A SHAPE LEMMA FOR IDEALS OF DIFFERENTIAL OPERATORS

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Abstract. We propose a version of the classical shape lemma for zero-dimensional ideals of a commutative multivariate polynomial ring to the noncommutative setting of zero-dimensional ideals in an algebra of differential operators.

1. Introduction

In the classical theory of Gröbner bases for commutative polynomial rings [3, 7, 1, 8, 4], the shape lemma makes a statement about the form of the Gröbner basis with respect to a lexicographic term order of an ideal of dimension zero. It was proposed by Gianni and Mora [9], and it is almost obvious.

Consider an ideal $I \subseteq K[x,y]$ in a commutative polynomial ring over a perfect field K. The ideal has dimension zero if and only if the corresponding algebraic set

$$V(I) = \{\, (\xi,\eta) \in \bar{K}^2 \mid \forall\, p \in I : p(\xi,\eta) = 0 \,\}$$

is finite. Here, $\bar{K}$ denotes the algebraic closure of K.

The finitely many points in $V(I)$ have only finitely many distinct x-coordinates, and if p is a generator of the elimination ideal $I \cap K[x]$, then the roots of p are precisely these x-coordinates. The shape lemma says that usually there is another polynomial $q \in K[x]$ with $\deg(q) < \deg(p)$ such that I is generated by $\{y - q, p\}$. This q is then the interpolating polynomial of the points in $V(I)$.

There may be no ideal basis of the required form if $V(I)$ contains two distinct points with the same x-coordinate. The ideal is said to be *in normal position* (w.r.t. x) if this is not the case, i.e., if any two distinct elements of $V(I)$ have distinct x-coordinates. If K is sufficiently large, then every ideal I of dimension zero can be brought into normal position by applying a change of variables.

Theorem 1. *(cf. Prop. 3.7.22 in* [12]*). Let $P = K[x, y_1, \ldots, y_n]$, let $I \subseteq P$ be an ideal of dimension zero, let $t = \dim_K P/I$, and suppose that $|K| > \binom{t}{2}$. Then there are constants $c_1, \ldots, c_n \in K$ such that mapping x to $x + c_1y_1 + c_2y_2 + \cdots + c_ny_n$ (and each y_i to itself) transforms I into an ideal in normal position w.r.t. x.*

A basis of the required form may also fail to exist if I is not a radical ideal. Recall that for a radical ideal I, we have $\dim_K K[x,y]/I = |V(I)|$. Also recall that if p is a generator of $I \cap K[x]$, then the equivalence classes $[1], [x], \ldots, [x^{\deg p - 1}]$ are linearly independent over K and $[1], [x], \ldots, [x^{\deg p-1}], [x^{\deg p}]$ are linearly dependent. Therefore, the following result is quite natural.

Theorem 2. *(cf. Thm. 3.7.23 in* [12]*). Let $P = K[x, y_1, \ldots, y_n]$ and let $I \subseteq P$ be a radical ideal of dimension zero. Let p be a generator of $I \cap K[x]$. Then the following conditions are equivalent:*

1. *I is in normal position w.r.t. x*
2. $\deg p = \dim_K P/I$
3. *$K[x]/\langle p\rangle$ and P/I are isomorphic as K-algebras.*

Finally, the shape lemma can be stated as follows.

Theorem 3. *(Shape Lemma; cf. Thm. 3.7.25 in* [12]*) Let $P = K[x, y_1, \ldots, y_n]$ and let $I \subseteq P$ be a radical ideal of dimension zero that is in normal position w.r.t. x. Let p be a generator of $I \cap K[x]$. Then there are polynomials $q_1, \ldots, q_{n-1} \in K[x]$ with $\deg(q_i) < \deg(p)$ for all i such that $\{y_1 - q_1, \ldots, y_n - q_n, p\}$ is a basis of I.*

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Here and elsewhere, by a "basis" of an ideal we understand just a set of generators, not necessarily minimal or independent in any sense.

The purpose of this note is to extend these well-known facts from commutative polynomial rings to rings of differential operators. This is motivated by recent developments in the area of symbolic integration for so-called D-finite functions [10]. Given such a function $f(x,y)$, the goal is to evaluate a definite integral

$$F(x) = \int_\Omega f(x,y)\,dy.$$

More precisely, given an ideal of annihilating operators for $f(x,y)$, we want to compute an ideal of annihilating operators for the integral $F(x)$. A general approach to this problem is known as creative telescoping [14, 15, 11, 6] and has been subject of intensive research during the past decades. There are several algorithms for creative telescoping, some of which assume that the ideal of operators for $f(x,y)$ has a basis of the form $\{D_y - M, L\}$, where M and L are operators in D_x only. Thanks to the shape lemma, this is a fair assumption.

The technique of creative telescoping also applies to summation problems. In this case, we have to deal with recurrence operators rather than differential operators. It would be interesting to have a version of the shape lemma also in this case. However, our results for the differential case do not seem to extend easily to the recurrence case.

2. Differential operators

The role of the field K in the commutative setting sketched in the introduction is now taken over by the field $C(x, y_1, \dots, y_n)$ of rational functions in x and $y_1, \dots, y_n$, with coefficients in some constant field C that we assume to have characteristic zero. Hence from now on, we let $K = C(x, y_1, \dots, y_n)$. We will also abbreviate $y_1, \dots, y_n$ by $\boldsymbol{y}$.

We use the symbols D_x and $D_{y_1}, \dots, D_{y_n}$ to denote the partial derivation operators, i.e., $D_x(f) = \frac{\partial f}{\partial x}$ and $D_{y_i}(f) = \frac{\partial f}{\partial y_i}$ $(i = 1, \dots, n)$. Note that $D_x(c) = D_{y_i}(c) = 0$ for all $c \in C$ and all i.

The action of D_x and $D_{y_1}, \dots, D_{y_n}$ turns K into a partial differential field. In general, if L is a field, a map $D\colon L \to L$ is called a *derivation* if $D(a+b) = D(a) + D(b)$ and $D(ab) = D(a)b + aD(b)$ for all $a, b \in L$. We call L a partial differential extension field of K if it is equipped with $n+1$ derivations that agree with the action of D_x and $D_{y_1}, \dots, D_{y_n}$ on the subfield K.

Let $K[D_x, \boldsymbol{D_y}] := K[D_x, D_{y_1}, \dots, D_{y_n}]$ denote the ring of linear differential operators with rational function coefficients, i.e.,

$$K[D_x, \boldsymbol{D_y}] = \left\{ \sum_{i,j_1,\dots,j_n=0}^{d} a_{i,j_1,\dots,j_n}(x,\boldsymbol{y}) D_x^i D_{y_1}^{j_1} \cdots D_{y_n}^{j_n} \;\middle|\; d \in \mathbb{N}, a_{i,j_1,\dots,j_n} \in K \right\}.$$

Because of the product rule, we have the commutation rules $D_x \cdot x = x \cdot D_x + 1$ and $D_{y_i} \cdot y = y \cdot D_{y_i} + 1$ for every i, so the ring $K[D_x, \boldsymbol{D_y}]$ is non-commutative. A linear partial differential equation can then be written as $L(f) = 0$ with $L \in K[D_x, \boldsymbol{D_y}]$.

Let $C[[x]]$ and $C[[x, \boldsymbol{y}]]$ denote, as usual, the rings of univariate and multivariate formal power series with coefficients in C, and let $C((x))$ and $C((x, \boldsymbol{y}))$ denote their respective quotient fields.

Let $L = \sum_{i=0}^r a_i(x) D_x^i \in C(x)[D_x]$ be a linear ordinary differential operator. An element $x_0 \in C$ is called a regular point (or ordinary point) of L if $a_r(x_0) \neq 0$ and $a_i(x_0)$ is defined for all $0 \le i \le r$, i.e., if no coefficient a_i has a pole at x_0. Via the change of variables $x \leftarrow x - x_0$ the point x_0 can be moved to the origin. Hence, without loss of generality, assume that 0 is a regular point of L. Then the set of power series solutions

$$V(L) = \{ f \in C[[x]] \mid L(f) = 0 \}$$

forms a C-vector space of dimension r.

For a power series $f(x, \boldsymbol{y}) \in C[[x, \boldsymbol{y}]]$, we define the ($K[D_x, \boldsymbol{D_y}]$-) annihilator of f as the set of all operators that annihilate f, that is $\{L \in K[D_x, \boldsymbol{D_y}] \mid L(f) = 0\}$. It is easily verified that this set forms a (left) ideal in $K[D_x, \boldsymbol{D_y}]$. The series f is called D-finite if $\dim_K(K[D_x, \boldsymbol{D_y}]/I) < \infty$, where I denotes the annihilator of f. Equivalently, f is called D-finite if I is an ideal of dimension zero.

Also in the multivariate setting we can make a similar statement about the dimension of the solution space, which directly follows from Thm. 3.7 in [5].

Theorem 4. *Let I be a zero-dimensional left ideal of $K[D_x, \boldsymbol{D_y}]$ and $r = \dim_K(K[D_x, \boldsymbol{D_y}]/I) \in \mathbb{N}$. If 0 is an ordinary point of I, then the set*

$$V(I) = \{f \in C[[x, \boldsymbol{y}]] \mid \forall L \in I : L(f) = 0\}$$

is a C-vector space of dimension r.

The definition of ordinary points proposed in [5] is a bit more complicated than the definition in the univariate case. We won't need it here, so we do not reproduce it. It suffices to know that almost every point is ordinary, so if 0 is not a ordinary point, we always have the option to get into the situation of Thm. 4 by making a change of variables.

For $f_1, \dots, f_r \in C((x, \boldsymbol{y}))$, their *Wronskian* (with respect to the variable x) is denoted and defined as follows:

$$\mathrm{Wr}_x(f_1, \dots, f_r) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_r \\ D_x(f_1) & D_x(f_2) & \cdots & D_x(f_r) \\ \vdots & \vdots & \ddots & \vdots \\ D_x^{r-1}(f_1) & D_x^{r-1}(f_2) & \cdots & D_x^{r-1}(f_r) \end{pmatrix}.$$

The Wronskian $\mathrm{Wr}_x(f_1, \dots, f_r)$ is equal to zero if and only if the f_i satisfy a linear relation with coefficients that do not depend on x, e.g., if $\sum_{i=1}^r a_i f_i = 0$ with $a_i \in C((\boldsymbol{y}))$ not all zero [2].

For later use, we state the following lemma.

Lemma 5. *If L is a partial differential field extension of K and I is an ideal in $L[D_x, \boldsymbol{D_y}]$ which has a basis in $K[D_x, \boldsymbol{D_y}]$, then also the elimination ideal $I \cap L[D_x]$ has a basis in $K[D_x]$.*

Proof. Let $P_1, \dots, P_m \in K[D_x, \boldsymbol{D_y}]$ be a basis of I, and let M be an element in the elimination ideal $I \cap K[D_x]$. Then there exist $Q_1, \dots, Q_m \in L[D_x, \boldsymbol{D_y}]$ such that $M = Q_1 P_1 + \dots + Q_m P_m$.

Clearly, L can be viewed as a K-vector space, of potentially infinite dimension. In any case, there exists a finite-dimensional K-subspace V of L that contains all the coefficients of the Q_i (note that each Q_i has only finitely many coefficients in L). Now let $B_1, \dots, B_d$ be a K-basis of V, which means that there are $Q_{i,j} \in K[D_x, \boldsymbol{D_y}]$ such that $Q_i = Q_{i,1} B_1 + \dots + Q_{i,d} B_d$ for all i. Hence we can write

$$M = \sum_{i=1}^{m} \Bigl(\sum_{j=1}^{d} Q_{i,j} B_j \Bigr) P_i = \sum_{j=1}^{d} \Bigl(\sum_{i=1}^{m} Q_{i,j} P_i \Bigr) B_j. \tag{1}$$

Since the B_j are linearly independent over K, it follows that for each j, the quantity $\sum_{i=1}^m Q_{i,j} P_i$ is free of $D_{y_1}, \dots, D_{y_n}$, because M is free of $D_{y_1}, \dots, D_{y_n}$ and because there cannot be a cancellation on the right-hand side of (1). Therefore, the coefficients $\sum_{i=1}^m Q_{i,j} P_i$ are in $K[D_x]$, which proves the claim. ☐

For readers familiar with the theory of Gröbner bases, we offer the following alternative proof: from a given basis of I with elements in $K[D_x, \boldsymbol{D_y}]$, we obtain a basis of $I \cap L[D_x]$ by computing a Gröbner basis with respect to an elimination order. Since Buchberger's algorithm never extends the ground field, the resulting basis must be a subset of $K[D_x]$.

3. The Shape Lemma

For an ideal $I \subseteq K[D_x, \boldsymbol{D_y}]$ of dimension zero, consider the quotient $K[D_x, \boldsymbol{D_y}]/I$ as a $K[D_x]$-module. Since its dimension as K-vector space is finite, this module must be cyclic [13, Prop. 2.9]. If $M \in K[D_x, \boldsymbol{D_y}]$ is such that the equivalence class $[M]$ is a generator of the module, then there is an $L \in K[D_x]$ such that $L \cdot [M] = [LM] = [1]$. Therefore, evaluating an integral

$$F(x) = \int_\Omega f(x, \boldsymbol{y})\, d\boldsymbol{y}$$

for a function $f(x, \boldsymbol{y})$ whose ideal of annihilating operators is I is the same as evaluating the integral

$$F(x) = \int_\Omega L \cdot g(x, \boldsymbol{y})\, d\boldsymbol{y}$$

where $g(x, \boldsymbol{y})$ is defined as $M(f(x, \boldsymbol{y}))$. The choice of M implies that the annihilating ideal J of $g(x, \boldsymbol{y})$ has a basis of the form $\{D_{y_1} - Q_1, \dots, D_{y_n} - Q_n, P\}$ for some operators $P, Q_1, \dots, Q_n$ in $K[D_x]$.

Transforming I to J is known as gauge transform and can be considered as a satisfactory solution to our problem: every ideal $I \subseteq K[D_x, \boldsymbol{D_y}]$ of dimension zero can brought to an ideal J to which the shape lemma applies by means of a gauge transform.

We shall propose an alternative approach here. Rather than applying a gauge transform, which amounts to applying an operator to the integrand, our question is whether we can also obtain an ideal basis of the required form by applying a linear change of variables, i.e., using

$$F(x) = \int_\Omega f(x, \boldsymbol{y})\, d\boldsymbol{y} = \int_{\tilde\Omega} f(x, y_1 + c_1 x, \dots, y_n + c_n x)\, d\boldsymbol{y}$$

for some constants $c_1, \dots, c_n$ (and an appropriately adjusted integration range). It turns out that this perspective leads to a shape lemma for differential operators that matches more closely the situation in the commutative case.

Note that

$$L(x, y_1, \dots, y_n, D_x, D_{y_1}, \dots, D_{y_n}) \in K[D_x, D_y]$$

is an annihilating operator of $f(x, \boldsymbol{y} + \boldsymbol{c}x) = f(x, y_1 + c_1 x, \dots, y_n + c_n x)$ if and only if

$$L(x, y_1 - c_1 x, \dots, y_n - c_n x, D_x + c_1 D_{y_1} + \dots + c_n D_{y_n}, D_{y_1}, \dots, D_{y_n})$$

is an annihilating operator of $f(x, \boldsymbol{y})$. In particular, the ideal of annihilating operators of $f(x, \boldsymbol{y})$ has dimension zero if and only this is the case for the ideal of annihilating operators of $f(x, \boldsymbol{y} + \boldsymbol{c}x)$.

We shall show (Thm. 11 below) that every zero-dimensional left ideal of $K[D_x, \boldsymbol{D_y}]$ can be brought to normal position by a change of variables $\boldsymbol{y} \leftarrow \boldsymbol{y} + \boldsymbol{c}x$. For the notion of being in normal position, we propose the following definition.

Definition 6. *Let $I \subseteq K[D_x, \boldsymbol{D_y}]$ be an ideal of dimension zero, so that $r = \dim_K K[D_x, \boldsymbol{D_y}]/I$ is finite. The ideal I is called* in normal position *(w.r.t. D_x) if for every choice of C-linearly independent solutions $f_1, \dots, f_r$ we have $\operatorname{Wr}_x(f_1, \dots, f_r) \neq 0$.*

Example 7. *For the ideal $I = \langle (D_x - 1)(D_x - 2), D_y \rangle$ we have $r = 2$. The solution space of I is generated by $\exp(x)$ and $\exp(2x)$. We have $\operatorname{Wr}_x(\exp(x), \exp(2x)) = \exp(3x)$. Therefore, I is in normal position w.r.t. D_x. However, as $D_y(\exp(x)) = D_y(\exp(2x)) = 0$, we also have $\operatorname{Wr}_y(\exp(x), \exp(2x)) = 0$, so I is* <u>not</u> *in normal position w.r.t. D_y.*

With this notion of being in normal position, we can state the following result.

Theorem 8. *(Shape Lemma; differential analog of Thms. 2 and 3) Let $I \subseteq K[D_x, \boldsymbol{D_y}]$ be an ideal of dimension zero. Let P be a generator of $I \cap K[D_x]$. Then the following conditions are equivalent:*

1. *I is in normal position w.r.t. D_x*
2. *$\operatorname{ord}(P) = \dim_K K[D_x, \boldsymbol{D_y}]/I$*
3. *$K[D_x]/\langle P \rangle$ and $K[D_x, \boldsymbol{D_y}]/I$ are isomorphic as $K[D_x]$-modules.*
4. *There are $Q_1, \dots, Q_n \in K[D_x]$ with $\operatorname{ord}(Q_i) < \operatorname{ord}(P)$ for all i such that $\{D_{y_1} - Q_1, \dots, D_{y_n} - Q_n, P\}$ is a basis of I.*

Proof. Let $r = \dim_K K[D_x, \boldsymbol{D_y}]/I$.

$1 \Rightarrow 2$ To show that $\operatorname{ord}(P) = r$, suppose that $\operatorname{ord}(P) < r$ and let $f_1, \dots, f_r$ be some C-linearly independent solutions of I. By Thm. 4, we may assume that such solutions exist. As no more than $\operatorname{ord}(P)$ solutions of P can be linearly independent over $C[[\boldsymbol{y}]]$, it follows that $f_1, \dots, f_r$ are linearly dependent over $C[[\boldsymbol{y}]]$. This implies $\operatorname{Wr}_x(f_1, \dots, f_r) = 0$, in contradiction to the assumption that I is in normal position.

$2 \Rightarrow 1$ Let $f_1, \dots, f_r$ be C-linearly independent solutions of I. We have to show that they are also linearly independent over $C((\boldsymbol{y}))$. Suppose otherwise. Then we may assume that f_r is a $C((\boldsymbol{y}))$-linear combination of $f_1, \dots, f_{r-1}$. The operator

$$Q = \begin{vmatrix} f_1 & \cdots & f_{r-1} & 1 \\ D_x(f_1) & \cdots & D_x(f_{r-1}) & D_x \\ \vdots & & \vdots & \vdots \\ D_x^{r-1}(f_1) & \cdots & D_x^{r-1}(f_{r-1}) & D_x^{r-1} \end{vmatrix} \in C((x, \boldsymbol{y}))[D_x]$$

has the solutions $f_1, \dots, f_{r-1}$ and f_r. It must therefore belong to the ideal generated by I in the larger ring $C((x, \boldsymbol{y}))[D_x, \boldsymbol{D_y}]$, for if it didn't, then $\dim_{C((x,\boldsymbol{y}))} C((x, \boldsymbol{y}))[D_x, \boldsymbol{D_y}]/(\langle I \rangle + \langle Q \rangle) < r$, which is impossible when the solution space has C-dimension r.

By Lemma 5, P is also a generator of the elimination ideal $\langle I \rangle \cap C((x, \boldsymbol{y}))[D_x]$, where $\langle I \rangle$ denotes the ideal generated by I in $C((x, \boldsymbol{y}))[D_x, \boldsymbol{D_y}]$. By assumption we have $\operatorname{ord}(P) = r > \operatorname{ord}(Q)$. This is a contradiction.

$2 \Rightarrow 3$ Consider the function $\phi\colon K[D_x]/\langle P\rangle \to K[D_x, \boldsymbol{D_y}]/I$ defined by $\phi([L]_{\langle P\rangle}) := [L]_I$. This function is well-defined because $\langle P\rangle \subseteq I$. The function is obviously a morphism of $K[D_x]$-modules, and it is injective, because if $L \in K[D_x]$ is such that $[L]_I = [0]_I$, then $L \in I$, so $L \in I \cap K[D_x] = \langle P\rangle$, so $[L]_{\langle P\rangle} = 0$. Being a morphism of $K[D_x]$-modules, ϕ is in particular a morphism of K-vector spaces. Therefore, since $\dim_K K[D_x]/\langle P\rangle = r = \dim_K K[D_x, \boldsymbol{D_y}]/I$ by assumption, injectivity implies bijectivity, and therefore ϕ is an isomorphism.

$3 \Rightarrow 2$ clear.

$2 \Rightarrow 4$ By assumption, the elements $[1], [D_x], \dots, [D_x^{r-1}]$ of $K[D_x, \boldsymbol{D_y}]/I$ are K-linearly independent and therefore form a vector space basis of $K[D_x, \boldsymbol{D_y}]/I$. Therefore, the element $[D_y]$ of $K[D_x, \boldsymbol{D_y}]/I$ can be expressed as a K-linear combination of $[1], [D_x], \dots, [D_x^{r-1}]$. This implies the existence of a Q.

$4 \Rightarrow 2$ By repeated addition of suitable multiples of basis elements, it can be seen that every element of $K[D_x, \boldsymbol{D_y}]$ is equivalent modulo I to an element of the form $q_0 + q_1 D_x + \cdots + q_{r-1} D_x^{r-1}$. Therefore, the elements $[1], \dots, [D_x^{r-1}]$ generate $K[D_x, \boldsymbol{D_y}]/I$ as a K-vector space. This implies $\dim_K K[D_x, \boldsymbol{D_y}]/I \le r$. At the same time, the dimension cannot be smaller than r, because if $[1], \dots, [D_x^{r-1}]$ were K-linearly dependent, then $I \cap K[D_x]$ would contain an element of order less than $\operatorname{ord}(P)$, which is impossible by the choice of P. □

Again, readers familiar with the theory of Gröbner bases will have no difficulty finding shorter arguments for some of the implications.

The similarity of Thm. 8 to the corresponding theorems for commutative polynomial rings is evident, but there are some subtle differences as well. One difference is that Thms. 2 and 3 require the ideal to be radical, while no such assumption is needed for Thm. 8.

However, it turns out that in order to also generalize Thm. 1 to differential operators, we do need to introduce a restriction. Note that Thm. 1 becomes wrong for non-radical ideals if we interpret their solutions as points with multiplicities. Indeed, in this sense, a non-radical ideal is never in normal position, and no linear change of variables will suffice to turn a non-radical ideal into a radical ideal.

Ideals of differential operators cannot have multiple solutions (cf. Thm. 4). Instead, it seems appropriate to adopt the following notion.

Definition 9. *A finite dimensional C-vector space V is called* linearly disjoint *with K (over C) if $\dim_K K \otimes_C V = \dim_C V$, or equivalently, if any C-basis of V is K-linearly independent. A zero-dimensional ideal $I \subseteq K[D_x, \boldsymbol{D_y}]$ is called* D-radical *if its solution space $V(I)$ is linearly disjoint with K.*

Observe the difference between Defs. 9 and 6. In both cases we require the absence of linear relations, but with respect to different coefficient domains. For normal position, the coefficients must be free of x but can depend in an arbitrary way on $\boldsymbol{y}$, and for D-radical the coefficients must be rational functions in x and $\boldsymbol{y}$.

Example 10.

1. *Let $I = \langle (D_x - 1)^2, D_y\rangle \subseteq K[D_x, D_y]$, then $V(I)$ contains the C-linearly independent solutions $\exp(x)$ and $x\exp(x)$. As these are not linearly independent over K, the ideal I is not D-radical.*
2. *The solution space of the ideal $\langle (D_x - 1)(D_x - 2), D_y\rangle$ has the basis $\{\exp(x), \exp(2x)\}$. Since $\exp(x)$ and $\exp(2x)$ are linearly independent over $K = C(x)$, the ideal is D-radical.*

Note that in both instances of Example 10 the generators of the ideal actually lie in the commutative ring $C[D_x, D_y]$. We observe that the corresponding ideal in $C[D_x, D_y]$ is radical (in the commutative sense) in Case (2), but not radical in Case (1). This is not a coincidence. Our definition of D-radicality specializes to the classical concept of radicality when differential operators with constant coefficients are considered, which justifies the choice of the name. More precisely: for a zero-dimensional ideal $I \subseteq C[D_x, \boldsymbol{D_y}]$ we have that I is D-radical if and only if it is radical (in the commutative sense). This can be understood by looking at the closed-form solutions of such constant-coefficient differential equations: the solution space of the operator $(D_x - \alpha_1)\cdots(D_x - \alpha_r)$ is spanned by $\exp(\alpha_1 x), \dots, \exp(\alpha_r x)$, which are linearly independent over K, whenever the α_i are pairwise disjoint. In contrast, the solution space of the operator $(D_x - \alpha)^r$ is spanned by $\exp(\alpha x), x\exp(\alpha x), \dots, x^{r-1}\exp(\alpha x)$, which are clearly linearly dependent over K. This argument applies analogously to the situation of several variables.

The correspondence between radical and D-radical also extends to Theorem 11 below, which reduces to Theorem 1 for ideals generated by operators with constant coefficients. In particular, the change of variable $\boldsymbol{y} \leftarrow \boldsymbol{y} + \boldsymbol{c}x$ keeps $D_{y_1}, \dots, D_{y_n}$ unchanged, and replaces D_x with $D_x + c_1 D_{y_1} + \cdots + c_n D_{y_n}$, and thus the theorem yields a change of variable with the same structure as Prop. 1.

Theorem 11. *(Differential analog of Thm. 1) Let $I \subseteq K[D_x, \boldsymbol{D_y}]$ be a zero-dimensional D-radical ideal. Then there are constants $c_1, \dots, c_n \in C$ such that the ideal J obtained from I by applying the linear change of variables $\boldsymbol{y} \leftarrow \boldsymbol{y} + \boldsymbol{c}x$ (where $\boldsymbol{c} = (c_1, \dots, c_n)$) is in normal position w.r.t. D_x.*

Proof. We show that whenever $f_1(x, \boldsymbol{y}), \dots, f_r(x, \boldsymbol{y})$ form a C-basis of $V(I)$ such that $\mathrm{Wr}_x\big(f_i(x, \boldsymbol{y} + \boldsymbol{c}x)\big)_{i=1}^r = 0$ for all $\boldsymbol{c} \in C^n$, then $f_1, \dots, f_r$ are K-linearly dependent. Since the ideal is D-radical, this implies that $f_1, \dots, f_r$ are C-linearly dependent, which is a contradiction.

Consider $c_1, \dots, c_n$ as an additional variables and recall that the assumption $\mathrm{Wr}_x\big(f_i(x, \boldsymbol{y} + \boldsymbol{c}x)\big)_{i=1}^r = 0$ implies that the $f_i(x, \boldsymbol{y} + \boldsymbol{c}x)$ are linearly dependent over the constant field with respect to x, i.e., $C((\boldsymbol{y}, \boldsymbol{c}))$-linearly dependent: thus we can assume that there exist $p_1, \dots, p_r \in C((\boldsymbol{y}, \boldsymbol{c}))$, not all 0, such that

$$\sum_{i=1}^r p_i(\boldsymbol{y}, \boldsymbol{c}) \cdot f_i(x, \boldsymbol{y} + \boldsymbol{c}x) = 0. \tag{2}$$

Each f_i has an expansion as a series in x:

$$f_i(x, \boldsymbol{y} + \boldsymbol{c}x) = \sum_{j=0}^{\infty} \underbrace{\frac{1}{j!} \frac{\partial^j f_i(x, \boldsymbol{y} + \boldsymbol{c}x)}{\partial x^j}\bigg|_{x=0}}_{=:\, f_{i,j}(\boldsymbol{y}, \boldsymbol{c})} \cdot x^j.$$

Note that the series coefficients $f_{i,j}$ are polynomials in $\boldsymbol{c}$, because

$$\frac{\partial^j f_i(x, y + cx)}{\partial x^j}\bigg|_{x=0} = \sum_{k_0 + \dots + k_n = j} \binom{j}{k_0, \dots, k_n} \cdot \underbrace{\Big[f_i^{(k_0, \dots, k_n)}(x, \boldsymbol{y} + \boldsymbol{c}x)\Big]_{x=0}}_{\in C((\boldsymbol{y}))} \cdot c_1^{k_1} \cdots c_n^{k_n} \in C((\boldsymbol{y}))[\boldsymbol{c}].$$

Here the notation $f_i^{(k_0, \dots, k_n)}$ refers to k_0-fold derivative w.r.t. the first argument, the k_1-fold derivative with respect to the second argument, etc.

It follows that Eq. (2) can be expanded as

$$\sum_{j=0}^{\infty} \left(\sum_{i=1}^r p_i(\boldsymbol{y}, \boldsymbol{c}) \cdot f_{i,j}(\boldsymbol{y}, \boldsymbol{c}) \right) \cdot x^j = 0$$

and therefore, for all $j \in \mathbb{N}$, $\sum_{i=1}^r p_i(\boldsymbol{y}, \boldsymbol{c}) \cdot f_{i,j}(\boldsymbol{y}, \boldsymbol{c}) = 0$.

Let M be the matrix $\big(f_{i,j}(\boldsymbol{y}, \boldsymbol{c})\big)_{j \geq 0, 1 \leq i \leq r}$ with infinitely many rows and r columns. From the above,

$$\big(p_i(\boldsymbol{y}, \boldsymbol{c})\big)_{i=1}^r \in \ker M,$$

and therefore M is rank-deficient; let $R < r$ denote the rank of M. Hence there exists an integer $m \in \mathbb{N}$ such that the rank of the $(m \times r)$-submatrix M', that is obtained by taking the first m rows of M, is also equal to R. It follows that $\ker(M') = \ker(M)$, and since $M' \in C((\boldsymbol{y}))[\boldsymbol{c}]^{m \times r}$ we have that $\ker(M')$ is a subspace of $C((\boldsymbol{y}))(\boldsymbol{c})^r$. Therefore, the coefficients $p_i(\boldsymbol{y}, \boldsymbol{c})$ can be chosen in $C((\boldsymbol{y}))(\boldsymbol{c})$. In fact, by clearing denominators, we can even assume them to belong to $C[[\boldsymbol{y}]][\boldsymbol{c}]$.

Now perform the substitution $\boldsymbol{c} \leftarrow \boldsymbol{c} - \boldsymbol{y}/x$ in (2) to get

$$\sum_{i=1}^r p_i(\boldsymbol{y}, \boldsymbol{c} - \boldsymbol{y}/x) \cdot f_i(x, \boldsymbol{c}x) = 0. \tag{3}$$

Each $p_i(\boldsymbol{y}, \boldsymbol{c} - \boldsymbol{y}/x)$ admits an expansion as a power series in $y_1, \dots, y_n$

$$p_i(\boldsymbol{y}, \boldsymbol{c} - \boldsymbol{y}/x) = \sum_{j_1, \dots, j_n = 0}^{\infty} q_{i, j_1, \dots, j_n}(\boldsymbol{c}, x) y_1^{j_1} \cdots y_n^{j_n}.$$

Eq. (3) then expands as

$$\sum_{j_1, \dots, j_n = 0}^{\infty} \big(\sum_{i=1}^r q_{i,j}(\boldsymbol{c}, x) \cdot f_i(x, \boldsymbol{c}x) \big) y_1^{j_1} \cdots y_n^{j_n} = 0$$

and therefore, for all $j_1, \dots, j_n \in \mathbb{Z}$,

$$\sum_{i=1}^r q_{i,j}(\boldsymbol{c}, x) \cdot f_i(x, \boldsymbol{c}x) = 0. \tag{4}$$

Since the p_i are not all 0, there must exist $i, j_1, \ldots, j_n$ with $q_{i,j_1,\ldots,j_n} \neq 0$, and therefore for such a choice of $j_1, \ldots, j_n$, the left-hand side of Eq. (4) is a non-trivial linear combination.

Furthermore, observe that since the p_i are rational in their second argument, the coefficients q_i are rational functions. So finally, substituting $\boldsymbol{c} \leftarrow \boldsymbol{y}/x$ yields the desired dependency with coefficients in K:

$$\sum_{i=1}^{r} q_i(\boldsymbol{y}/x, x) f_i(x, \boldsymbol{y}) = 0. \qquad \square$$

Example 12. *The annihilator* I_1 *of* $\exp(x)$ *and* $y \exp(x)$ *is not D-radical. The annihilator* I_2 *of* $\exp(x)$ *and* $\exp(x+y)$ *is D-radical but not in normal position w.r.t.* D_x*. Setting* y *to* $y + cx$ *in* I_1 *gives the annihilator of* $\exp(x), (y + cx)\exp(x)$*, which is still not D-radical. However, setting* y *to* $y + cx$ *in* I_2 *gives the annihilator of* $\exp(x), \exp((1+c)x + y)$*, which is in normal* ∂_x*-position for every choice* $c \neq 0$*.*

Recall that our motivation was the computation of an ideal of annihilating operators for an integral

$$\int_\Omega f(x, \boldsymbol{y})\, d\boldsymbol{y}$$

where $f(x, \boldsymbol{y})$ is a D-finite function. Let $I \subseteq K[D_x, \boldsymbol{D_y}]$ be the annihilating ideal of $f(x, \boldsymbol{y})$, and assume that it is D-radical. According to Theorem 11, there exists $\boldsymbol{c}$ such that, after the change of variables $\boldsymbol{y} \leftarrow \boldsymbol{y} + \boldsymbol{c}x$, the ideal I is in normal position. According to the Shape Lemma 8, this implies that, after change of variables, the ideal I is generated by P, and $D_{y_1} - Q_1, \ldots, D_{y_n} - Q_n$, for certain $P, Q_1, \ldots, Q_n \in K[D_x]$. This means that $g(x, \boldsymbol{y}) = f(x, \boldsymbol{y} + \boldsymbol{c}x)$ is such that $P(g) = 0$ and $D_{y_i}(g) = Q_i(g)$ for all i, and we can use creative telescoping to compute an annihilating ideal J for

$$\int_{\tilde{\Omega}} g(x, \boldsymbol{y})\, d\boldsymbol{y} = \int_\Omega f(x, \boldsymbol{y})\, d\boldsymbol{y},$$

where $\tilde{\Omega}$ is the inverse image of Ω under the change of variables.

It remains open how these results extend to the recurrence case. While the restriction to the differential case does not seem essential for the Shape Lemma itself (Theorem 8), there is a substantial difference as far as the effect of a linear change of variables $y_i \leftarrow y_i + c_i x$ on annihilating operators is concerned: While D_x gets replaced by $D_x + c_1 D_{y_1} + \cdots + c_n D_{y_n}$ in the differential case, the shift operator S_x would have to be replaced by $S_x S_{y_1}^{c_1} \cdots S_{y_n}^{c_n}$. Even if we restrict $c_1, \ldots, c_n$ to nonnegative integers in order to make this meaningful, it is not clear how Theorem 11 could be adapted to this situation.

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