APÉRY LIMITS FOR ELLIPTIC L-VALUES

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Abstract. For an (irreducible) recurrence equation with coefficients from $\mathbb{Z}[n]$ and its two linearly independent rational solutions u_n, v_n, the limit of u_n/v_n as $n \to \infty$, when exists, is called the Apéry limit. We give a construction that realises certain quotients of L-values of elliptic curves as Apéry limits.

Apéry's famous proof [10] of the irrationality of $\zeta(3)$ displayed a particular phenomenon (which could have been certainly dismissed if discussed in the arithmetic context of some *boring* quantities). One considers the recurrence equation

$$(n+1)^3 v_{n+1} - (2n+1)(17n^2+17n+5)v_n + n^3 v_{n-1} = 0 \quad \text{for } n = 1, 2, \dots \tag{1}$$

and its two *rational* solutions u_n and v_n, where $n \ge 0$, originating from the initial data $u_0 = 0$, $u_1 = 6$ and $v_0 = 1$, $v_1 = 5$. Then v_n are in fact integral for any $n \ge 0$ and the denominators of u_n have a moderate growth with n — certainly not like $n!^3$ as suggested by the recursion — but $O(C^n)$ as $n \to \infty$, for some $C > 1$. Namely, $D_n^3 u_n \in \mathbb{Z}$ for all $n \ge 1$, where D_n denotes the least common multiple of $1, 2, \dots, n$; the asymptotics $D_n^{1/n} \to e$ as $n \to \infty$ is a consequence of the prime number theorem. An important additional property is that the quotient $u_n/v_n \to \zeta(3)$ as $n \to \infty$ (and also $u_n/v_n \ne \zeta(3)$ for *all* n); even sharper: $v_n\zeta(3) - u_n \to 0$ as $n \to \infty$; and at the highest level of sharpness we have $D_n^3(v_n\zeta(3) - u_n) \to 0$ as $n \to \infty$. It is the latter sharpest form that leads to the conclusion $\zeta(3) \notin \mathbb{Q}$. But already the arithmetic properties of u_n, v_n coupled with the 'irrational' limit relation $u_n/v_n \to \zeta(3)$ as $n \to \infty$ are phenomenal.

One way to prove all the above claims in one shot is to cast the sequence $I_n = v_n\zeta(3) - u_n$ as the Beukers triple integral [4]

$$I_n = \frac{1}{2}\int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad \text{for } n = 1, 2, \dots.$$

A routine use of creative telescoping machinery, based on the Almkvist–Zeilberger algorithm [2] (in fact, its multivariable version [3]), then shows that I_n indeed satisfies (1), while the evaluations $I_0 = \zeta(3)$ and $I_1 = 5\zeta(3) - 6$ are straightforward. The arithmetic and analytic properties follow from the analysis of the integrals I_n performed in [4]; more *practically*, they can be predicted/checked numerically based on the recurrence equation (1).

A common belief is that we have a better understanding of the phenomenon these days. Namely, we possess some (highly non-systematic!) recipes and strategies (see,

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for example, [1, 6, 7, 13, 15, 16]) for getting other meaningful constants c as *Apéry limits* — in other words, there are (irreducible) recurrence equations with coefficients from $\mathbb{Z}[n]$ such that for two *rational* solutions u_n, v_n we have $u_n/v_n \to c$ as $n \to \infty$ and the denominators of u_n, v_n are growing at most exponentially in n. (We may also consider *weak* Apéry limits when the latter condition on the growth of denominators is dropped.) Though one would definitely like to draw some conclusions about the irrationality of those constants c, this constraint for the arithmetic to be in the sharpest form would severly shorten the existing list of known Apéry limits; for example, it would throw out Catalan's constant from the list. A very basic question is then as follows.

Question. What real numbers can be realised as Apéry limits?

Without going at any depth into this direction, we present here a ('weak') construction of Apéry limits which are related to the L-values of elliptic curves (or of weight 2 modular forms). The construction emanates from identities, most of which remain conjectural, between the L-values and Mahler measures.

Consider the family of double integrals

$$\begin{aligned} J_n(z) &= \int_0^1\int_0^1 \frac{x^{n-1/2}(1-x)^{n-1/2}y^{n-1/2}(1-y)^n}{(1-zxy)^{n+1/2}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{\Gamma(n+\frac12)^3\Gamma(n+1)}{\Gamma(2n+1)\Gamma(2n+\frac32)}\cdot {}_3F_2\biggl(\begin{matrix} n+\frac12,\, n+\frac12,\, n+\frac12 \\ 2n+1,\, 2n+\frac32 \end{matrix}\biggm| z\biggr). \end{aligned}$$

Thanks to the nice hypergeometric representation, a recurrence equation satisfied by the double integral can be computed using Zeilberger's fast summation algorithm [3, 14], which is based on the method of creative telescoping. It leads to the following third-order recurrence equation:

$$\begin{aligned} &4z^4(2n+1)^2(n+1)^2\bigl(16(27z-32)n^4-16(69z-86)n^3 \\ &\quad +8(108z-143)n^2-4(55z-76)n+3(7z-10)\bigr)J_{n+1} \\ &+z^2\bigl(256(3z+8)(27z-32)n^8-256(3z+8)(15z-22)n^7 \\ &\quad -64(651z^2+661z-1744)n^6+192(59z^2-186)n^5 \\ &\quad +16(1503z^2+697z-3610)n^4-16(79z^2-290z+116)n^3 \\ &\quad -4(569z^2-381z-580)n^2+4(11z^2-44z+18)n+3(4z+3)(7z-10)\bigr)J_n \\ &+4n\bigl(64(3z^2-20z+16)(27z-32)n^7-384(3z^2-20z+16)(7z-9)n^6 \\ &\quad -16(411z^3-2698z^2+3988z-1696)n^5+64(183z^3-1372z^2+2339z-1134)n^4 \\ &\quad +4(531z^3-1400z^2-424z+1240)n^3-8(571z^3-4001z^2+6532z-3060)n^2 \\ &\quad +(151z^3-4742z^2+11596z-6888)n+12(14z^2-29z-30)(z-1)\bigr)J_{n-1} \\ &+4n(n-1)(2n-3)^2(z-1)\bigl(16(27z-32)n^4+48(13z-14)n^3 \\ &\quad +8(18z-11)n^2-4(19z-24)n-(7z+6)\bigr)J_{n-2}=0. \end{aligned}$$

Furthermore, if we take

$$\lambda(z) = J_0(z) = 2\pi\, {}_3F_2\biggl(\begin{matrix} \frac12,\, \frac12,\, \frac12 \\ 1,\, \frac32 \end{matrix}\biggm| z\biggr) = \int_0^1\int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{x(1-x)y(1-zxy)}},$$

$$\rho_1(z)=\pi\,{}_2F_1\biggl(\begin{matrix}\frac12,\ \frac12\\ 1\end{matrix}\biggm| z\biggr)=\int_0^1\frac{\mathrm{d}x}{\sqrt{x(1-x)(1-zx)}},$$
$$\rho_2(z)=\pi\,{}_2F_1\biggl(\begin{matrix}-\frac12,\ \frac12\\ 1\end{matrix}\biggm| z\biggr)=\int_0^1\frac{\sqrt{1-zx}}{\sqrt{x(1-x)}}\,\mathrm{d}x,$$

then $J_0(z)=\lambda(z)$,

$$J_1(z)=-\frac{3+4z}{4z^2}\,\lambda-\frac{5(1-z)}{z^2}\,\rho_1+\frac{13}{2z^2}\,\rho_2,$$
$$J_2(z)=\frac{105+480z+64z^2}{64z^4}\,\lambda+\frac{3151-2167z-984z^2}{144z^4}\,\rho_1-\frac{7247+3452z}{288z^4}\,\rho_2;$$

in other words, each $J_n(z)$ is a $\mathbb{Q}(z)$-linear combination of $\lambda(z),\rho_1(z),\rho_2(z)$. For $z^{-1}\in\mathbb{Z}\setminus\{\pm1\}$ we find out experimentally that the coefficients a_n,b_n,c_n (depending, of course, on this z^{-1}) in the representation

$$J_n(z)=a_n\lambda(z)+b_n\rho_1(z)+c_n\rho_2(z)$$

satisfy

$$z^n2^{4n}a_n,\ z^n2^{4n}D_{2n}^2b_n,\ z^n2^{4n}D_{2n}^2c_n\in\mathbb{Z}\qquad\text{for}\quad n=0,1,2,\dots.$$

Now observe that

$$\det\begin{pmatrix}J_n & J_{n+1}\\ c_n & c_{n+1}\end{pmatrix}=\det\begin{pmatrix}a_n & a_{n+1}\\ c_n & c_{n+1}\end{pmatrix}\cdot\lambda(z)+\det\begin{pmatrix}b_n & b_{n+1}\\ c_n & c_{n+1}\end{pmatrix}\cdot\rho_1(z)$$

for $n=0,1,2,\dots$. The sequences

$$A_n=\det\begin{pmatrix}a_n & a_{n+1}\\ c_n & c_{n+1}\end{pmatrix}\quad\text{and}\quad B_n=-\det\begin{pmatrix}b_n & b_{n+1}\\ c_n & c_{n+1}\end{pmatrix}$$

satisfy the following third-order (again!) recurrence equation which is the exterior square of the recurrence for J_n:

$$\begin{aligned}
&4(n+1)(n+2)^2(2n+1)^2(2n+3)^2z^8p_0(n)p_0(n-1)A_{n+1}\\
&\quad-4(n+1)^2(2n+1)^2z^4p_0(n-1)\bigl(64(3z^2-20z+16)(27z-32)n^7\\
&\qquad+64(3z^2-20z+16)(147z-170)n^6+16(3369z^3-26678z^2+44012z-20576)n^5\\
&\qquad+16(2457z^3-20918z^2+34376z-15896)n^4\\
&\qquad+4(843z^3-16808z^2+29432z-13736)n^3-4(1445z^3-6794z^2+9600z-4144)n^2\\
&\qquad-(741z^3-6922z^2+10772z-4728)n+z^2(131z-66)\bigr)A_n\\
&\quad-n(2n-1)^2(1-z)z^2p_0(n+1)\bigl(256(3z+8)(27z-32)n^8\\
&\qquad-256(3z+8)(15z-22)n^7-64(651z^2+661z-1744)n^6+192(59z^2-186)n^5\\
&\qquad+16(1503z^2+697z-3610)n^4-16(79z^2-290z+116)n^3\\
&\qquad-4(569z^2-381z-580)n^2+4(11z^2-44z+18)n+3(4z+3)(7z-10)\bigr)A_{n-1}\\
&\quad-4(n-1)n^2(2n-3)^2(2n-1)^2(1-z)^2p_0(n)p_0(n+1)A_{n-2}=0,
\end{aligned}$$

where

$$p_0(n)=16(27z-32)n^4+48(13z-14)n^3+8(18z-11)n^2-4(19z-24)n-(7z+6)$$

and

$$A_0=\frac{13}{2z^2},\quad A_1=\frac{395z^2-1051z+591}{72z^6},$$
$$A_2=\frac{15196z^4-201551z^3+548091z^2-543600z+183120}{3600z^{10}},$$

and

$$B_0=0,\quad B_1=\frac{1117z^2-2299z+1182}{72z^6},$$
$$B_2=\frac{6867z^4-65547z^3+156430z^2-143530z+45780}{450z^{10}}.$$

Furthermore, by construction

$$\lim_{n\to\infty}\frac{B_n}{A_n}=\frac{\lambda}{\rho_1}$$

and, still only experimentally and for $z^{-1}\in\mathbb{Z}\setminus\{\pm1\}$,

$$z^{2n+2}2^{2n}D_{2n}(n+1)(2n+1)^2A_n,\ z^{2n+2}2^{2n}D_{2n}^2(n+1)(2n+1)^2B_n\in\mathbb{Z}$$

for $n=0,1,2,\dots$. In other words, the number λ/ρ_1 (but also the quotients λ/ρ_2 and ρ_1/ρ_2) are (weak) Apéry limits for the values of z in consideration.

For real $k>0$ with $k^2\in\mathbb{Z}\setminus\{0,16\}$, the Mahler measure

$$\begin{aligned}\mu(k)&=\mathrm{m}(X+X^{-1}+Y+Y^{-1}+k)\\&=\frac{1}{(2\pi i)^2}\iint\limits_{|X|=|Y|=1}\log|X+X^{-1}+Y+Y^{-1}+k|\,\frac{\mathrm{d}X}{X}\,\frac{\mathrm{d}Y}{Y}\end{aligned}$$

is expected to be rationally proportional to the L-value

$$L'(E,0)=\frac{N}{(2\pi)^2}\,L(E,2)$$

of the elliptic curve $E=E_k: X+X^{-1}+Y+Y^{-1}+k=0$ of conductor $N=N_k=N(E_k)$. This is actually proven [5] when $k=1$, $\sqrt2$, 2, $2\sqrt2$ and 3 for the corresponding elliptic curves `15a8`, `56a1`, `24a4`, `32a1` and `21a4` labeled in accordance with the database [9]; the first number in the label indicates the conductor.

For the range $0<k<4$ we have the formula

$$\mu(k)=\frac{k}{4}\cdot{}_3F_2\biggl(\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ \frac32\end{matrix}\biggm|\frac{k^2}{16}\biggr),$$

thus linking $\mu(k)$ to $z^{-1/2}\lambda(z)/\pi$ at $z=k^2/16$. Furthermore, the quantity $z^{-1/2}\rho_1(z)$ in this case is rationally proportional to the imaginary part of the nonreal period of the same curve, while $z^{-1/2}\rho_2(z)$ is a $\mathbb{Q}$-linear combination of the imaginary parts of the nonreal period and the corresponding quasi-period. It means that in many cases we can record $z^{-1/2}\rho_1(z)$ as a rational multiple of the central L-value of a quadratic twist of the curve E. For example, when $k=2\sqrt2$ (hence $z=1/2$) the quadratic twist of the CM elliptic curve of conductor 32 coincides with itself and we have

$$\lambda\Bigl(\frac12\Bigr)=2\sqrt2\,\pi L'(E,0)=16\sqrt2\,\frac{L(E,2)}{\pi}\quad\text{and}\quad\rho_1\Bigl(\frac12\Bigr)=4\sqrt2\,L(E,1),$$

so that the recursion above with the choice $z = 1/2$ realises the quotient $L(E,2)/(\pi L(E,1))$ as an Apéry limit for an elliptic curve of conductor 32. When $k = 1$ we get

$$\lambda\Big(\frac{1}{16}\Big) = 8\pi L'(E,0) = 30\,\frac{L(E,2)}{\pi} \quad \text{and} \quad \rho_1\Big(\frac{1}{16}\Big) = \frac{1}{2}\,L(E,\chi_{-4},1)$$

for the twist of the elliptic curve by the quadratic character $\chi_{-4} = \big(\frac{-4}{\cdot}\big)$; this means that the quotient $L(E,2)/(\pi L(E,\chi_{-4},1))$ for an elliptic curve of conductor 15 is realised as an Apéry limit.

Clearly, the range $0 < k < 4$ has a limited supply of elliptic L-values. When $k > 4$, one can write

$$\mu(k) = \frac{1}{2\pi}\,f\bigg(\frac{16}{k^2}\bigg),$$

where

$$\begin{aligned} f(z) &= -\pi\Bigg(\log\frac{z}{16} + \frac{z}{4}\,{}_4F_3\bigg(\begin{matrix}\frac32,\,\frac32,\,1,\,1\\ 2,\,2,\,2\end{matrix}\;\bigg|\; z\bigg)\Bigg)\\ &= -\int_0^1 x^{-1/2}(1-x)^{-1/2}\log\frac{1-\sqrt{1-zx}}{1+\sqrt{1-zx}}\,\mathrm{d}x\\ &= \int_0^1\!\!\int_0^1 \frac{x^{-1/2}(1-x)^{-1/2}(1-zx)^{1/2}y^{-1/2}}{1-(1-zx)y}\,\mathrm{d}x\,\mathrm{d}y\\ &= Z\int_0^1\!\!\int_0^1 \frac{x^{-1/2}(1-x)^{-1/2}(1-x/Z)^{1/2}(1-y)^{-1/2}}{x(1-y)+yZ}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

with $Z = z^{-1} > 1$. At this point we see that the integrals resemble the integrals

$$Z^{-l-m}\int_0^1\!\!\int_0^1 \frac{x^j(1-x)^h y^k(1-y)^l}{(x(1-y)+yZ)^{j+k-m+1}}\,\mathrm{d}x\,\mathrm{d}y,$$

with h, j, k, l, m non-negative integers, appearing in the linear independence results for the dilogarithm [11, 12]. This similarity suggests looking at the family

$$L_n(Z) = \int_0^1\!\!\int_0^1 \frac{x^{n-1/2}(1-x)^{2n-1/2}(1-x/Z)^{1/2}y^n(1-y)^{n-1/2}}{(x(1-y)+yZ)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y,$$

where $Z = z^{-1}$ is a large (positive) integer. We tackle this double integral by iterated applications of creative telescoping: while the first integration (no matter whether one starts with x or with y) can be done with the Almkvist–Zeilberger algorithm, the second one requires more general holonomic methods, since the integrand is not any more hyperexponential. Using the `Mathematica` package `HolonomicFunctions` [8], where these algorithms are implemented, we find that the integral $L_n(Z)$ satisfies a lengthy fourth-order recurrence equation. Moreover, it turns out that $L_n(Z)$ is a $\mathbb{Q}(Z)$-linear combination of $\rho_1 = \rho_1(1/Z)$, $\rho_2 = \rho_2(1/Z)$, $\sigma_1 = L_0(Z)$ and

$$\sigma_2 = \sigma_2(Z) = \int_0^1\!\!\int_0^1 \frac{x^{-1/2}(1-x)^{1/2}(1-x/Z)^{1/2}(1-y)^{1/2}}{x(1-y)+yZ}\,\mathrm{d}x\,\mathrm{d}y.$$

One can produce a recurrence equation out of the one for $L_n(Z)$ to cast, for example, σ_1/ρ_1 as an Apéry limit. Because this finding does not meet any reasonable aesthetic requirements and does not imply anything (to be claimed) irrational, we leave it outside this note.

References

[1] G. Almkvist, D. van Straten and W. Zudilin, Apéry limits of differential equations of order 4 and 5, in *Modular forms and string duality* (Banff, June 3–8, 2006), N. Yui, H. Verrill and C. F. Doran (eds.), Fields Inst. Commun. Ser. **54** (Amer. Math. Soc., Providence, RI, 2008), 105–123.

[2] G. Almkvist and D. Zeilberger, The method of differentiating under the integral sign, *J. Symbolic Comput.* **10**:6 (1990), 571–591.

[3] M. Apagodu and D. Zeilberger, Multi-variable Zeilberger and Almkvist–Zeilberger algorithms and the sharpening of Wilf–Zeilberger theory, *Adv. in Appl. Math.* **37**:2 (2006), 139–152.

[4] F. Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* **11**:3 (1979), 268–272.

[5] F. Brunault and W. Zudilin, *Many variations of Mahler measures: a lasting symphony*, Aust. Math. Soc. Lecture Ser. **28** (Cambridge University Press, Cambridge, 2020).

[6] M. Chamberland and A. Straub, Apéry limits: experiments and proofs, *Amer. Math. Monthly* **128**:9 (2021), 811–824 .

[7] R. Dougherty-Bliss, C. Koutschan and D. Zeilberger, Tweaking the Beukers integrals in search of more miraculous irrationality proofs à la Apéry, *Ramanujan J.* (to appear); *Preprint* `arXiv:2101.08308 [math.NT]` (2021).

[8] C. Koutschan, HolonomicFunctions (user's guide), *RISC Report* 10-01 (2010), `http://www.risc.jku.at/research/combinat/software/HolonomicFunctions/`.

[9] The LMFDB Collaboration, *The L-functions and modular forms database*, `http://www.lmfdb.org`, 2021 (online; accessed 12 October 2021).

[10] A. van der Poorten, A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$. An informal report, *Math. Intelligencer* **1**:4 (1978/79), 195–203.

[11] G. Rhin and C. Viola, The permutation group method for the dilogarithm, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (5) **4**:3 (2005), 389–437.

[12] C. Viola and W. Zudilin, Linear independence of dilogarithmic values, *J. Reine Angew. Math.* **736** (2018), 193–223.

[13] D. B. Zagier, Integral solutions of Apéry-like recurrence equations, in *Groups and symmetries*, CRM Proc. Lecture Notes **47** (Amer. Math. Soc., Providence, RI, 2009), 349–366.

[14] D. Zeilberger, A fast algorithm for proving terminating hypergeometric identities, *Disc. Math.* **80**:2 (1990), 207–211.

[15] W. Zudilin, The birthday boy problem, *Preprint* `arXiv:2108.06586 [math.NT]` (2021).

[16] W. Zudilin, Apéry limits and Mahler measures, *Preprint* `arXiv:2109.12972 [math.NT]` (2021).

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