017108466619178896n^{3} + 9901273828612752684n^{2}\\
&+5411908796200065936n + 1358800904704763520)a_{n+1}\\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10}\\
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&-485718284438018172n - 132150596906568240)a_{n+2}\\
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&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10}\\
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$092056n^{10}$
$6714761n^{7}$
$4219813n^{4}$
$2752684n^{2}$
$3520)a_{n+1}$
$208164n^{10}$
$4981077n^{7}$
$3691226n^{4}$
$6910820n^{2}$
$8240)a_{n+2}$
$207552n^{10}$
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$7181535n^{4}$
$7585736n^{2}$
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Neil Sloane (05.03.2022, about A189281): In the text of the paper you say the coefficients are small! Au contraire. In fact the amount of data in the g.f. is comparable with the data in the original 35-term b-file for the sequence.

If you print the g.f. and then print the data, the number of digits in the two printouts look about the same. When this happens, surely you should be worried. I am very worried, and I think the g.f. needs more justification.

In fact the g.f. looks wrong. I use gfun all the time, and when the g.f. looks like this, like something you would find in the dumpster behind a restaurant, then I would not even consider it :D

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&+92743576895010081600n + 19927056990544704000)a_{n} \\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10} \\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7} \\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4} \\
&+11017108466619178896n^{3} + 9901273828612752684n^{2} \\
&+5411908796200065936n + 1358800904704763520)a_{n+1} \\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10} \\
&-990927551678562n^{9} - 6445641158908164n^{8} - 30971993224981077n^{7} \\
&-111314492026841106n^{6} - 299240095376493090n^{5} - 594271149013691226n^{4} \\
&-847459848696773373n^{3} - 821800045816910820n^{2} \\
&-485718284438018172n - 132150596906568240)a_{n+2} \\
&+(-34192224n^{13} - 1709611200n^{12} - 39348646744n^{11} - 551960207552n^{10} \\
&-5264405804862n^{9} - 36048494147578n^{8} - 182315015737541n^{7} \\
&-689472630263907n^{6} - 1949560872565283n^{5} - 4070539427181535n^{4} \\
&-6099491170412670n^{3} - 6211013227585736n^{2} \\
&-3851899366258336n - 1098712786184832)a_{n+3} \\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10} \\
&+702635490n^{9} + 5025358332n^{8} + 26510256652n^{7} \\
&+104430770292n^{6} + 307166340054n^{5} \\
&+666220125600n^{4} + 1035598237875n^{3} \\
&+1092435142500n^{2} + 700889050000n \\
&+206542200000)a_{n+4}
\end{aligned}
$$

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
97212368765632828873597857210432906828323036261620913199779764525314490735250548751871 0000

Result for A172671

$$\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}\\
&+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7}\\
&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4}\\
&+248660072114197834440n^{3} + 195845152107619591920n^{2}\\
&+92743576895010081600n + 19927056990544704000)a_{n}\\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10}\\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7}\\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4}\\
&+11017108466619178896n^{3} + 9901273828612752684n^{2}\\
&+5411908796200065936n + 1358800904704763520)a_{n+1}\\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10}\\
&-990927551678562n^{9} - 6445641158908164n^{8} - 30971993224981077n^{7}\\
&-111314492026841106n^{6} - 299240095376493090n^{5} - 594271149013691226n^{4}\\
&-847459848696773373n^{3} - 821800045816910820n^{2}\\
&-485718284438018172n - 132150596906568240)a_{n+2}\\
&+(-34192224n^{13} - 1709611200n^{12} - 39348646744n^{11} - 551960207552n^{10}\\
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&-689472630263907n^{6} - 1949560872565283n^{5} - 4070539427181535n^{4}\\
&-6099491170412670n^{3} - 6211013227585736n^{2}\\
&-3851899366258336n - 1098712786184832)a_{n+3}\\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10}\\
&+702635490n^{9} + 5025358332n^{8} + 26510256652n^{7}\\
&+104430770292n^{6} + 307166340054n^{5}\\
&+666220125600n^{4} + 1035598237875n^{3}\\
&+1092435142500n^{2} + 700889050000n\\
&+206542200000)a_{n+4}
\end{aligned}$$

Trustworthy?

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
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68378027287127596101538933052599954448793862727300484972893130374083314936140639370265791902301600
576696135477018756656097310539308206595297137655178128559217331447163987622287690653154248117571110400
4880259952199292008921826526312609348825249147788374851144194565482906902131493688167790266738802504840000
41428792196488801486282127539417868379239611007329360384215118568533632449531545568541320235439941375576624000
352722570320243675250582868120044596892546971115353276044155030127971499536247490129735774654399339137472372179200

Result for A172671

Trustworthy !

$$\begin{aligned}
&416745(n+2)(3n+4)(3n+5)(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(3784n^4+62436n^3+384549n^2+1047914n+1066254)a_n\\
&+9(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(29681696n^7+712360704n^6+7253307424n^5\\
&\quad+40621828312n^4+135172900470n^3+267337368752n^2\\
&\quad+291083104767n+134667010044)a_{n+1}\\
&-9(n+3)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(10844944n^8+309080904n^7+3833838118n^6\\
&\quad+27035659722n^5+118560795930n^4+331121212914n^3\\
&\quad+575194973415n^2+568260550317n+244478848756)a_{n+2}\\
&-(n+3)(n+4)^3(3n+13)(3n+14)\\
&\quad\times(3799136n^7+98777536n^6+1092573240n^5\\
&\quad+6662600832n^4+24184813590n^3+52244190090n^2\\
&\quad+62174897623n+31442101253)a_{n+3}\\
&+(n+3)(n+4)^3(n+5)^5\\
&\quad\times(3784n^4+47300n^3+219945n^2\\
&\quad+450988n+344237)a_{n+4}
\end{aligned}$$

```
90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
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245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
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207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000
8137021686675518646293987429238146291939698206862646669804019655299405410612322521011793199840
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57669613547701875665609731053930820659529713765517812855921733144716398762228769065315424811757111040000
4880259952199292008921826526312609348825249147788374851144194565482906902131493688167790266738802504840000
414287921964888014862821275394178683792396110073293603842151185685336324495315455685413202354399413755766240000
352722570320243675250582868120044596892546971115353276044155030127971499536247490129735774654399339137472372179200
```
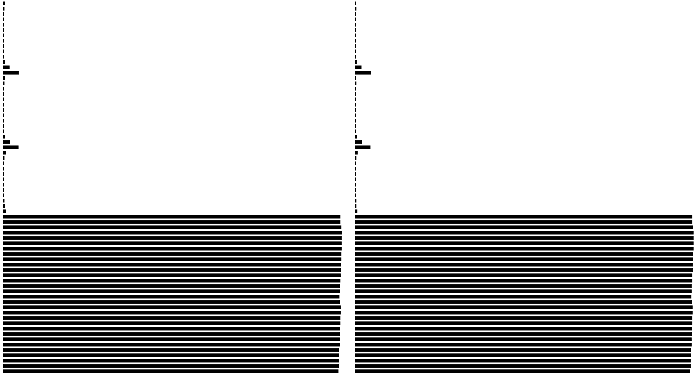
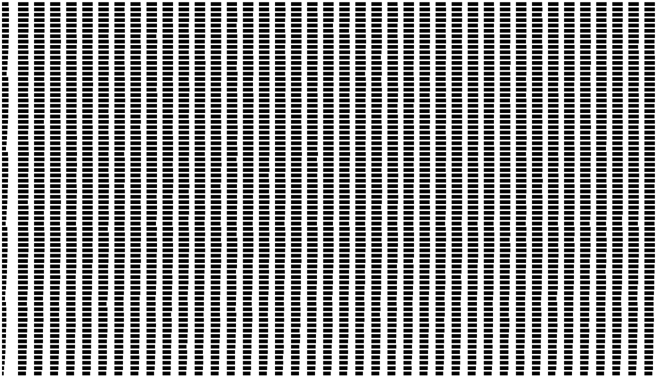

Result for A172671

First two vectors of $\ker_{\mathbb{Z}} M$:



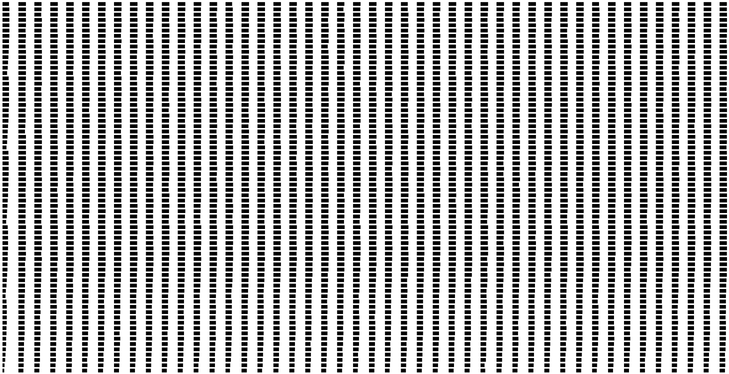
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 33$:



Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 28$:



Result for A172671

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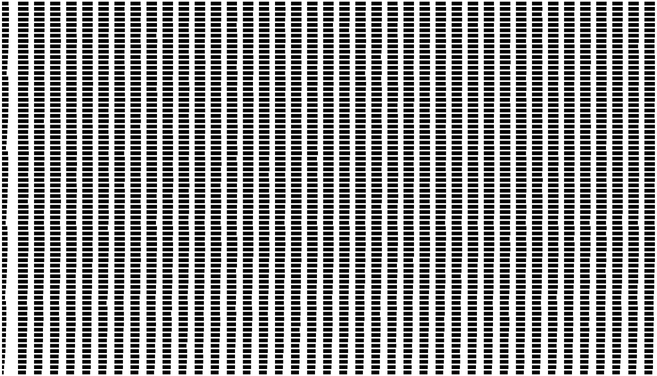
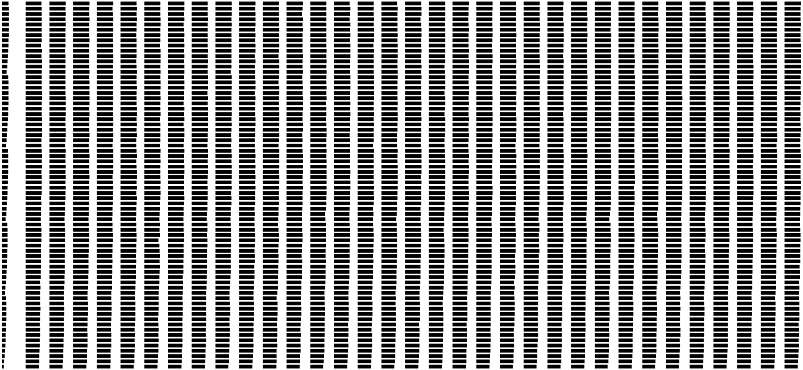


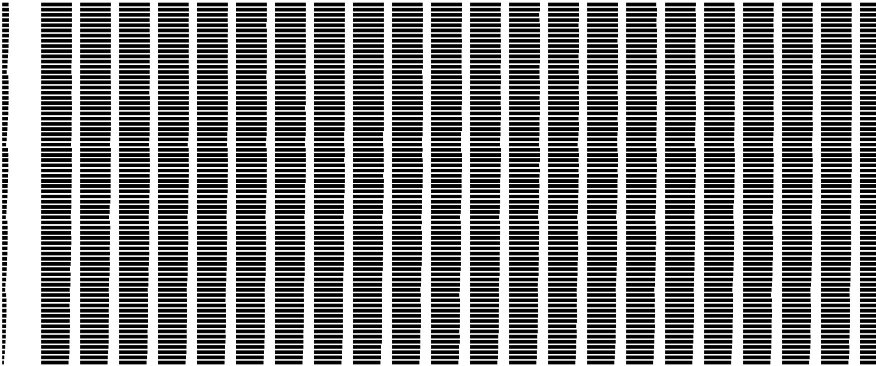
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 40$:



Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 50$:



Human Insight

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Indeed, let
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- Recurrence for b_n has the same order (namely, 4), but the coefficient degree drops from 13 to 8.
- With the classical linear algebra guessing, we can find this simplified recurrence using only 48 terms, instead of 75 terms.
- But the LLL-based guessing required only 24 terms for a_n, and it requires only 17 terms to find the simpler recurrence for b_n.

A189281

$a_n := \#\big\{\pi \in S_n \;\big|\; \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\big\}.$

A189281

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Example: $a_3 = 5$ because only one $\pi \in S_3$ is excluded.

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$$

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$$\begin{matrix} \xcancel{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}} & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \end{matrix}$$

A189281

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Only $a_0, \ldots, a_{35}$ are known:

1
1
2
5
18
75
410
2729
20906
181499
1763490
18943701
222822578
2847624899
39282739034
581701775369
9202313110506
154873904848803
2762800622799362
52071171437696453
1033855049655584786
21567640717569135515
471630531427793184474
10787660036599729160073
257590656485400508526570
6409633590481106885238443
165928838963556686281573922
4462073606461933066205164757
124470791290376112779747519538
3597058248632667485834774744787
107559658152025736992729145688602
3324154021716716493547315823808809
106067493846954075776733869818571690
3490771207487802026912252686947947027
118383998479651470880820236769742970626
4133478159186775319059453592629838113797

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1
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2
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18
75
410
2729
20906
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1763490
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222822578
2847624899
39282739034
581701775369
9202313110506
154873904848803
2762800622799362
52071171437696453
1033855049655584786
21567640717569135515
471630531427793184474
10787660036599729160073
257590656485400508526570
6409633590481106885238443
165928838963556686281573922
4462073606461933066205164757
124470791290376112779747519538
3597058248632667485834774744787
107559658152025736992729145688602
3324154021716716493547315823808809
106067493846954075776733869818571690
3490771207487802026912252686947947027
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We found a trustworthy recurrence of order 10 and degree 6.

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- The first vector in the LLL-basis is much shorter than the other vectors.

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- The first vector in the LLL-basis is much shorter than the other vectors.
- Even with just 29 of the 36 terms the recurrence can be detected.

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- The first vector in the LLL-basis is much shorter than the other vectors.
- Even with just 29 of the 36 terms the recurrence can be detected.
- The guessed operator has a right factor of order 8 and degree 11.
- We computed $a_{36}, a_{37}, a_{38}, a_{39}$ (!) and found that these terms were correctly predicted by the guessed recurrence.

A189282

$a_n := \#\{\pi \in S_n \mid \pi(k+3) - \pi(k) \neq 3 \text{ for all } k\}.$

- Only the terms $a_0, \ldots, a_{30}$ were given in the OEIS (by Vaclav Kotesovec).

A189282

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- We computed $a_{31}, \ldots, a_{36}$, taking some weeks of computation time, but could not find a recurrence.
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- Still: too little data to guess a recurrence with our method!

A199250

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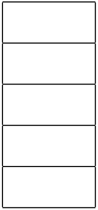

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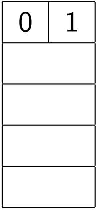

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$\longrightarrow$ Sequence can be computed with the transfer matrix method!

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Define a 12×12 transfer matrix

$$M=\begin{pmatrix}
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0 & 0 & 0 & xy & 0 & y & xz & yz & z & x & y & 0\\
0 & 0 & 0 & xy & yz & 0 & xz & yz & 0 & x & y & z\\
xy & xz & x & 0 & 0 & 0 & 0 & yz & z & 0 & y & z\\
xy & 0 & x & 0 & 0 & 0 & xz & yz & z & x & y & 0\\
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and initial vector $v_{\text{init}} = (xy,0,0,0,0,0,0,0,0,0,0,0)$. Compute

$$p(x,y,z) = v_{\text{init}} \cdot M^{n-1} \cdot (1,1,1,1,1,1,1,1,1,1,1,1)^\top.$$

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Generating function for the full counting sequence:

$$\begin{aligned} f(t,x,y,z) &= \frac{txy\sum_{i=1}^{12}(-1)^{i+1}\ \det(I_{12}-tM)^{[1,i]}}{\det(I_{12}-tM)} \\ &= \frac{txy(tz+1)}{1-tx-ty-txy-tz-txz-tyz-7t^2xyz}. \end{aligned}$$

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We can obtain the desired recurrence via creative telescoping. For example, for even n, we compute a recurrence for

$$\operatorname{res}_{t,x,y,z} \frac{1}{txyz}\frac{f(t,x,y,z)}{t^{2n}\,(xyz)^n}.$$

A339987

Number of labeled graphs on $2n$ vertices s.t. $n-1$ vertices have degree 3 and the remaining $n+1$ vertices have degree 1:

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- How difficult would it be to compute more terms?

Minimal Number of Terms for Guessing

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Toy example: The central Delannoy numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ satisfy a recurrence of order 2 and degree 1:

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- Thus, $(*)$ can be recovered from the single term $D_8 = 265729$.

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- We were not able to find a recurrence for the notorious Av(1324) sequence. . .