

Binomial Determinants for Tiling Problems Yield to the Holonomic Ansatz

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Séminaire de Combinatoire de Lyon à l'ENS



Families of Binomial Determinants

Definition: For $n \in \mathbb{N}$, for $s, t \in \mathbb{Z}$, and for μ an indeterminate, define the following $(n \times n)$ -determinants:

$$D_{s,t}^{\mu}(n) := \det_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \left(\binom{\mu + i + j + s + t - 4}{j + t - 1} + \delta_{i+s, j+t} \right),$$

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History: $D_{0,0}^{\mu}(n)$ was introduced in the work of Andrews in 1979–1980 in the context of descending plane partitions:

Inventiones math. 53, 193–225 (1979)

*Inventiones
mathematicae*

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Plane Partitions (III): The Weak Macdonald Conjecture

George E. Andrews*

The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

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Example: $D_{4,6}^{\mu}(5)$ is the determinant of the matrix

$$\begin{pmatrix} \binom{\mu+8}{6} & \binom{\mu+9}{7} & \binom{\mu+10}{8} & \binom{\mu+11}{9} & \binom{\mu+12}{10} \\ \binom{\mu+9}{6} & \binom{\mu+10}{7} & \binom{\mu+11}{8} & \binom{\mu+12}{9} & \binom{\mu+13}{10} \\ \binom{\mu+10}{6} + 1 & \binom{\mu+11}{7} & \binom{\mu+12}{8} & \binom{\mu+13}{9} & \binom{\mu+14}{10} \\ \binom{\mu+11}{6} & \binom{\mu+12}{7} + 1 & \binom{\mu+13}{8} & \binom{\mu+14}{9} & \binom{\mu+15}{10} \\ \binom{\mu+12}{6} & \binom{\mu+13}{7} & \binom{\mu+14}{8} + 1 & \binom{\mu+15}{9} & \binom{\mu+16}{10} \end{pmatrix}$$

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Example: $D_{3,5}^{\mu+2}(5)$ is the determinant of the matrix

$$\begin{pmatrix} \binom{\mu+8}{5} & \binom{\mu+9}{6} & \binom{\mu+10}{7} & \binom{\mu+11}{8} & \binom{\mu+12}{9} \\ \binom{\mu+9}{5} & \binom{\mu+10}{6} & \binom{\mu+11}{7} & \binom{\mu+12}{8} & \binom{\mu+13}{9} \\ \binom{\mu+10}{5} + 1 & \binom{\mu+11}{6} & \binom{\mu+12}{7} & \binom{\mu+13}{8} & \binom{\mu+14}{9} \\ \binom{\mu+11}{5} & \binom{\mu+12}{6} + 1 & \binom{\mu+13}{7} & \binom{\mu+14}{8} & \binom{\mu+15}{9} \\ \binom{\mu+12}{5} & \binom{\mu+13}{6} & \binom{\mu+14}{7} + 1 & \binom{\mu+15}{8} & \binom{\mu+16}{9} \end{pmatrix}$$

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Example: $D_{5,6}^{\mu-1}(5)$ is the determinant of the matrix

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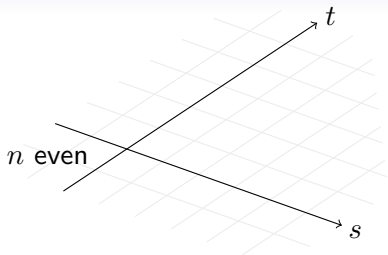
$$\begin{pmatrix} \binom{\mu+8}{6} & \binom{\mu+9}{7} & \binom{\mu+10}{8} & \binom{\mu+11}{9} & \binom{\mu+12}{10} \\ \binom{\mu+9}{6} - 1 & \binom{\mu+10}{7} & \binom{\mu+11}{8} & \binom{\mu+12}{9} & \binom{\mu+13}{10} \\ \binom{\mu+10}{6} & \binom{\mu+11}{7} - 1 & \binom{\mu+12}{8} & \binom{\mu+13}{9} & \binom{\mu+14}{10} \\ \binom{\mu+11}{6} & \binom{\mu+12}{7} & \binom{\mu+13}{8} - 1 & \binom{\mu+14}{9} & \binom{\mu+15}{10} \\ \binom{\mu+12}{6} & \binom{\mu+13}{7} & \binom{\mu+14}{8} & \binom{\mu+15}{9} - 1 & \binom{\mu+16}{10} \end{pmatrix}$$

“Conjecture 37” (Lascoux/Krattenthaler 2005)

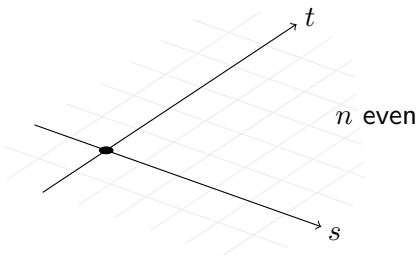
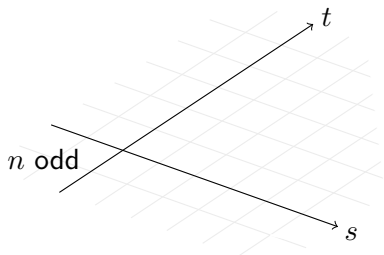
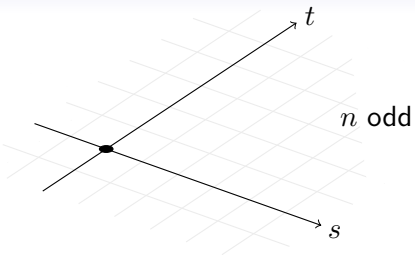
Let μ be an indeterminate and $m, r \in \mathbb{Z}$. If $m \geq r \geq 1$, then

$$\begin{aligned}
 E_{1,2r-1}^{\mu}(2m-1) &= (-1)^{m-r} \cdot 2^{4m+(m-r)(m-r-1)-3r} \\
 &\times \left(\prod_{i=0}^{2r-3} i! \right) \cdot \left(\prod_{i=0}^{m-1} \frac{i!(i+1)!}{(2i)!(2i+2)!} \right) \\
 &\times (\mu-1) \cdot \left(\frac{\mu}{2} + r - \frac{1}{2} \right)_{m-r} \cdot \left(\prod_{i=1}^{2r-2} (\mu+i-1)_{2m+2r-2i-1} \right) \\
 &\times \left(\prod_{i=0}^{r-2} \frac{((2m-2i-3)!)^2}{((m-i-2)!)^2 (2m+2i-1)!(2m+2i+1)!} \right) \\
 &\times \left(\prod_{i=0}^{\lfloor \frac{m-r-1}{2} \rfloor} \left(\frac{\mu}{2} + 3i + 3r - \frac{1}{2} \right)_{m-r-2i-1}^2 \left(-\frac{\mu}{2} - 3m + 3i + 3 \right)_{m-r-2i}^2 \right)
 \end{aligned}$$

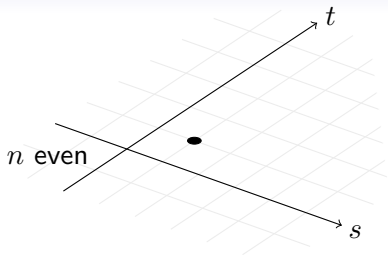
$E_{s,t}^\mu(n)$ Family



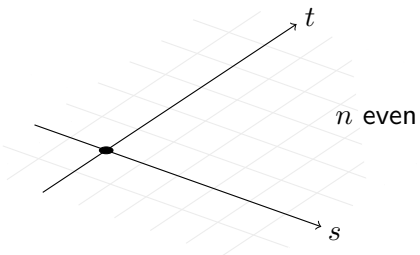
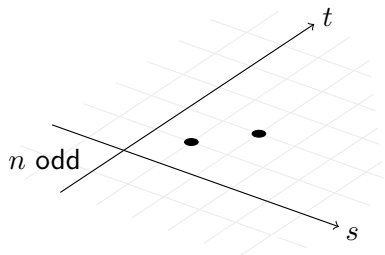
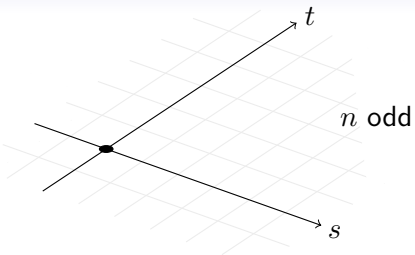
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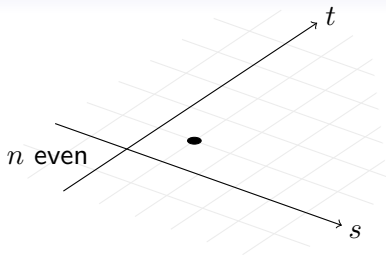
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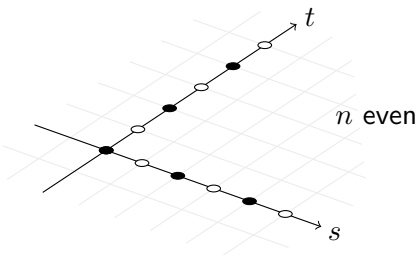
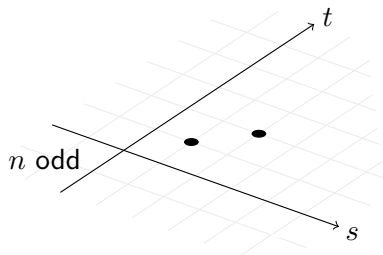
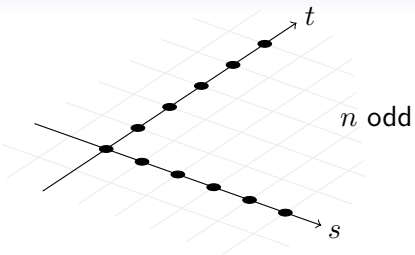
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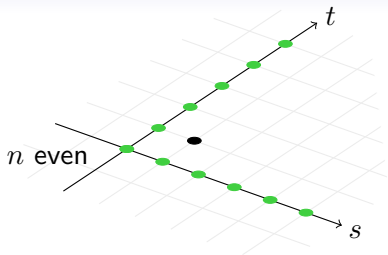
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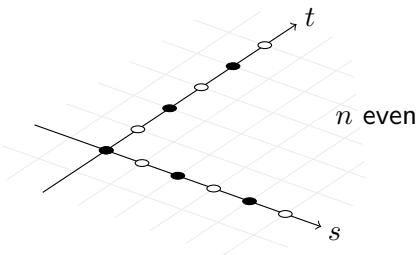
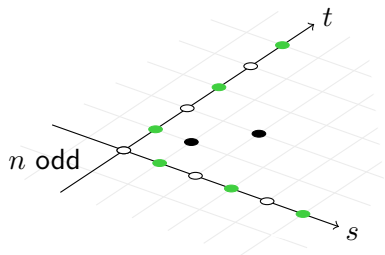
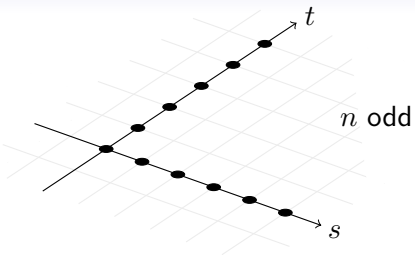
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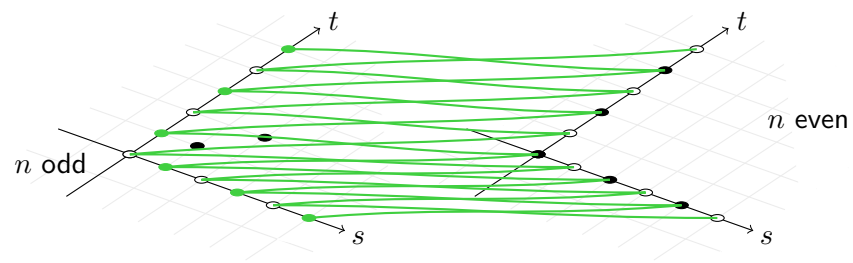
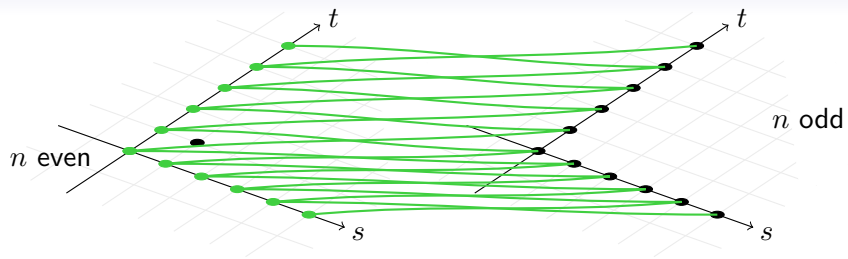


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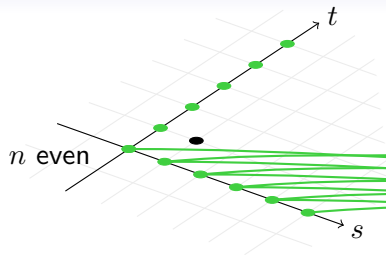


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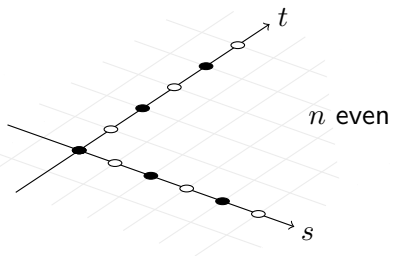
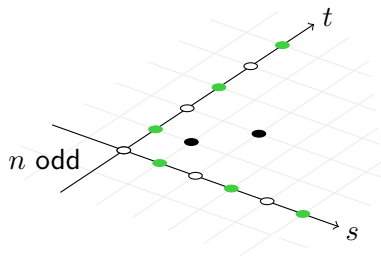
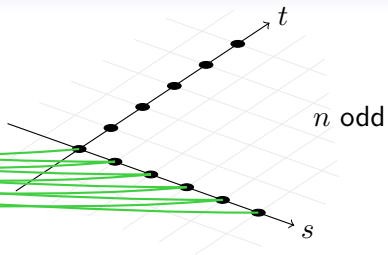
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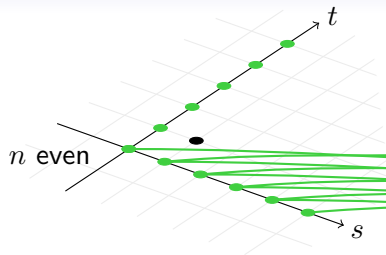
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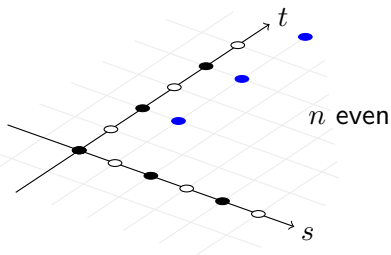
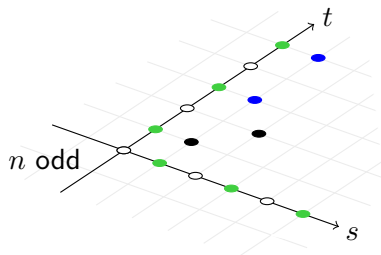
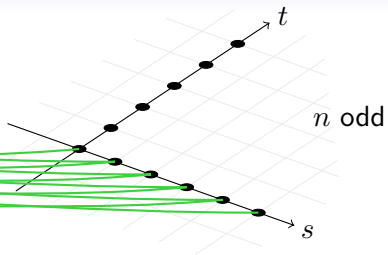
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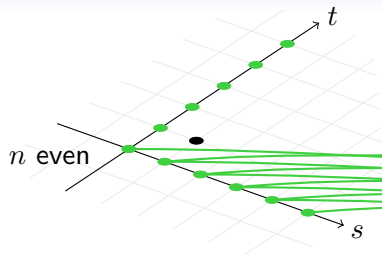
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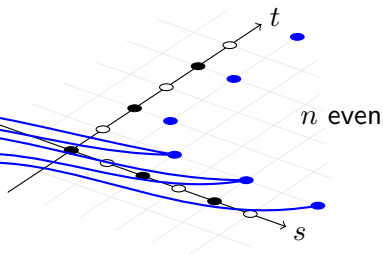
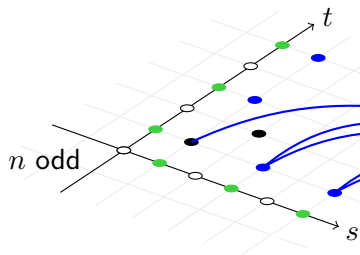
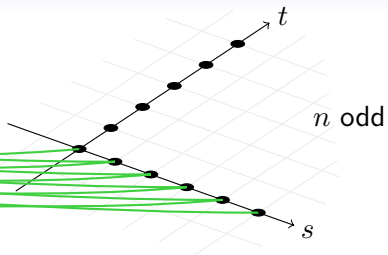
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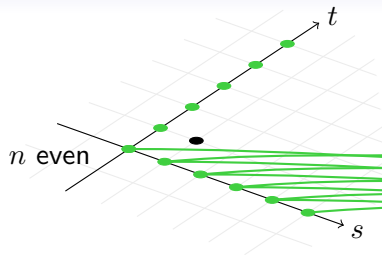
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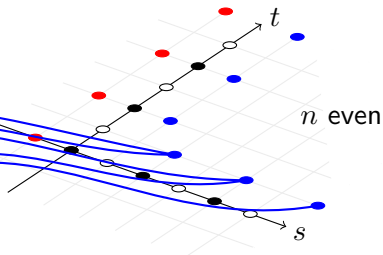
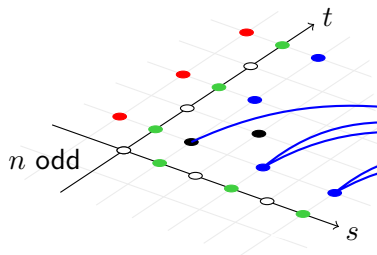
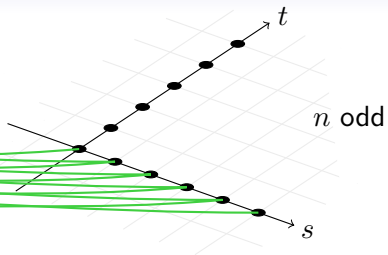
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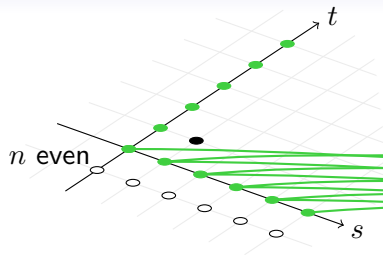
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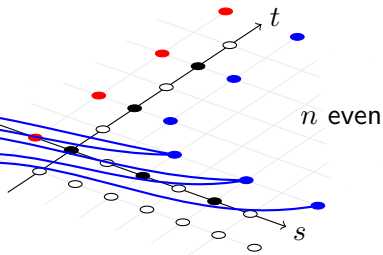
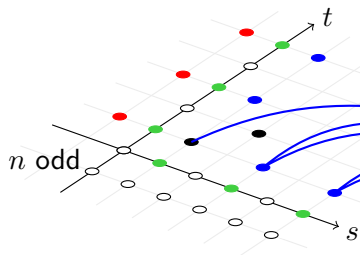
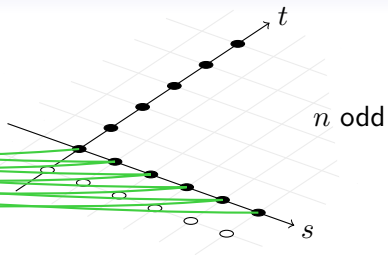
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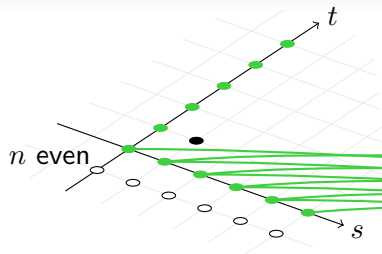
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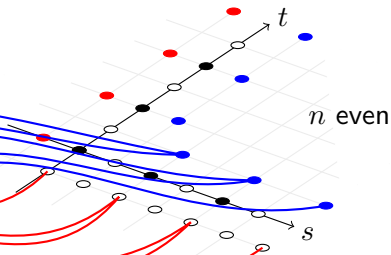
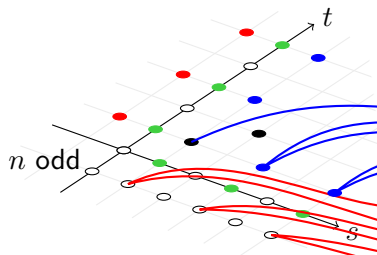
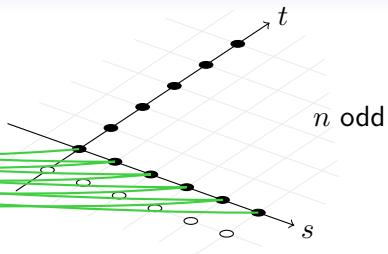
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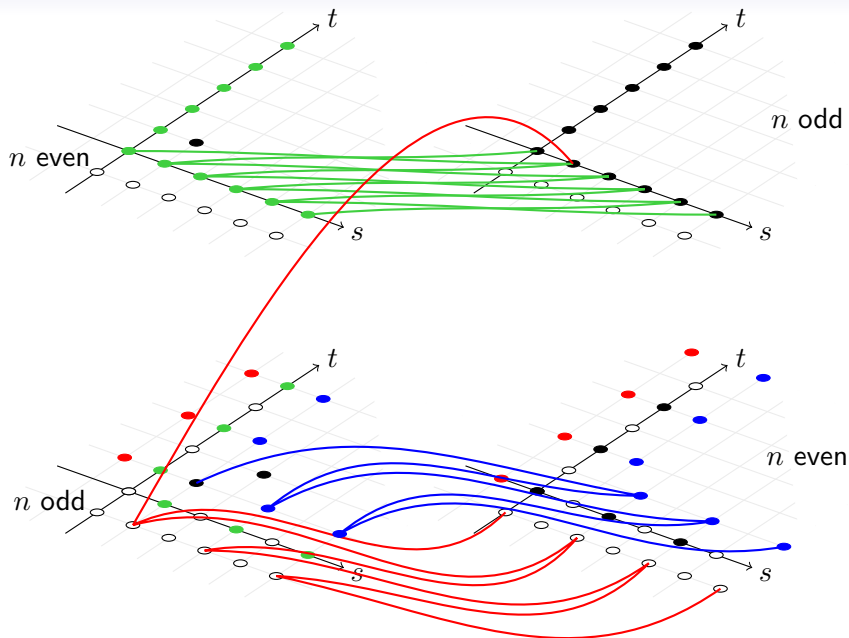


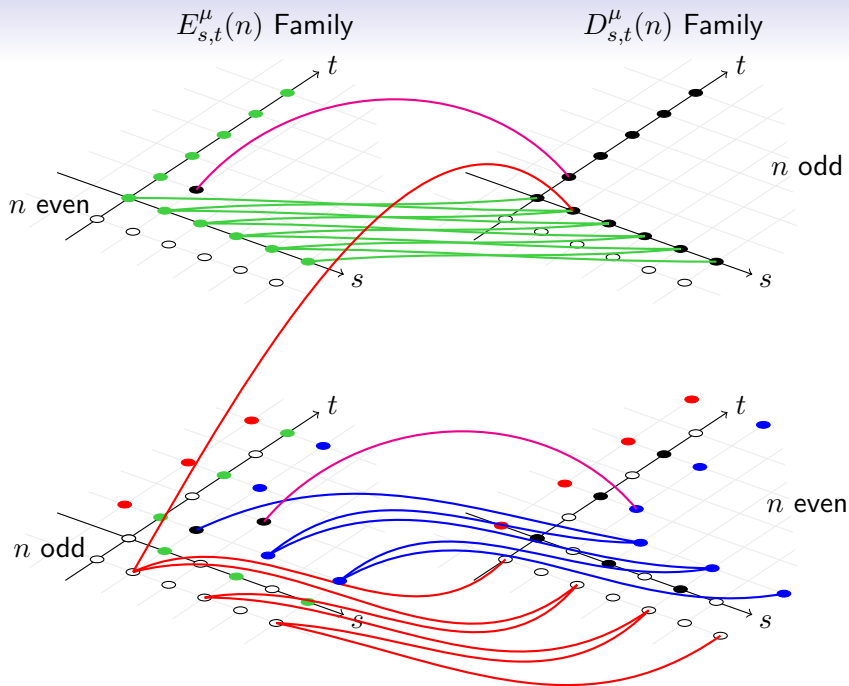
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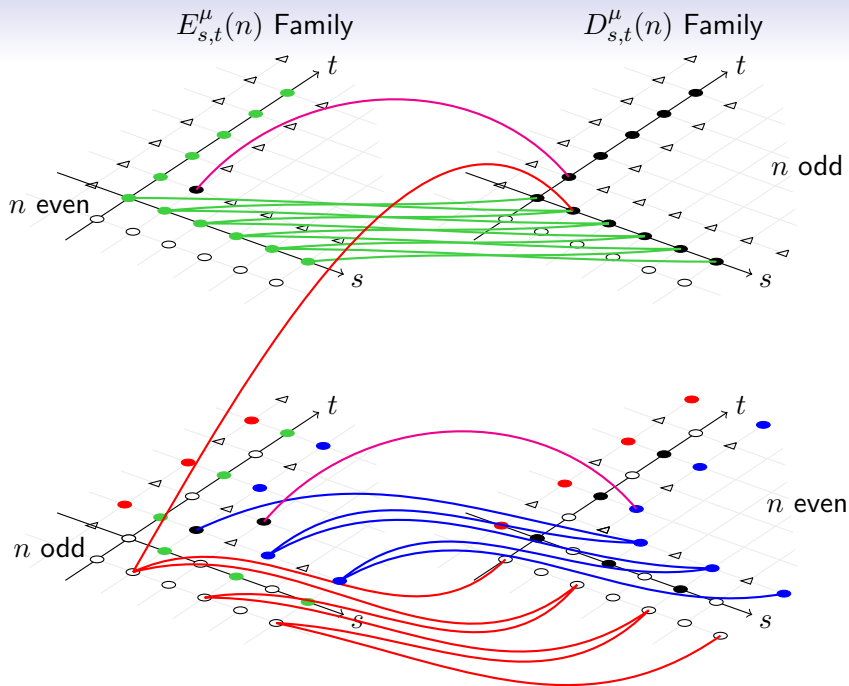


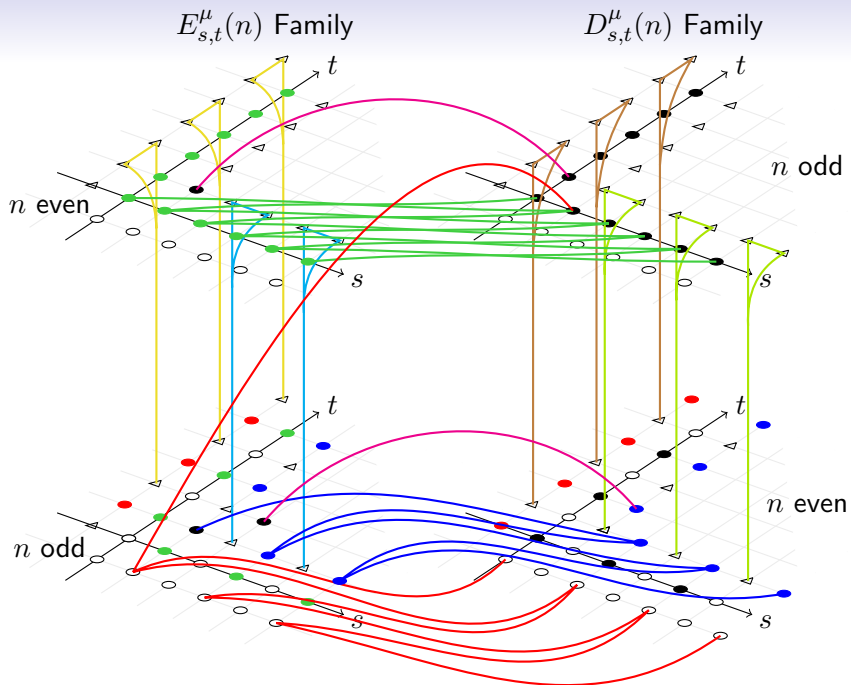
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Lindström-Gessel-Viennot Lemma

Let G be a directed acyclic graph and consider base vertices $A = \{a_1, \dots, a_n\}$ and destination vertices $B = \{b_1, \dots, b_n\}$.

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$$e(a, b) = \sum_{P:a \rightarrow b} \omega(P) \quad \text{and}$$
$$M = \begin{pmatrix} e(a_1, b_1) & e(a_1, b_2) & \cdots & e(a_1, b_n) \\ e(a_2, b_1) & e(a_2, b_2) & \cdots & e(a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n, b_1) & e(a_n, b_2) & \cdots & e(a_n, b_n) \end{pmatrix}.$$

Lindström-Gessel-Viennot Lemma

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Then the determinant of M is the signed sum over all n -tuples $P = (P_1, \dots, P_n)$ of non-intersecting paths from A to B :

$$\det(M) = \sum_{(P_1, \dots, P_n): A \rightarrow B} \text{sign}(\sigma(P)) \prod_{i=1}^n \omega(P_i).$$

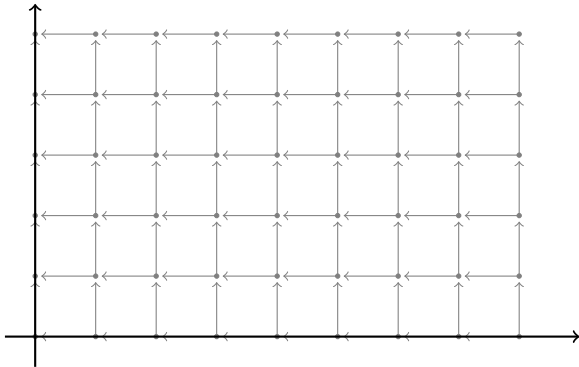
where σ denotes a permutation that is applied to B .

Lindström-Gessel-Viennot Lemma

In our context, it implies that the determinant without the Kronecker delta

$$\det_{1 \leq i, j \leq n} \begin{pmatrix} \mu + i + j + s + t - 4 \\ j + t - 1 \end{pmatrix}$$

counts n -tuples of non-intersecting paths in the lattice \mathbb{N}^2 :

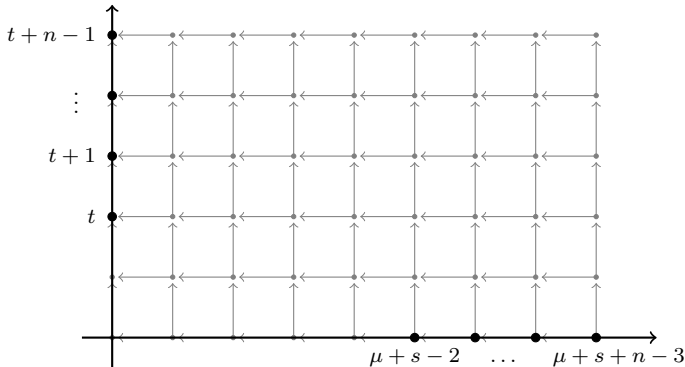


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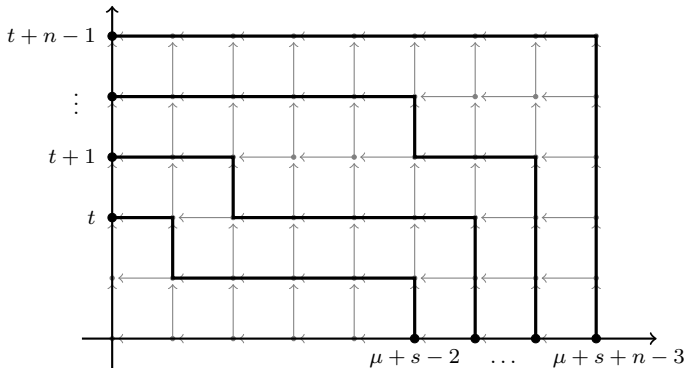


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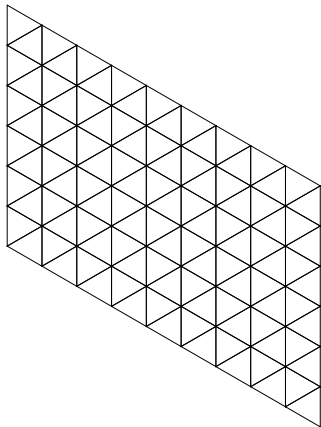
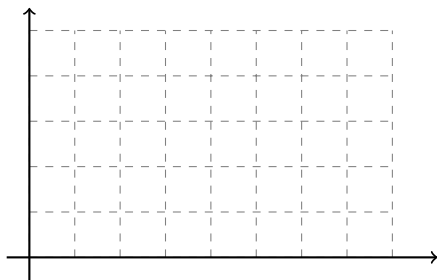
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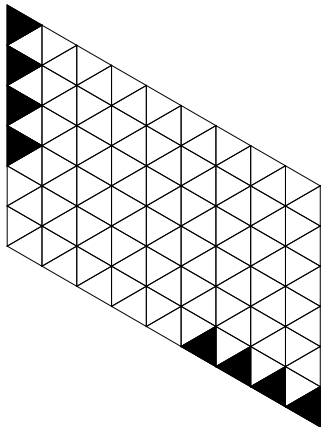
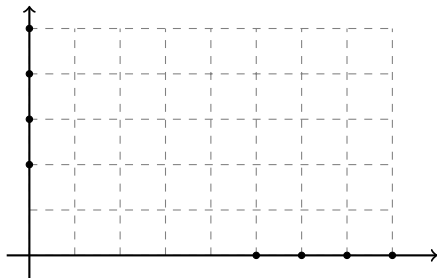
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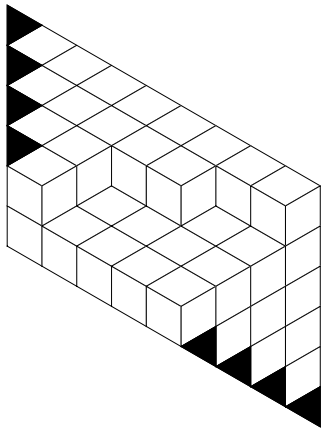
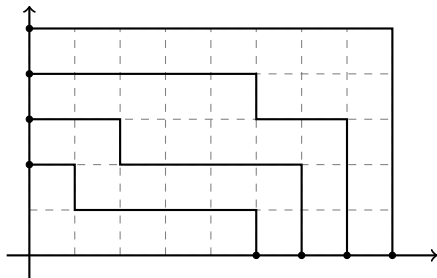
Lattice Paths \longleftrightarrow Rhombus Tilings



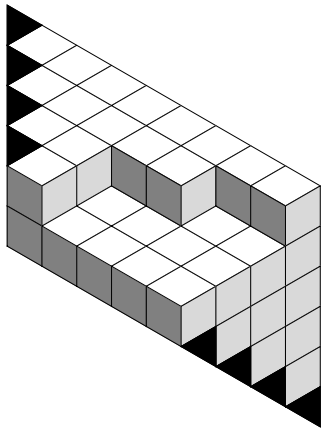
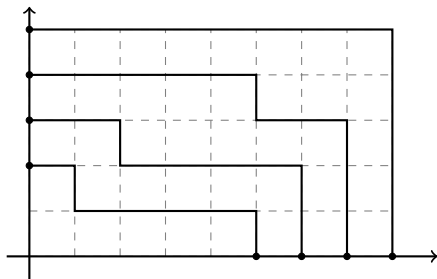
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Expansion of Kronecker Deltas

$$\begin{aligned}
 \text{Laplace expansion:} \quad & \begin{vmatrix} \cdots & b_{1,6} & b_{1,7} + 1 & b_{1,8} & \cdots \\ \cdots & b_{2,6} & b_{2,7} & b_{2,8} + 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{vmatrix} = \\
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By applying this procedure recursively, one obtains

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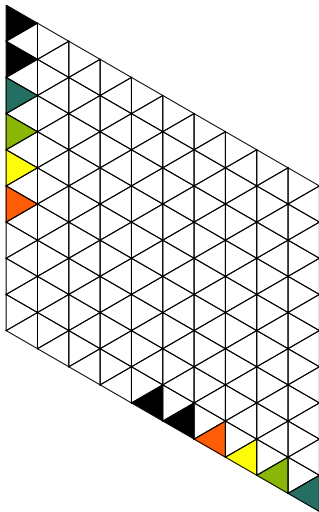
Hexagonal Fusion

$$s = 1$$

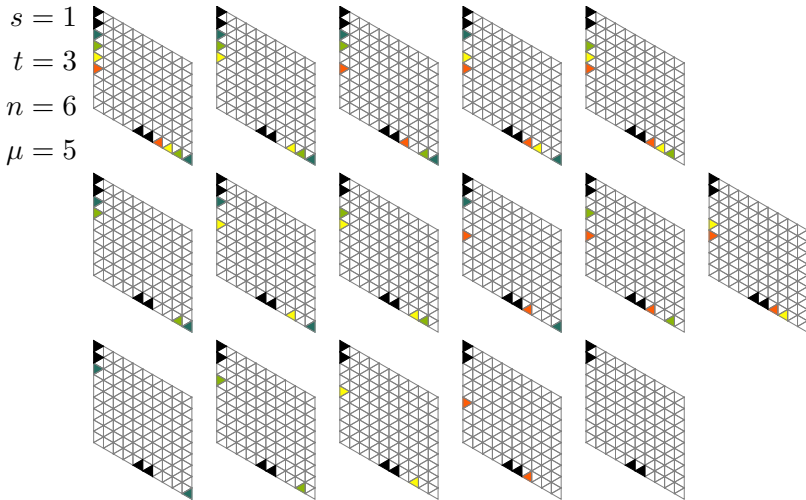
$$t = 3$$

$$n = 6$$

$$\mu = 5$$



Hexagonal Fusion



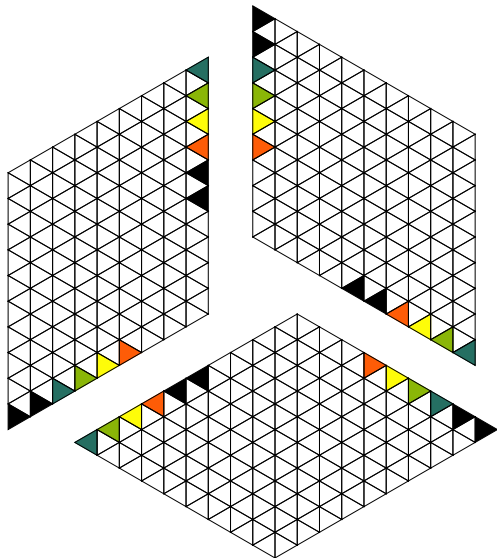
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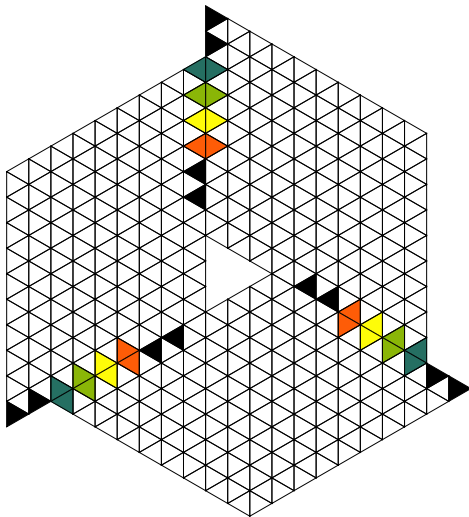
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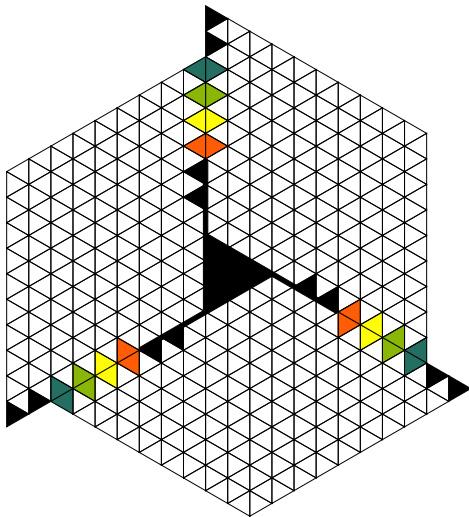
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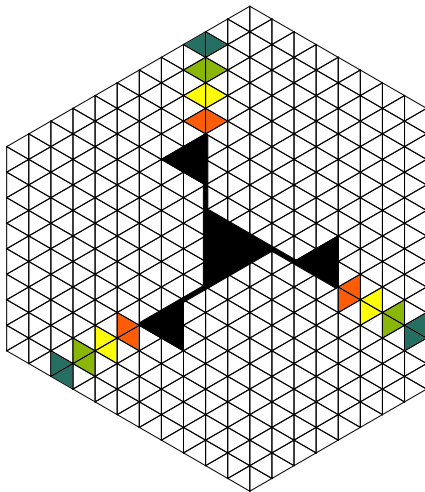
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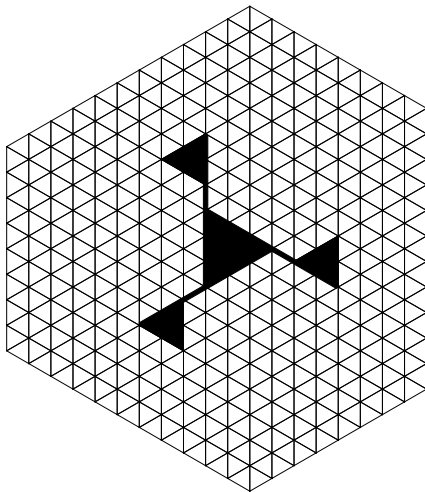
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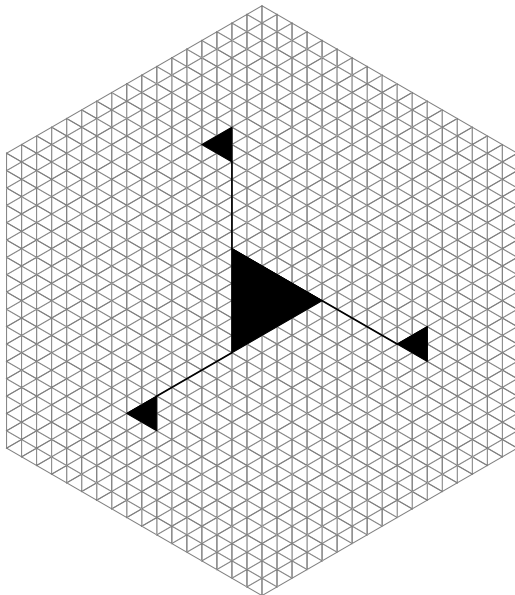
Combinatorial Interpretation: Holey Hexagon

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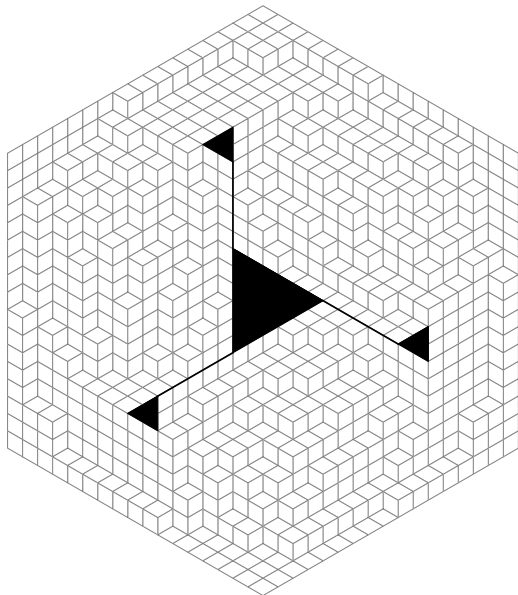
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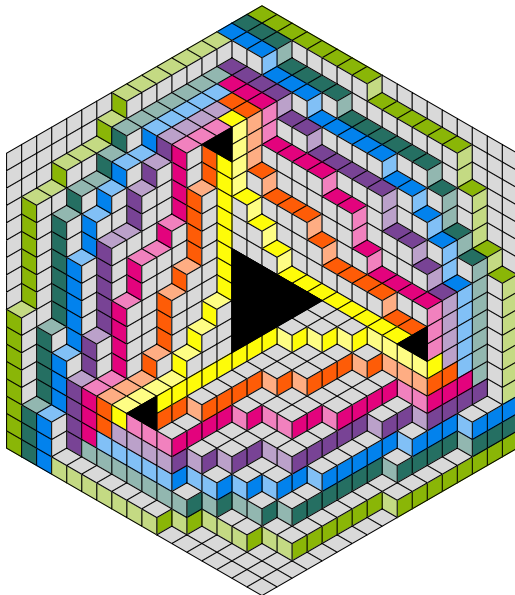
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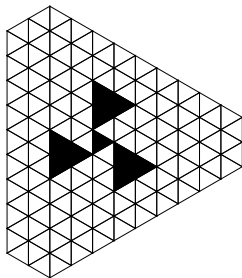
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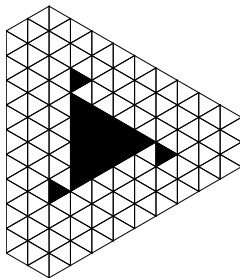
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Proof (by example):

$$D_{2,0}^3(4)$$



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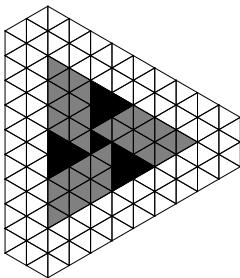
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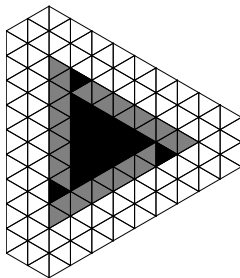
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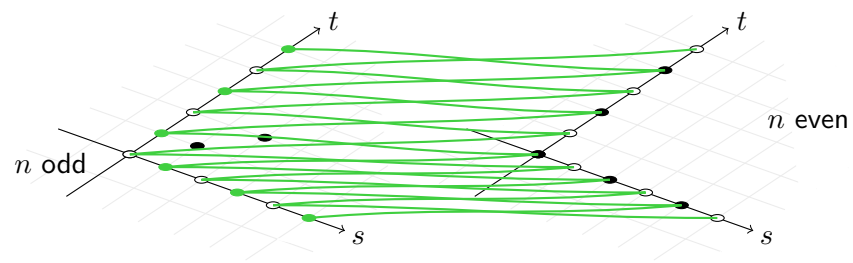
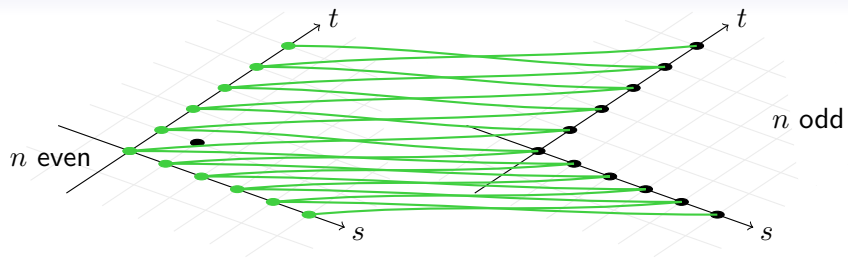


$E_{1,0}^6(3)$



$E_{s,t}^\mu(n)$ Family

$D_{s,t}^\mu(n)$ Family



Switching Lemma and DJD

Lemma: Let $A_{s,t}^\mu(n)$ be either $D_{s,t}^\mu(n)$ or $E_{s,t}^\mu(n)$. For real numbers $s, t \notin \{-1, -2, \dots\}$ with $t - s \in \mathbb{N}$ and $n \in \mathbb{Z}^+$,

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Theorem (Desnanot-Jacobi-Dodgson). Let $(m_{i,j})_{i,j \in \mathbb{Z}}$ and $M_{s,t}(n) := \det_{s \leq i < s+n, t \leq j < t+n} (m_{i,j})$. Then:

$$M_{s,t}(n)M_{s+1,t+1}(n-2) = \\ M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

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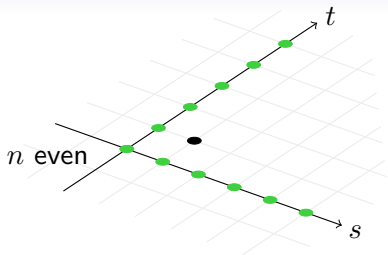
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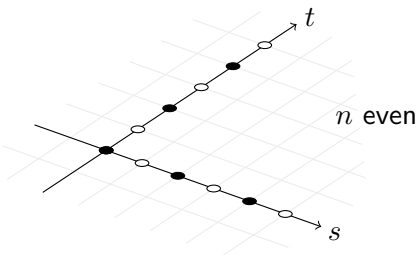
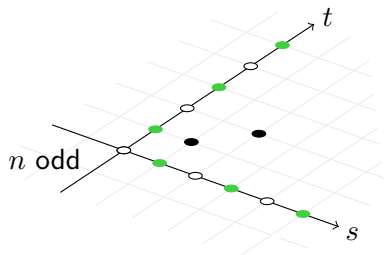
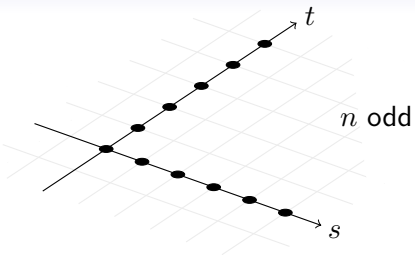
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Schematically:

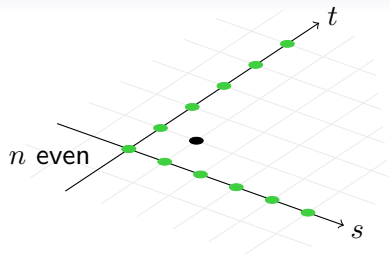
$E_{s,t}^\mu(n)$ Family



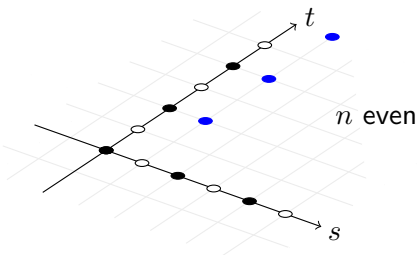
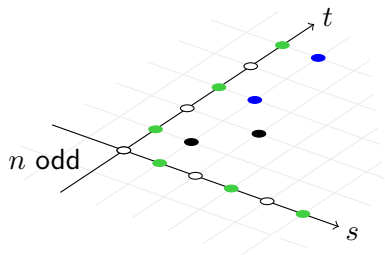
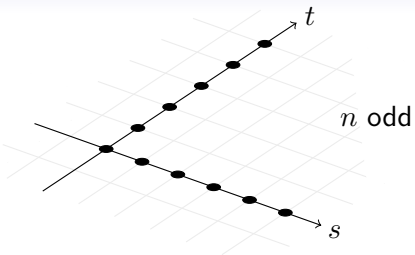
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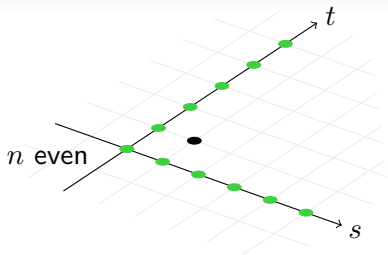
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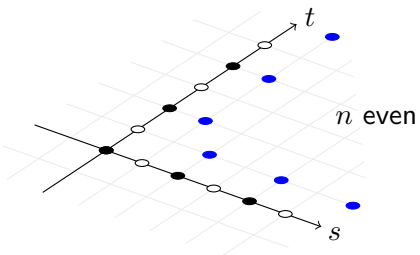
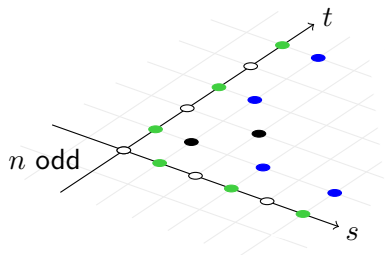
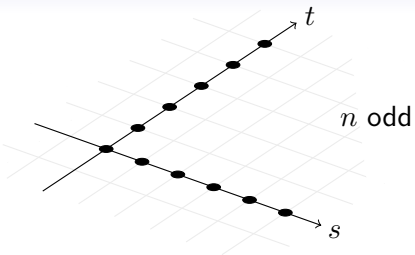
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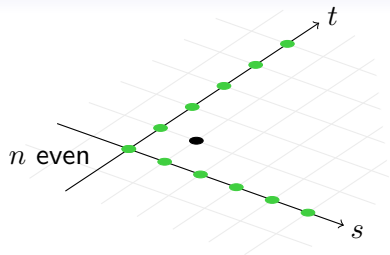
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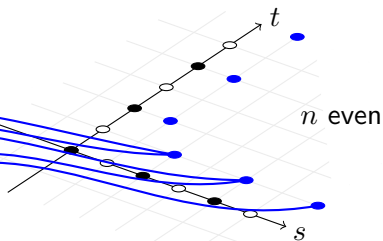
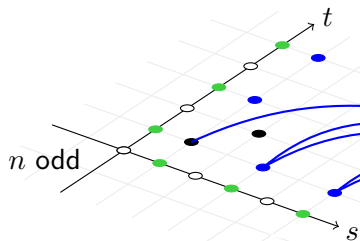
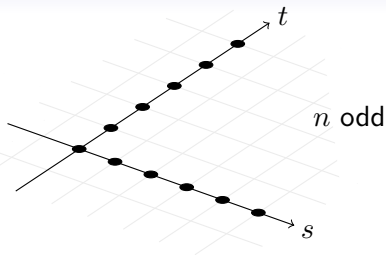
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$D_{s,t}^\mu(n)$ Family



Proof of the Lascoux/Krattenthaler Conjecture

Lemma: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$\frac{D_{2r,1}^{\mu}(2m)}{E_{2r-1,1}^{\mu+3}(2m-1)} = \frac{(m+r-1)(\mu-1)(\mu+2m+1)(\mu+2r)}{2m(2r-1)(\mu+2)(\mu+2m+2r-1)},$$
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Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{2r-1,1}^{\mu}(2m-1) = \frac{(-1)^{m-r} (\mu-1) (\mu+2r-1)_{2m-2}}{(2r-2)! (m+r-1)_{m-r+1} \left(\frac{\mu}{2} + r\right)_{m-r}} \\ \times \prod_{i=1}^{m-r} \frac{(\mu+2i+6r-5)_{i-1}^2 \left(\frac{\mu}{2} + 2i + 3r - 2\right)_i^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 2\right)_{i-1}^2}.$$

Proof of the Lascoux/Krattenthaler Conjecture

Lemma: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$\frac{D_{2r,1}^\mu(2m)}{E_{2r-1,1}^{\mu+3}(2m-1)} = \frac{(m+r-1)(\mu-1)(\mu+2m+1)(\mu+2r)}{2m(2r-1)(\mu+2)(\mu+2m+2r-1)},$$
$$\frac{E_{2r+1,1}^\mu(2m+1)}{D_{2r,1}^{\mu+3}(2m)} = \frac{(m+r)(\mu-1)(\mu+2m+2)(\mu+2r+1)}{2r(2m+1)(\mu+2)(\mu+2m+2r+1)}.$$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{2r-1,1}^\mu(2m-1) = \frac{(-1)^{m-r} (\mu-1) (\mu+2r-1)_{2m-2}}{(2r-2)! (m+r-1)_{m-r+1} \left(\frac{\mu}{2} + r\right)_{m-r}}$$
$$\times \prod_{i=1}^{m-r} \frac{(\mu+2i+6r-5)_{i-1}^2 \left(\frac{\mu}{2} + 2i + 3r - 2\right)_i^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 2\right)_{i-1}^2}.$$

Corollary: Apply switching lemma to obtain $E_{1,2r-1}^\mu(2m-1)$.

Proof of the Lascoux/Krattenthaler Conjecture

Lemma: For $n, s \in \mathbb{Z}$ and $n \geq s \geq 1$,

$$\frac{A_{s,1}^{\mu}(n)}{B_{s-1,1}^{\mu+3}(n-1)} = \frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)},$$

where (A, B, s, n) is $(D, E, 2r, 2m)$ or $(E, D, 2r+1, 2m+1)$.

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{2r-1,1}^{\mu}(2m-1) = \frac{(-1)^{m-r} (\mu-1) (\mu+2r-1)_{2m-2}}{(2r-2)! (m+r-1)_{m-r+1} \left(\frac{\mu}{2} + r\right)_{m-r}} \\ \times \prod_{i=1}^{m-r} \frac{(\mu+2i+6r-5)_{i-1}^2 \left(\frac{\mu}{2} + 2i+3r-2\right)_i^2}{(i)_i^2 \left(\frac{\mu}{2} + i+3r-2\right)_{i-1}^2}.$$

Corollary: Apply switching lemma to obtain $E_{1,2r-1}^{\mu}(2m-1)$.

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^\mu(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+2}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} + 1 & \binom{\mu+5}{4} \\ \binom{\mu+3}{1} & \binom{\mu+4}{2} & \binom{\mu+5}{3} & \binom{\mu+6}{4} + 1 \\ \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} & \binom{\mu+7}{4} \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+2}{1} - \binom{\mu+1}{1} & \binom{\mu+3}{2} - \binom{\mu+2}{2} - 1 & \binom{\mu+4}{3} - \binom{\mu+3}{3} + 1 & \binom{\mu+5}{4} - \binom{\mu+4}{4} \\ \binom{\mu+3}{1} - \binom{\mu+2}{1} & \binom{\mu+4}{2} - \binom{\mu+3}{2} & \binom{\mu+5}{3} - \binom{\mu+4}{3} - 1 & \binom{\mu+6}{4} - \binom{\mu+5}{4} + 1 \\ \binom{\mu+4}{1} - \binom{\mu+3}{1} & \binom{\mu+5}{2} - \binom{\mu+4}{2} & \binom{\mu+6}{3} - \binom{\mu+5}{3} & \binom{\mu+7}{4} - \binom{\mu+6}{4} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + \binom{\mu+1}{1} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1} + \binom{\mu+1}{0} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} + \binom{\mu+2}{0} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} + \binom{\mu+3}{0} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + \binom{\mu+2}{1} & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1} + \binom{\mu+2}{0} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} + \binom{\mu+3}{0} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} + \binom{\mu+4}{0} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^\mu(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+3}{3} + \binom{\mu+3}{2} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+3}{2} + \binom{\mu+3}{1} & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+4}{2} + \binom{\mu+4}{1} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+5}{2} + \binom{\mu+5}{1} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+4}{4} + \binom{\mu+4}{3} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+4}{3} + \binom{\mu+4}{2} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+5}{3} + \binom{\mu+5}{2} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+6}{3} + \binom{\mu+6}{2} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+5}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+5}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+6}{3} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+7}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^\mu(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+5}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+5}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+6}{3} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+7}{3} - 1 \end{pmatrix}$$

$$\mathcal{L} \cdot \mathcal{D}_{2,1}^\mu(4) \cdot \mathcal{R} = \begin{pmatrix} * & * & * & * \\ - & - & - & - \\ 1 & & & \\ 1 & & \mathcal{E}_{1,1}^{\mu+3}(3) & \\ 1 & & & \end{pmatrix}$$

Proof via the Holonomic Ansatz

To show:

$$\frac{A_{s,1}^{\mu}(n)}{B_{s-1,1}^{\mu+3}(n-1)} = \frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)}$$

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Laplace expansion:

$$A_{s,1}^{\mu}(n) = \det \begin{pmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \cdots & \tilde{a}_{1,n} \\ 1 & & & \\ 1 & & \mathcal{B}_{s-1,1}^{\mu+3}(n-1) & \\ 1 & & & \end{pmatrix}$$
$$= \tilde{a}_{1,1} \cdot \text{Cof}_{1,1}(n-1) + \dots + \tilde{a}_{1,n} \cdot \text{Cof}_{1,n}(n-1).$$

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$$= \tilde{a}_{1,1} \cdot \text{Cof}_{1,1}(n-1) + \dots + \tilde{a}_{1,n} \cdot \text{Cof}_{1,n}(n-1).$$

With $c_{n,j} := \text{Cof}_{1,j}(n-1)/\text{Cof}_{1,1}(n-1)$, we obtain

$$\frac{A_{s,1}^{\mu}(n)}{B_{s-1,1}^{\mu+3}(n-1)} = \sum_{j=1}^n \tilde{a}_{1,j} \cdot c_{n,j}$$

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Laplace expansion:

$$A_{s,1}^{\mu}(n) = \det \begin{pmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \cdots & \tilde{a}_{1,n} \\ 1 & & & \\ & & \mathcal{B}_{s-1,1}^{\mu+3}(n-1) & \\ & & & 1 \end{pmatrix}$$
$$= \tilde{a}_{1,1} \cdot \text{Cof}_{1,1}(n-1) + \dots + \tilde{a}_{1,n} \cdot \text{Cof}_{1,n}(n-1).$$

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$$\frac{A_{s,1}^{\mu}(n)}{B_{s-1,1}^{\mu+3}(n-1)} = \sum_{j=1}^n \tilde{a}_{1,j} \cdot c_{n,j} \stackrel{!}{=} R_{s,1}^{\mu}(n).$$

Proof via the Holonomic Ansatz

Guess: $c_{n,j}$ satisfies a holonomic system of recurrence equations.

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$$\begin{aligned} p_{0,2}^{[1]} \cdot c_{n,j+2} + p_{1,0}^{[1]} \cdot c_{n+1,j} + p_{0,1}^{[1]} \cdot c_{n,j+1} + p_{0,0}^{[1]} \cdot c_{n,j} &= 0 \\ p_{1,1}^{[2]} \cdot c_{n+1,j+1} + p_{1,0}^{[2]} \cdot c_{n+1,j} + p_{0,1}^{[2]} \cdot c_{n,j+1} + p_{0,0}^{[2]} \cdot c_{n,j} &= 0 \\ p_{2,0}^{[3]} \cdot c_{n+2,j} + p_{1,0}^{[3]} \cdot c_{n+1,j} + p_{0,1}^{[3]} \cdot c_{n,j+1} + p_{0,0}^{[3]} \cdot c_{n,j} &= 0 \end{aligned}$$

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$$\begin{aligned}p_{2,0}^{[3]} = & -(j - 2n - 4)(j - 2n - 3)(\mu + 6n + 5)(\mu + 6n + 7)(\mu + \\& 6n + 9)(n + r - 1)(n + r)(j + \mu + 2n + 3)(j + \mu + 2n + 4)(2j^4 + \\& 3j^3\mu - 6j^3n + j^3 + j^2\mu^2 - 12j^2\mu n - 3j^2\mu + 12j^2n^2 - 30j^2n - \\& 8j^2 - 4j\mu^2n - 2j\mu^2 + 24j\mu n^2 - 8j\mu n - 6j\mu + 72jn^2 + 12jn - 4j + \\& 8\mu^2n^2 + 4\mu^2n + 40\mu n^2 + 20\mu n + 48n^2 + 24n)(\mu + 2n + 2r)(\mu + \\& 2n + 2r + 1)(\mu + 2n + 2r + 2)(\mu + 2n + 2r + 3)(\mu + 4n + 2r + 1)\end{aligned}$$

Proof via the Holonomic Ansatz

Prove: in the case where (A, B, s, n) is $(D, E, 2r, 2m)$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R_{2r,1}^{\mu}(2m).$$

Proof via the Holonomic Ansatz

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- ▶ Abandon the original definition $c_{n,j} := \frac{\text{Cof}_{1,j}(n-1)}{\text{Cof}_{1,1}(n-1)}$.

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- ▶ Abandon the original definition $c_{n,j} := \frac{\text{Cof}_{1,j}(n-1)}{\text{Cof}_{1,1}(n-1)}$.
- ▶ Use the conjectured holonomic description for $c_{n,j}$ instead.

Proof via the Holonomic Ansatz

Prove: in the case where (A, B, s, n) is $(D, E, 2r, 2m)$

$$c_{2m,1} = 1,$$

$$\sum_{j=1}^{2m} \binom{\mu + i + j + 2r - 3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} = 0, \quad (2 \leq i \leq 2m),$$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R_{2r,1}^{\mu}(2m).$$

- ▶ Abandon the original definition $c_{n,j} := \frac{\text{Cof}_{1,j}(n-1)}{\text{Cof}_{1,1}(n-1)}$.
- ▶ Use the conjectured holonomic description for $c_{n,j}$ instead.
- ▶ The first two identities prove $c_{n,j} = \frac{\text{Cof}_{1,j}(n-1)}{\text{Cof}_{1,1}(n-1)}$.

Proof via the Holonomic Ansatz

Prove: in the case where (A, B, s, n) is $(D, E, 2r, 2m)$

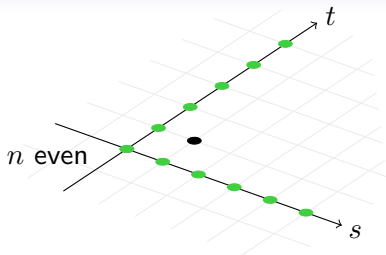
$$c_{2m,1} = 1,$$

$$\sum_{j=1}^{2m} \binom{\mu + i + j + 2r - 3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} = 0, \quad (2 \leq i \leq 2m),$$

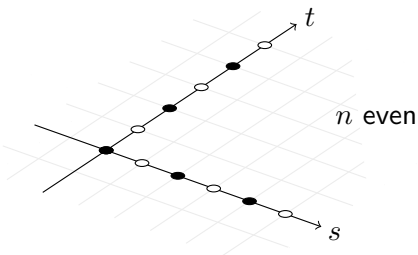
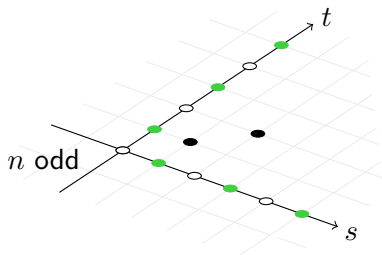
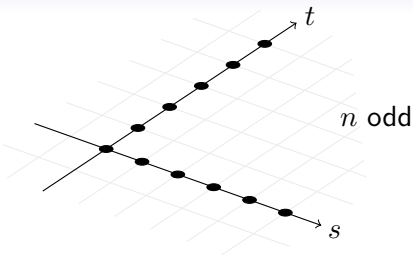
$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R_{2r,1}^{\mu}(2m).$$

- ▶ Abandon the original definition $c_{n,j} := \frac{\text{Cof}_{1,j}(n-1)}{\text{Cof}_{1,1}(n-1)}$.
- ▶ Use the conjectured holonomic description for $c_{n,j}$ instead.
- ▶ The first two identities prove $c_{n,j} = \frac{\text{Cof}_{1,j}(n-1)}{\text{Cof}_{1,1}(n-1)}$.
- ▶ The third identity proves the claimed quotient of determinants.

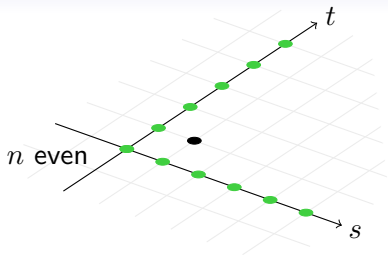
$E_{s,t}^\mu(n)$ Family



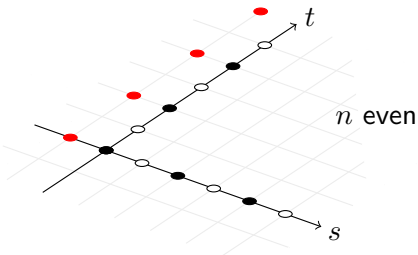
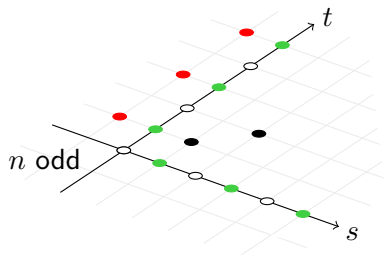
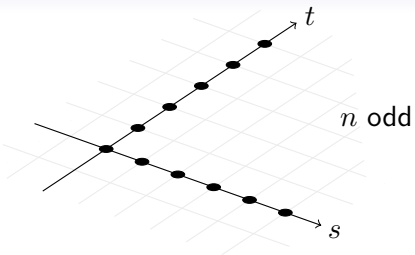
$D_{s,t}^\mu(n)$ Family



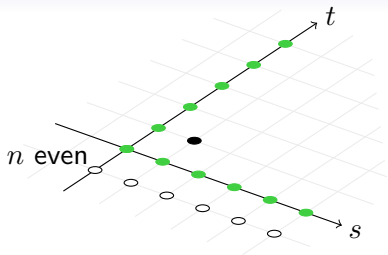
$E_{s,t}^\mu(n)$ Family



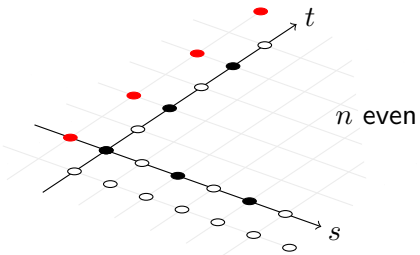
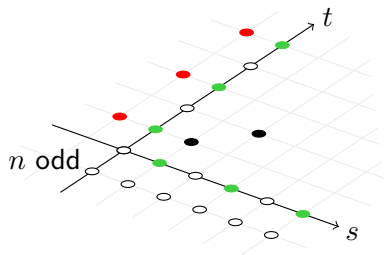
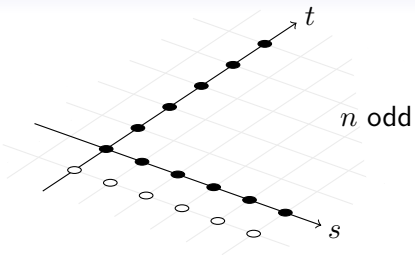
$D_{s,t}^\mu(n)$ Family



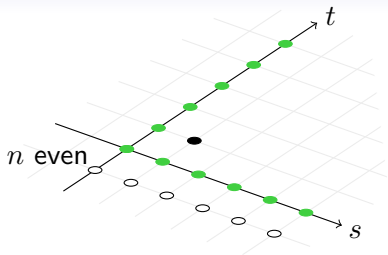
$E_{s,t}^\mu(n)$ Family



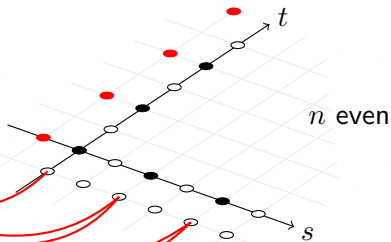
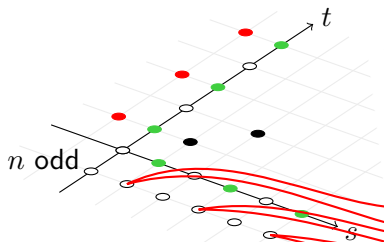
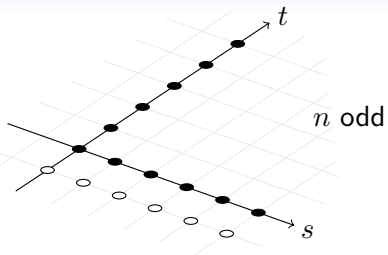
$D_{s,t}^\mu(n)$ Family



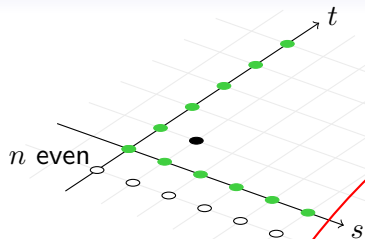
$E_{s,t}^\mu(n)$ Family



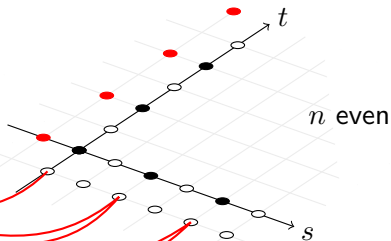
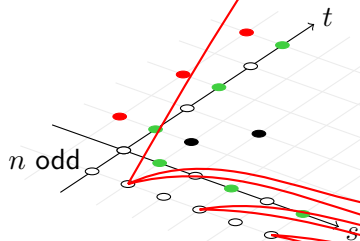
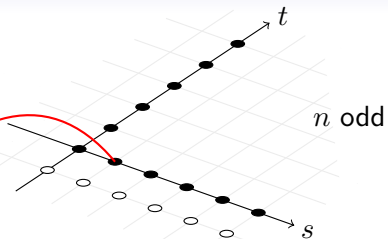
$D_{s,t}^\mu(n)$ Family



$E_{s,t}^\mu(n)$ Family



$D_{s,t}^\mu(n)$ Family



Proof of a Conjecture for $s = -1$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{-1, 2r-1}^{\mu}(2m+1) = \frac{(-1)^{m-r} (3-\mu) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r - \frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}} \right) \\ \times \left(\prod_{i=1}^{m-r} \frac{(\mu+2i+6r-3)_i^2 \left(\frac{\mu}{2} + 2i + 3r - 1\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 1\right)_{i-1}^2} \right).$$

Proof of a Conjecture for $s = -1$

$$\frac{D_{-1,2r}^{\mu}(2m)}{E_{-1,2r-1}^{\mu+3}(2m-1)}$$
$$\frac{E_{-1,2r+1}^{\mu}(2m+1)}{D_{-1,2r}^{\mu+3}(2m)}$$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{-1,2r-1}^{\mu}(2m+1) =$$
$$\frac{(-1)^{m-r} (3-\mu) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r - \frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}} \right)$$
$$\times \left(\prod_{i=1}^{m-r} \frac{(\mu+2i+6r-3)_i^2 \left(\frac{\mu}{2} + 2i + 3r - 1\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 1\right)_{i-1}^2} \right).$$

Proof of a Conjecture for $s = -1$

$$\frac{D_{2r,-1}^\mu(2m)}{E_{2r-1,-1}^{\mu+3}(2m-1)}$$
$$\frac{E_{2r+1,-1}^\mu(2m+1)}{D_{2r,-1}^{\mu+3}(2m)}$$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{-1,2r-1}^\mu(2m+1) =$$
$$\frac{(-1)^{m-r} (3-\mu) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r - \frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}} \right)$$
$$\times \left(\prod_{i=1}^{m-r} \frac{(\mu+2i+6r-3)_i^2 \left(\frac{\mu}{2} + 2i + 3r - 1\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 1\right)_{i-1}^2} \right).$$

Proof of a Conjecture for $s = -1$

$$\frac{D_{2r,-1}^\mu(2m)}{E_{2r-1,-1}^{\mu+3}(2m-1)} = \frac{0}{0}$$

$$\frac{E_{2r+1,-1}^\mu(2m+1)}{D_{2r,-1}^{\mu+3}(2m)} = \frac{0}{0}$$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{-1,2r-1}^\mu(2m+1) = \frac{(-1)^{m-r} (3-\mu) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r - \frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}} \right) \\ \times \left(\prod_{i=1}^{m-r} \frac{(\mu+2i+6r-3)_i^2 \left(\frac{\mu}{2} + 2i + 3r - 1\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 1\right)_{i-1}^2} \right).$$

Proof of a Conjecture for $s = -1$

$$\frac{D_{2r+\varepsilon, -1+\varepsilon}^{\mu}(2m)}{E_{2r-1+\varepsilon, -1+\varepsilon}^{\mu+3}(2m-1)} =$$
$$\frac{E_{2r+1+\varepsilon, -1+\varepsilon}^{\mu}(2m+1)}{D_{2r+\varepsilon, -1+\varepsilon}^{\mu+3}(2m)} =$$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{-1, 2r-1}^{\mu}(2m+1) =$$
$$\frac{(-1)^{m-r} (3-\mu) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r - \frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}} \right)$$
$$\times \left(\prod_{i=1}^{m-r} \frac{(\mu+2i+6r-3)_i^2 \left(\frac{\mu}{2} + 2i + 3r - 1\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 1\right)_{i-1}^2} \right).$$

Proof of a Conjecture for $s = -1$

Lemma: For $m, r \in \mathbb{Z}$ and $m > r \geq 1$,

$$\lim_{\varepsilon \rightarrow 0} \frac{D_{2r+\varepsilon, -1+\varepsilon}^{\mu}(2m)}{E_{2r-1+\varepsilon, -1+\varepsilon}^{\mu+3}(2m-1)} = \frac{2r(2m-1)(\mu-3)(\mu+2m+2r-2)}{\mu(m+r)(\mu+2m-3)(\mu+2r-2)},$$

$$\lim_{\varepsilon \rightarrow 0} \frac{E_{2r+1+\varepsilon, -1+\varepsilon}^{\mu}(2m+1)}{D_{2r+\varepsilon, -1+\varepsilon}^{\mu+3}(2m)} = \frac{2m(2r+1)(\mu-3)(\mu+2m+2r)}{\mu(m+r+1)(\mu+2m-2)(\mu+2r-1)}.$$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geq r \geq 1$,

$$E_{-1, 2r-1}^{\mu}(2m+1) =$$

$$\frac{(-1)^{m-r} (3-\mu) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r - \frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}} \right)$$

$$\times \left(\prod_{i=1}^{m-r} \frac{(\mu+2i+6r-3)_i^2 \left(\frac{\mu}{2} + 2i + 3r - 1\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r - 1\right)_{i-1}^2} \right).$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{c} \mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) \\ \downarrow \text{det} \\ A_{s+\varepsilon, -1+\varepsilon}^\mu(n) \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^{\mu}(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccc} A_{s+\varepsilon, -1+\varepsilon}^{\mu}(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} \\ \downarrow \det & & \\ A_{s+\varepsilon, -1+\varepsilon}^{\mu}(n) & & \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccc} A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} \\ \downarrow \det & & \downarrow \det \\ A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccccc} A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{A} \\ \downarrow \det & & \downarrow \det & & \\ A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & & \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccccc} A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{A} \\ \downarrow \det & & \downarrow \det & & \downarrow \det \\ A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & \xrightarrow[\text{determinant}]{\text{linearity of}} & A + O(\varepsilon^2) \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccccccc} A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{A} & \xrightarrow[\text{truncation}]{\varepsilon = 0 \text{ Taylor}} & \tilde{A}^{\text{tr}} \\ \downarrow \det & & \downarrow \det & & \downarrow \det & & \\ A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & \xrightarrow[\text{determinant}]{\text{linearity of}} & A + O(\varepsilon^2) & & \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccccccc}
 A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{A} & \xrightarrow[\text{truncation}]{\varepsilon = 0 \text{ Taylor}} & \tilde{A}^{\text{tr}} \\
 \downarrow \det & & \downarrow \det & & \downarrow \det & & \downarrow \det \\
 A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & \xrightarrow[\text{determinant}]{\text{linearity of}} & A + O(\varepsilon^2) & \xrightarrow[\text{mod } \varepsilon^2]{=} & A + O(\varepsilon^2)
 \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccccccc}
 A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{A} & \xrightarrow[\text{truncation}]{\varepsilon = 0 \text{ Taylor}} & \tilde{A}^{\text{tr}} \\
 \downarrow \det & & \downarrow \det & & \downarrow \det & & \downarrow \det \\
 A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & \xrightarrow[\text{determinant}]{\text{linearity of}} & A + O(\varepsilon^2) & \xrightarrow[\text{mod } \varepsilon^2]{=} & A + O(\varepsilon^2) \\
 & & & & & & \uparrow = \\
 & & & & & & \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \tilde{B}^{\text{tr}} \right)
 \end{array}$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)}$$

$$\begin{array}{ccccccc}
 A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{A} & \xrightarrow[\text{truncation}]{\varepsilon = 0 \text{ Taylor}} & \tilde{A}^{\text{tr}} \\
 \downarrow \det & & \downarrow \det & & \downarrow \det & & \downarrow \det \\
 A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & \xrightarrow[\text{determinant}]{\text{linearity of}} & A + O(\varepsilon^2) & \xrightarrow[\text{mod } \varepsilon^2]{=} & A + O(\varepsilon^2) \\
 & & & & & & \downarrow \text{Laplace} \\
 & & & & & & \sum_{i=2}^n \tilde{a}_{i,1} \cdot \text{Cof}_{i,1}(n-1)
 \end{array}$$

$$\left(\begin{array}{c} \vdots \\ \tilde{B}^{\text{tr}} \\ \vdots \end{array} \right)$$

=

Proof Layout

$$\begin{array}{c}
 \frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)} \\
 \\
 \begin{array}{ccccccc}
 \mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{\mathcal{A}} & \xrightarrow[\text{truncation}]{\varepsilon = 0 \text{ Taylor}} & \tilde{\mathcal{A}}^{\text{tr}} \\
 \downarrow \det & & \downarrow \det & & \downarrow \det & & \downarrow \det \\
 A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & \xrightarrow[\text{determinant}]{\text{linearity of}} & A + O(\varepsilon^2) & \xrightarrow[\text{mod } \varepsilon^2]{=} & A + O(\varepsilon^2) \\
 & & & & & & \downarrow \text{Laplace} \\
 & & & & & & \sum_{i=2}^n \tilde{a}_{i,1} \cdot \text{Cof}_{i,1}(n-1)
 \end{array} \\
 \\
 \begin{array}{c}
 \left(\begin{array}{c} \vdots \\ \tilde{\mathcal{B}}^{\text{tr}} \\ \vdots \end{array} \right) \\
 \uparrow = \\
 \tilde{\mathcal{A}}^{\text{tr}}
 \end{array}
 \end{array}$$

$$c_{n,i} := \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon \cdot \text{Cof}_{i,1}(n-1)}{\text{Cof}_{1,1}(n-1)}$$

$$\sum_{i=2}^n \tilde{a}_{i,1} \cdot \text{Cof}_{i,1}(n-1)$$

Proof Layout

$$\frac{A_{s+\varepsilon, -1+\varepsilon}^\mu(n)}{B_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1)} = \sum_{i=2}^n \tilde{a}_{i,1} \cdot c_{n,i} + O(\varepsilon) \quad \left(\begin{array}{c} \vdots \\ \tilde{B}^{\text{tr}} \\ \vdots \end{array} \right)$$

$$\begin{array}{ccccccc} A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow[\text{operations}]{\text{row/column}} & \mathcal{LAR} & \xrightarrow[\text{column}]{\text{adapt 2nd}} & \tilde{A} & \xrightarrow[\text{truncation}]{\varepsilon = 0 \text{ Taylor}} & \tilde{A}^{\text{tr}} \\ \downarrow \det & & \downarrow \det & & \downarrow \det & & \downarrow \det \\ A_{s+\varepsilon, -1+\varepsilon}^\mu(n) & \xrightarrow{=} & A & \xrightarrow[\text{determinant}]{\text{linearity of}} & A + O(\varepsilon^2) & \xrightarrow[\text{mod } \varepsilon^2]{=} & A + O(\varepsilon^2) \end{array}$$

$$c_{n,i} := \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon \cdot \text{Cof}_{i,1}(n-1)}{\text{Cof}_{1,1}(n-1)}$$

$$\sum_{i=2}^n \tilde{a}_{i,1} \cdot \text{Cof}_{i,1}(n-1)$$

Laplace

Matrix Transformations

$$\mathcal{A}_{s+\varepsilon, -1+\varepsilon}^{\mu}(n) = \left(\begin{array}{c} \binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{s, j-i-1} \\ (1 \leq i, j \leq n) \end{array} \right)$$

Matrix Transformations

$$\mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) = \begin{pmatrix} \binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{s, j-i-1} \\ (1 \leq i, j \leq n) \end{pmatrix}$$

$$\mathcal{L}_n \cdot \mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) \cdot \mathcal{R}_n =$$

$$\begin{pmatrix} \binom{\mu+s-3+2\varepsilon}{-1+\varepsilon} & \binom{\mu+j+s-3+2\varepsilon}{j-2+\varepsilon} - \binom{\mu+s-3+2\varepsilon}{-2+\varepsilon} \pm \sum_{k=1}^j \delta_{s, k-2} \\ (2 \leq j \leq n) \\ \hline \binom{\mu+i+s-5+2\varepsilon}{-2+\varepsilon} & \binom{\mu+i+j+s-5+2\varepsilon}{j-3+\varepsilon} - \binom{\mu+i+s-5+2\varepsilon}{-3+\varepsilon} \mp \delta_{s, j-i} \\ (2 \leq i \leq n) & (2 \leq i, j \leq n) \end{pmatrix}$$

Matrix Transformations

$$\mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) = \left(\begin{array}{c} \binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{s, j-i-1} \\ (1 \leq i, j \leq n) \end{array} \right)$$

$$\mathcal{L}_n \cdot \mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) \cdot \mathcal{R}_n =$$

$$\left(\begin{array}{c|c} \binom{\mu+s-3+2\varepsilon}{-1+\varepsilon} & \binom{\mu+j+s-3+2\varepsilon}{j-2+\varepsilon} - \binom{\mu+s-3+2\varepsilon}{-2+\varepsilon} \pm \sum_{k=1}^j \delta_{s, k-2} \\ & (2 \leq j \leq n) \\ \hline \binom{\mu+i+s-5+2\varepsilon}{-2+\varepsilon} & \binom{\mu+i+j+s-5+2\varepsilon}{j-3+\varepsilon} - \binom{\mu+i+s-5+2\varepsilon}{-3+\varepsilon} \mp \delta_{s, j-i} \\ & (2 \leq i, j \leq n) \end{array} \right)$$

$$\left(\begin{array}{c|c|c} \frac{\varepsilon}{\mu+s-2} + O(\varepsilon^2) & 1 + O(\varepsilon) & \binom{\mu+j+s-3}{j-2} \pm \sum_{k=1}^j \delta_{s, k-2} + O(\varepsilon) \\ & & (3 \leq j \leq n) \\ \hline \frac{-\varepsilon}{(\mu+i+s-4)_2} + O(\varepsilon^2) & \frac{\varepsilon}{\mu+i+s-2} + O(\varepsilon^2) & \binom{\mu+i+j+s-5}{j-3} \mp \delta_{s, j-i} + O(\varepsilon) \\ & (2 \leq i \leq n) & (2 \leq i \leq n, 3 \leq j \leq n) \end{array} \right)$$

Matrix Transformations

$$\mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) = \begin{pmatrix} \binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{s, j-i-1} \\ (1 \leq i, j \leq n) \end{pmatrix}$$

$$\mathcal{L}_n \cdot \mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) \cdot \mathcal{R}_n =$$

$$\begin{pmatrix} \binom{\mu+s-3+2\varepsilon}{-1+\varepsilon} & \binom{\mu+j+s-3+2\varepsilon}{j-2+\varepsilon} - \binom{\mu+s-3+2\varepsilon}{-2+\varepsilon} \pm \sum_{k=1}^j \delta_{s, k-2} \\ (2 \leq j \leq n) \\ \hline \binom{\mu+i+s-5+2\varepsilon}{-2+\varepsilon} & \binom{\mu+i+j+s-5+2\varepsilon}{j-3+\varepsilon} - \binom{\mu+i+s-5+2\varepsilon}{-3+\varepsilon} \mp \delta_{s, j-i} \\ (2 \leq i, j \leq n) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\varepsilon}{\mu+s-2} + O(\varepsilon^2) & 1 + O(\varepsilon) & \binom{\mu+j+s-3}{j-2} \pm \sum_{k=1}^j \delta_{s, k-2} + O(\varepsilon) \\ (3 \leq j \leq n) \\ \hline \frac{-\varepsilon}{(\mu+i+s-4)_2} + O(\varepsilon^2) & \frac{\varepsilon}{\mu+i+s-2} + O(\varepsilon^2) & \binom{\mu+i+j+s-5}{j-3} \mp \delta_{s, j-i} + O(\varepsilon) \\ (2 \leq i \leq n) & (2 \leq i \leq n) & (2 \leq i \leq n, 3 \leq j \leq n) \end{pmatrix}$$

$$\mathcal{B}_{s-1+\varepsilon, -1+\varepsilon}^{\mu+3}(n-1) = \begin{pmatrix} \binom{\mu+i+j+s-3+2\varepsilon}{j-2+\varepsilon} \mp \delta_{s, j-i} \\ (1 \leq i, j \leq n-1) \end{pmatrix}$$

Matrix Transformations

$$\mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) = \begin{pmatrix} \binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{s, j-i-1} \\ (1 \leq i, j \leq n) \end{pmatrix}$$

$$\mathcal{L}_n \cdot \mathcal{A}_{s+\varepsilon, -1+\varepsilon}^\mu(n) \cdot \mathcal{R}_n =$$

$$\begin{pmatrix} \binom{\mu+s-3+2\varepsilon}{-1+\varepsilon} & \binom{\mu+j+s-3+2\varepsilon}{j-2+\varepsilon} - \binom{\mu+s-3+2\varepsilon}{-2+\varepsilon} \pm \sum_{k=1}^j \delta_{s, k-2} \\ (2 \leq j \leq n) \\ \hline \binom{\mu+i+s-5+2\varepsilon}{-2+\varepsilon} & \binom{\mu+i+j+s-5+2\varepsilon}{j-3+\varepsilon} - \binom{\mu+i+s-5+2\varepsilon}{-3+\varepsilon} \mp \delta_{s, j-i} \\ (2 \leq i, j \leq n) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\varepsilon}{\mu+s-2} + O(\varepsilon^2) & 1 + O(\varepsilon) & \binom{\mu+j+s-3}{j-2} \pm \sum_{k=1}^j \delta_{s, k-2} + O(\varepsilon) \\ (3 \leq j \leq n) \\ \hline \frac{-\varepsilon}{(\mu+i+s-4)_2} + O(\varepsilon^2) & \frac{\varepsilon}{\mu+i+s-2} + O(\varepsilon^2) & \binom{\mu+i+j+s-5}{j-3} \mp \delta_{s, j-i} + O(\varepsilon) \\ (2 \leq i \leq n) & (2 \leq i \leq n) & (2 \leq i \leq n, 3 \leq j \leq n) \end{pmatrix}$$

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The Holonomic Ansatz

$$\sum_{i=2}^n \frac{-1}{(\mu + i + s - 4)_2} \cdot c_{n,i} = \frac{2s(n-1)(\mu-3)(\mu+n+s-2)}{\mu(n+s)(\mu+n-3)(\mu+s-2)}$$

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- ▶ The third identity proves the claimed quotient of determinants.

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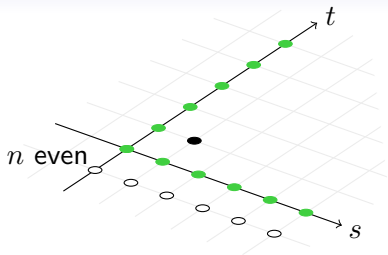
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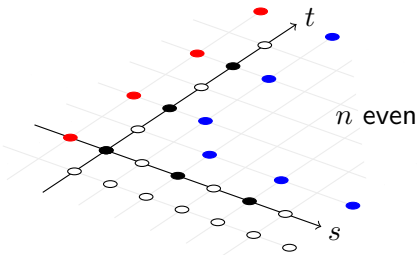
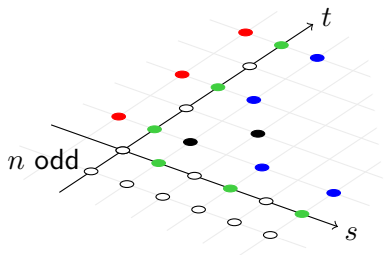
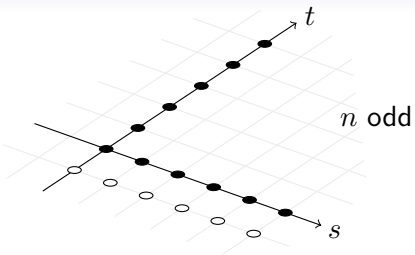
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- ▶ intermediate results are quite large (several 100 MB) due to the extra two parameters μ and r
- ▶ some of the certificates had poles close to the summation boundaries
- ▶ additional sums coming from the Kronecker deltas in combination with the row and column operations, some of which having non-natural boundaries

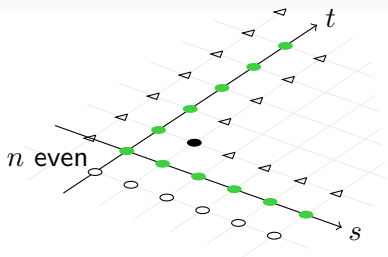
$E_{s,t}^\mu(n)$ Family



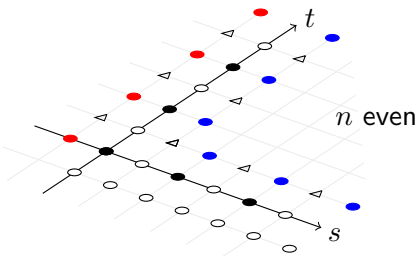
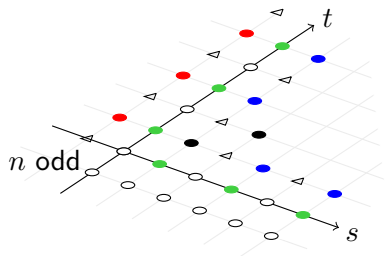
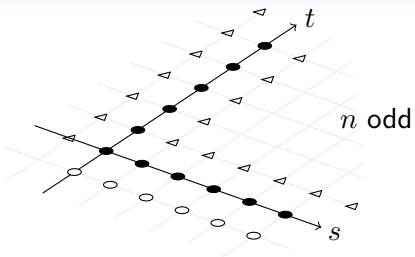
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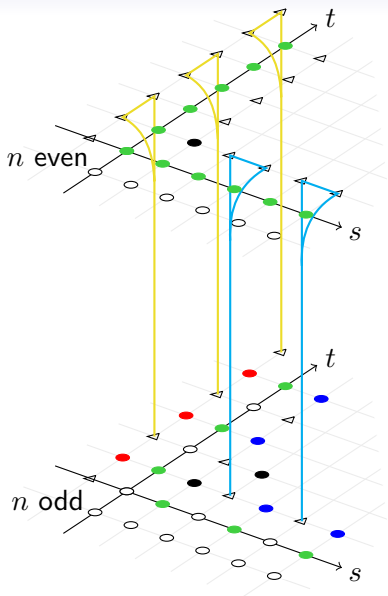
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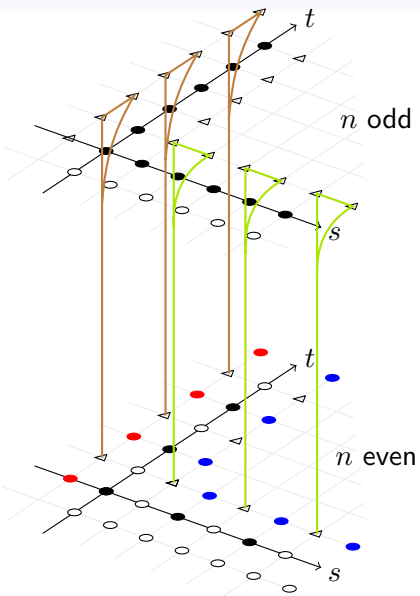
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Triangle Relations

Corollary: Let μ be an indeterminate, and let $m, r \in \mathbb{Z}$.
If $m > r \geq 1$, then

$$\frac{E_{2r,1}^{\mu}(2m+1)}{E_{2r,1}^{\mu}(2m)} =$$

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$$\frac{E_{2r,1}^{\mu}(2m+1)}{E_{2r+1,1}^{\mu}(2m)} = \frac{(-1)^{m-r} \left(\frac{\mu}{2} + 2m + r + 1\right)_{m-r} \left(\frac{\mu}{2} + 3r - \frac{1}{2}\right)_{m-r+1}}{\left(\frac{3}{2}\right)_{m-r} (m-r)_{m-r}},$$

$$\frac{E_{2r+1,1}^{\mu}(2m)}{E_{2r,1}^{\mu}(2m)} = \frac{(-1)^{m-r} \left(\frac{1}{2}\right)_{m-r+1} (\mu + 2m + 4r - 1)_{m-r+1}}{(2m - 2r + 1) \left(\frac{\mu}{2} + m + 2r\right)_{m-r} \left(\frac{\mu}{2} + 3r - \frac{1}{2}\right)_{m-r+1}}.$$

Conjecture for $D_{1,1}^\mu(n)$

$D_{1,1}(n) = C(n) F(n) G(\lfloor \frac{1}{2}(n+1) \rfloor)$, where

$$C(n) = \frac{(-1)^n + 3}{2} \prod_{i=1}^n \frac{\lfloor \frac{i}{2} \rfloor!}{i!},$$

$$E(n) = (\mu + 1)_n \left(\prod_{i=1}^{\lfloor \frac{3}{2} \lfloor \frac{1}{2}(n-1) \rfloor - 2 \rfloor} (\mu + 2i + 6)^{2 \lfloor \frac{1}{3}(i+2) \rfloor} \right) \cdot \left(\prod_{i=1}^{\lfloor \frac{3}{2} \lfloor \frac{n}{2} \rfloor - 2 \rfloor} (\mu + 2i + 2 \lfloor \frac{3}{2} \lfloor \frac{n}{2} \rfloor + 1 \rfloor - 1) \right)^{2 \lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor - \frac{1}{3}(i-1) \rfloor - 1},$$

$$F_m(n) = \left(\prod_{i=1}^{\lfloor \frac{1}{4}(n-1) \rfloor} (\mu + 2i + n + m)^{1-2i-m} \right) \cdot \left(\prod_{i=1}^{\lfloor \frac{n}{4} - 1 \rfloor} (\mu - 2i + 2n - 2m + 1)^{1-2i-m} \right),$$

$$F(n) = \begin{cases} E(n)F_0(n), & \text{if } n \text{ is even,} \\ E(n)F_1(n) \prod_{i=1}^{\frac{1}{2}(n-5)} (\mu + 2i + 2n - 1), & \text{if } n \text{ is odd,} \end{cases}$$

$$T(k) = 55296k^6 + 41472(\mu - 1)k^5 + 384(30\mu^2 - 66\mu + 53)k^4 + 96(\mu - 1)(15\mu^2 - 42\mu + 61)k^3 \\ + 4(19\mu^4 - 122\mu^3 + 419\mu^2 - 544\mu + 72)k^2 + (\mu - 1)(\mu^4 - 14\mu^3 + 101\mu^2 - 160\mu - 84)k \\ + 2(\mu - 3)(\mu - 2)(\mu - 1)(\mu + 1),$$

$$S_1(n) = \sum_{k=1}^{n-1} \frac{2^{6k}(\mu + 8k - 1) \left(\frac{1}{2}\right)_{2k-1}^2 \left(\frac{1}{2}(\mu + 5)\right)_{2k-3} \left(\frac{1}{2}(\mu + 4k + 2)\right)_{k-2} \left(\frac{1}{2}(\mu + 4k + 2)\right)_{2n-2k-2}}{(2k)! \left(\frac{1}{2}(\mu + 6k - 3)\right)_{3k+4}} \cdot T(k),$$

$$S_2(n) = \sum_{k=1}^{n-1} \frac{2^{6k}(\mu + 8k + 3) \left(\frac{1}{2}\right)_{2k}^2 \left(\frac{1}{2}(\mu + 5)\right)_{2k-2} \left(\frac{1}{2}(\mu + 4k + 4)\right)_{k-2} \left(\frac{1}{2}(\mu + 4k + 4)\right)_{2n-2k-2}}{(2k+1)! \left(\frac{1}{2}(\mu + 6k + 1)\right)_{3k+5}} \cdot T(k + \frac{1}{2}),$$

$$P_1(n) = 2^{3n-1} \frac{\left(\frac{1}{2}(\mu + 6n - 3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu + 5)\right)_{2n-3}} \left(\frac{\left(\frac{1}{2}(\mu + 2)\right)_{2n-2}}{(\mu + 3)^2} + \frac{\mu(\mu - 1)S_1(n)}{2^{13}} \right),$$

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$$G(n) = \begin{cases} P_1\left(\frac{1}{2}(n+1)\right), & \text{if } n \text{ is odd,} \\ P_2\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Theorem for $D_{1,1}^\mu(n)$

Let ρ_k be defined as $\rho_0(a, b) = a$ and $\rho_k(a, b) = b$ for $k > 0$.

If n is an odd positive integer then

$$D_{1,1}^\mu(n) = \sum_{k=0}^{(n+1)/2} \rho_k \left(4(\mu - 2), \frac{1}{(2k-1)!} \right) \frac{(\mu-1)_{3k-2}}{2 \left(\frac{\mu}{2} + k - \frac{1}{2}\right)_{k-1}} \left(\prod_{j=1}^{k-1} \frac{(\mu+2j+1)_{j-1} \left(\frac{\mu}{2} + 2j + \frac{1}{2}\right)_{j-1}}{(j)_{j-1} \left(\frac{\mu}{2} + j + \frac{1}{2}\right)_{j-1}} \right)^2$$

$$F_m(n) = \left(\prod_{i=1}^{(n-1)/2} (\mu+2i)^2 \left(\frac{\mu}{2} + 2i - \frac{1}{2}\right)_i \left(\frac{\mu}{2} + 2i + \frac{3}{2}\right)_{j+1} \right)$$

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$$D_{1,1}^\mu(n) = \sum_{k=0}^{n/2} \rho_k \left(4(\mu - 2), \frac{1}{(2k+1)!} \right) \frac{(\mu-1)_{3k-2}}{2 \left(\frac{\mu}{2} + k + \frac{1}{2}\right)_{k-1}} \left(\prod_{j=1}^{k-1} \frac{(\mu+2j+1)_{j-1} \left(\frac{\mu}{2} + 2j + \frac{1}{2}\right)_{j-1}}{(j)_{j-1} \left(\frac{\mu}{2} + j + \frac{1}{2}\right)_{j-1}} \right)^2$$

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(can be extended to recurrences, q -difference equations, etc.)

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Many special functions can be characterized as solutions to systems of linear differential equations and recurrences, and in fact are holonomic.

Table of Integrals by Gradshteyn and Ryzhik

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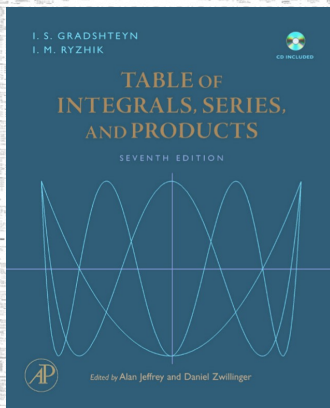


Table of Integrals by Gradshteyn and Ryzhik

The image displays a comprehensive table of integrals, organized into columns and rows. Each entry typically includes a mathematical expression involving variables like x , a , b , and n , and a corresponding integral result. The table is densely packed with formulas, covering a wide range of mathematical functions and operations. The layout is consistent, with each formula occupying a small, uniform space within its respective cell. The text is small and black on a white background, typical of a technical reference work. The overall structure is a grid of approximately 20 columns and 20 rows of formulas.

Table of Integrals by Gradshteyn and Ryzhik

7.20	7.21	7.22	7.23	7.24	7.25	7.26	7.27	7.28	7.29	7.30	7.31	7.32	7.33	7.34	7.35	7.36	7.37	7.38	7.39	7.40	7.41	7.42	7.43	7.44	7.45	7.46	7.47	7.48	7.49	7.50	7.51	7.52	7.53	7.54	7.55	7.56	7.57	7.58	7.59	7.60	7.61	7.62	7.63	7.64	7.65	7.66	7.67	7.68	7.69	7.70	7.71	7.72	7.73	7.74	7.75	7.76	7.77	7.78	7.79	7.80	7.81	7.82	7.83	7.84	7.85	7.86	7.87	7.88	7.89	7.90	7.91	7.92	7.93	7.94	7.95	7.96	7.97	7.98	7.99	8.00
7.20	7.21	7.22	7.23	7.24	7.25	7.26	7.27	7.28	7.29	7.30	7.31	7.32	7.33	7.34	7.35	7.36	7.37	7.38	7.39	7.40	7.41	7.42	7.43	7.44	7.45	7.46	7.47	7.48	7.49	7.50	7.51	7.52	7.53	7.54	7.55	7.56	7.57	7.58	7.59	7.60	7.61	7.62	7.63	7.64	7.65	7.66	7.67	7.68	7.69	7.70	7.71	7.72	7.73	7.74	7.75	7.76	7.77	7.78	7.79	7.80	7.81	7.82	7.83	7.84	7.85	7.86	7.87	7.88	7.89	7.90	7.91	7.92	7.93	7.94	7.95	7.96	7.97	7.98	7.99	8.00

Table of Integrals by Gradshteyn and Ryzhik

7.319

$$1. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda(\gamma x^{1/2}) dx = (-1)^n \frac{\Gamma(\lambda+n)\Gamma(\mu)\Gamma(\nu)}{n!\Gamma(\lambda)\Gamma(\mu+\nu)} {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{ET II 191(41)a}$$

$$2. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda(\gamma x^{1/2}) dx = \frac{(-1)^n 2\gamma \Gamma(\mu)\Gamma(\lambda+n+1)\Gamma(\nu+\frac{1}{2})}{n!\Gamma(\lambda)\Gamma(\mu+\nu+\frac{1}{2})} \\ \times {}_3F_2\left(-n, n+\lambda+1, \nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 191(42)}$$

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

$$7.321 \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n!\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 281(7), MO 99a}$$

$$7.322 \int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu\left(\frac{x}{a}-1\right) e^{-bx} dx = (-1)^n \frac{\pi \Gamma(2\nu+n)}{n!\Gamma(\nu)} \left(\frac{a}{2b}\right)^\nu e^{-ab} I_{\nu+n}(ab) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 171(9)}$$

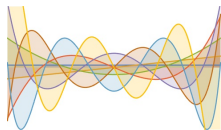
7.323

$$1. \int_0^\pi C_n^\nu(\cos \varphi) (\sin \varphi)^{2\nu} d\varphi = 0 \quad [n = 1, 2, 3, \dots]$$

Table of Integrals by Gradshteyn and Ryzhik

$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

Table of Integrals by Gradshteyn and Ryzhik



Gegenbauer
polynomials $C_n^{(\alpha)}(x)$


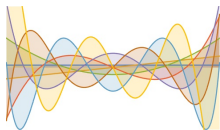
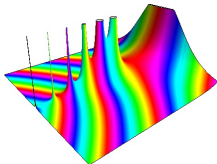

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Table of Integrals by Gradshteyn and Ryzhik



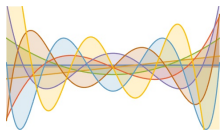
Gegenbauer
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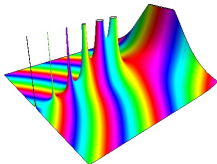
Gamma
function $\Gamma(x)$

$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

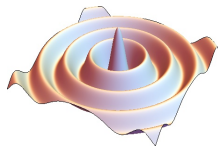
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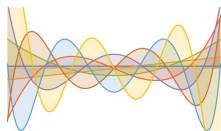
Gamma
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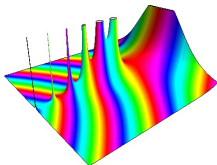
Bessel
function $J_\nu(x)$

$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

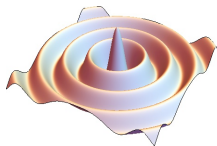
Table of Integrals by Gradshteyn and Ryzhik



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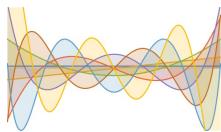


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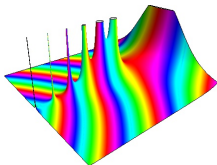
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- ▶ A large portion of such identities can be proven via the holonomic systems approach.

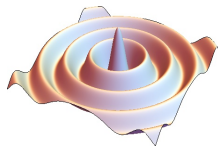
Table of Integrals by Gradshteyn and Ryzhik



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- ▶ A large portion of such identities can be proven via the holonomic systems approach.
- ▶ Algorithms are implemented in the HolonomicFunctions package.

The HolonomicFunctions Package

Example: Holonomic system, satisfied by both sides of the identity:

$$\begin{aligned}ia(n + 2\nu)f'_n(a) + a(n + 1)f_{n+1}(a) - in(n + 2\nu)f_n(a) &= 0, \\a(n + 1)(n + 2)f_{n+2}(a) - 2i(n + 1)(n + \nu + 1)(n + 2\nu + 1)f_{n+1}(a) \\- a(n + 2\nu)(n + 2\nu + 1)f_n(a) &= 0.\end{aligned}$$

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The HolonomicFunctions package is written in Mathematica, has been developed since 2007, and consists of $\sim 12,000$ lines of code.

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Special function identities is one application area, but there are also many holonomic functions without a name!