A Generalized Apagodu-Zeilberger Algorithm

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23 July 2014
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Different approaches to creative telescoping:

Elimination approach:

Zeilberger's slow algorithm (1990), Takayama's algorithm (1990)

$\longrightarrow$ works for general ∂-finite holonomic functions

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Prediction approach:

Apagodu-Zeilberger algorithms (2005, 2006)
$\longrightarrow$ generalization to ∂-finite functions (NEW!)

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4. Derivations: K-linear maps $\delta_x, \delta_y \colon K(x,y) \to K(x,y)$ s.t.

$$\delta_x(ab) = \delta_x(a)b + \sigma_x(a)\delta_x(b), \quad \delta_y(ab) = \delta_y(a)b + \sigma_y(a)\delta_y(b)$$

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Additional assumptions: For all $p \in K[x,y]$:

$$\sigma_x(p), \sigma_y(p), \delta_x(p), \delta_y(p) \in K[x,y],$$

$$\deg_x(\sigma_x(p)) = \deg_x(p), \qquad \deg_y(\sigma_x(p)) = \deg_y(p),$$

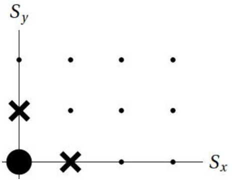
$$\deg_x(\delta_x(p)) \leq \deg_x(p) - 1, \quad \deg_y(\delta_x(p)) \leq \deg_y(p),$$

and likewise for σ_y, δ_y.

Hypergeometric term:

$f(x,y)$ is hg. if $f(x+1,y)/f(x,y), f(x,y+1)/f(x,y) \in K(x,y)$.

$\longrightarrow f(x,y)$ satisfies first-order recurrence equations in x and y.



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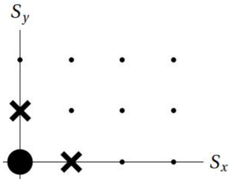

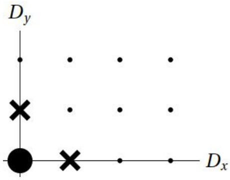

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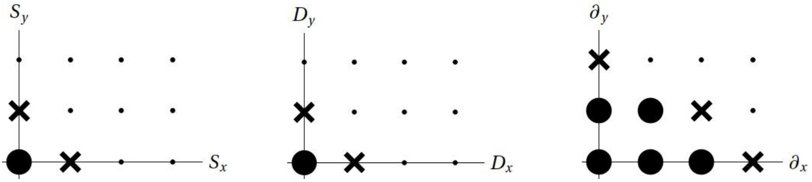
∂-finite function:

$f(x,y)$ is ∂-finite if the annihilator $\operatorname{ann}_{\mathbb{A}}(f) := \{P \in \mathbb{A} \mid P \cdot f = 0\}$ is a zero-dimensional left ideal, i.e., $\dim_{K(x,y)}(\mathbb{A}/\operatorname{ann}_{\mathbb{A}}(f)) < \infty$.
$\longrightarrow$ $f(x,y)$ satisfies a higher-order system of linear equations.



The Apagodu-Zeilberger Algorithm

Setting: Work in the Ore algebra $\mathbb{A} = K(x, y)[\partial_x, \partial_y]$ where

- ∂_x denotes the x-shift operator ($\sigma_x(x) = x + 1$, $\delta_x = 0$)
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Problem: Find

- telescoper $T = t_0 + t_1\partial_x + \cdots + t_r\partial_x^r \in K(x)[\partial_x] \setminus \{0\}$
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3. The condition #unknowns $>$ #equations yields an upper bound for r, the order of the telescoper.

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Consider the simple example $h(x, y) := \dfrac{1}{\Gamma(ax + by)}, \quad a, b \in \mathbb{N}$.

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$\longrightarrow$ For $r \geq b$ we get a nontrivial solution.

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Factorials:

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$\longrightarrow$ Analogously: $(a;i)_x,\ p\lceil_x,\ p\rceil_x$.

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- every $w \in \mathbb{A}/\mathfrak{a}$ can be written uniquely $wb = \sum_{i=1}^{n} w_i b_i$ for $w = (w_1, \ldots, w_n) \in K(x,y)^n$ and $b = (b_1, \ldots, b_n)^T$.

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- $\partial_x(wb) = \big(\sigma_x(w)\partial_x + \delta_x(w)\big)b = \big(\sigma_x(w)M + \delta_x(w)\big)b$
- Similarly, there exists a matrix $N \in K(x,y)^{n \times n}$ such that $\partial_y b = Nb$ and $\partial_y(wb) = \big(\sigma_y(w)N + \delta_y(w)\big)b$.

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Admissible basis:
$1 \in \mathbb{A}/\mathfrak{a}$ is represented by a polynomial vector $e \in K(x)[y]^n$.

Telescoper Part

Ansatz: $T = t_0 + t_1\partial_x + \cdots + t_r\partial_x^r \in K(x)[\partial_x], \quad t_i \in K(x)$.

Task: Predict the shape of the vector $Te \in K(x,y)^n$.

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Lemma. Let $e \in K(x)[y]^n$ be some polynomial vector. For every $i \geq 0$ we have $\partial_x^i e = w/(u;i)_x$ for some vector $w \in K(x)[y]^n$ with

$$\deg_y(w) \leq \deg_y(e) + i\max\{\deg_y(u), \deg_y(U)\}$$

where $\deg_y$ refers to the maximum degree of all components.

Proof. By induction on i.

Telescoper Part

Ansatz: $T = t_0 + t_1\partial_x + \cdots + t_r\partial_x^r \in K(x)[\partial_x], \quad t_i \in K(x)$.

Task: Predict the shape of the vector $Te \in K(x,y)^n$.

Lemma. Let $e \in K(x)[y]^n$ be some polynomial vector. For every $i \geq 0$ we have $\partial_x^i e = w/(u;i)_x$ for some vector $w \in K(x)[y]^n$ with

$$\deg_y(w) \leq \deg_y(e) + i\max\{\deg_y(u), \deg_y(U)\}$$

where $\deg_y$ refers to the maximum degree of all components.

Proof. By induction on i.

$\longrightarrow$ Thus we obtain $Te = w/(u;r)_x$ for some polynomial vector w

- whose entries are $K(x)[y]$-linear combinations of $t_0, \ldots, t_r$,
- whose degree is bounded by $\deg_y(e) + r\max\{\deg_y(u), \deg_y(U)\}$.

Certificate Part

Task: Characterize those certificates $C \in \mathbb{A}$ for which the vector $\partial_y C e$ matches a prescribed numerator degree and a prescribed denominator $d \in K(x)[y]$ (coming from the telescoper part).

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Observation: Common factors of d and v behave slightly different than other factors. This motivates the decomposition

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\begin{aligned}
d &= (f_1; p_1)_y \cdots (f_m; p_m)_y \, g, && p_1, \ldots, p_m > 0, \\
v &= (f_1; q_1)_y \cdots (f_m; q_m)_y \, \sigma_y(h), && q_1, \ldots, q_m > 0.
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(no coprimeness conditions on the f_i's with g and h is imposed.)

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$\longrightarrow$ W.l.o.g. assume $p_i \geq q_i$, otherwise move some factors to $\sigma_y(h)$.

Certificate Part (2)

For convenience, set $c := Ce \in K(x,y)^n$.

Lemma. Assume that $p_i \geq q_i \geq 1$ for $i = 1, \ldots, m$ and let

$$z = \sigma_y^{-1}\left(\frac{(f_1; p_1)_y \cdots (f_m; p_m)_y}{(f_1; q_1)_y \cdots (f_m; q_m)_y}\right)\frac{g}{g\rceil_y} \in K(x)[y].$$

Let $w \in K(x)[y]^n$ be any polynomial vector and consider $c = \frac{h}{z}w$. Then $\partial_y c = \frac{1}{d}\tilde{w}$ for some polynomial vector $\tilde{w} \in K(x)[y]^n$ with $\deg_y(\tilde{w}) \leq \deg_y(w) + \deg_y(g\lceil_y) + \max\{\deg_y(v) - 1, \deg_y(V)\}$.

Proof. By "straight-forward" calculation, but a bit technical.

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Solution:

1. Differential case: no problem here since $\sigma_x = \sigma_y = \mathrm{id}$.
2. Hypergeometric case: admit only proper hypergeometric terms.
3. General case: impose certain conditions on the input ideal $\mathfrak{a}$; this leads to our definition of proper ∂-finite ideals:
   - It generalizes the notion of proper hypergeometric terms.
   - It refines properness by distinguishing the free variable x from the summation/integration variable y.

Proper ∂-Finite Ideals

Definition.

1. A polynomial $u \in K[x, y]$ is called y-proper (w.r.t. σ_x, σ_y) if $\deg_y\big((u; r)_x\lceil_y\big) = \mathrm{O}(1)$ as $r \to \infty$.

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Examples.

1. Let $u \in K[x, y]$ and $\sigma_x = \sigma_y = \mathrm{id}$. Then trivially we get $(u;r)_x = u^r = (u;r)_y$ and $(u;r)_x\lceil_y = \mathrm{sfp}(u)$ for all $r \geq 1$.

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2. Let $\sigma_x(x) = x+1$, $\sigma_y(y) = y+1$, and take $u = x+2y$. Then

$$(u;r)_x = \prod_{i=0}^{r-1}(x+2y+i) = \prod_{i=0}^{(r-1)/2}(x+2(y+i)) \prod_{i=0}^{r/2-1}(x+2(y+i)+1)$$
$$= \big(x+2y; \lfloor \tfrac{r-1}{2} \rfloor\big)_y \big(x+2y+1; \lfloor \tfrac{r}{2} \rfloor - 1\big)_y$$

and hence $(u;r)_x\lceil_y = (x+2y)(x+2y+1)$ for all $r \geq 2$.

Height of ∂-Finite Ideals

Definition.

1. Let $\eta \in \mathbb{N}$ be the smallest number such that for all $r \geq 1$ there exist $f_1, \ldots, f_m, g, h \in K[x, y]$, $p_1, \ldots, p_m, q_1, \ldots, q_m \in \mathbb{N}$, $p_i \geq q_i \geq 1$ for $i = 1, \ldots, m$, with

$$v = \sigma_y(h) \prod_{i=1}^{m} (f_i; q_i)_y \quad \text{and} \quad (u; r)_x = g \prod_{i=1}^{m} (f_i; p_i)_y$$

and $\deg_y(g\lceil_y) \leq \eta$. Then

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2. Let $\mathfrak{a} \subseteq \mathbb{A}$ be a proper ∂-finite ideal. The height of $\mathfrak{a}$ is defined as the minimum height of $\mathfrak{a}$ with respect to all admissible bases of $\mathbb{A}/\mathfrak{a}$.

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Shift case: consider the bivariate sequence $f = 1/\left(x^2 + y^2\right)$.

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- However, this basis is not admissible since $1 \in \mathbb{A}/\mathfrak{a}$ is not represented by a polynomial vector.

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Again shift case: consider $f(x, y) = 1/(x + y)! + 1/(x - y)!$, which is not hypergeometric but ∂-finite with holonomic rank 2.

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- This example is proper ∂-finite.

Main Theorem

Theorem. Assume that $\mathfrak{a} \subseteq \mathbb{A} = K(x,y)[\partial_x, \partial_y]$ is proper ∂-finite w.r.t. y. Let ϱ be the height of $\mathfrak{a}$, let $n = \dim_{K(x,y)} \mathbb{A}/\mathfrak{a}$, and let

$$\phi = \dim_{K(x)}\big\{W \in \mathbb{A}/\mathfrak{a} \mid \partial_y W = 0\big\}.$$

Then there exist $T \in K(x)[\partial_x] \setminus \{0\}$ and $C \in \mathbb{A}$ such that $T - \partial_y C \in \mathfrak{a}$ and $\operatorname{ord}(T) \leq n\varrho + \phi$.

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Note: The quantity ϕ ensures solutions with nonzero telescoper. Apagodu and Zeilberger excluded rational functions as input.

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Proposition. If $\mathfrak{a} \subseteq \mathbb{A}$ is ∂-finite, B is a basis of $\mathbb{A}/\mathfrak{a}$ and the multiplication matrices are $\frac{1}{u}U, \frac{1}{v}V$, then the squarefree part of u in $K(x)[y]$ divides the squarefree part of v in $K(x)[y]$.

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Consider the bivariate function $f(x,y) = p(x,y)^{-1/3} + p(x,y)^{-1/5}$ where p is a random polynomial of y-degree 2.

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- $n = \dim_{\mathbb{Q}(x,y)} \mathbb{A}/\mathfrak{a} = 2$
- $\phi = \dim_{\mathbb{Q}(x)} \{W \in \mathbb{A}/\mathfrak{a} \mid \partial_y W = 0\} = 0$
- $B = \{p^{-1/3}, p^{-1/5}\}$
- $1 \in \mathbb{A}/\mathfrak{a}$ is represented by the vector $(1,1) \in \mathbb{K}(x)[y]^2$.
- $M = \frac{D_x(p)}{p} \begin{pmatrix} -1/3 & 0 \\ 0 & -1/5 \end{pmatrix}, \; N = \frac{D_y(p)}{p} \begin{pmatrix} -1/3 & 0 \\ 0 & -1/5 \end{pmatrix}$
- $u = v = p$
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- More generally, consider $f = p^{e_1} + \cdots + p^{e_n}$ with random polynomial p of y-degree d; our theorem produces the bound $n(d-1)$ which is exact for $d = 2, \ldots, 5$ and $n = 1, \ldots, 4$.

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Proposition. A ∂-finite ideal $\mathfrak{a}$ is proper if and only if there exists an admissible basis B of $\mathbb{A}/\mathfrak{a}$ for which the multiplication matrices $\frac{1}{u}U$, $\frac{1}{v}V$ are such that u is a product of integer-linear polynomials.

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Note: This implies that a function $f(x,y)$ is proper hypergeometric if and only if its annihilating ideal is proper ∂-finite with respect to both x and y.

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- The minimal telescoper is

$$T = (\partial_x^\varrho - 1)(\partial_x^\varrho - 2) \cdots (\partial_x^\varrho - n).$$

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The family $f_k(x, y)$ involving the Bessel function of the first kind

$$f_k(x, y) = (y + 1)^{-k} J_y(x), \quad k \in \mathbb{N},$$

is ∂-finite w.r.t. $\mathbb{A} = K(x, y)[\partial_x, \partial_y]$.

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- $n = \dim_{K(x,y)}(\mathbb{A}/\mathfrak{a}) = 2$
- As a basis B for $\mathbb{A}/\mathfrak{a}$ choose the monomials 1 and ∂_x.

Example (Mixed Case)

- multiplication matrices:

$$U = \begin{pmatrix} 0 & x^2 \\ y^2 - x^2 & -x \end{pmatrix}$$

$$V = \begin{pmatrix} xy(y+1)^k - x^2(y+2)^k & -x^2(y+1)^k \\ (y+1)^k(x^2 - y^2 - y) & x(y+1)^{k+1} - x^2(y+2)^k \end{pmatrix}$$

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$\longrightarrow$ Our theorem produces the bound $2(k+2)$ for the order of T.

$\longrightarrow$ The minimal telescoper (conjecturally) has order $2k+1$.

Conclusion and Outlook