Twisting q-holonomic sequences by complex roots of unity

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Motivation

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- One of the central questions is to decide whether two knots are equivalent or not.
- For this purpose **knot invariants** are studied.
- Example: the **colored Jones polynomial** $J_{K,n}(q)$ of a knot K; it is a q-holonomic sequence of Laurent polynomials (Garoufalidis+Lê 2005).
- The **Kashaev invariant** $\langle K\rangle_n$ of a knot K is defined as

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$\longrightarrow$ Twisting q-holonomic sequences by complex roots of unity

Definition: q-Holonomic Sequence

Notation:

- $\mathbb{K}$: field of characteristic zero
- q: indeterminate, transcendental over $\mathbb{K}$

A univariate sequence $\big(f_n(q)\big)_{n\in\mathbb{N}}$ is called **q-holonomic** if it satisfies a nontrivial linear recurrence with coefficients that are polynomials in q and q^n:

$$\sum_{j=0}^{d} c_j(q,q^n)f_{n+j}(q) = 0 \qquad (n \in \mathbb{N})$$

where d is a nonnegative integer and $c_j(u,v) \in \mathbb{K}[u,v]$ are bivariate polynomials for $j = 0,\ldots,d$ with $c_d(u,v) \neq 0$.

(Zeilberger 1990)

Closure Properties for q-Holonomic Sequences

Let $f_n(q)$ and $g_n(q)$ be two q-holonomic sequences.
Then:

1. The sum $f_n(q) + g_n(q)$ is q-holonomic.
2. The product $f_n(q) \cdot g_n(q)$ is q-holonomic.
3. The sequence $f_{an+b}(q)$ with $a, b \in \mathbb{N}_0$ is q-holonomic.

(Chyzak 1998), (Koepf+Rajkovic+Marinkovic 2007)

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These closure properties can be executed algorithmically,
on the level of recurrence equations.

Software:

- `qGeneratingFunctions` for Mathematica (Kauers+K. 2009)
- `qFPS` for Maple (Koepf+Sprenger 2010)

Multivariate q-Holonomy, ∂-Finiteness

A generalization of q-holonomy to a multivariate setting was introduced by (Sabbah 1990).

A different generalization of univariate q-holonomic sequences to several variables was given by **∂-finite functions** (Chyzak 2000).

Definition: ∂-Finite Sequence (in the q-Setting)

A multivariate sequence $f_{\mathbf{n}}(q)$ is ∂-finite if for every variable $\mathbf{n} = n_1, \ldots, n_r$ it satisfies a linear recurrence of the form

$$\sum_{j=0}^{d_k} c_{k,j}(q, q^{n_1}, \ldots, q^{n_r}) f_{\mathbf{n}+j\mathbf{e}_k}(q) = 0$$

for $k = 1, \ldots, r$, where

- the d_k's are nonnegative integers,
- the $c_{k,j}$'s are multivariate polynomials in $\mathbb{K}[\mathbf{u}, \mathbf{v}]$ with $c_{k,d_k} \neq 0$,
- and $\mathbf{e}_k$ denotes the k-th unit vector of length r.

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for $k = 1, \ldots, r$, where

- the d_k's are nonnegative integers,
- the $c_{k,j}$'s are multivariate polynomials in $\mathbb{K}[\mathbf{u}, \mathbf{v}]$ with $c_{k,d_k} \neq 0$,
- and $\mathbf{e}_k$ denotes the k-th unit vector of length r.
- The indeterminates $\mathbf{q} = q_1, \ldots, q_s$ with $1 \leq s \leq r$ are transcendental over $\mathbb{K}$
- and the indices $a_1, \ldots, a_r$ are between 1 and s.

Closure Properties for ∂-Finite Sequences

Like q-holonomic sequences, the class of ∂-finite sequences is closed under

- addition,
- multiplication,
- integer-linear substitution.

Again, these closure properties can be executed algorithmically, on the level of recurrence equations.

Software:

- `Mgfun` for Maple (Chyzak 1998)
- `HolonomicFunctions` for Mathematica (K. 2009)

Twisting by Roots of Unity

We're now going to establish two new closure properties:

1. **Twisting by roots of unity:**
   For complex numbers $\boldsymbol{\omega} = \omega_1, \ldots, \omega_s \in \mathbb{C}$, we call $f_{\mathbf{n}}(\omega_1 q_1, \ldots, \omega_s q_s)$ the **twist** of the sequence $f_{\mathbf{n}}(\mathbf{q})$ by $\boldsymbol{\omega}$; we will show that ∂-finiteness is preserved under twisting by complex roots of unity.

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2. **Taking n-th roots of q:**
   For rational numbers $\alpha_1, \ldots, \alpha_s \in \mathbb{Q}$, we consider the sequence $f_{\mathbf{n}}(q_1^{\alpha_1}, \ldots, q_s^{\alpha_s})$; ∂-finiteness is also preserved under this substitution.

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Convention: For sake of simplicity, we will assume from now on that the ground field $\mathbb{K}$ contains all roots of unity.

Operator Notation

Write recurrences as operators, using the following notation: let the operators L and M act on a sequence $f_n(q)$ by

$$Lf_n(q) = f_{n+1}(q),$$
$$Mf_n(q) = q^n f_n(q),$$

and which satisfy the q-commutation relation $LM = qML$.

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Analogously in the multivariate setting $(1 \leq k \leq r)$:

$$L_k f_{\mathbf{n}}(\mathbf{q}) = f_{\mathbf{n}+\mathbf{e}_k}(\mathbf{q}),$$
$$M_k f_{\mathbf{n}}(\mathbf{q}) = q_{a_k}^{n_k} f_{\mathbf{n}}(\mathbf{q}),$$

with

$$L_k M_k = q_{a_k} M_k L_k,$$
$$L_j M_k = M_k L_j \quad \text{for } j \neq k.$$

Left Ideals: Dimension and Rank

We denote by $\mathbb{O}$ the Ore algebra $\mathbb{K}(\mathbf{q}, \mathbf{M})\langle \mathbf{L} \rangle$.

Given a multivariate sequence $f_{\mathbf{n}}(\mathbf{q})$, the set

$$\mathrm{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} \mid Pf = 0\},$$

the annihilator of f, is a left ideal in $\mathbb{O}$.

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Using this terminology:

1. A multivariate sequence $f_{\mathbf{n}}(\mathbf{q})$ is ∂-finite with respect to $\mathbb{O}$ if $\mathrm{Ann}_{\mathbb{O}}(f)$ is a zero-dimensional left ideal in $\mathbb{O}$.

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2. The dimension $\dim_{\mathbb{K}} \mathbb{O}/I$ is called the **rank** of the ideal I.

Theorem 1

Theorem

Let $f_{\mathbf{n}}(\mathbf{q}) = f_{n_1,\ldots,n_r}(q_1,\ldots,q_s)$ be a multivariate ∂-finite sequence, and let $\omega_j \in \mathbb{C}$ be an m_j-th root of unity ($1 \leq j \leq s$). Then the twisted sequence $g_{\mathbf{n}}(\mathbf{q}) = f_{\mathbf{n}}(\omega_1 q_1,\ldots,\omega_s q_s)$ is ∂-finite as well.
Moreover, let I be a zero-dimensional left ideal of rank R such that $If = 0$. From a generating set of I, a Gröbner basis of a zero-dimensional left ideal J with $Jg = 0$ can be obtained and its rank is at most $R \cdot m_{a_1} \cdots m_{a_r}$.

Corollary

Let $f_n(q)$ be a q-holonomic sequence that satisfies a recurrence of order d. Then for any root of unity $\omega \in \mathbb{C}$ of order m the sequence $f_n(\omega q)$ is q-holonomic as well and satisfies a recurrence of order at most $m \cdot d$.

Idea of the Proof (Univariate Setting)

Naive approach: substitute $q \to \omega q$ in the recurrence.

Example: $(q^{2n} + q^{n+1} - 1)f_{n+1}(q) - q^2 f_n(q) = 0$ leads to

$$(\omega^{2n}q^{2n} + \omega^{n+1}q^{n+1} - 1)f_{n+1}(\omega q) - \omega^2 q^2 f_n(\omega q) = 0.$$

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Idea: Let m be the order of ω; find a recurrence for $f_n(q)$ in which all powers of $M = q^n$ are divisible by m.

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Idea: Let m be the order of ω; find a recurrence for $f_n(q)$ in which all powers of $M = q^n$ are divisible by m.

Strategy:

- Rewrite M^{am+b} into $N^a M^b$ where $b < m$ and $N = M^m$ is a new variable.
- Eliminate M.
- This can be done by pure linear algebra (no Gröbner basis calculation is necessary)!

Algorithm (Input)

Input:

- $\mathbb{O} = \mathbb{K}(\mathbf{q}, \mathbf{M})\langle \mathbf{L} \rangle = \mathbb{K}(q_1, \ldots, q_s, M_1, \ldots, M_r)\langle L_1, \ldots, L_r \rangle$
- a monomial order $\prec$ for $\mathbb{O}$
- a finite set $F \subset \mathbb{O}$ such that F is a left Gröbner basis w.r.t. $\prec$ and the left ideal ${}_{\mathbb{O}}\langle F \rangle$ is zero-dimensional
- for $1 \leq j \leq s$:
  $m_j \in \mathbb{N}$, $\omega_j \in \mathbb{C}$ with $\omega_j^{m_j} = 1$ and $\omega_j^{\ell} \neq 1$ for all $\ell < m_j$

Notation:

- let U denote the set of monomials under the stairs of F,
- write $m(k)$ for m_{a_k}.

Algorithm

$G = \emptyset$, $V = \emptyset$, $T = \{1\}$

while $T \neq \emptyset$

  $T_0 = \min_\prec T$, $T = T \setminus \{T_0\}$

  $A = c_0 T_0 + \sum_{j=1}^{|V|} c_j V_j$

  $A' = A$ reduced with F

  clear denominators of A'

  substitute $M_k^a \to M_k^{a \bmod m(k)} N_k^{\lfloor a/m(k) \rfloor}$ in A'

  write A' as $\sum_{i=1}^{|U|} \sum_{j_1=0}^{m(1)-1} \cdots \sum_{j_r=0}^{m(r)-1} d_{i,\mathbf{j}} M_1^{j_1} \cdots M_r^{j_r} U_i$

  equate all $d_{i,\mathbf{j}}$ to zero

  solve this linear system for $c_0, \ldots, c_{|V|}$ over $\mathbb{K}(\mathbf{q}, \mathbf{N})$

  if a solution exists **then**

    substitute the solution into A

    $G = G \cup \{A\}$

    $T = T \cup \{T_0 L_k : 1 \leq k \leq r\}$

    $T = T \setminus \{T_j : 1 \leq j \leq |T| \wedge \exists_k \operatorname{lm}_\prec(G_k) \mid T_j\}$

  else

    $V = V \cup \{T_0\}$

Algorithm (Final Steps)

$\vdots$

substitute $N_k \to M_k^{m(k)}$ and $q_j \to \omega_j q_j$ in G

return G

Example

Recall the definition for the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Let $f_n(q)$ be the central q-binomial coefficient $\begin{bmatrix} 2n \\ n \end{bmatrix}_q$.
It satisfies the recurrence

$$(1-q^{n+1})f_{n+1}(q) = (1+q^{n+1}-q^{2n+1}-q^{3n+2})f_n(q)$$

which translates to the operator

$$(qM-1)L - q^2M^3 - qM^2 + qM + 1.$$

The twisted sequence $f_n(-q)$ is annihilated by the operator

$$\left(q^4M^2-1\right)L^2 + \left(\left(q^7-q^6\right)M^4 - q + 1\right)L - \\ q^7M^6 - \left(q^6-q^5+q^4\right)M^4 + \left(q^4-q^3+q^2\right)M^2 + q.$$

Computation with HolonomicFunctions

```
qbin = Annihilator[QBinomial[2n, n, q], QS[M, q^n]]
```

$$\left\{(qM-1)S_{M,q}+(-q^2M^3-qM^2+qM+1)\right\}$$

```
DFiniteQSubstitute[qbin, {q, 2}]
```

$$\left\{(q^4M^2-1)S_{M,q}^2+(q^7M^4-q^6M^4-q+1)S_{M,q}+\right.$$
$$\left.(-q^7M^6-q^6M^4+q^5M^4-q^4M^4+q^4M^2-q^3M^2+q^2M^2+q)\right\}$$

Example 2

The q-Pochhammer symbol $(q;q)_n := \prod_{k=1}^{n}(1-q^k)$ satisfies the simple recurrence

$$(q;q)_{n+1} = (1-q^{n+1})(q;q)_n.$$

We want to study the twisted sequence $(\omega q;\omega q)_n$ for ω being a third root of unity. Therefore we have to compute a recurrence for $(q;q)_n$ in which all exponents of $M = q^n$ are divisible by 3:

$$\begin{aligned}(q;q)_{n+3} - \left(q^2+q+1\right)(q;q)_{n+2} &+ \\ \left(q^3+q^2+q\right)(q;q)_{n+1} &+ \left(q^{3n+6}-q^3\right)(q;q)_n = 0.\end{aligned}$$

Substituting $q \to \omega q$ delivers a recurrence for the twist $(\omega q;\omega q)_n$.

Computation with HolonomicFunctions

```
qp = Annihilator[QPochhammer[q, q, n], QS[M, q^n]]
```

$$\{S_{M,q} + (qM - 1)\}$$

```
DFiniteQSubstitute[qp, {q, 3}, Return -> Backsubstitution]
```

$$\{S_{M,q}^3 + (-q^2 - q - 1)S_{M,q}^2 + (q^3 + q^2 + q)S_{M,q} + (q^6 M^3 - q^3)\}$$

Theorem 2

Theorem

Let $f_{\mathbf{n}}(\mathbf{q}) = f_{n_1,\ldots,n_r}(q_1,\ldots,q_s)$ be a multivariate ∂-finite sequence, and let $\alpha_1,\ldots,\alpha_s \in \mathbb{Q}$. Then the sequence $g_{\mathbf{n}}(\mathbf{q}) = f_{\mathbf{n}}(q_1^{\alpha_1},\ldots,q_s^{\alpha_s})$ is ∂-finite as well.
Moreover, let I be a zero-dimensional left ideal of rank R such that $If = 0$. From a generating set of I, a Gröbner basis of a zero-dimensional left ideal J with $Jg = 0$ can be obtained and its rank is at most $R \cdot m_1 \cdots m_s \cdot m_{a_1} \cdots m_{a_r}$, where $m_j \in \mathbb{N}$ denotes the denominator of α_j.

Corollary

Let $f_n(q)$ be a q-holonomic sequence that satisfies a recurrence of order d. Then for $\alpha \in \mathbb{Q}$ the sequence $f_n(q^\alpha)$ is q-holonomic as well and satisfies a recurrence of order at most $m^2 \cdot d$, where $m \in \mathbb{N}$ is the denominator of α.

Idea of the Proof

Write $\alpha_j = \ell_j / m_j$ for all $1 \leq j \leq s$.

Idea: Find recurrences in I in which all powers of q_j are divisible by m_j, as well as all powers of M_k for which $a_k = j$.

Then the substitutions $q_j \to q_j^{\alpha_j}$ can be safely performed, i.e., the resulting recurrences will have polynomial coefficients in $q_1, \ldots, q_s$ and $M_1, \ldots, M_r$.

Example 3

The substitution $q \to \sqrt{q}$ is performed on the q-Pochhammer symbol $(q;q)_n$.

Theorem 2 predicts that the resulting recurrence is of order at most 4. As an intermediate result, the operator

$$L^4 - (q^2+1)L^3 - (q^8M^2 + q^6M^2 - q^4 - q^2)L \\ - q^{10}M^4 + q^8M^2 + q^6M^2 - q^4$$

is found in ${}_{\mathbb{O}}\langle L + qM - 1\rangle$, the annihilator of $(q;q)_n$.

The final result for $f_n = \left(\sqrt{q};\sqrt{q}\right)_n$ is the recurrence

$$f_{n+4} - (q+1)f_{n+3} - (q^{n+4} + q^{n+3} - q^2 - q)f_{n+1} \\ + \left(-q^{2n+5} + q^{n+4} + q^{n+3} - q^2\right) f_n = 0.$$

Computation with HolonomicFunctions

```
qp = Annihilator[QPochhammer[q, q, n], QS[M, q^n]]
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$$\{S_{M,q} + (qM - 1)\}$$

```
DFiniteQSubstitute[qp, {q, 1, 2}]
```

$$\{S_{M,q}^4 - (q+1)S_{M,q}^3 + (-q^4M - q^3M + q^2 + q)S_{M,q} + (-q^5M^2 + q^4M + q^3M - q^2)\}$$