Creative Telescoping for Holonomic Functions

Christoph Koutschan

Abstract The aim of this article is twofold: on the one hand it is intended to serve as a gentle introduction to the topic of creative telescoping, from a practical point of view; for this purpose its application to several problems is exemplified. On the other hand, this chapter has the flavour of a survey article: the developments in this area during the last two decades are sketched and a selection of references is compiled in order to highlight the impact of creative telescoping in numerous contexts.

1 Introduction

The method of *creative telescoping* is a widely used paradigm in computer algebra, in order to treat symbolic sums and integrals in an algorithmic way. Its modus operandi is to derive, from an implicit description of the summand resp. integrand, e.g., in terms of recurrences or differential equations, an implicit description for the sum resp. integral. The latter can be used for proving an identity or for finding a closed form for the expression in question. Algorithms that use this idea are nowadays implemented in all major computer algebra systems. Meanwhile, they have been successfully applied to many problems from various areas of mathematics and physics, see Section 7 for a selection of such applications.

The key idea of creative telescoping is rather simple and works for summation problems as well as for integrals. For example, consider the problem of evaluating a

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sum of the form $F(n) = \sum_{k=a}^{b} f(k,n)$ for $a,b \in \mathbb{Z}$ and some bivariate sequence f. If one succeeds to find another bivariate sequence g and univariate sequences c_0 and c_1 such that the equation

$$c_1(n)f(k,n+1) + c_0(n)f(k,n) = g(k+1,n) - g(k,n) \tag{1}$$

holds, then a recurrence for the sum F is obtained by summing (1) with respect to k from a to b, and then telescoping the right-hand side:

$$c_1(n)F(n+1) + c_0(n)F(n) = g(b+1,n) - g(a,n).$$

For this reasoning to be nontrivial, one stipulates that the sequence g is given as a closed-form expression in terms of the input (this will be made precise later). Note that on the left-hand side of (1) one can have a longer linear combination of $f(k,n), \ldots, f(k,n+d)$, giving rise to a higher-order recurrence for F. This procedure works similarly for integrals, see Section 4 for a detailed exposition. In order to guarantee that a creative telescoping equation, like (1), exists, one requires that the summand f satisfies sufficiently many equations. This requirement leads to the concepts of *holonomic functions* and *∂-finite functions*; they will be introduced in Section 3.

The class of holonomic functions is quite rich and thus the method of creative telescoping applies to a wide variety of summation and integration problems. Just to give the reader an impression of this diversity, we list a random selection of identities that can be proven by the methods described in this article (where $P_n^{(a,b)}(x)$ denotes the Jacobi polynomials, $L_n^a(x)$ the Laguerre polynomials, $J_n(x)$ the Bessel function of the first kind, $H_n(x)$ the Hermite polynomials, $C_n^{(\lambda)}(x)$ the Gegenbauer polynomials, $\Gamma(n)$ the Gamma function, and $y_n(x)$ the spherical Bessel function of the second kind):

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3,$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x = \frac{\pi P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}},$$

$$\int_0^\infty e^{-t} t^{\frac{a}{2}+n} J_a\left(2\sqrt{tx}\right)\,\mathrm{d}t = e^{-x}x^{a/2} n! L_n^a(x),$$

$$\sum_{n=0}^{\infty} \frac{(-t)^n y_{n-1}(z)}{n!} = \frac{1}{z}\sin\left(\sqrt{z^2+2tz}\right),$$

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!n!}\,\mathrm{d}x = \sqrt{\pi}e^{2rs},$$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi 2^{1-\nu} i^n \Gamma(n+2\nu) a^{-\nu} J_{n+\nu}(a)}{n!\Gamma(\nu)}.$$

Further examples are discussed in Section 6 where we also demonstrate the usage of our Mathematica package HolonomicFunctions:

In[1]:= << HolonomicFunctions.m

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.6 (12.04.2012)

For further reading, we recommend the following textbooks: the classic source for hypergeometric summation is the wonderful book [80], although Zeilberger's algorithm made it already into the second edition of Concrete Mathematics [45], as well as into its recent "algorithmic supplement" [52]. A book that is completely dedicated to hypergeometric summation is [57]. We also would like to point the reader to the excellent survey articles [24, 59, 78, 100, 33] and to the theses [32, 61] for more detailed introductions to the topic of creative telescoping in the context of holonomic functions.

2 History and Developments

The notion *creative telescoping* was first coined by van der Poorten in his essay [92] on Apéry's proof of the irrationality of $\zeta(3)$. But certainly, the underlying principle was known and used long before as an ad hoc trick to tackle sums and integrals. The most famous example is the practice of *differentiating under the integral sign*, that was made popular by Feynman in his enjoyable book "Surely You're Joking, Mr. Feynman!" [40], see also [4]. It was Zeilberger who equipped creative telescoping with a concrete well-defined meaning and connected it to an algorithmic method [99].

The seminal paper that initiated all the developments presented here is Zeilberger's 1990 *holonomic systems approach* paper [98]. It sketches an algorithmic proof theory for identities among a large class of elementary and special functions, involving summation quantifiers and integrals. The main theorems are based on the theory of D-modules [13, 38], as well as the creative telescoping algorithm which uses a general, but inefficient, elimination procedure. Therefore, it was not really suited to be applied to real problems, except from some toy examples, and was later called "the slow algorithm" by Zeilberger, see Section 5.1. But very quickly, one realized the big potential that lied in these ideas. Takayama designed a method that is still based on elimination, but in a more sophisticated way using modules [90], see Section 5.2. In the same year—we're still in 1990—more efficient creative telescoping algorithms for special cases were formulated: Zeilberger's celebrated "fast algorithm" for hypergeometric single sums [97] and its differential analogue, the Almkvist-Zeilberger algorithm for the integration of hyperexponential functions [4]. The theory on which these two algorithms are built was developed by Wilf and Zeilberger [94] and was named *WZ theory* after its inventors, who were awarded the Leroy P. Steele Prize in 1998 for this seminal work.

In the following years the main focus of research in this field concentrated on hypergeometric summation. Certain extensions [55] and optimizations [83] of Zeilberger's algorithm and its q-analogue [76] were published. The problem of dealing with multiple sums was studied in more detail [93, 10, 31], also for q-hypergeometric terms [85]. Based on estimates on the order of the output recurrence and the largest integer root of its leading coefficient, Yen derived an a priori bound for the number of instances one has to check in order to get a rigorous proof of a (q-) hypergeometric summation identity [95, 96]; although these bounds are too large for real applications, this in principle allows to prove such identities by just verifying them on a finite set of special cases, without executing Zeilberger's algorithm explicitly. This bound was later improved drastically in [47]. Sharp bounds for the order of the telescoper that is computed by Zeilberger's algorithm and its q-analogue were derived in [72]. Abramov considered the question for which inputs the algorithm succeeds [3, 2].

In the late 1990s a return to the original ideas of Zeilberger started, namely to consider general holonomic functions instead of only (q-) hypergeometric / hyperexponential expressions. This development was initiated by Chyzak and Salvy [36, 32] and culminated in a generalization of Zeilberger's algorithm to holonomic functions [34] that is now known as Chyzak's algorithm, see Section 5.3. This work was picked up in [61] where several nontrivial applications of creative telescoping were presented. A fast but heuristic approach to the computation of creative telescoping relations for general holonomic functions was then given in [63], see Section 5.4.

During the last few years, a new interest in creative telescoping algorithms arose. The main motivation was to understand the complexity of such algorithms, a question that had been neglected during the two preceding decades. This research finally also led to new algorithmic ideas. A first attempt to study the complexity of creative telescoping was made in [18], but this investigation was restricted to bivariate rational functions as inputs. The problem of predicting the order and the degree of the coefficients of the output was largely solved in [28] for the hyperexponential case and in [27] for the hypergeometric case. Both articles also discuss the trading of order for degree, i.e., the option of computing an equation with lower coefficient degree at the cost of a larger order and vice versa; this trade-off can be used to reduce the complexity of the algorithms. The question of existence criteria for creative telescoping relations for mixed hypergeometric terms was answered in [26]. Concerning new creative telescoping algorithms, the use of residues for the computation of telescopers has been investigated in [30] for rational functions and in [29] for algebraic functions. Further innovations include an algorithm for hyperexponential functions based on Hermite reduction [19] and new algorithm for rational functions [22] using the Griffiths-Dwork method.

Since our focus is on creative telescoping for holonomic functions, we mention only briefly some other settings in which this method can be realized. The first algorithm for a class of non-holonomic sequences was given in [71], where Abel-type sums were considered. An algorithm for summation of expressions involving Stirling numbers and similar non-holonomic bivariate sequences was invented in [50]. Closure properties and creative telescoping for general non-holonomic functions were presented in [35]. In the setting of difference fields, Schneider developed a sophisticated symbolic summation theory [86] whose core again is creative telescoping. For more information on this topic we refer to the book chapter [87]. Similarly, see [82] for creative telescoping in differential fields.

We have already mentioned that algorithms based on creative telescoping are part of many computer algebra systems. For example, Zeilberger's fast algorithm [97] for hypergeometric summation has been implemented in Maple [59, 80], shortly after its invention. In current Maple versions it is available by the command SumTools[Hypergeometric][Zeilberger]. Other implementations of Zeilberger's algorithm are in Mathematica [77], in Reduce [56], and in Macsyma [25]. Its differential analogue, the Almkvist-Zeilberger algorithm [4], can be called by DEtools[Zeilberger] in Maple. For the q-analogue, Zeilberger's algorithm for q-hypergeometric summation, there exist implementations in Mathematica [84, 76] and in Maple [16], see also the command QDifferenceEquations[Zeilberger] there. Packages for multiple sums have been written in Mathematica, namely MultiSum [93] for hypergeometric summands and its q-version qMultiSum [85] for q-hypergeometric multi-sums. Multiple integrals can be treated with the Maple package MultInt [91]. Finally, there are two software packages for creative telescoping of general holonomic functions, which are not restricted to (q-) hypergeometric / hyperexponential inputs, i.e., expressions satisfying first-order equations: Mgfun [32] for Maple and HolonomicFunctions [64] for Mathematica.

3 Holonomic and ∂-Finite Functions

In order to state, in an algebraic language, the concepts that are introduced in this section, and for writing mixed difference-differential equations in a concise way, the following operator notation is employed: let D_x denote the partial derivative operator with respect to x (x is then called a *continuous variable*) and S_n the forward shift operator with respect to n (n is then called a *discrete variable*); they act on a function f by

$$D_x f = \frac{\partial f}{\partial x} \quad \text{and} \quad S_n f = f\big|_{n \to n+1}.$$

They allow us to write linear homogeneous difference-differential equations in terms of operators, e.g.,

$$\frac{\partial}{\partial x} f(k,n+1,x,y) + n\frac{\partial}{\partial y} f(k,n,x,y) + x f(k+1,n,x,y) - f(k,n,x,y) = 0$$

turns into

$$\big(D_x S_n + n D_y + x S_k - 1\big) f(k,n,x,y) = 0,$$

in other words, such equations are represented by polynomials in the operator symbols D_x, S_n, etc., with coefficients in some field $\mathbb{F}$ which we assume to be of characteristic 0. Note that the polynomial ring $\mathbb{F}\langle D_x, S_n, \ldots\rangle$ is not necessarily commutative, a fact that is indicated by the angle brackets. Its multiplication is subject to the rules

$$D_x \cdot a(x) = a(x) \cdot D_x + a'(x) \quad \text{and} \quad S_n \cdot a(n) = a(n+1) \cdot S_n.$$

Typically, $\mathbb{F}$ is a rational function field in the variables x, n, etc. over $\mathbb{Q}$ or over some other field $\mathbb{K}$. Such non-commutative rings of operators were introduced in [73] and are called *Ore algebras*. We use the symbol ∂ to denote an arbitrary operator symbol from an Ore algebra, so that ∂_w may stand for S_w or D_w, for example. Thus, a generic Ore algebra can be written as $\mathbb{O} = \mathbb{F}\langle \boldsymbol{\partial_w} \rangle$ with, e.g., $\mathbb{F} = \mathbb{Q}(\boldsymbol{w})$, where $\boldsymbol{w} = w_1, \ldots, w_\ell$ and $\boldsymbol{\partial_w} = \partial_{w_1}, \ldots, \partial_{w_\ell}$. We define the *annihilator* (w.r.t. some Ore algebra $\mathbb{O}$) of a function f:

$$\mathrm{Ann}_{\mathbb{O}}(f) := \{P \in \mathbb{O} \mid P(f) = 0\}.$$

It can easily be seen that $\mathrm{Ann}_{\mathbb{O}}(f)$ is a left ideal in $\mathbb{O}$. Every left ideal $I \subseteq \mathrm{Ann}_{\mathbb{O}}(f)$ is called an *annihilating ideal* for f. In the holonomic systems approach, functions are represented by annihilating ideals (plus initial values) as a data structure. When working with left ideals, we use *left Gröbner bases* [23, 49] which are an important tool for executing certain operations algorithmically (e.g., for deciding the ideal membership problem).

Definition 1. Let $\mathbb{O} = \mathbb{F}\langle \boldsymbol{\partial_w} \rangle$ be an Ore algebra. A function f is called *∂-finite* or D-finite w.r.t. $\mathbb{O}$ if $\mathbb{O}/\mathrm{Ann}_{\mathbb{O}}(f)$ is a finite-dimensional $\mathbb{F}$-vector space. Its dimension is called the *rank* of f w.r.t. $\mathbb{O}$.

Example 1. Consider the family of Laguerre polynomials $L_n^a(x)$ as an example of a ∂-finite function w.r.t. $\mathbb{O} = \mathbb{Q}(n,a,x)\langle S_n, S_a, D_x \rangle$. The left ideal $I = \mathrm{Ann}_{\mathbb{O}}(L_n^a(x))$ is generated by the following three operators that can be easily obtained with the HolonomicFunctions package:

In[2]:= $\mathrm{Annihilator}[\mathrm{LaguerreL}[n,a,x],\{\mathrm{S}[n],\mathrm{S}[a],\mathrm{Der}[x]\}]$

Out[2]= $\{S_a + D_x - 1, (n+1)S_n - xD_x + (-a-n+x-1), xD_x^2 + (a-x+1)D_x + n\}$

These operators represent well-known identities for Laguerre polynomials. Moreover, they are a left Gröbner basis of I with respect to the degree-lexicographic order. Thus, from the leading monomials S_a, S_n, and D_x^2, one can easily read off that the dimension of the $\mathbb{Q}(n,a,x)$-vector space $\mathbb{O}/I$ is two, in other words: $L_n^a(x)$ is ∂-finite w.r.t. $\mathbb{O}$ of rank 2.

Without proof we state the following theorem about *closure properties* of ∂-finite functions; its proof can be found in [61, Chap. 2.3]. We remark that all of them are algorithmically executable, and the algorithms work with the above mentioned data structure.

Theorem 1. *Let $\mathbb{O}$ be an Ore algebra and let f and g be ∂-finite w.r.t. $\mathbb{O}$ of rank r and s, respectively. Then*
(a) $f+g$ is ∂-finite of rank $\leqslant r+s$.
(b) $f\cdot g$ is ∂-finite of rank $\leqslant r\cdot s$.
(c) Pf is ∂-finite of rank $\leqslant r$ for any $P\in\mathbb{O}$.
(d) $f|_{x\to A(x,y,\dots)}$ is ∂-finite of rank $\leqslant r\cdot d$ if $x,y,\dots$ are continuous variables and if A satisfies a polynomial equation of degree d.
(e) $f|_{n\to A(m,n,\dots)}$ is ∂-finite of rank $\leqslant r$ if A is an integer-linear expression in the discrete variables $m,n,\dots$.

If we want to consider integration and summation problems, then the function in question needs to be *holonomic*, a concept that is closely related to ∂-finiteness. The precise definition is a bit technical and therefore skipped here; the interested reader can find it, e.g., in [98, 38, 61]. The closure properties for ∂-finite functions are also valid for holonomic functions. Additionally, the following theorem establishes the closure of holonomic functions with respect to sums and integrals; for its proof, we once again refer to [98, 61].

Theorem 2. *Let the function f be holonomic w.r.t. D_x (resp. S_n). Then also $\int_a^b f\,\mathrm{d}x$ (resp. $\sum_{n=a}^b f$) is holonomic.*

All holonomic functions that appear in this article are also ∂-finite and vice versa; therefore we will not continue to care about this subtle distinction, but only talk about holonomic functions from now on. A more elaborate introduction to holonomic and ∂-finite functions is given in [51].

4 Creative Telescoping for Holonomic Functions

In order to treat a sum of the form $F(\boldsymbol{w})=\sum_{k=a}^b f(k,\boldsymbol{w})$ with creative telescoping, one has to find an operator P which annihilates f, i.e., $Pf=0$, and which is of the form

$$P=T(\boldsymbol{w},\boldsymbol{\partial_w})+(S_k-1)\cdot C(k,\boldsymbol{w},S_k,\boldsymbol{\partial_w}) \tag{2}$$

where $\boldsymbol{\partial_w}$ stands for some operators that act on the variables $\boldsymbol{w}=w_1,\dots,w_\ell$. The operator T is called the *telescoper*, and we will refer to C as the *certificate* or *delta part*. Written as an equation, (2) turns into $-Tf(k,\boldsymbol{w})=g(k+1,\boldsymbol{w})-g(k,\boldsymbol{w})$ with $g(k,\boldsymbol{w})=Cf(k,\boldsymbol{w})$, compare also with (1). With such an operator P we can immediately derive a relation for $F(\boldsymbol{w})$:

$$
\begin{aligned}
0 &= \sum_{k=a}^{b} P(k,\boldsymbol{w},S_k,\boldsymbol{\partial_w}) f(k,\boldsymbol{w}) \\
&= \sum_{k=a}^{b} T(\boldsymbol{w},\boldsymbol{\partial_w}) f(k,\boldsymbol{w}) + \sum_{k=a}^{b} \big((S_k-1)C(k,\boldsymbol{w},S_k,\boldsymbol{\partial_w})\big) f(k,\boldsymbol{w}) \\
&= T(\boldsymbol{w},\boldsymbol{\partial_w}) \underbrace{\sum_{k=a}^{b} f(k,\boldsymbol{w})}_{F(\boldsymbol{w})} + \underbrace{\Big[C(k,\boldsymbol{w},S_k,\boldsymbol{\partial_w}) f(k,\boldsymbol{w})\Big]_{k=a}^{k=b+1}}_{\text{inhomogeneous part}}.
\end{aligned} \tag{3}
$$

If the *inhomogeneous part* evaluates to zero then T is an annihilating operator for the sum, otherwise we get an inhomogeneous relation. In the latter case, one can homogenize it by multiplying an annihilating operator for the inhomogeneous part to T from the left. Note that in general, the summation bounds a and b may depend on $\boldsymbol{w}$ in which case some correction terms need to be added which are created when the operator T is pulled in front of the sum.

In terms of closure properties for holonomic functions, see Theorem 2, this reads as follows: the summand $f(k,\boldsymbol{w})$ is given by an annihilating ideal and the operator P must be a member of this ideal. The goal is to compute an annihilating ideal for the function $F(\boldsymbol{w})$ that is sufficiently large (to testify its holonomicity). We have seen that every operator P with the above properties yields an annihilating operator for F, so one continues to compute such creative telescoping operators until the left ideal generated by them is large enough.

Multiple sums can be done by iteratively applying the above procedure. Alternatively, one can use creative telescoping operators of the form

$$
T(\boldsymbol{w},\boldsymbol{\partial_w}) + (S_{k_1}-1)\cdot C_1(\boldsymbol{k},\boldsymbol{w},\boldsymbol{S_k},\boldsymbol{\partial_w}) + \cdots + (S_{k_j}-1)\cdot C_j(\boldsymbol{k},\boldsymbol{w},\boldsymbol{S_k},\boldsymbol{\partial_w}) \tag{4}
$$

where $\boldsymbol{k} = k_1,\ldots,k_j$ are the summation variables.

Similarly one derives annihilating operators for an integral $I(\boldsymbol{w}) = \int_a^b f(x,\boldsymbol{w})\,\mathrm{d}x$. In this case we look for creative telescoping operators that annihilate f and that are of the form

$$
P = T(\boldsymbol{w},\boldsymbol{\partial_w}) + D_x \cdot C(x,\boldsymbol{w},D_x,\boldsymbol{\partial_w}). \tag{5}
$$

Again, it is straightforward to deduce a relation for the integral

$$
\begin{aligned}
0 &= \int_a^b P(x,\boldsymbol{w},D_x,\boldsymbol{\partial_w}) f(x,\boldsymbol{w})\,\mathrm{d}x \\
&= \int_a^b T(\boldsymbol{w},\boldsymbol{\partial_w}) f(x,\boldsymbol{w})\,\mathrm{d}x + \int_a^b \big(D_x C(x,\boldsymbol{w},D_x,\boldsymbol{\partial_w})\big) f(x,\boldsymbol{w})\,\mathrm{d}x \\
&= T(\boldsymbol{w},\boldsymbol{\partial_w}) \underbrace{\int_a^b f(x,\boldsymbol{w})\,\mathrm{d}x}_{I(\boldsymbol{w})} + \underbrace{\Big[C(x,\boldsymbol{w},D_x,\boldsymbol{\partial_w}) f(x,\boldsymbol{w})\Big]_{x=a}^{x=b}}_{\text{inhomogeneous part}}
\end{aligned} \tag{6}
$$

which may be homogeneous or inhomogeneous, as before. Analogously to the summation case, multiple integrals can be treated iteratively or by creative telescoping operators of the form

$$T(\boldsymbol{w},\boldsymbol{\partial_w}) + D_{x_1}\cdot C_1(\boldsymbol{x},\boldsymbol{w},\boldsymbol{D_x},\boldsymbol{\partial_w}) + \cdots + D_{x_j}\cdot C_j(\boldsymbol{x},\boldsymbol{w},\boldsymbol{D_x},\boldsymbol{\partial_w}). \tag{7}$$

where now $\boldsymbol{x} = x_1,\ldots,x_j$ are the integration variables.

In practice it happens very often that the inhomogeneous part vanishes. The reason for that is because many sums and integrals run over *natural boundaries*. This concept is often used, e.g., in Takayama's algorithm, to argue a priori that there will be no inhomogeneous parts after telescoping. For that purpose, we define that $\sum_{k=a}^{b} f$ resp. $\int_a^b f\,\mathrm{d}x$ has natural boundaries if for any arbitrary operator $P \in \mathbb{O}$ for a suitable Ore algebra $\mathbb{O}$ the expression $\big[Pf\big]_{k=a}^{k=b+1}$ resp. $\big[Pf\big]_{x=a}^{x=b}$ evaluates to zero. Typical examples for natural boundaries are sums with finite support, or integrals over the whole real line that involve something like $\exp(-x^2)$. Likewise contour integrals along a closed path do have natural boundaries.

5 Algorithms for Computing Creative Telescoping Relations

In this section some algorithms for computing creative telescoping relations are described briefly; for a detailed exposition see [61]. We focus on algorithms that are applicable to general holonomic functions and omit those which are designed for special cases of holonomic functions—like rational, hypergeometric, or hyperexponential functions—and refer to Section 2 and the references given there. In the following, the summation and integration variables are denoted by $\boldsymbol{v} = v_1,\ldots,v_j$ whereas $\boldsymbol{w} = w_1,\ldots,w_\ell$ are the surviving parameters. So the most general case to consider is a holonomic function $f(\boldsymbol{v},\boldsymbol{w})$ which has to be summed and integrated several times, thus some of the $\boldsymbol{v}$ may be discrete variables and the others continuous ones. The task is to find operators in the (given) annihilating ideal of f which can be written in the form

$$T(\boldsymbol{w},\boldsymbol{\partial_w}) + \Delta_{v_1}\cdot C_1(\boldsymbol{v},\boldsymbol{w},\boldsymbol{\partial_v},\boldsymbol{\partial_w}) + \cdots + \Delta_{v_j}\cdot C_j(\boldsymbol{v},\boldsymbol{w},\boldsymbol{\partial_v},\boldsymbol{\partial_w}) \tag{8}$$

where $\Delta_v = S_v - 1$ if v is a discrete variable and $\Delta_v = D_v$ if v is a continuous variable; compare also with (4) and (7).

5.1 Zeilberger's Slow Algorithm

In [98] Zeilberger suggested to approach holonomic sums or integrals by finding operators whose coefficients are completely free of the summation and integration variables $\boldsymbol{v}$. Once such an operator is found, it is immediate to rewrite it into the form (8) using division with remainder, since the corresponding operators $\boldsymbol{\partial_v}$ now commute with all remaining variables $\boldsymbol{w}$ and with all other operators $\boldsymbol{\partial_w}$. The theory of holonomic D-modules answers the question whether this elimination is possible at all in an affirmative way. The same argument justifies the termination of all other algorithms described in this section. Operators that are free of some variables can be found, e.g., by a Gröbner basis computation in $\mathbb{K}(\boldsymbol{w})[\boldsymbol{v}]\langle\boldsymbol{\partial_v},\boldsymbol{\partial_w}\rangle$ or by ansatz and coefficient comparison. In any case, this algorithm searches for creative telescoping operators that are not as general as possible—also the certificates are free of $\boldsymbol{v}$ in contrast to what is indicated in (8)—and therefore is very slow in practice and often does not find the minimal telescoper.

5.2 Takayama's Algorithm

In order to avoid the overhead that results in a complete elimination of the $\boldsymbol{v}$, Takayama came up with an algorithm that he termed an "infinite dimensional analog of Gröbner basis" [90]. He formulated it only in the differential setting and in a quite theoretical fashion. Chyzak and Salvy [36] later presented optimizations that are relevant in practice and extended it to the more general setting of Ore operators. Compared to Zeilberger's slow algorithm, Takayama's algorithm is faster and delivers better results, i.e., larger annihilating ideals.

The idea in a nutshell is the following: while in Zeilberger's slow algorithm first the $\boldsymbol{v}$ were eliminated and then the certificates were divided out, the order is now reversed. In Takayama's algorithm one first reduces modulo the right ideals $\partial_{v_1}\mathbb{O},\dots,\partial_{v_j}\mathbb{O}$ and then performs the elimination of the $\boldsymbol{v}$. The consequence is that the certificates $C_1,\dots,C_j$ are not computed at all because everything that would contribute to them is thrown away in the first step. Hence one has to assume a priori that the inhomogeneous parts vanish, e.g., in the case of natural boundaries.

There is one technical complication in this approach: one starts with a left ideal and then divides out some right ideals. After that there is no ideal structure any more and therefore, one is not allowed to multiply by either of the variables $\boldsymbol{v}$ from the left. In order to solve this problem one enlarges, at the very beginning, the set of generators of the input annihilating ideal by some of their left multiples by $\boldsymbol{v}$-powers and, at the end, computes a Gröbner basis w.r.t. to POT ordering (position over term) in the module that is generated by the power products of $\boldsymbol{v}$.

5.3 Chyzak's Algorithm

Chyzak presented his algorithm [34] as an extension of Zeilberger's algorithm to general holonomic functions. Like the latter, Chyzak's algorithm can only find creative telescoping operators for single sums or single integrals. Hence the goal is to find operators of the form

$$T(\boldsymbol{w}, \boldsymbol{\partial_w}) + \Delta_v \cdot C(v, \boldsymbol{w}, \partial_v, \boldsymbol{\partial_w}) \tag{9}$$

in the annihilating ideal $I \subseteq \mathbb{K}(v, \boldsymbol{w})\langle \partial_v, \boldsymbol{\partial_w} \rangle$ of the summand or integrand $f(v, \boldsymbol{w})$. The idea of the algorithm is to make an ansatz with undetermined coefficients for T and C. Since we may assume that C is in normal form w.r.t. I, its ansatz is as follows:

$$C(v, \boldsymbol{w}, \partial_v, \boldsymbol{\partial_w}) = c_1(v, \boldsymbol{w})U_1 + \cdots + c_r(v, \boldsymbol{w})U_r \tag{10}$$

where $U_1, \ldots, U_r$ are the monomials which cannot be reduced by I. Given a Gröbner basis for I, these are exactly the monomials under its staircase and r is the rank of f. The ansatz for T is of the form

$$T(\boldsymbol{w}, \boldsymbol{\partial_w}) = t_1(\boldsymbol{w})\boldsymbol{\partial_w}^{\boldsymbol{\alpha}_1} + \cdots + t_s(\boldsymbol{w})\boldsymbol{\partial_w}^{\boldsymbol{\alpha}_s} \tag{11}$$

where $\boldsymbol{\alpha}_i \in \mathbb{N}^\ell$ for $1 \leqslant i \leqslant s$. The ansatz $T + \Delta_v \cdot C$ is reduced with the Gröbner basis of I which leads to a system of equations for the unknown rational functions $c_1, \ldots, c_r, t_1, \ldots, t_s$. In the summation (resp. integration) case, this is a parametrized linear first-order system of difference (resp. differential) equations in the unknown functions $c_1 \ldots, c_r$ and with parameters $t_1, \ldots, t_s$. One has to find rational function solutions of this system and for the parameters, a problem for which several algorithms exist. Finally, Chyzak's algorithm proceeds by increasing the support of T in (11) until the ansatz yields a solution; doing this in a certain systematic way guarantees that the computed telescopers form a Gröbner basis in $\mathbb{K}(\boldsymbol{w})\langle \boldsymbol{\partial_w} \rangle$.

5.4 A Heuristic Approach

In [63] a variant of Chyzak's algorithm was developed that is based on a refined ansatz for the unknown rational functions $c_1, \ldots, c_r$. The motivation comes from the fact that the bottleneck in Chyzak's algorithm is to solve the coupled first-order system. The key observation is that good candidates for the denominators of the c_i can be obtained from the leading coefficients of the input Gröbner basis. Thus the ansatz (10) is refined in the following way:

$$c_i(v, \boldsymbol{w}) = \frac{c_{i,0}(\boldsymbol{w}) + c_{i,1}(\boldsymbol{w})v + \cdots + c_{i,e_i}(\boldsymbol{w})v^{e_i}}{d_i(v, \boldsymbol{w})}, \quad 1 \leqslant i \leqslant r,$$

where the d_i are explicit polynomials and the e_i are degree bounds for the numerator; both quantities are determined heuristically. In many examples this approach is faster than Chyzak's algorithm, but due to its heuristics it may not always succeed. Note also that this approach can be generalized to multiple sums and integrals, see Section 6.5.

6 Demonstration of the HolonomicFunctions Package

6.1 Differential Equations for Bivariate Hypergeometric Functions

The most studied concept in the area of special functions are hypergeometric functions, whose most prominent representative is the Gauss hypergeometric function ${}_2F_1$. We consider here the Appell hypergeometric function F_1 defined by

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^m y^n (\beta)_m (\beta')_n (\alpha)_{m+n}}{m!n!(\gamma)_{m+n}} \tag{12}$$

for $|x| < 1$ and $|y| < 1$. Classical mathematical tables like [44] list systems of differential equations for such functions, e.g., entry 9.181 for the Appell functions. The nature of this example is that no closed form is desired, but a system of partial differential equations. These equations are now derived completely automatically from (12) using Takayama's algorithm.

The input for Takayama's algorithm is an annihilating ideal for the summand which is obtained by the command Annihilator. We need to introduce the shift operators S_m and S_n for the summation variables and the partial derivatives D_x and D_y since we are interested in PDEs w.r.t. x and y. The computation of the annihilating ideal is direct since the summand is hypergeometric in all discrete variables and hyperexponential in all continuous variables:

In[3]:= $\mathrm{ann} = \mathrm{Annihilator}\big[\mathrm{Pochhammer}[\alpha, m+n] * \mathrm{Pochhammer}[\beta, m] * \mathrm{Pochhammer}[b, n] \,/\, (\mathrm{Pochhammer}[\gamma, m+n] * m! * n!) * x\hat{\ }m * y\hat{\ }n, \{\mathrm{S}[m], \mathrm{S}[n], \mathrm{Der}[x], \mathrm{Der}[y]\}\big]$

Out[3]= $\big\{yD_y - n, xD_x - m, (mn+m+n^2+n\gamma+n+\gamma)S_n - (bmy+bny+by\alpha+mny+n^2y+ny\alpha), (m^2+mn+m\gamma+m+n+\gamma)S_m - (m^2x+mnx+mx\alpha+mx\beta+nx\beta+x\alpha\beta)\big\}$

Next the double summation is performed and a Gröbner basis for the left ideal containing partial differential equations satisfied by the series F_1 is computed:

In[4]:= $\mathrm{pde} = \mathrm{Takayama}[\mathrm{ann}, \{m, n\}]$

Out[4]= $\big\{(xy^2 - xy - y^3 + y^2)D_y^2 + (bx^2 - bx)D_x + (bxy - by^2 + xy\alpha - xy\beta + xy + x\beta - x\gamma - y^2\alpha - y^2 + y\gamma)D_y + (bx\alpha - by\alpha), (x-y)D_xD_y - bD_x + \beta D_y, (x^3 - x^2y - x^2 + xy)D_x^2 + (bxy - by + x^2\alpha + x^2\beta + x^2 - xy\alpha - xy\beta - xy - x\gamma + y\gamma)D_x + (y\beta - y^2\beta)D_y + (x\alpha\beta - y\alpha\beta)\big\}$

Observe that the two equations given in [44, 9.181] do not appear in the above result. To verify that they are nevertheless correct, one has to show that they are members of the derived annihilating ideal. This is achieved by reducing them with the Gröbner basis and check whether the remainder is zero:

In[5]:= $\mathrm{OreReduce}[(x(y-1)){*}{*}(\mathrm{Der}[x]\mathrm{Der}[y]) + (y(y-1)){*}{*}\mathrm{Der}[y]^2 + (bx){*}{*}\mathrm{Der}[x] + (y(\alpha+b+1)-\gamma){*}{*}\mathrm{Der}[y] + \alpha b, \mathrm{pde}]$

Out[5]= 0

On the other hand, the desired equations can be produced automatically by observing that the first is free of β' and the second does not involve β. The command FindRelation finds operators in a given annihilating ideal that satisfy certain properties, to be specified by options:

In[6]:= $\mathrm{FindRelation}[\mathrm{pde}, \mathrm{Eliminate} \to \beta]$

Out[6]= $\{(xy-x)D_xD_y + (y^2-y)D_y^2 + bxD_x + (by+y\alpha+y-\gamma)D_y + b\alpha\}$

This is precisely the form in which the first partial differential equation appears in [44] and an analogous computation yields the second one.

6.2 An Integral Involving Chebyshev Polynomials

It has been pointed out that creative telescoping does not deliver closed-form solutions. The next example demonstrates how it can be used to prove an identity, in this case the evaluation of a definite integral which appears in [44, 7.349]:

$$\int_{-1}^{1} \left(1-x^2\right)^{-1/2} T_n\left(1-x^2y\right) \mathrm{d}x = \frac{\pi}{2}\left(P_{n-1}(1-y)+P_n(1-y)\right). \tag{13}$$

Here $T_n(x)$ denotes the Chebyshev polynomials of the first kind defined by

$$T_n(x) = \cos(n \arccos x)$$

and the evaluation is given in terms of Legendre polynomials $P_n(x)$ defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n.$$

This relatively simple example is chosen not only to demonstrate Chyzak's algorithm but also to enlighten the concept of closure properties.

The starting point is the computation of an annihilating ideal for the integrand $f(n,x,y) = (1-x^2)^{-1/2}\,T_n(1-x^2y)$ in (13) which, in this instance, we will discuss in some more detail. For this purpose, recall the three-term recurrence

$$T_{n+2}(z) - 2zT_{n+1}(z) + T_n(z) = 0 \tag{14}$$

and the second-order differential equation

$$(z^2-1)T_n''(z) + zT_n'(z) - n^2T_n(z) = 0 \tag{15}$$

for the Chebyshev polynomials which are both classic and well-known. The HolonomicFunctions package has these relations stored in a kind of database. Clearly, the integrand f also satisfies the recurrence (14) if z is replaced by $1-x^2y$. The same substitution is performed in (15) and considering $T_n(1-x^2y)$ as a function in y yields

$$\frac{(1-x^2y)^2-1}{x^4}\frac{\partial^2}{\partial y^2}T_n(1-x^2y)+\frac{1-x^2y}{-x^2}\frac{\partial}{\partial y}T_n(1-x^2y)-n^2T_n(1-x^2y)=0.$$

Multiplying with x^2 produces another annihilating operator

$$(x^2y^2-2y)D_y^2+(x^2y-1)D_y-n^2x^2$$

for the integrand f. Note that the square root term can be ignored since it is free of y. Finally, observe that

$$\begin{aligned}\frac{df}{dx} &= \frac{-2xy}{\sqrt{1-x^2}}T_n'(1-x^2y)+\frac{x}{(1-x^2)^{3/2}}T_n(1-x^2y)\\ \frac{df}{dy} &= \frac{-x^2}{\sqrt{1-x^2}}T_n'(1-x^2y)\end{aligned}$$

giving rise to the operator

$$xD_x-2yD_y-\frac{x^2}{1-x^2}$$

which also annihilates f. The above ad hoc derivation of annihilating operators for a compound expression can be turned into an algorithmic method, and this is implemented in the Annihilator command:

In[7]:= $\text{Annihilator}\big[\text{ChebyshevT}[n,1-x^2y]/\text{Sqrt}[1-x^2],\ \{\text{S}[n],\text{Der}[x],\text{Der}[y]\}\big]$

Out[7]= $$\begin{aligned}\big\{&(x^3-x)D_x+(2y-2x^2y)D_y+x^2,\\ &nS_n+(x^2y^2-2y)D_y+(nx^2y-n),\\ &(x^2y^2-2y)D_y^2+(x^2y-1)D_y-n^2x^2\big\}\end{aligned}$$

The above operators form a left Gröbner basis, and therefore differ slightly from the ones that were derived by hand; but the latter can be obtained as simple linear combinations of the previous ones.

Now we are ready to perform creative telescoping: we apply Chyzak's algorithm to find operators of the form $T_i+D_xC_i$ in the annihilating ideal. Our implementation returns two such operators, with the property that $\{T_1,T_2\}$ is a Gröbner basis:

In[8]:= $\{\{T_1,T_2\},\{C_1,C_2\}\}=\text{CreativeTelescoping}[\%,\text{Der}[x]]$

Out[8]= $$\begin{aligned}\Big\{&\big\{(2n^2+2n)S_n+(2ny^2-4ny+y^2-2y)D_y+(2n^2y-2n^2+ny-2n),\\ &(y^2-2y)D_y^2+(y-2)D_y-n^2\big\},\\ &\Big\{\frac{y\left(x^4y-x^2y-2x^2+2\right)}{x}D_y+y\left(nx^3-nx\right),\frac{x^2-1}{x}D_y\Big\}\Big\}\end{aligned}$$

With the help of Mathematica, it is easily verified that the inhomogeneous part, see (6), vanishes:

In[9]:= $\text{Limit}\big[\text{ApplyOreOperator}\big[C_1, \text{ChebyshevT}[n, 1-x^2y]/\text{Sqrt}[1-x^2]\big], x \to 1\big]$

Out[9]= 0

(Similar checks have to be done for the lower bound and for C_2.) It follows that T_1 and T_2 generate an annihilating ideal for the integral. For the convenience of the user, all the previous steps can be performed at once by typing a single command:

In[10]:= $\text{Annihilator}\big[\text{Integrate}[\text{ChebyshevT}[n, 1-x^2y]/\text{Sqrt}[1-x^2], \{x, -1, 1\}],$
$\{\text{S}[n], \text{Der}[y]\}\big]$

Out[10]= $\big\{(2n^2+2n)S_n + (2ny^2-4ny+y^2-2y)D_y + (2n^2y-2n^2+ny-2n),$
$(y^2-2y)D_y^2 + (y-2)D_y - n^2\big\}$

The next step is to compute an annihilating ideal for the right-hand side of (13). Instead of applying the Annihilator command to the expression itself which would produce an annihilating ideal of rank 4 by assertion (a) of Theorem 1, the fact that the sum of the two Legendre polynomials can be written as $Q(P_{n-1}(1-y))$ with $Q = S_n + 1$ is employed. This observation produces an annihilating ideal of rank 2, see part (c) of Theorem 1:

In[11]:= $\text{rhs} = \text{Annihilator}\big[\text{ApplyOreOperator}[\text{S}[n]+1, \text{LegendreP}[n-1, 1-y]],$
$\{\text{S}[n], \text{Der}[y]\}\big]$

Out[11]= $\big\{(2n^2+2n)S_n + (2ny^2-4ny+y^2-2y)D_y + (2n^2y-2n^2+ny-2n),$
$(y^2-2y)D_y^2 + (y-2)D_y - n^2\big\}$

Finally, one realizes that the annihilating ideals for both sides of the identity coincide. The proof is completed by comparing two initial values, e.g., for $n = 0$ and $n = 1$. This has to be done by hand (of course, with the help of the computer algebra system), but is not part of the functionality of the HolonomicFunctions package.

6.3 A q-Holonomic Summation Problem from Knot Theory

The colored Jones function is a powerful knot invariant; it is a q-holonomic sequence of Laurent polynomials [42]. Its recurrence equation is of interest since it seems to be closely related with the A-polynomial of a knot. The recurrence for the colored Jones function $J_{7_4,n}(q)$ of the knot 7_4 was derived in [41] using creative telescoping, starting from the sum representation

$$J_{7_4,n}(q) = \sum_{k=0}^{n-1} (-1)^k \big(c_k(q)\big)^2 q^{-kn-\frac{k(k+3)}{2}} (q^{n-1}; q^{-1})_k (q^{n+1}; q)_k \tag{16}$$

where $(x;q)_n$ denotes the q-Pochhammer symbol defined as $\prod_{j=0}^{n-1}(1-xq^j)$ and where the sequence $c_k(q)$ satisfies a second-order recurrence:

$$c_{k+2}(q)+(q^{k+3}+q^{k+4}-q^{2k+5}+q^{3k+7})c_{k+1}(q)+(q^{2k+6}-q^{3k+7})c_k(q)=0. \quad (17)$$

Note that the summand in (16) is not q-hypergeometric and therefore the q-version of Zeilberger's algorithm cannot be applied.

Again, we start by constructing an annihilating ideal for the summand. The one for the sequence $c_k(q)$ is given by its definition (17), we just have to add the trivial relation w.r.t. n and convert everything to operator form (note the usage of q-shift operators):

In[12]:= $\text{annc} = \text{ToOrePolynomial}\big[\big\{\text{QS}[\text{qn},q^n]-1, \text{QS}[\text{qk},q^k]^2 + \big(q^{k+3}(1+q-q^{k+2}+q^{2k+4})\big){*}{*}\text{QS}[\text{qk},q^k] + q^{2k+6}(1-q^{k+1})\big\}\big]$

Out[12]= $\big\{S_{\text{qn},q}-1, S_{\text{qk},q}^2 + \big(q^7\text{qk}^3 - q^5\text{qk}^2 + q^4\text{qk} + q^3\text{qk}\big)S_{\text{qk},q} + \big(q^6\text{qk}^2 - q^7\text{qk}^3\big)\big\}$

Next, the closure property "multiplication", see Theorem 1 (b), is applied (the result is about 2 pages long and therefore not displayed here):

In[13]:= $\text{annSmnd} = \text{DFiniteTimes}[\text{annc}, \text{annc}, \text{Annihilator}[(-1)^k\, q\,\hat{}\,(-kn-k(k+3)/2)\, \text{QPochhammer}[q^{n-1}, 1/q, k]\ \text{QPochhammer}[q^{n+1}, q, k], \{\text{QS}[\text{qk},q^k], \text{QS}[\text{qn},q^n]\}]];$

The stage is now prepared for calling Chyzak's algorithm which delivers a pair (T,C) consisting of telescoper and certificate:

In[14]:= $\{T,C\} = \text{CreativeTelescoping}[\text{annSmnd}, \text{QS}[\text{qk},q^k]-1]$

This computation takes about two minutes and the result is again too large to be printed here. We remark that the inhomogeneous part does not vanish so that we obtain an inhomogeneous recurrence for the function $J_{7_4,n}(q)$. The result is in accordance with the AJ conjecture and the previously known A-polynomial of the knot 7_4.

6.4 A Double Integral Related to Feynman Diagrams

We study the double integral

$$\int_0^1 \int_0^1 \frac{w^{-1-\varepsilon/2}(1-z)^{\varepsilon/2}z^{-\varepsilon/2}}{(z+w-wz)^{1-\varepsilon}}\left(1-w^{n+1}-(1-w)^{n+1}\right)\mathrm{d}w\,\mathrm{d}z \quad (18)$$

than can be found in [54, (J.17)]. The task is to compute a recurrence in n where ε is just a parameter. We are aware of the fact that (18) is not a hard challenge for physicists, and we use it only as a proof of concept here. We are going to apply Chyzak's algorithm iteratively.

For computing an annihilating ideal for the inner integral, we simply use the command Annihilator that takes care of the inhomogeneous part automatically:

In[15]:= $f = w\,\hat{}\,(-1-\varepsilon/2)\,(1-z)\,\hat{}\,(\varepsilon/2)\,z\,\hat{}\,(-\varepsilon/2)/(w+z-wz)\,\hat{}\,(1-\varepsilon)\ (1-w\,\hat{}\,(n+1)-(1-w)\,\hat{}\,(n+1));$

In[16]:= $\text{ann} = \text{Annihilator}\big[\text{Integrate}[f, \{w,0,1\}], \{\text{S}[n], \text{Der}[z]\}\big];$

This result is quite large so that we do not want to display it here. But it can be used again as input to Chyzak's algorithm, in order to treat the outer integral.

In[17]:= $\{\{T\},\{C\}\} = \text{CreativeTelescoping}[\text{ann}, \text{Der}[z], \text{S}[n]];$

It is a little bit tricky to handle the inhomogeneous part of the outer integral since it involves an integral itself:

$$\left[C\int_0^1 f\,\mathrm{d}w\right]_{z=0}^{z=1} = \int_0^1 \left[Cf\right]_{z=0}^{z=1}\mathrm{d}w. \tag{19}$$

It turns out that the right-hand side of (19) is preferable to show that the inhomogeneous part evaluates to zero. Therefore the operator T annihilates the double integral, and this is the desired recurrence in n (which is of order 3):

In[18]:= $\text{Factor}[T]$

Out[18]= $-(\varepsilon-n-3)(\varepsilon-n-2)(\varepsilon+2n+4)(\varepsilon+2n+6)S_n^3 +$
$(\varepsilon-n-2)(\varepsilon+2n+4)(\varepsilon^2+2\varepsilon n+5\varepsilon-6n^2-28n-34)S_n^2 -$
$(n+2)(\varepsilon^3-3\varepsilon^2 n-6\varepsilon^2-8\varepsilon n^2-30\varepsilon n-28\varepsilon+12n^3+64n^2+116n+72)S_n -$
$2(n+1)(n+2)^2(\varepsilon-2n-2)$

6.5 A Hypergeometric Double Sum

We finally turn to a binomial double sum which was investigated in [8]:

$$\sum_i\sum_j \binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2. \tag{20}$$

We apply the heuristic approach from Section 5.4 to it. The corresponding command in the HolonomicFunctions package is FindCreativeTelescoping:

In[19]:= $\text{FindCreativeTelescoping}[\text{Binomial}[i+j,i]\verb|^|2\,\text{Binomial}[4n-2i-2j,2n-2i],$
$\{\text{S}[i]-1,\text{S}[j]-1\}, \text{S}[n]]$

Out[19]= $\left\{\{1\},\left\{\left\{\frac{-2i^2j+i^2n-i^2-2ij^2+3ijn-2ij+3in}{(j+1)(i+j-2n)},\right.\right.\right.$
$\left.\left.\left.\frac{-2i^2j-2ij^2+3ijn-2ij+j^2n-j^2+3jn}{(i+1)(i+j-2n)}\right\}\right\}\right\}$

The output consists of the telescoper and the two certificates. At first glance it may seem contradictory that the telescoper is 1, but there are contributions from the certificates that make the recurrence for the double sum inhomogeneous. So we don't claim that the operator 1 annihilates the double sum, which would imply that it is zero.

7 Selected Applications of Creative Telescoping

In this section we want to give an extensive, but certainly not complete, collection of examples which show the beneficial use of creative telescoping in diverse areas of mathematics and physics.

Zeilberger's algorithm for hypergeometric sums is a meanwhile so classic tool that it is impossible to list all papers where it has been used to prove some binomial sum identity. We therefore restrict ourselves to publications where this algorithm plays a more or less central role. In [39] it was used to prove Ramanujan's famous formula for π, and in [46] for some formulas of similar type. The whole paper [89] is dedicated to binomial identities that arise in combinatorics and how to prove them algorithmically. Two proofs of the notorious binomial double sum identity (20) are given in [8] where, due to the lack of multi-summation software packages at that time, the problem was reduced in a tricky way to a single sum identity. A "triumph of computer algebra" is celebrated in [81] where the computation of factorial moments and probability generating functions for heap ordered trees is based on Zeilberger's algorithm. In [7] it is used to derive formulas for hypergeometric series acceleration, among them a pretty formula for $\zeta(3)$ that allowed to evaluate this constant to a large number of digits. In the article [60], Zeilberger's algorithm is combined with asymptotic estimates in order to give automated proofs of non-terminating series identities of Saalschütz type. Applications in the context of orthogonal polynomials are given in [58]. A fast way of computing Catalan's constant is derived in [103] by means of creative telescoping. While the recurrence that plays a crucial role in Apéry's proof of the irrationality of $\zeta(3)$ is nowadays a popular example for demonstrating these techniques, they were not available to Apéry when he came up with his proof. A new, elementary proof, still using Zeilberger's algorithm, is given in [104]. We conclude this paragraph by mentioning [5] where a binomial identity that arose in the study of a certain integral is investigated.

We turn to applications of creative telescoping that go beyond Zeilberger's algorithm. As an application of its q-analogue we cite [74] where computer proofs for the Rogers-Ramanujan identities are constructed. Multi-summation techniques for q-hypergeometric terms were used in [12] to prove a partition theorem of Göllnitz. Computer proofs for summation identitites involving Stirling numbers are given in [53]. In [20] creative telescoping was used to obtain bounds on the order and degree of differential equations satisfied by algebraic functions. Chyzak's algorithm was applied to the generating function of 3-dimensional rook paths [21] in order to derive an explicit formula. Creative telescoping proofs for a selection of special function identities, mostly involving integrals, are presented in [68]. Another application to the evaluation of integrals is [6].

In [101] Zeilberger proposed an approach how to evaluate determinants of matrices with holonomic entries with the method of creative telescoping. This approach applies to determinants of the form $\det_{1\leqslant i,j\leqslant n}(a_{i,j})$ whose entries are bivariate holonomic sequences, not depending on the dimension n. The so-called "holonomic ansatz" celebrated its greatest success so far when it was employed to prove the qTSPP conjecture [66], a long-standing prominent problem in enumerative combinatorics, which previously had been reduced to a certain determinant evaluation of the above type. This conjecture is the q-analogue of what is known as Stembridge's theorem about the enumeration of totally symmetric plane partitions. Based on creative telescoping, this theorem was re-proved twice, both times using the formulation as a determinant evaluation: the first time by applying symbolic summation techniques to a decomposition of the matrix [9], the second time following the holonomic ansatz [62]. Some extensions of the holonomic ansatz were presented in [69] and were applied to solve several conjectures about determinants. An analogous method for the evaluation of Pfaffians was developed in [48].

In the field of quantum topology and knot theory, a prominent object of interest is the so-called *colored Jones function* of a knot. This function is actually an infinite sequence of Laurent polynomials and in [42] it has been shown that this sequence is always q-holonomic, by establishing an explicit multisum representation with proper q-hypergeometric summand. The corresponding minimal-order recurrence is called the *non-commutative A-polynomial* of the knot. Creative telescoping was used to compute it for a family of twist knots [43] and for a few double twist knots [41].

We are turning to applications in the area of numerical analysis. A widely used method for computer simulations of real-world phenomena described by partial differential equations is the *finite element method* (FEM). A short motivation of using symbolic summation techniques in this area is given in [75], and a concrete application where hypergeometric summation algorithms deliver certain recurrence equations which allow for a fast evaluation of the basis functions, is described in [11]. Further examples, where creative telescoping is used for verifying identities arising in the context of FEM or for finding identities that help to speed up the numerical simulations, can be found in [14, 15, 67].

Last but not least we want to point out that creative telescoping has extensively supported computations in physics. We will not detail on the very fruitful interaction of summation methods in difference fields with the computation of Feynman integrals in particle physics [1], but refer to the survey [87], and the references therein. The estimation of the entropy of a certain process [70] was supported by computer algebra. In the study of generalized two-Qubit Hilbert-Schmidt separability probabilities [88] creative telescoping was employed to simplify a complicated expression involving generalized hypergeometric functions. The authors of [17] underline the particular importance that creative telescoping may play in the evaluation of the n-fold integrals $\chi^{(n)}$ of the magnetic susceptibility of the Ising model. Also relativistic Coulomb integrals have been treated with the holonomic systems approach [79]. Likewise it was used in the proof of a third-order integrability criterion for homogeneous potentials of degree -1 [37]. One branch of statistical physics deals with random walks on lattices; some results in this area [102, 65] were obtained by creative telescoping.

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