Twisting $q$-holonomic sequences 
by complex roots of unity

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Motivation

In quantum topology the properties of knots are studied.
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• One of the central questions is to decide whether two knots are equivalent or not.

• For this purpose knot invariants are studied.

• Example: the colored Jones polynomial $J_{K,n}(q)$ of a knot $K$; it is a $q$-holonomic sequence of Laurent polynomials (Garoufalidis+$Lê$ 2005).

• The Kashaev invariant $\langle K \rangle_n$ of a knot $K$ is defined as

$$\langle K \rangle_n = J_{K,n}(e^{2\pi i/n}).$$
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- The Kashaev invariant $\langle K \rangle_n$ of a knot $K$ is defined as
  \[ \langle K \rangle_n = J_{K,n}(e^{2\pi i/n}). \]

→ Twisting $q$-holonomic sequences by complex roots of unity
Definition: $q$-Holonomic Sequence

Notation:

- $\mathbb{K}$: field of characteristic zero
- $q$: indeterminate, transcendental over $\mathbb{K}$

A univariate sequence $(f_n(q))_{n \in \mathbb{N}}$ is called $q$-holonomic if it satisfies a nontrivial linear recurrence with coefficients that are polynomials in $q$ and $q^n$:

$$\sum_{j=0}^{d} c_j(q, q^n) f_{n+j}(q) = 0 \quad (n \in \mathbb{N})$$

where $d$ is a nonnegative integer and $c_j(u, v) \in \mathbb{K}[u, v]$ are bivariate polynomials for $j = 0, \ldots, d$ with $c_d(u, v) \neq 0$.

(Zeilberger 1990)
Closure Properties for $q$-Holonomic Sequences

Let $f_n(q)$ and $g_n(q)$ be two $q$-holonomic sequences. Then:

1. The sum $f_n(q) + g_n(q)$ is $q$-holonomic.
2. The product $f_n(q) \cdot g_n(q)$ is $q$-holonomic.
3. The sequence $f_{an+b}(q)$ with $a, b \in \mathbb{N}_0$ is $q$-holonomic.

(Chyzak 1998), (Koepf+Rajkovic+Marinkovic 2007)
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These closure properties can be executed algorithmically, on the level of recurrence equations.

Software:

- qGeneratingFunctions for Mathematica (Kauers+K. 2009)
- qFPS for Maple (Koepf+Sprenger 2010)
A generalization of $q$-holonomy to a multivariate setting was introduced by (Sabbah 1990).

A different generalization of univariate $q$-holonomic sequences to several variables was given by $\partial$-finite functions (Chyzak 2000).
Definition: \( \partial \)-Finite Sequence (in the \( q \)-Setting)

A multivariate sequence \( f_n(q) \) is \( \partial \)-finite if for every variable \( n = n_1, \ldots, n_r \) it satisfies a linear recurrence of the form

\[
\sum_{j=0}^{d_k} c_{k,j}(q^n_1, \ldots, q^n_r) f_{n+j} e_k(q) = 0
\]

for \( k = 1, \ldots, r \), where

- the \( d_k \)'s are nonnegative integers,
- the \( c_{k,j} \)'s are multivariate polynomials in \( \mathbb{K}[u, v] \) with \( c_k, d_k \neq 0 \),
- and \( e_k \) denotes the \( k \)-th unit vector of length \( r \).
Definition: $\partial$-Finite Sequence (in the $q$-Setting)

A multivariate sequence $f_n(q)$ is $\partial$-finite if for every variable $n = n_1, \ldots, n_r$ it satisfies a linear recurrence of the form

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- the $d_k$'s are nonnegative integers,
- the $c_{k,j}$'s are multivariate polynomials in $\mathbb{K}[u, v]$ with $c_{k,d_k} \neq 0$,
- and $e_k$ denotes the $k$-th unit vector of length $r$.
- The indeterminates $q = q_1, \ldots, q_s$ with $1 \leq s \leq r$ are transcendental over $\mathbb{K}$
- and the indices $a_1, \ldots, a_r$ are between 1 and $s$. 


Closure Properties for $\partial$-Finite Sequences

Like $q$-holonomic sequences, the class of $\partial$-finite sequences is closed under

- addition,
- multiplication,
- integer-linear substitution.

Again, these closure properties can be executed algorithmically, on the level of recurrence equations.

Software:
- Mgfun for Maple (Chyzak 1998)
- HolonomicFunctions for Mathematica (K. 2009)
Twisting by Roots of Unity

We’re now going to establish two new closure properties:

1. **Twisting by roots of unity:**
   For complex numbers \( \omega = \omega_1, \ldots, \omega_s \in \mathbb{C} \), we call
   \( f_n(\omega_1 q_1, \ldots, \omega_s q_s) \) the **twist** of the sequence \( f_n(q) \) by \( \omega \);
   we will show that \( \partial \)-finiteness is preserved under twisting by complex roots of unity.
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2. **Taking n-th roots of q:**
   For rational numbers \( \alpha_1, \ldots, \alpha_s \in \mathbb{Q} \), we consider the sequence \( f_n(q_1^{\alpha_1}, \ldots, q_s^{\alpha_s}) \); \( \partial \)-finiteness is also preserved under this substitution.
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**Convention:** For sake of simplicity, we will assume from now on that the ground field \( \mathbb{K} \) contains all roots of unity.
Write recurrences as operators, using the following notation:

let the operators $L$ and $M$ act on a sequence $f_n(q)$ by

\[ L f_n(q) = f_{n+1}(q), \]
\[ M f_n(q) = q^n f_n(q), \]

and which satisfy the $q$-commutation relation $LM = qML$. 

Analogously in the multivariate setting ($1 \leq k \leq r$):

\[ L_k f_n(q) = f_{n+e_k}(q), \]
\[ M_k f_n(q) = q^{nk} f_n(q), \]

with $L_k M_k = q^{a_k} M_k L_k$, $L_j M_k = M_k L_j$ for $j \neq k$. 

Operator Notation

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$$L_k M_k = q_{a_k} M_k L_k,$$
$$L_j M_k = M_k L_j \quad \text{for } j \neq k.$$
Left Ideals: Dimension and Rank

We denote by $\mathcal{O}$ the Ore algebra $\mathbb{K}(q, M)\langle L \rangle$.

Given a multivariate sequence $f_n(q)$, the set

$$\text{Ann}_{\mathcal{O}}(f) = \{ P \in \mathcal{O} \mid Pf = 0 \},$$

the annihilator of $f$, is a left ideal in $\mathcal{O}$. 

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Using this terminology:

1. A multivariate sequence $f_n(q)$ is $\partial$-finite with respect to $\mathcal{O}$ if $\text{Ann}_\mathcal{O}(f)$ is a zero-dimensional left ideal in $\mathcal{O}$. 
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2. The dimension $\dim_{\mathbb{K}} \mathcal{O}/I$ is called the rank of the ideal $I$. 
Theorem

Let $f_n(q) = f_{n_1,\ldots,n_r}(q_1,\ldots,q_s)$ be a multivariate $\partial$-finite sequence, and let $\omega_j \in \mathbb{C}$ be an $m_j$-th root of unity $(1 \leq j \leq s)$. Then the twisted sequence $g_n(q) = f_n(\omega_1q_1,\ldots,\omega_sq_s)$ is $\partial$-finite as well.

Moreover, let $I$ be a zero-dimensional left ideal of rank $R$ such that $I\cdot f = 0$. From a generating set of $I$, a Gröbner basis of a zero-dimensional left ideal $J$ with $J\cdot g = 0$ can be obtained and its rank is at most $R \cdot m_{a_1} \cdots m_{a_r}$.

Corollary

Let $f_n(q)$ be a $q$-holonomic sequence that satisfies a recurrence of order $d$. Then for any root of unity $\omega \in \mathbb{C}$ of order $m$ the sequence $f_n(\omega q)$ is $q$-holonomic as well and satisfies a recurrence of order at most $m \cdot d$. 
Idea of the Proof (Univariate Setting)

Naive approach: substitute $q \rightarrow \omega q$ in the recurrence.

Example: $(q^{2n} + q^{n+1} - 1)f_{n+1}(q) - q^2 f_n(q) = 0$ leads to

$$(\omega^{2n} q^{2n} + \omega^{n+1} q^{n+1} - 1)f_{n+1}(\omega q) - \omega^2 q^2 f_n(\omega q) = 0.$$

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**Idea:** Let $m$ be the order of $\omega$; find a recurrence for $f_n(q)$ in which all powers of $M = q^n$ are divisible by $m$. 
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**Idea:** Let $m$ be the order of $\omega$; find a recurrence for $f_n(q)$ in which all powers of $M = q^n$ are divisible by $m$.

**Strategy:**

- Rewrite $M^{am+b}$ into $N^aM^b$ where $b < m$ and $N = M^m$ is a new variable.
- Eliminate $M$.
- This can be done by pure linear algebra (no Gröbner basis calculation is necessary)!
Algorithm (Input)

Input:

- \( \mathcal{O} = \mathbb{K}(q, M)\langle L \rangle = \mathbb{K}(q_1, \ldots, q_s, M_1, \ldots, M_r)\langle L_1, \ldots, L_r \rangle \)
- a monomial order \( \prec \) for \( \mathcal{O} \)
- a finite set \( F \subset \mathcal{O} \) such that \( F \) is a left Gröbner basis w.r.t. \( \prec \) and the left ideal \( \mathcal{O}\langle F \rangle \) is zero-dimensional
- for \( 1 \leq j \leq s \):
  - \( m_j \in \mathbb{N} \), \( \omega_j \in \mathbb{C} \) with \( \omega_j^{m_j} = 1 \) and \( \omega_j^{\ell} \neq 1 \) for all \( \ell < m_j \)

Notation:

- let \( U \) denote the set of monomials under the stairs of \( F \),
- write \( m(k) \) for \( m_{a_k} \).
Algorithm

\[ G = \emptyset, \quad V = \emptyset, \quad T = \{1\} \]

while \( T \neq \emptyset \)

\[ T_0 = \min T, \quad T = T \setminus \{T_0\} \]

\[ A = c_0 T_0 + \sum_{j=1}^{|V|} c_j V_j \]

\[ A' = A \text{ reduced with } F \]

clear denominators of \( A' \)

substitute \( M_k^a \rightarrow M_k^a \mod m(k) N_k^{|a/m(k)|} \) in \( A' \)

write \( A' \) as \( \sum_{i=1}^{|U|} \sum_{j_1=0}^{m(1)-1} \cdots \sum_{j_r=0}^{m(r)-1} d_{i,j_1 \cdots j_r} M_1^{j_1} \cdots M_r^{j_r} U_i \)


equate all \( d_{i,j} \) to zero

solve this linear system for \( c_0, \ldots, c_{|V|} \) over \( \mathbb{K}(q, N) \)

if a solution exists then

substitute the solution into \( A \)

\[ G = G \cup \{A\} \]

\[ T = T \cup \{T_0 L_k : 1 \leq k \leq r\} \]

\[ T = T \setminus \{T_j : 1 \leq j \leq |T| \wedge \exists_k \Im(G_k) \mid T_j\} \]

else

\[ V = V \cup \{T_0\} \]
Algorithm (Final Steps)

\begin{itemize}
  \item substitute $N_k \rightarrow M_k^{m(k)}$ and $q_j \rightarrow \omega_j q_j$ in $G$
\end{itemize}

\textbf{return} $G$
Example

Recall the definition for the $q$-binomial coefficient

$$\binom{n}{k}_q := \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-k+1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}.$$

Let $f_n(q)$ be the central $q$-binomial coefficient $\binom{2n}{n}_q$. It satisfies the recurrence

$$(1 - q^{n+1})f_{n+1}(q) = (1 + q^{n+1} - q^{2n+1} - q^{3n+2}) f_n(q)$$

which translates to the operator

$$(qM - 1)L - q^2M^3 - qM^2 + qM + 1.$$  

The twisted sequence $f_n(-q)$ is annihilated by the operator

$$(q^4M^2 - 1)L^2 + ((q^7 - q^6)M^4 - q + 1)L - q^7M^6 - (q^6 - q^5 + q^4)M^4 + (q^4 - q^3 + q^2)M^2 + q.$$
Computation with HolonomicFunctions

\[ q\text{bin} = \text{Annihilator}[\text{QBinomial}[2n, n, q], \text{QS}[M, q^n]] \]

\[ \{(qM - 1)S_{M,q} + (-q^2 M^3 - q M^2 + q M + 1)\} \]

\[ \text{DFiniteQSubstitute}[q\text{bin}, \{q, 2\}] \]

\[ \{(q^4 M^2 - 1)S^2_{M,q} + (q^7 M^4 - q^6 M^4 - q + 1)S_{M,q} + \\
(-q^7 M^6 - q^6 M^4 + q^5 M^4 - q^4 M^4 + q^4 M^2 - q^3 M^2 + q^2 M^2 + q)\} \]
Example 2

The $q$-Pochhammer symbol $(q; q)_n := \prod_{k=1}^{n} (1 - q^k)$ satisfies the simple recurrence

$$(q; q)_{n+1} = (1 - q^{n+1})(q; q)_n.$$  

We want to study the twisted sequence $(\omega q; \omega q)_n$ for $\omega$ being a third root of unity. Therefore we have to compute a recurrence for $(q; q)_n$ in which all exponents of $M = q^n$ are divisible by $3$:

$$(q; q)_{n+3} - (q^2 + q + 1) (q; q)_{n+2} +$$

$$ (q^3 + q^2 + q) (q; q)_{n+1} + (q^{3n+6} - q^3) (q; q)_n = 0.$$  

Substituting $q \rightarrow \omega q$ delivers a recurrence for the twist $(\omega q; \omega q)_n$. 
Computation with HolonomicFunctions

\[ \text{qp} = \text{Annihilator}[\text{QPochhammer}[q, q, n], QS[M, q^n]] \]

\[ \{S_{M,q} + (qM - 1)\} \]

\[ \text{DFiniteQSubstitute[qp, \{q, 3\}, Return \rightarrow \text{Backsubstitution}]} \]

\[ \{S^3_{M,q} + (-q^2 - q - 1)S^2_{M,q} + (q^3 + q^2 + q)S_{M,q} + (q^6M^3 - q^3)\} \]
Theorem 2

Theorem

Let \( f_n(q) = \overline{f}_{n_1, \ldots, n_r}(q_1, \ldots, q_s) \) be a multivariate \( \partial \)-finite sequence, and let \( \alpha_1, \ldots, \alpha_s \in \mathbb{Q} \). Then the sequence \( g_n(q) = f_n(q_{\alpha_1}^1, \ldots, q_{\alpha_s}^s) \) is \( \partial \)-finite as well.

Moreover, let \( I \) be a zero-dimensional left ideal of rank \( R \) such that \( I f = 0 \). From a generating set of \( I \), a Gröbner basis of a zero-dimensional left ideal \( J \) with \( J g = 0 \) can be obtained and its rank is at most \( R \cdot m_1 \cdot \ldots \cdot m_s \cdot m_{a_1} \cdot \ldots \cdot m_{a_r} \), where \( m_j \in \mathbb{N} \) denotes the denominator of \( \alpha_j \).

Corollary

Let \( f_n(q) \) be a \( q \)-holonomic sequence that satisfies a recurrence of order \( d \). Then for \( \alpha \in \mathbb{Q} \) the sequence \( f_n(q^\alpha) \) is \( q \)-holonomic as well and satisfies a recurrence of order at most \( m^2 \cdot d \), where \( m \in \mathbb{N} \) is the denominator of \( \alpha \).
Idea of the Proof

Write $\alpha_j = \ell_j/m_j$ for all $1 \leq j \leq s$.

**Idea:** Find recurrences in $I$ in which all powers of $q_j$ are divisible by $m_j$, as well as all powers of $M_k$ for which $a_k = j$.

Then the substitutions $q_j \rightarrow q_j^{\alpha_j}$ can be safely performed, i.e., the resulting recurrences will have polynomial coefficients in $q_1, \ldots, q_s$ and $M_1, \ldots, M_r$. 
Example 3

The substitution \( q \rightarrow \sqrt{q} \) is performed on the \( q \)-Pochhammer symbol \((q; q)_n\).

Theorem 2 predicts that the resulting recurrence is of order at most 4. As an intermediate result, the operator

\[
L^4 - (q^2 + 1)L^3 - (q^8 M^2 + q^6 M^2 - q^4 - q^2)L \\
- q^{10} M^4 + q^8 M^2 + q^6 M^2 - q^4
\]

is found in \( \mathcal{O}(L + qM - 1) \), the annihilator of \((q; q)_n\).

The final result for \( f_n = (\sqrt{q}; \sqrt{q})_n \) is the recurrence

\[
f_{n+4} - (q + 1)f_{n+3} - (q^{n+4} + q^{n+3} - q^2 - q)f_{n+1} \\
+ (-q^{2n+5} + q^{n+4} + q^{n+3} - q^2) f_n = 0.
\]
Computation with HolonomicFunctions

\[ \text{qp} = \text{Annihilator}[\text{QPochhammer}[q, q, n], \text{QS}[M, q^n]] \]
\[ \{ S_{M,q} + (qM - 1) \} \]

\[ \text{DFiniteQSubstitute}[\text{qp}, \{q, 1, 2\}] \]
\[ \{ S_{M,q}^4 - (q + 1)S_{M,q}^3 + (-q^4M - q^3M + q^2 + q)S_{M,q} + \]
\[ (-q^5M^2 + q^4M + q^3M - q^2) \} \]