Guessing with Little Data

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ÖAW RICAM

OEIS A172671

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- Recurrence? Efficient method for computing next terms?
- Nature of the generating function?

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1, \dots, c_{21} \leqslant n \\ +\text{lin. constraints}}} \binom{3n}{c_1, c_2, \dots, c_{21}}$$

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Answer 2: Interpret arrays as walks in $\mathbb{Z}^6_{\geqslant 0}$ ending at $(n, \ldots, n)$:

$$\sum_{n=0}^{\infty} a_n x^n = \mathrm{Diag}\left(\frac{1}{1 - x_1x_2 - x_1x_3 - \cdots - x_5x_6 - x_1^2 - \cdots - x_6^2} \right)$$

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How to construct this recurrence / ODE?

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Ansatz: $c_0 a_n + c_1 a_{n+1} + \cdots + c_r a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

$$M = \begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

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- For sequence A172671 we know only 33 terms, i.e. we can try $(r,d) = (1,15), (2,9), (3,6), (4,5), (6,3), (7,2), (10,1), (16,0)$.

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- The coefficients of the recurrence involve small integers.

$\longrightarrow$ Employ a lattice reduction algorithm (LLL, BKZ, ...).

Lattice Basis

Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \ldots, c_\ell \in \mathbb{Z}\}.$$

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A lattice reduction algorithm computes a basis $w_1, \ldots, w_\ell$ of $\mathcal{L}$ that consists of short vectors.

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- It can be computed, e.g. using the Hermite normal form.

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$
Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

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if $\ell = 0$, then return FAIL.

Algorithm

Input: $a_0, \dots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

```
Compute ℤ-module basis v_1, ..., v_ℓ ∈ ℤ^{(r+1)(d+1)} of ker_ℤ M.
if ℓ = 0, then return FAIL.
Apply LLL to v_1, ..., v_ℓ, and call the result w_1, ..., w_ℓ.
```

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

Compute $\mathbb{Z}$-module basis $v_1, \ldots, v_\ell \in \mathbb{Z}^{(r+1)(d+1)}$ of $\ker_{\mathbb{Z}} M$.
if $\ell = 0$, then return FAIL.
Apply LLL to $v_1, \ldots, v_\ell$, and call the result $w_1, \ldots, w_\ell$.
Return the recurrence corresponding to the vector w_1.

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Improvements and Variations:

- Use different basis than standard monomials (binomials, ...)

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Improvements and Variations:

- Use different basis than standard monomials (binomials, ...)
- Incorporate homomorphic images and Chinese remaindering
- Recycle LLL-output when trying a range of degrees d

The Generic Case

Theorem [Bombieri–Vaaler, 1983] Let $M \in \mathbb{Z}^{k \times m}$ with $k < m$, and let g be the $\gcd$ of all $k \times k$ minors of M.
Then $\ker_{\mathbb{Z}} M$ contains a nonzero element $x \in \mathbb{Z}^m$ with

$$||x||_\infty \leqslant \left(\frac{1}{g}\sqrt{\det\bigl(MM^\top\bigr)}\right)^{1/(m-k)}$$

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$$N \lesssim \frac{1}{2}\Big(\sqrt{8(r+1)(d+1)+49} - 7\Big) + r - 1.$$

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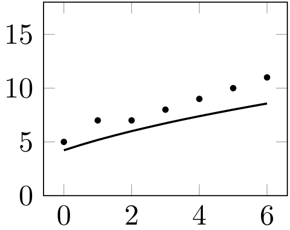
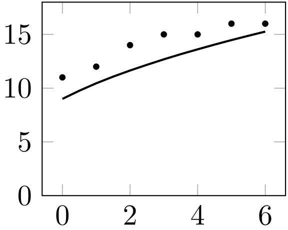


$r = 4$



$r = 8$

Implications:

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The Non-Generic Case

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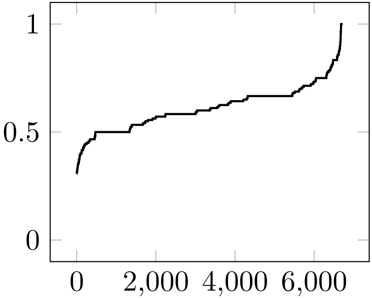

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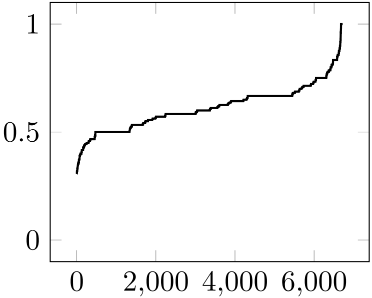

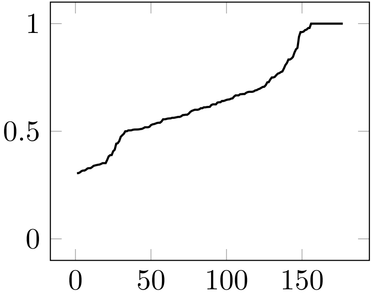

Result for A172671

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$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}\\
&+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7}\\
&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4}\\
&+248660072114197834440n^{3} + 195845152107619591920n^{2}\\
&+92743576895010081600n + 19927056990544704000)a_n\\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10}\\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7}\\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4}\\
&+11017108466619178896n^{3} + 9901273828612752684n^{2}\\
&+5411908796200065936n + 1358800904704763520)a_{n+1}\\
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&-847459848696773373n^{3} - 821800045816910820n^{2}\\
&-485718284438018172n - 132150596906568240)a_{n+2}\\
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&-689472630263907n^{6} - 1949560872565283n^{5} - 4070539427181535n^{4}\\
&-6099491170412670n^{3} - 6211013227585736n^{2}\\
&-3851899366258336n - 1098712786184832)a_{n+3}\\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10}\\
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972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

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Result for A172671

Trustworthy?

$$
\begin{aligned}
&416745(n+2)(3n+4)(3n+5)(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(3784n^4+62436n^3+384549n^2+1047914n+1066254)a_n\\
&+9(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(29681696n^7+712360704n^6+7253307424n^5\\
&\quad+40621828312n^4+135172900470n^3+267337368752n^2\\
&\quad+291083104767n+134667010044)a_{n+1}\\
&-9(n+3)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(10844944n^8+309080904n^7+3833838118n^6\\
&\quad+27035659722n^5+118560795930n^4+331121212914n^3\\
&\quad+575194973415n^2+568260550317n+244478848756)a_{n+2}\\
&-(n+3)(n+4)^3(3n+13)(3n+14)\\
&\quad\times(3799136n^7+98777536n^6+1092573240n^5\\
&\quad+6662600832n^4+24184813590n^3+52244190090n^2\\
&\quad+62174897623n+31442101253)a_{n+3}\\
&+(n+3)(n+4)^3(n+5)^5\\
&\quad\times(3784n^4+47300n^3+219945n^2\\
&\quad+450988n+344237)a_{n+4}
\end{aligned}
$$

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245885257930209910994050195049583660
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13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
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390229178478432343758493287708395462786699986146463590205462500
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1703218238481833503830053446085753316816923905337688679320940617430053026793000
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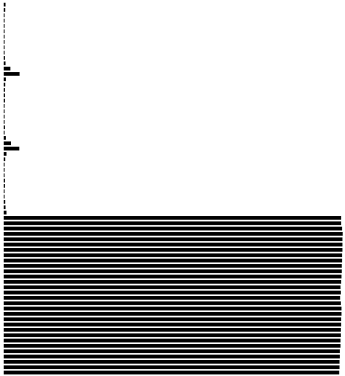
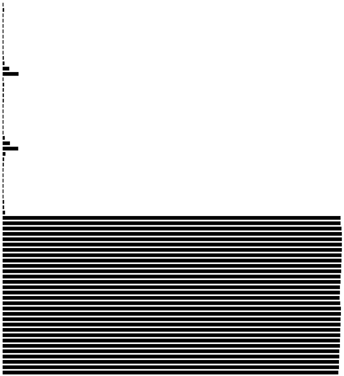
4880259952199292008921826526312609348825249147788374851144194565482906902131493688167790266738802504840000

Result for A172671

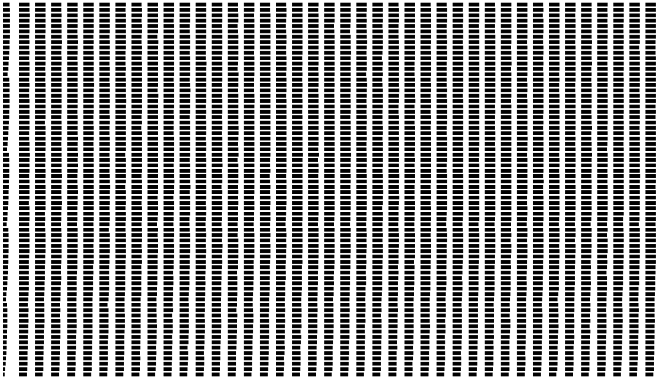
First two vectors of $\ker_{\mathbb{Z}} M$:



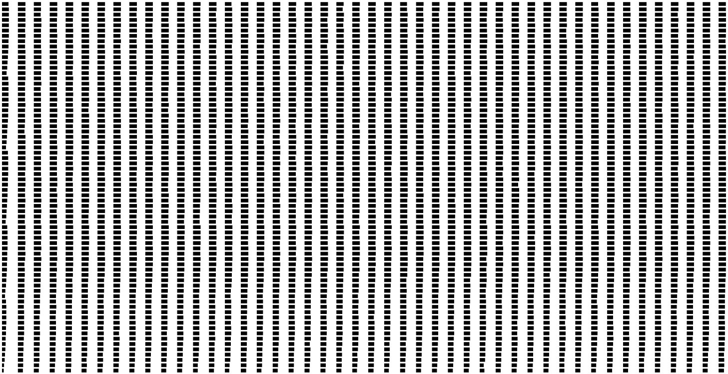


Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 33$:

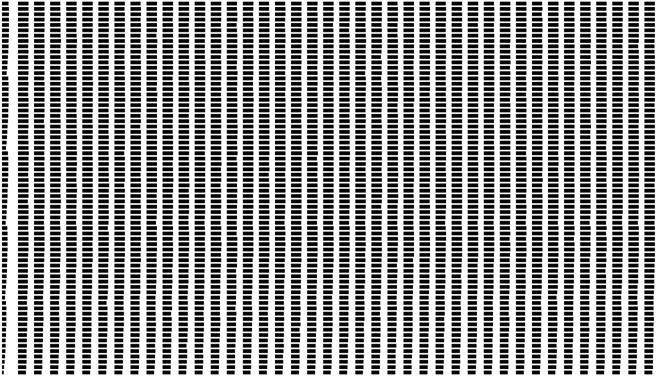


LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 28$:



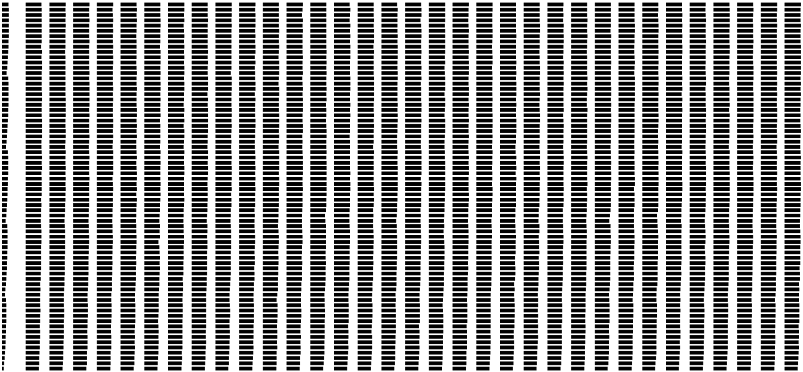
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 33$:



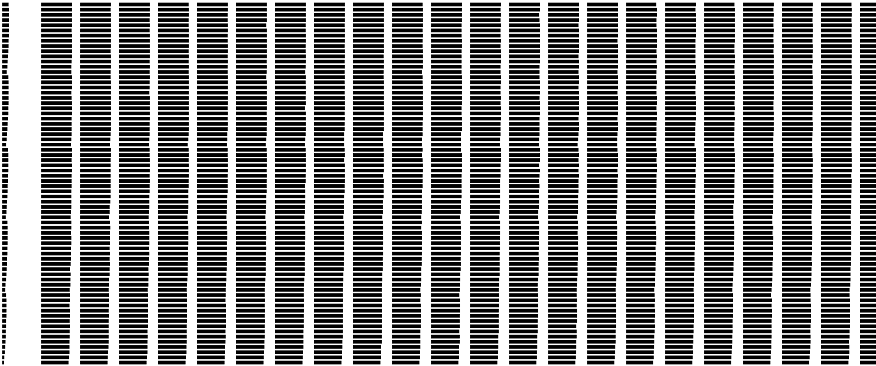
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 40$:



Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 50$:



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- We were not able to find a recurrence for the notorious Av(1324) sequence. . .

Encore: Guessing from a Single Term

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#	85994	48240	3056	853	25	8	1	1	$\cdots$	1

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Hence, the recurrence can be recovered from the single sequence term $D_8 = 265729$.