Motion Polynomials and Planar Linkages

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PlanarLinkages.m

Problem: Given a bounded rational curve in the plane, construct a planar linkage with rotational joints that draws this curve.

PlanarLinkages.m is a Mathematica package for

- computing with motion polynomials (arithmetic, factorization),
- constructing a planar linkage for a given curve,
- and for visualizing these objects.

Webpage: `http://www.koutschan.de/data/link/`.

The underlying theory and algorithms were developed in

- M. Gallet, C. Koutschan, Z. Li, G. Regensburger, J. Schicho, N. Villamizar: *Planar linkages following a prescribed motion.* Mathematics of Computation (10.1090/mcom/3120), 2016.

Some references:

- Kempe (1876), Jordan+Steiner (1999), King (1999), Gao+Zhu (2001), Kapovich+Millson (2002), Abbott (2008), Kobel (2008), Abel+Demaine^2+Eisenstat+Lynch+Schardl (2016)

Strategy for Constructing Linkages

1. Algebraic representation of direct isometries: Construct $\mathbb{K}$, a ring whose multiplication corresponds to composition in SE_2.
2. A motion polynomial $P \in \mathbb{K}[t]$ describes a (rational) motion, i.e., a one-parameter family of direct isometries.
3. Compute a factorization of P into linear factors (each linear factor represents a revolution and can be realized by a joint).
4. A factorization of P gives rise to an open chain of links, which has high mobility and hence "weakly" realizes the motion.
5. Insert more links in order to restrain the mobility of the linkage to one, so that it "strongly" realizes the given motion.

Algebraic representation of SE_2

Embedding: We embed SE_2 in $\mathbb{P}^3_{\mathbb{R}}$ as the open subset

$$\mathbb{P}^3_{\mathbb{R}} \setminus \left\{(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_{\mathbb{R}} \mid x_1^2 + x_2^2 = 0\right\}.$$

The action of $(x_1 : x_2 : y_1 : y_2)$ on a point $(x, y) \in \mathbb{R}^2$ is given by

$$\frac{1}{x_1^2 + x_2^2}\left[\begin{pmatrix} x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix}\right].$$

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Notation. Write $(x_1 : x_2 : y_1 : y_2)$ as a **dual complex number** $x_1 + i\,x_2 + \eta\,(y_1 + i\,y_2) \in \mathbb{K}$, where the symbol η satisfies

$$z\,\eta = \eta\,\overline{z} \quad \text{for all } z \in \mathbb{C} \quad \text{and} \quad \eta^2 = 0.$$

Hence for $z, z', w, w' \in \mathbb{C}$ we have

$$(z + \eta\,w) \cdot (z' + \eta\,w') = z\,z' + \eta\big(\overline{z}\,w' + z'\,w\big).$$

Rational motions and motion polynomials

Definition. A **rational motion** is a map $\mathbb{R} \to \mathbb{P}^3_{\mathbb{R}}$ given by four real polynomials $X_1, X_2, Y_1, Y_2 \in \mathbb{R}[t]$ such that $X_1^2 + X_2^2 \not\equiv 0$.

Notation. A rational motion is written as a polynomial $P \in \mathbb{K}[t]$

$$P(t) = Z(t) + \eta\, W(t), \qquad Z, W \in \mathbb{C}[t],$$

where $Z = X_1 + i\, X_2$ and $W = Y_1 + i\, Y_2$.

- A polynomial in $\mathbb{K}[t]$ is called a **motion polynomial**.
- For $P \in \mathbb{K}[t]$ and a real polynomial $R \in \mathbb{R}[t]$, the motion polynomial RP describes the **same** rational motion as P.

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Proposition. Motion polynomials of degree 1 correspond to either a translational or a rotational motion.

Example: ellipse

We want to construct a linkage drawing an ellipse with radii 1 and $\frac{1}{2}$.

- Consider the ellipse that is defined by the equation

  $$(x+1)^2 + 4y^2 = 1$$

  (the ellipse passes through the origin).
- A parametrization of this curve is given by

  $$\varphi\colon \mathbb{R} \to \mathbb{R}^2, \quad \varphi(t) = \frac{1}{t^2+1}\begin{pmatrix} -2 \\ t \end{pmatrix}.$$

- The motion polynomial $t^2 + 1 + \eta\,(-2 + i\,t)$ represents a (translational) motion under which $(0,0)$ traces the ellipse.

Factorization into linear factors

Factor $P = Z + \eta W \in \mathbb{K}[t]$ into monic linear motion polynomials:

$$P = P_1 \cdot P_2 \cdots P_n, \quad P_i = t - z_i + \eta\, w_i, \quad w_i, z_i \in \mathbb{C}.$$

Recall: $(z + \eta\, w) \cdot (z' + \eta\, w') = z\, z' + \eta\, (\overline{z}\, w' + z'\, w)$.

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By expanding the ansatz we obtain:

$$P = (t - z_1 + \eta\, w_1) \cdot (t - z_2 + \eta\, w_2) \cdots (t - z_n + \eta\, w_n) =$$

$$= \prod_{i=1}^{n} (t - z_i) \ + \ \eta \sum_{k=1}^{n} \Bigg(\prod_{j=1}^{k-1} (t - \overline{z_j}) \Bigg) \Bigg(\prod_{j=k+1}^{n} (t - z_j) \Bigg) w_k.$$

- The z_i are precisely the complex roots of $Z(t)$.
- The w_i can be found by ansatz and solving a linear system.
- The order of $\mathbf{z} = (z_1, \ldots, z_n)$ matters.

Factorization

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$\longrightarrow$ This motion polynomial cannot be factored!

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Solution: Multiply P by some real polynomial $R \in \mathbb{R}[t]$.
$\longrightarrow$ Note that this doesn't change the motion itself.

Example (continued): For the ellipse we can take $R = t^2 + 1$:

$$\begin{aligned} R \cdot P &= \bigl(t^2 + 1\bigr) \cdot \bigl(t^2 + 1 + \eta\,(-2 + i\,t)\bigr) \\ &= t^4 + 2t^2 + 1 + \eta\,\bigl(i\,t^3 - 2t^2 + i\,t - 2\bigr) \\ &= \bigl(t + i - \eta\,i\bigr) \cdot \bigl(t - i + \tfrac{1}{2}\eta\,i\bigr) \cdot \bigl(t - i + \tfrac{3}{2}\eta\,i\bigr) \cdot \bigl(t + i\bigr) \\ &= \bigl(t - i + \eta\,i\bigr) \cdot \bigl(t + i + \tfrac{1}{2}\eta\,i\bigr) \cdot \bigl(t + i - \eta\,i\bigr) \cdot \bigl(t - i + \tfrac{1}{2}\eta\,i\bigr). \end{aligned}$$

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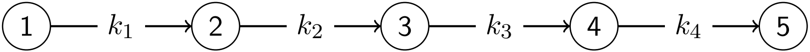
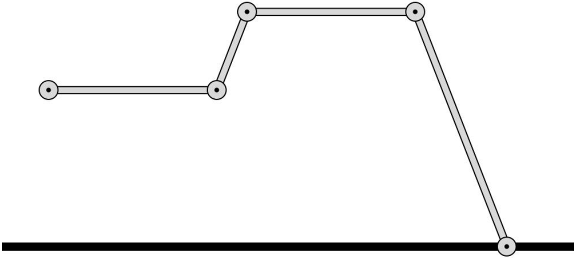
Definition. A motion polynomial $P = Z + \eta\,W \in \mathbb{K}[t]$ is called bounded if it is monic and if Z does not have real roots.

Theorem. Let $P \in \mathbb{K}[t]$ be a bounded motion polynomial. Then there exists a real polynomial $R \in \mathbb{R}[t]$ such that RP can be factored into linear polynomials.

Rigidification by flip procedure





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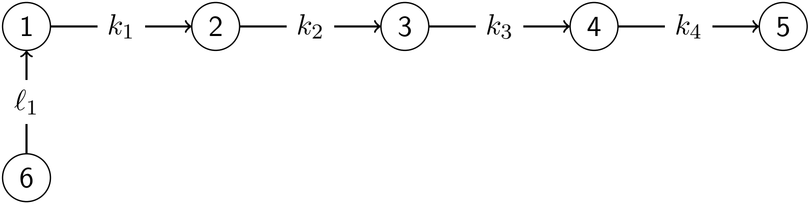

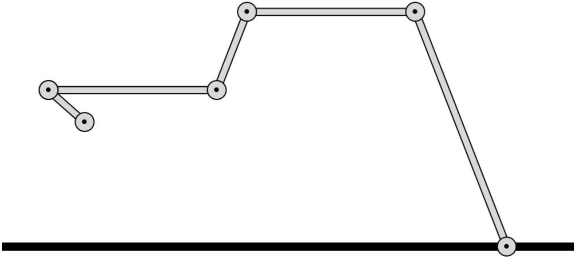

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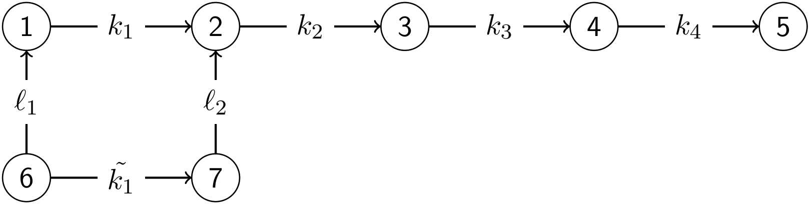

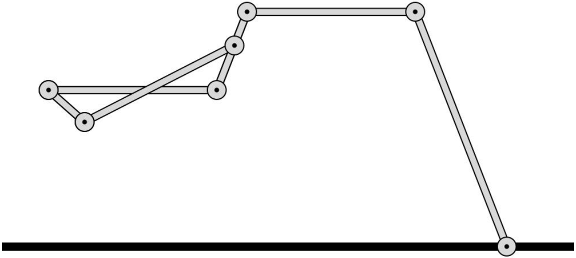

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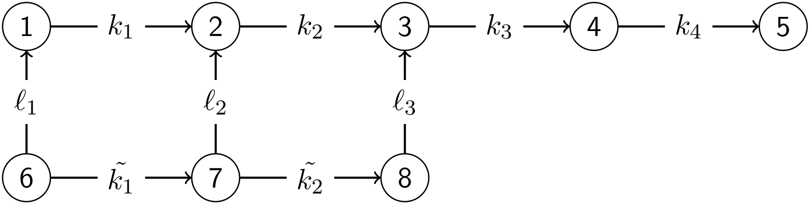

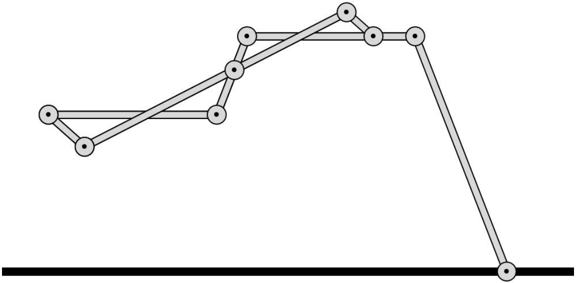

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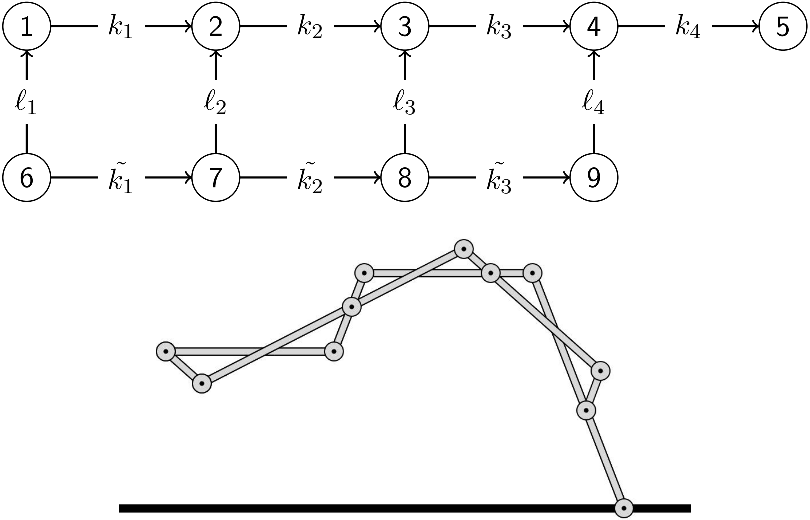

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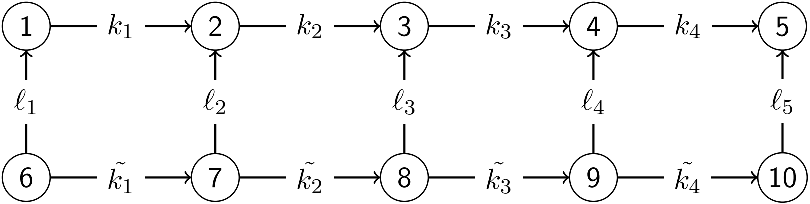

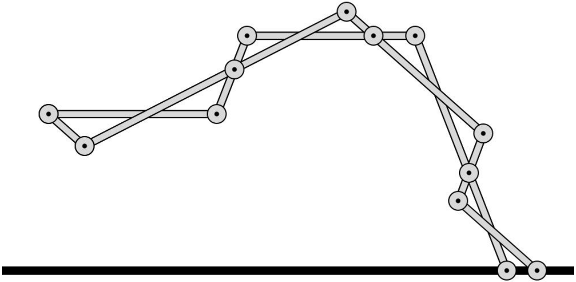

Construction of linkages

Algorithm (sketch).

1. Given a motion $\phi\colon \mathbb{R} \to \mathrm{SE}_2$ via a polynomial $P \in \mathbb{K}[t]$.
2. Compute $R \in \mathbb{R}[t]$ such that RP factors into linear factors.
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Theorem. The linkage obtained in this way has mobility one and realizes the motion ϕ in the strong sense. It has at most $3d+2$ links and $\frac{9}{2}d+1$ joints (d is the degree of the curve).

Consequence: For drawing a given curve we construct a linkage that is driven by a single motor operating at constant speed.

Detect self-collisions

Collision: happens if for some $t \in \mathbb{R} \cup \{\infty\}$ and $0 \leqslant s \leqslant 1$

$$x_1(t) = s \cdot x_2(t) + (1-s) \cdot x_3(t)$$
$$y_1(t) = s \cdot y_2(t) + (1-s) \cdot y_3(t)$$

- $i < k < j$ are links (assume that labels correspond to heights)
- $\big(x_1(t), y_1(t)\big)$ = position of joint (i, j)
- $\big(x_2(t), y_2(t)\big)$ = position of some joint connected to k
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The above equations can be solved with reasonably small effort. In contrast to general linkages, our linkages allow for a relatively simple collision detection.