Software for Special Functions

Christoph Koutschan

Research Institute for Symbolic Computation
Johannes Kepler Universität Linz, Austria

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tool for manipulating ∂-finite/holonomic functions

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This software is available for free at

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Plan of the talk:

1. short introduction into the underlying theory
2. main part: present several examples from different areas of mathematics to demonstrate the applicability of this software

What problems can be addressed?

Various problems over the class of ∂-finite/holonomic functions:

- compute a Gröbner basis of annihilating operators for a given expression
- prove identities (possibly involving integrals and summations)
- help to evaluate sums and integrals
- find certain identities (e.g. contiguous relations)
- determinant evaluations

Some Notation

The following operator symbols will be used:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
- q-shift operator $S_{v,q}$: $S_{v,q} f(v) = f(qv)$
- arbitrary operator: ∂_v any of the two above

Note: a variable v is considered to be either discrete or continuous.

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All operators live in an Ore algebra of the form

$$\mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle,$$

i.e., polynomials in the ∂'s with rational function coefficients (where $\mathbb{K}$ is a field of characteristic 0 containing $\mathbb{Q}$).

Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function f is ∂-finite w.r.t. $\mathbb{O}$ if all its shifts and derivatives

$$\mathbb{O}(f) = \{P(f) : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space.

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Note: we consider only equations of the form $P(f) = 0$, i.e., linear and homogeneous equations.

Example: Legendre is ∂-Finite

$$P_n^{(4)}(x) =$$

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$$P_n^{(4)}(x) =$$

$$-\tfrac{6x}{x^2-1}P_n^{(3)}(x) + \tfrac{n^2+n-6}{x^2-1}P_n''(x)$$

$$(x^2-1)P_n^{(4)}(x) + 6xP_n^{(3)}(x) - (n^2+n-6)P_n''(x) = 0$$

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Example: Legendre is ∂-Finite

$$P_n^{(4)}(x) =$$

$$\frac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x)$$

$$-\frac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$

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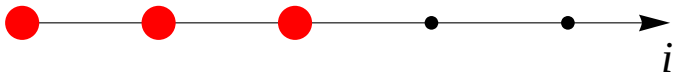
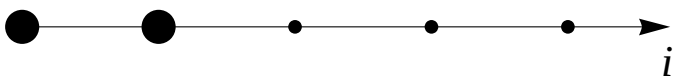


$$(x^2-1)P_n''(x) + 2xP_n'(x) - (n^2+n)P_n(x) = 0$$

Example: Legendre is ∂-Finite

$$P_n^{(4)}(x) =$$
$$-\frac{8x(n^2x^2-n^2+nx^2-n+3x^2+3)}{(x^2-1)^3}P_n'(x)$$
$$+\frac{n(n+1)(n^2x^2-n^2+nx^2-n+18x^2+6)}{(x^2-1)^3}P_n(x)$$



$\longrightarrow$ $P_n(x)$ is ∂-finite w.r.t. $\mathbb{Q}(n)(x)\langle D_x\rangle$.

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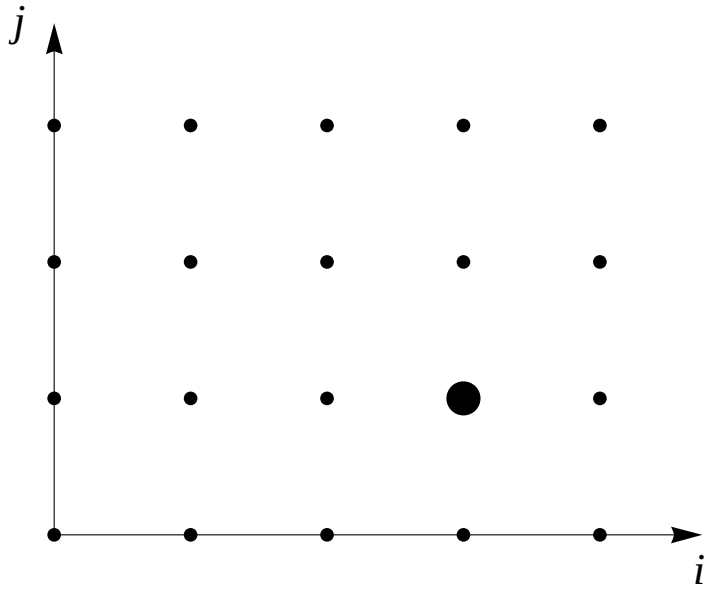

$$P_{n+1}^{(3)}(x) =$$

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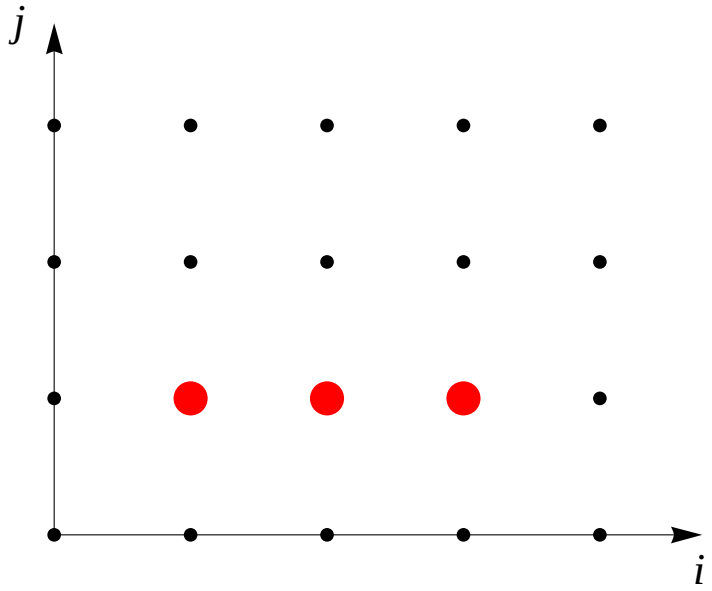

$$P^{(3)}_{n+1}(x) =$$

$$-\frac{4x}{x^2-1}P''_{n+1}(x) + \frac{(n^2+3n)}{x^2-1}P'_{n+1}(x)$$

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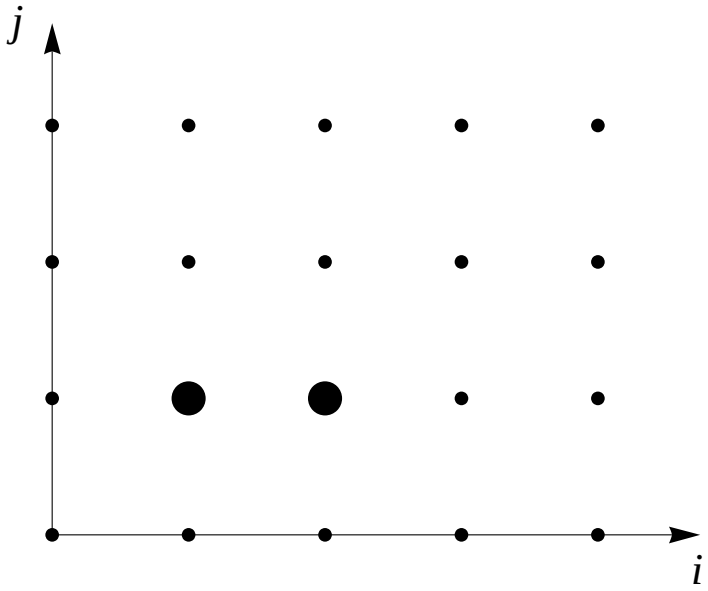

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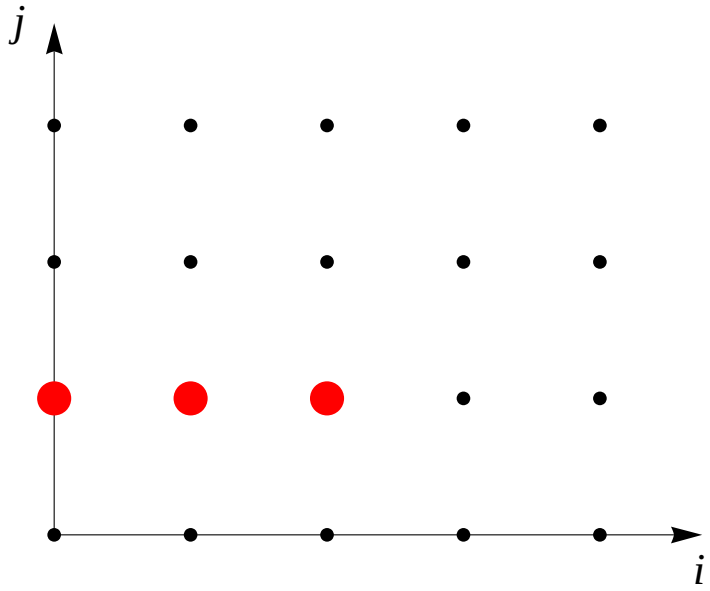

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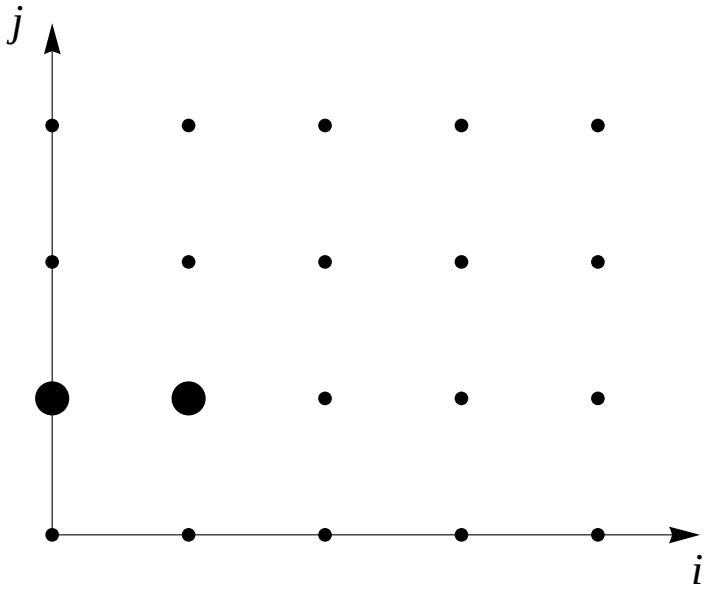

$$P^{(3)}_{n+1}(x) = \frac{(n^2x^2-n^2+3nx^2-3n+8x^2)}{(x^2-1)^2} P'_{n+1}(x) - \frac{4(n^2x+3nx+2x)}{(x^2-1)^2} P_{n+1}(x)$$

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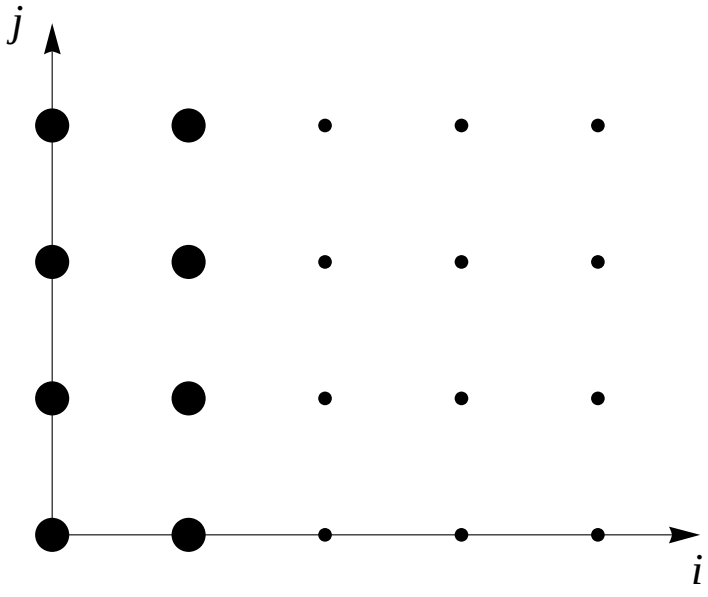

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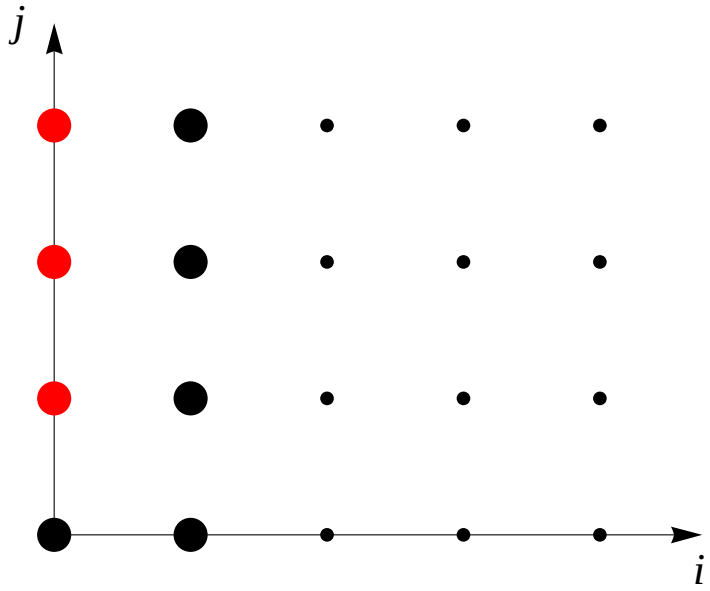

$$P_{n+3}(x) = \frac{(2n+5)x}{n+3} P_{n+2}(x) - \frac{n+2}{n+3} P_{n+1}(x)$$

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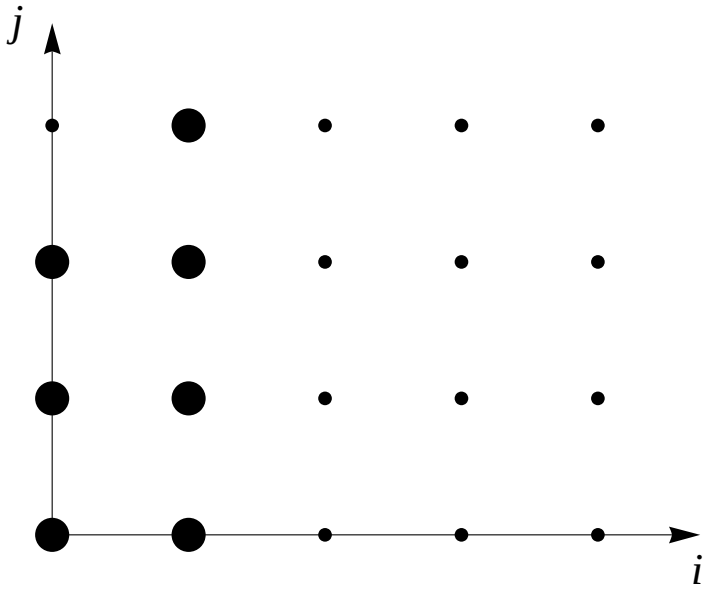

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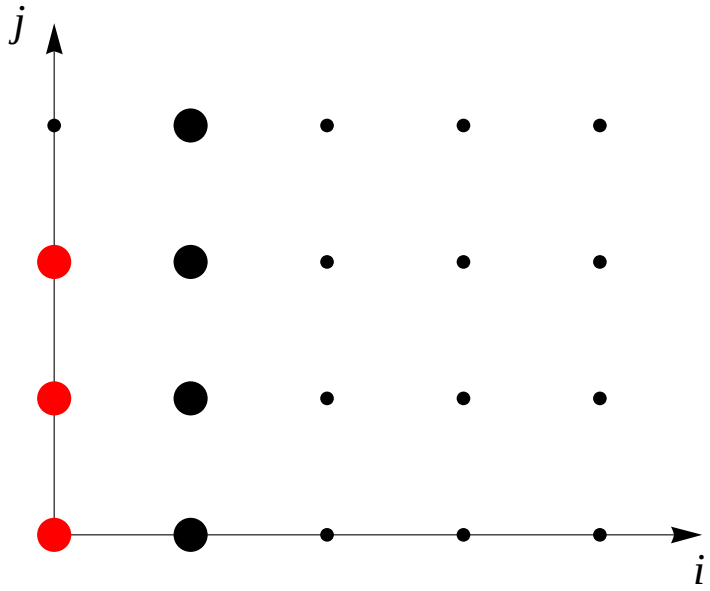

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$$(n+2)P_{n+2}(x) - (2n+3)xP_{n+1}(x) + (n+1)P_n(x) = 0$$

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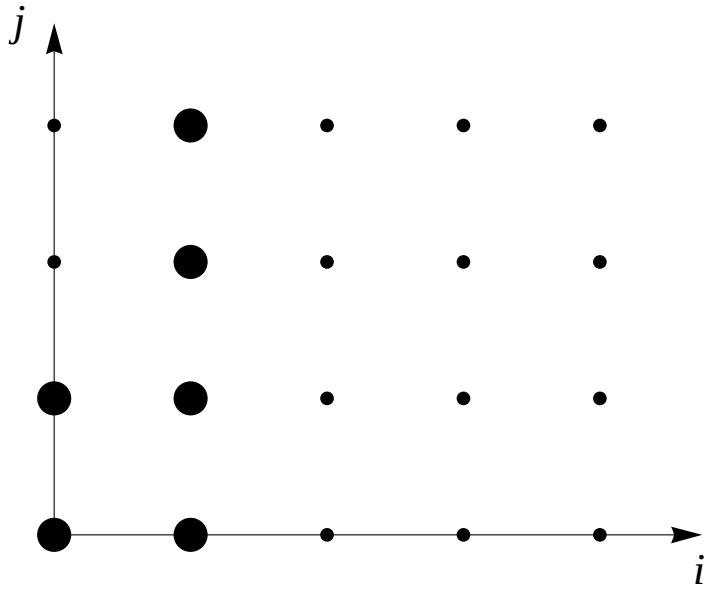

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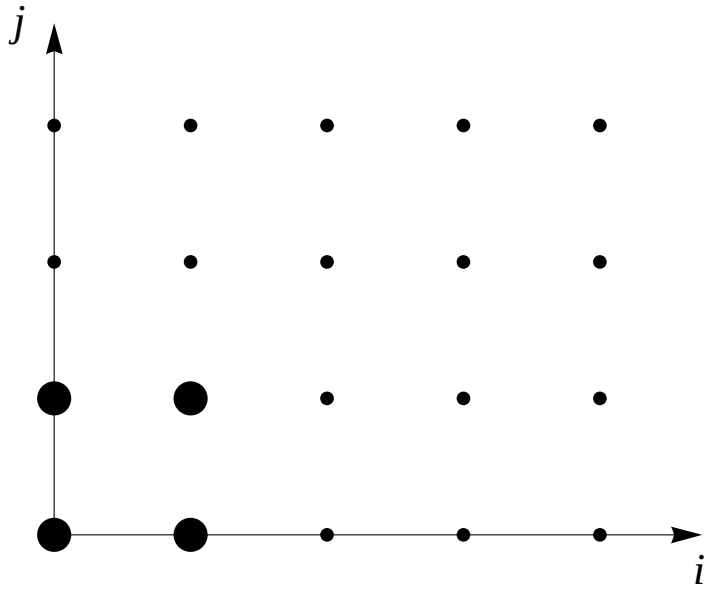

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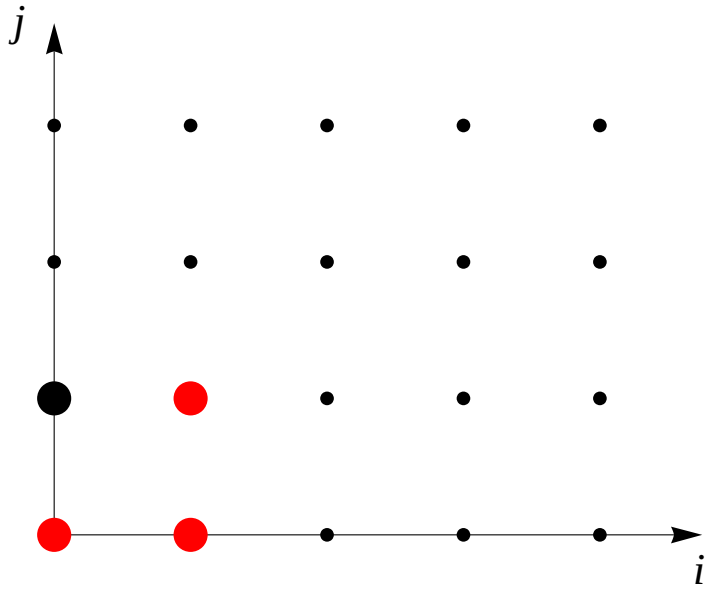

$$P'_{n+1}(x) - xP'_n(x) - (n+1)P_n(x) = 0$$

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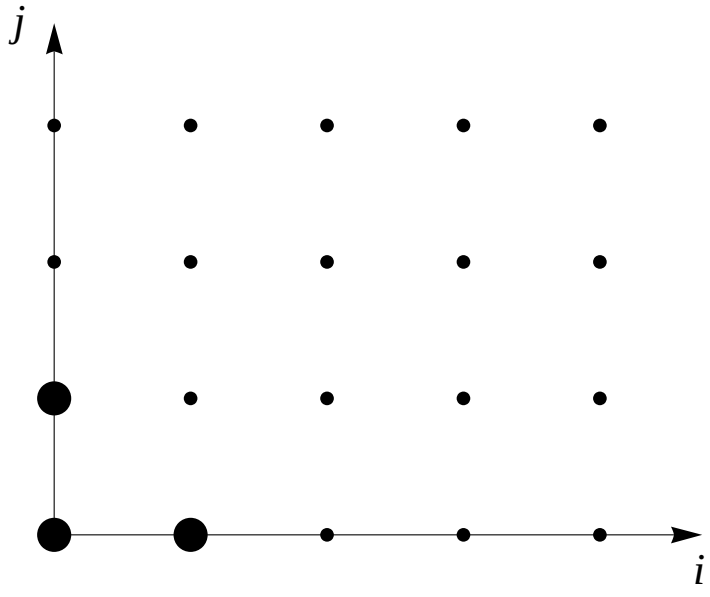

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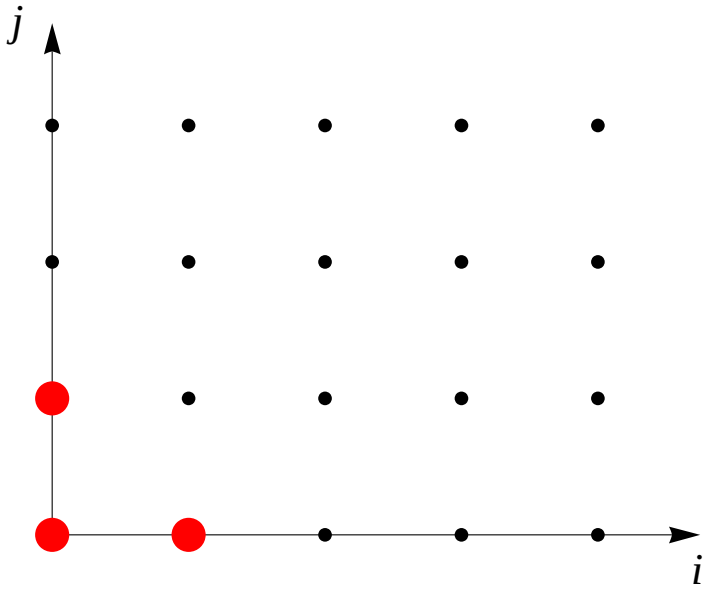

$$(n+1)P_{n+1}(x) + (1-x^2)P_n'(x) - (n+1)xP_n(x) = 0$$

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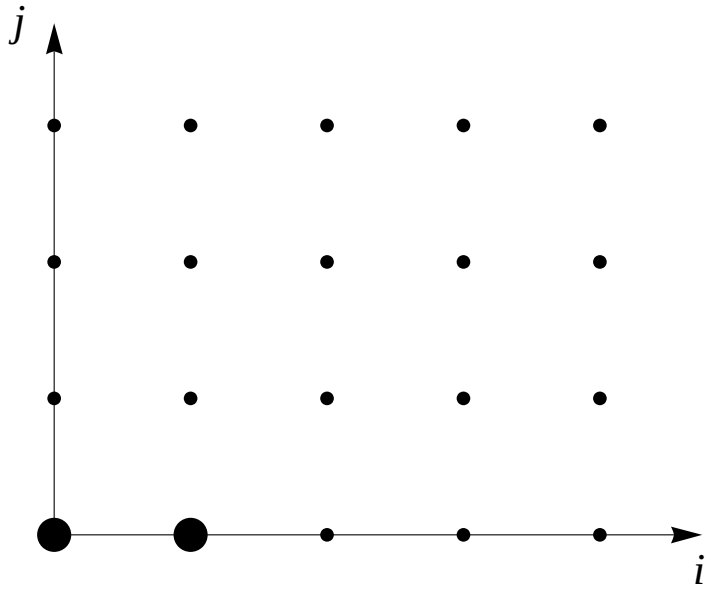

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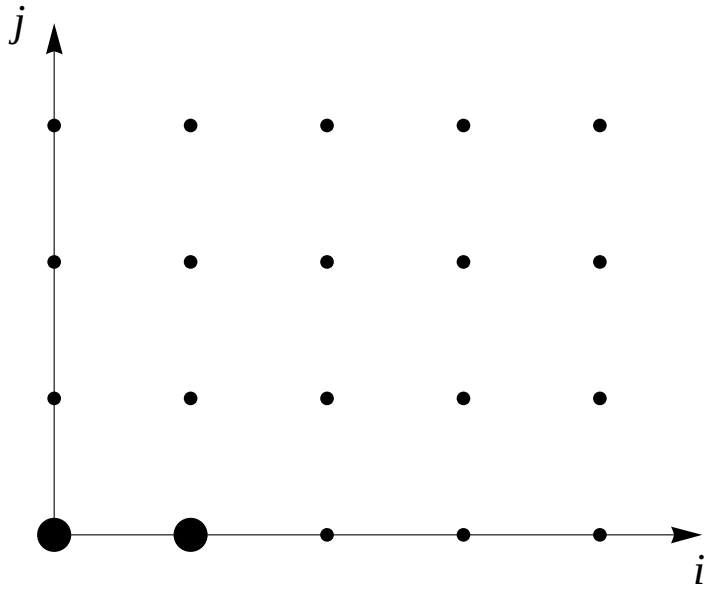
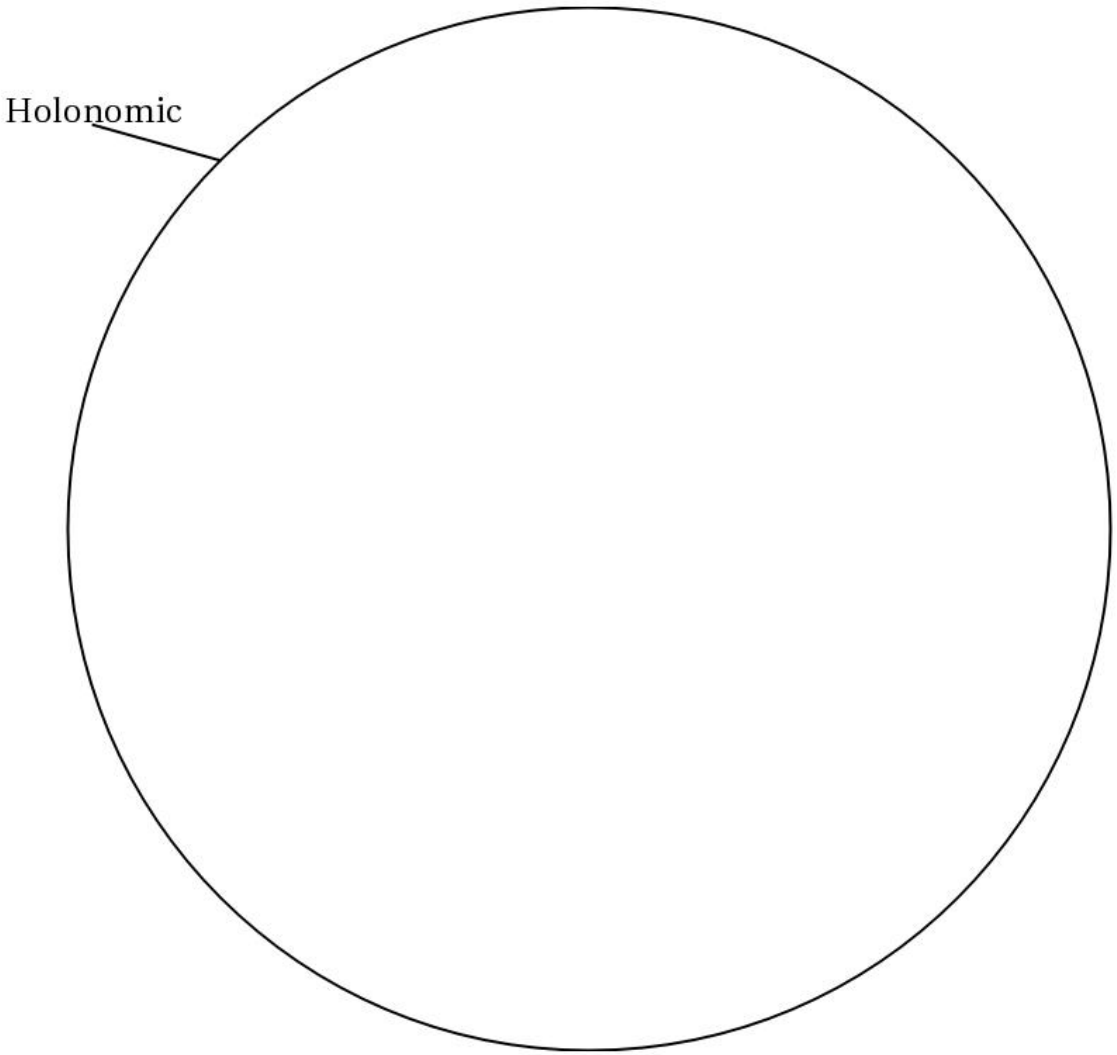


The definition of "holonomic" is a bit technical, so we skip it here.

Holonomic and ∂-finite functions



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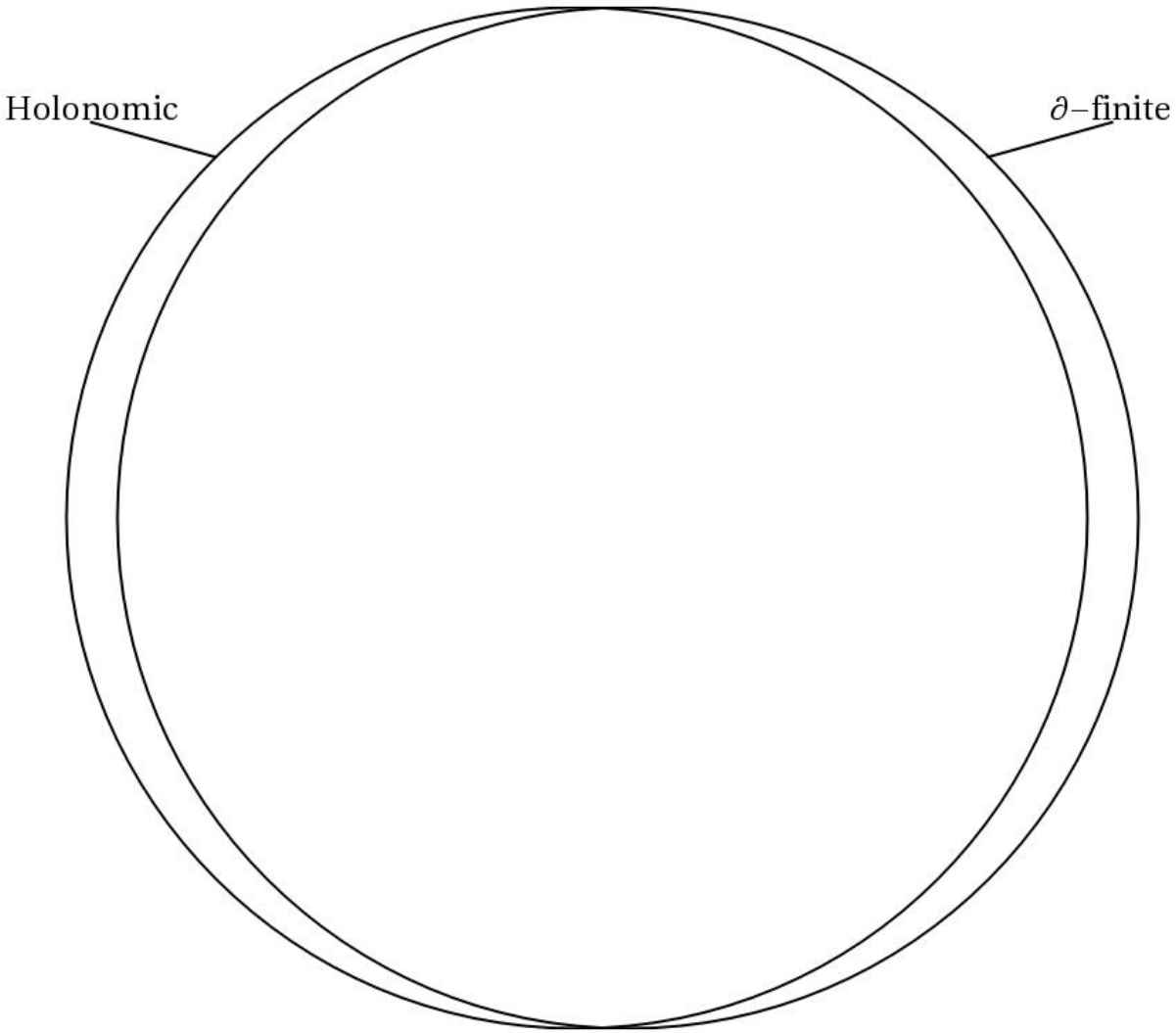

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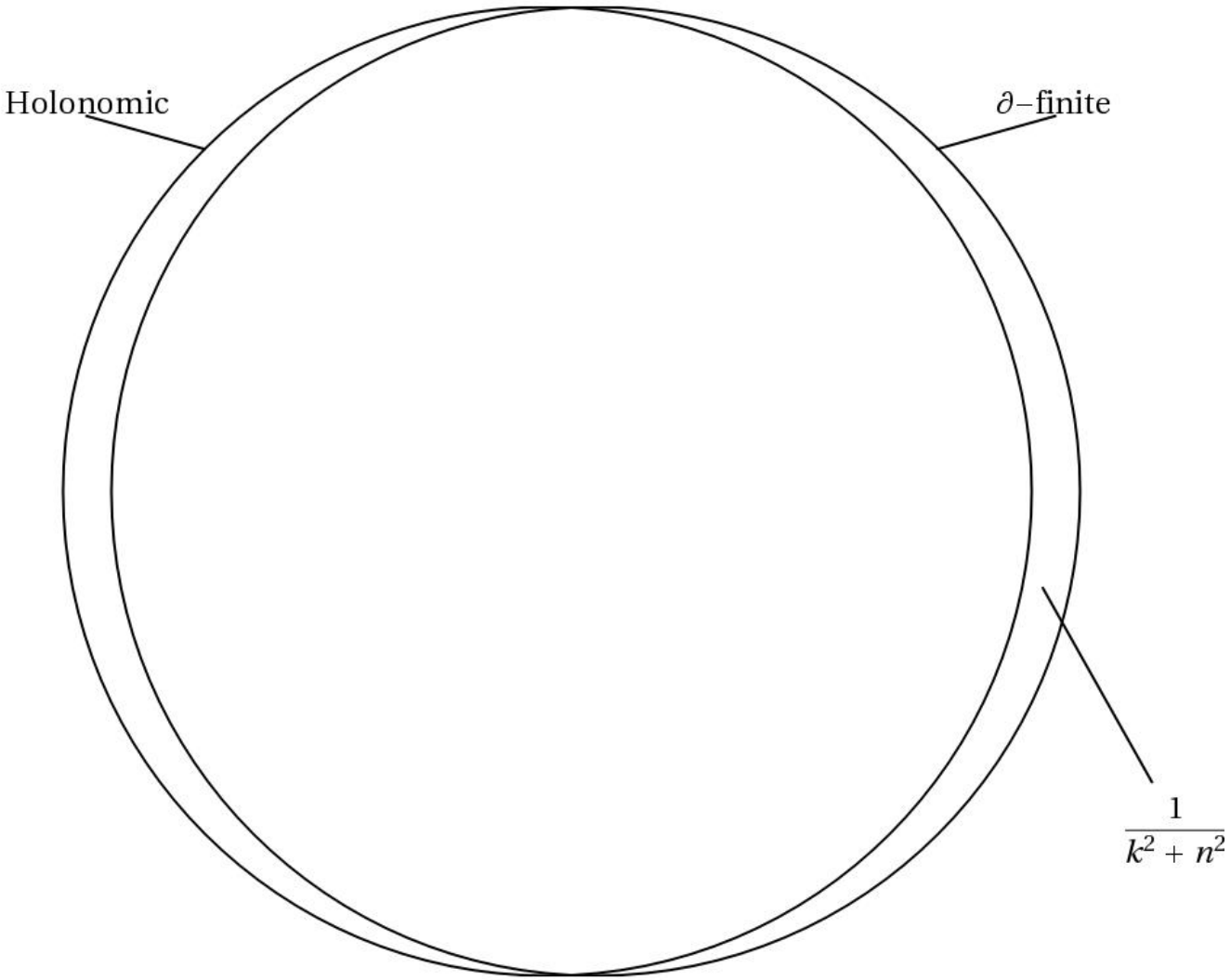

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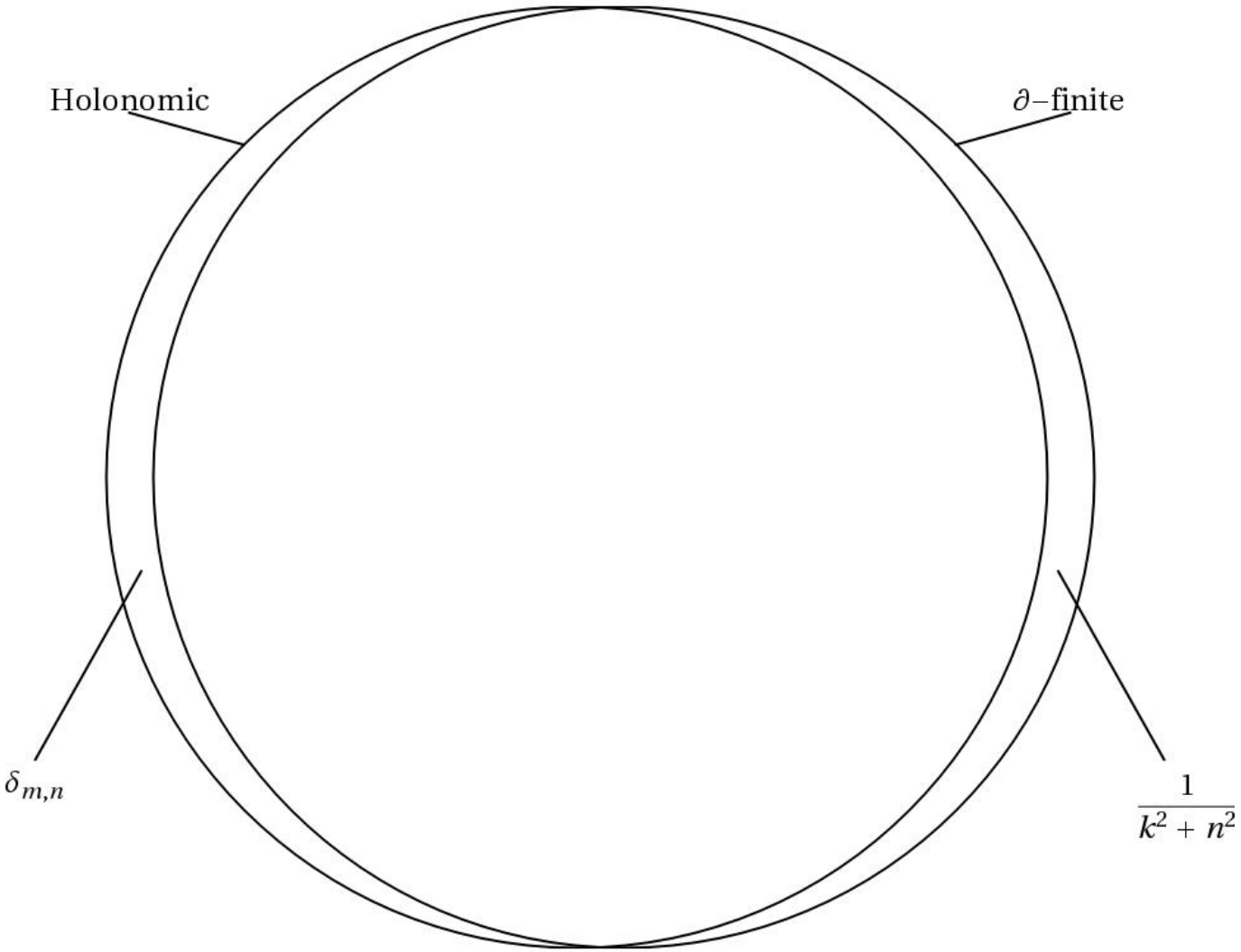

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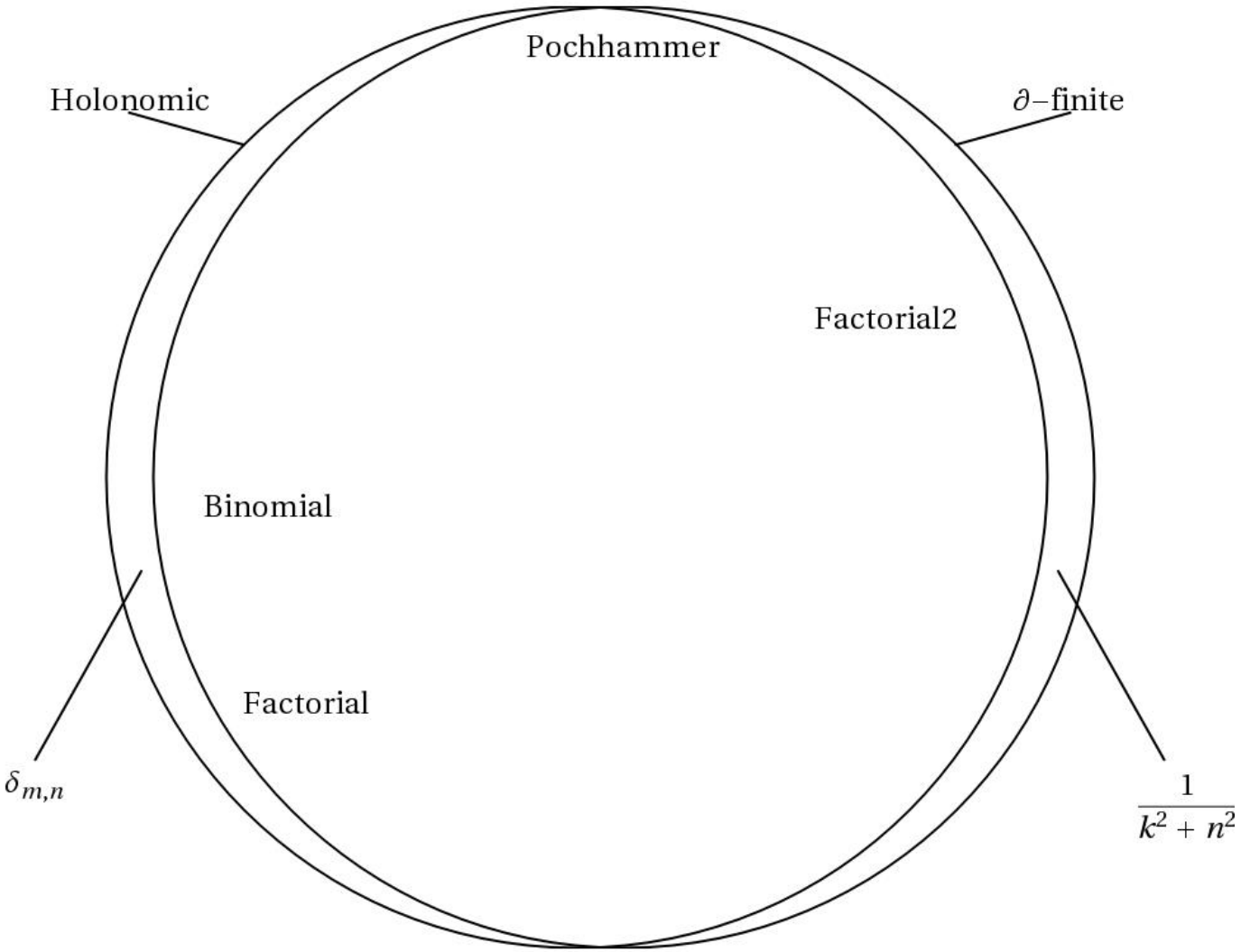

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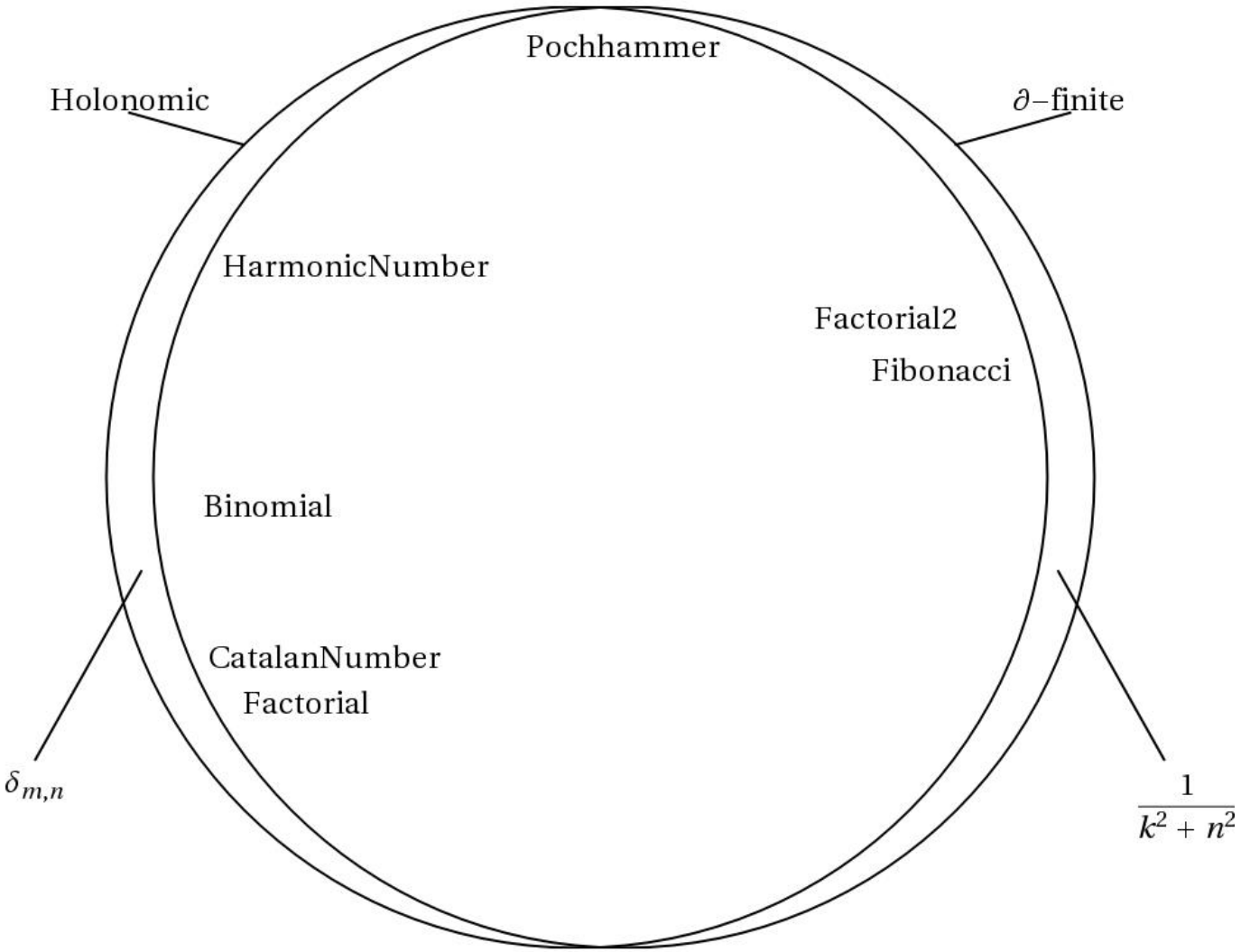

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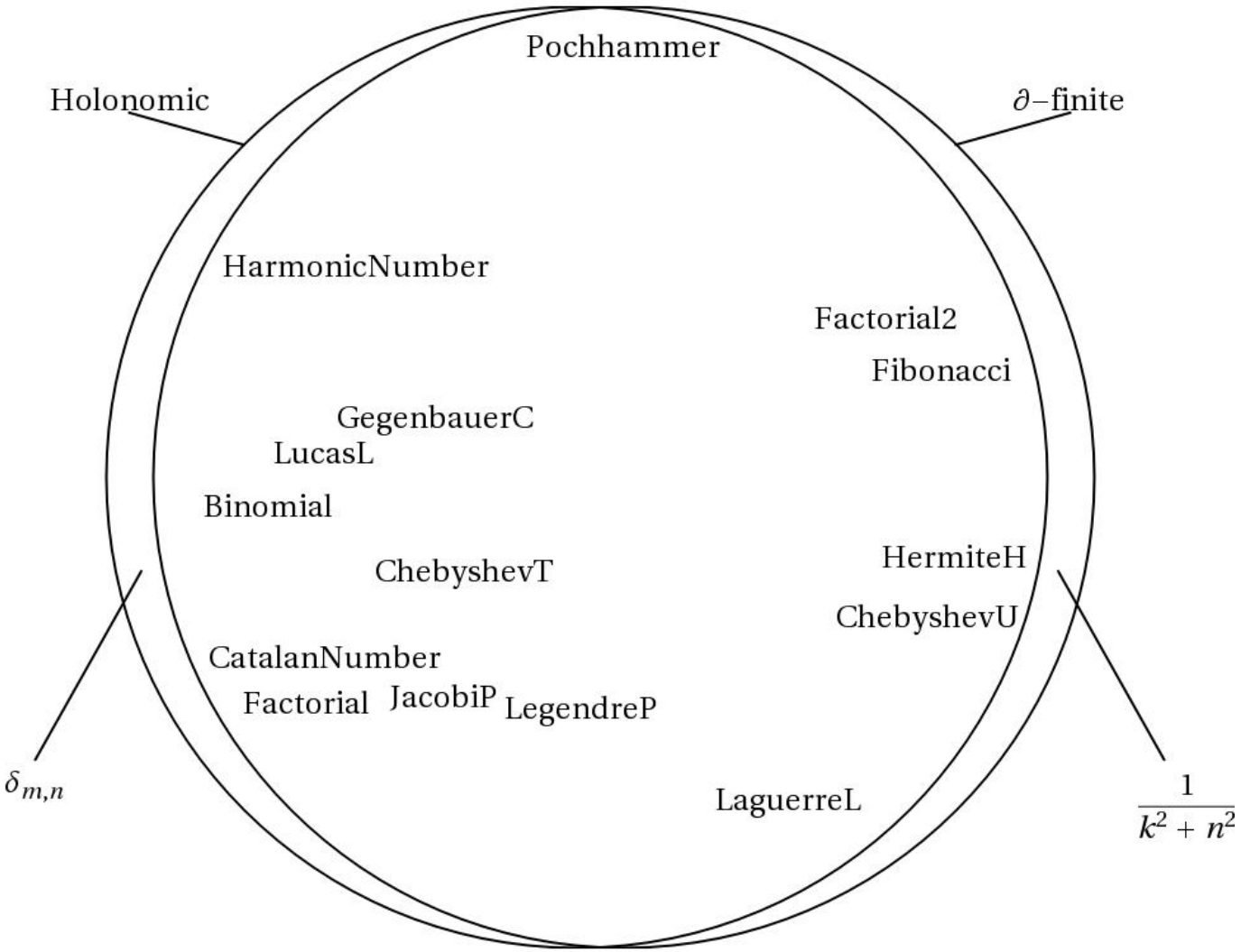

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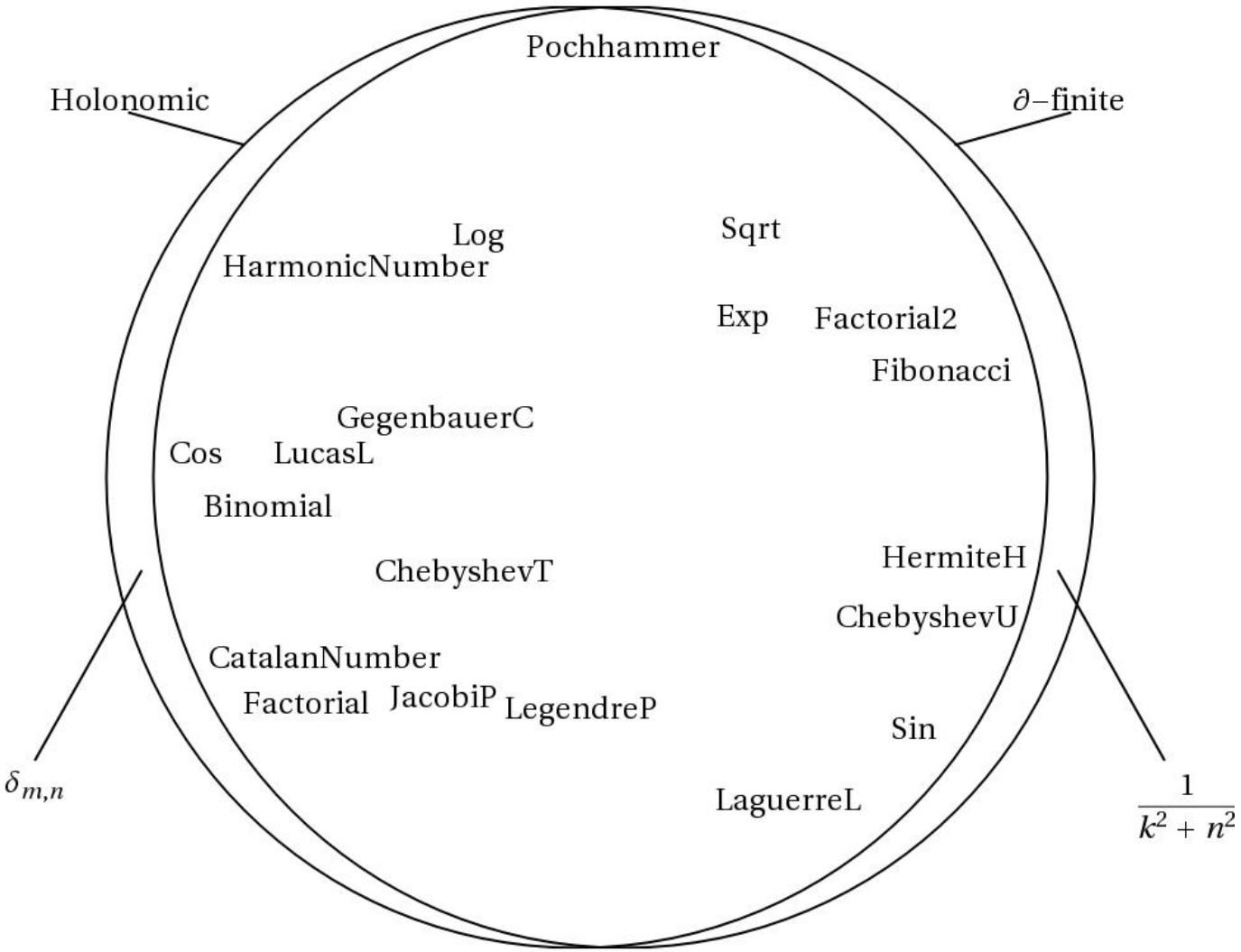

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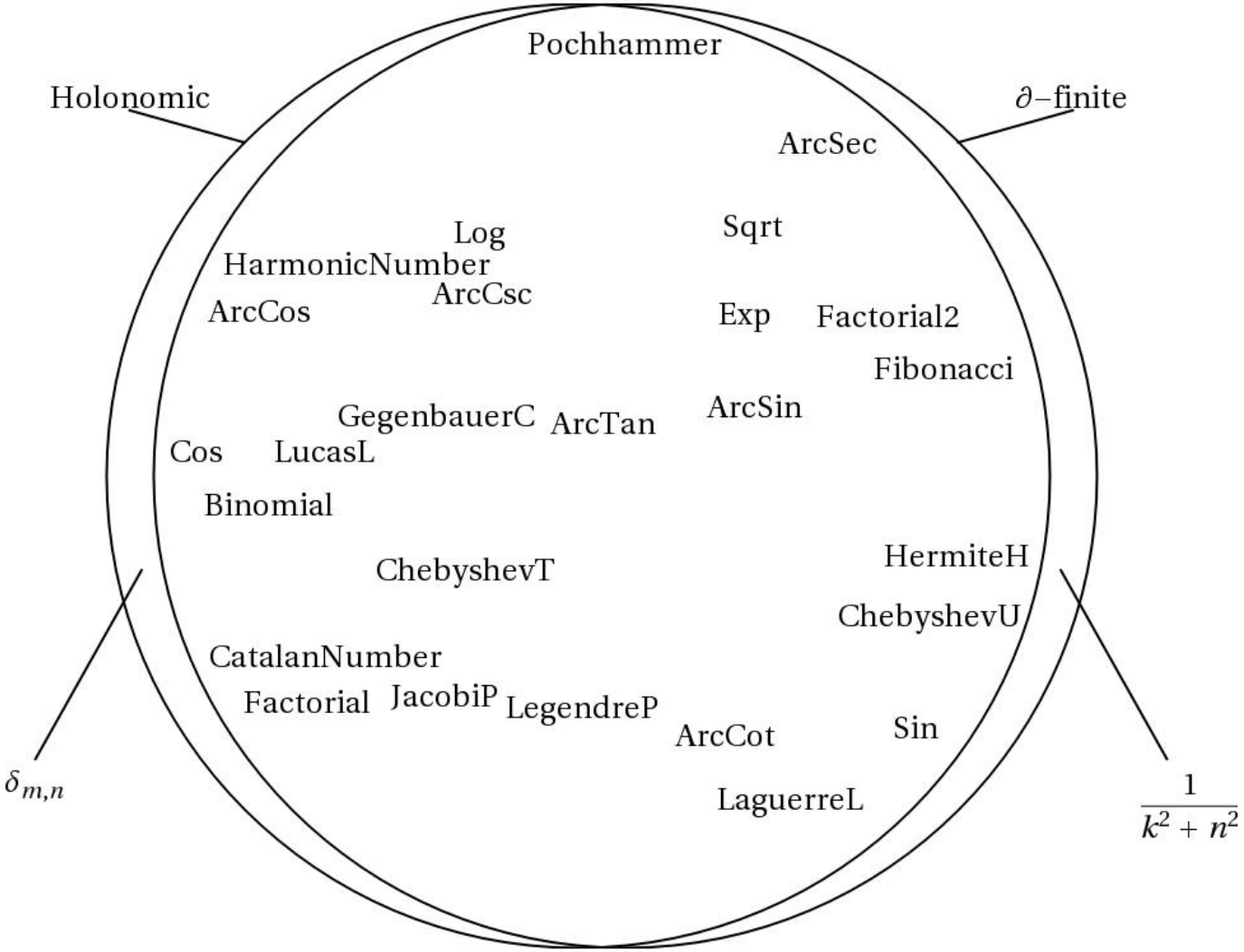

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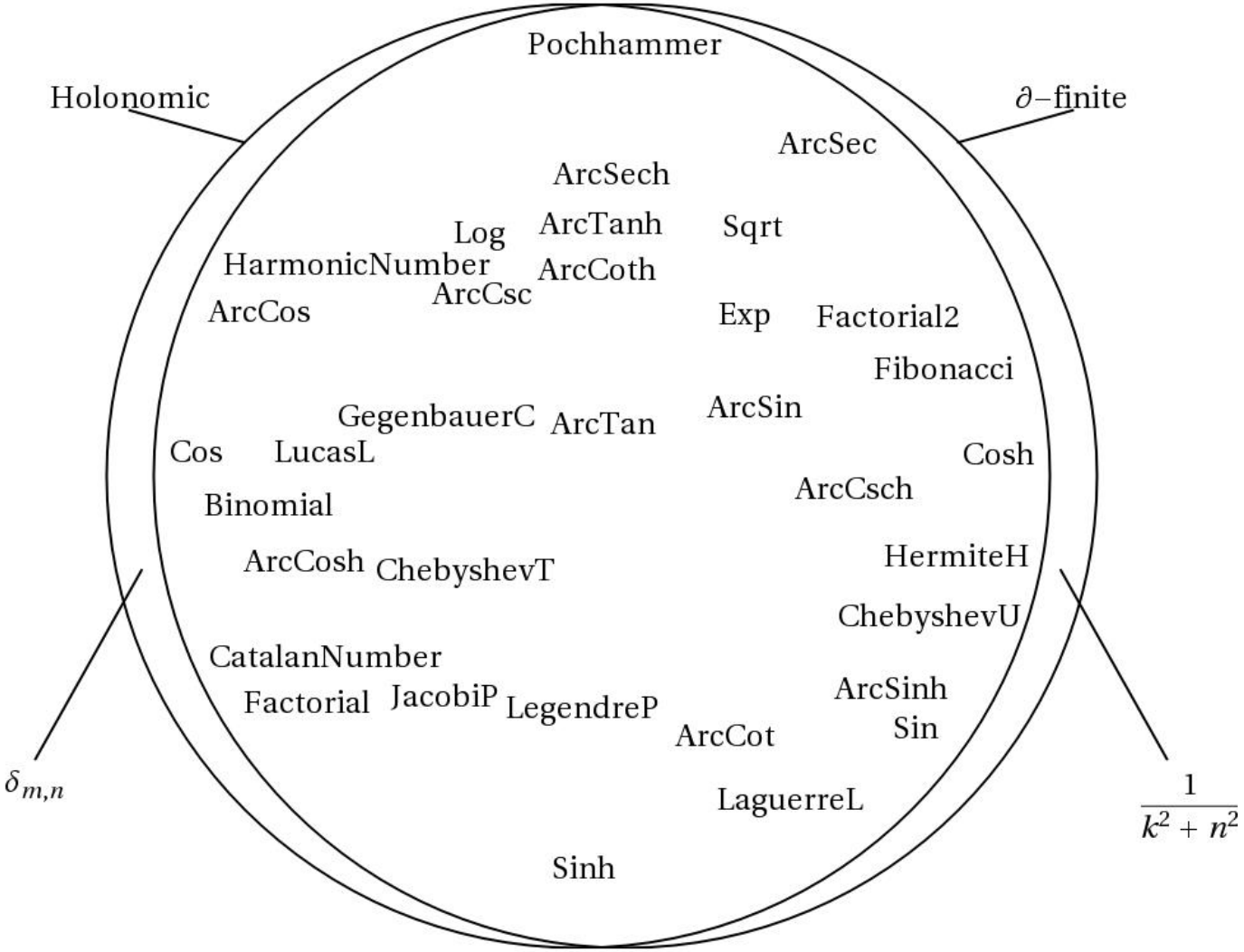

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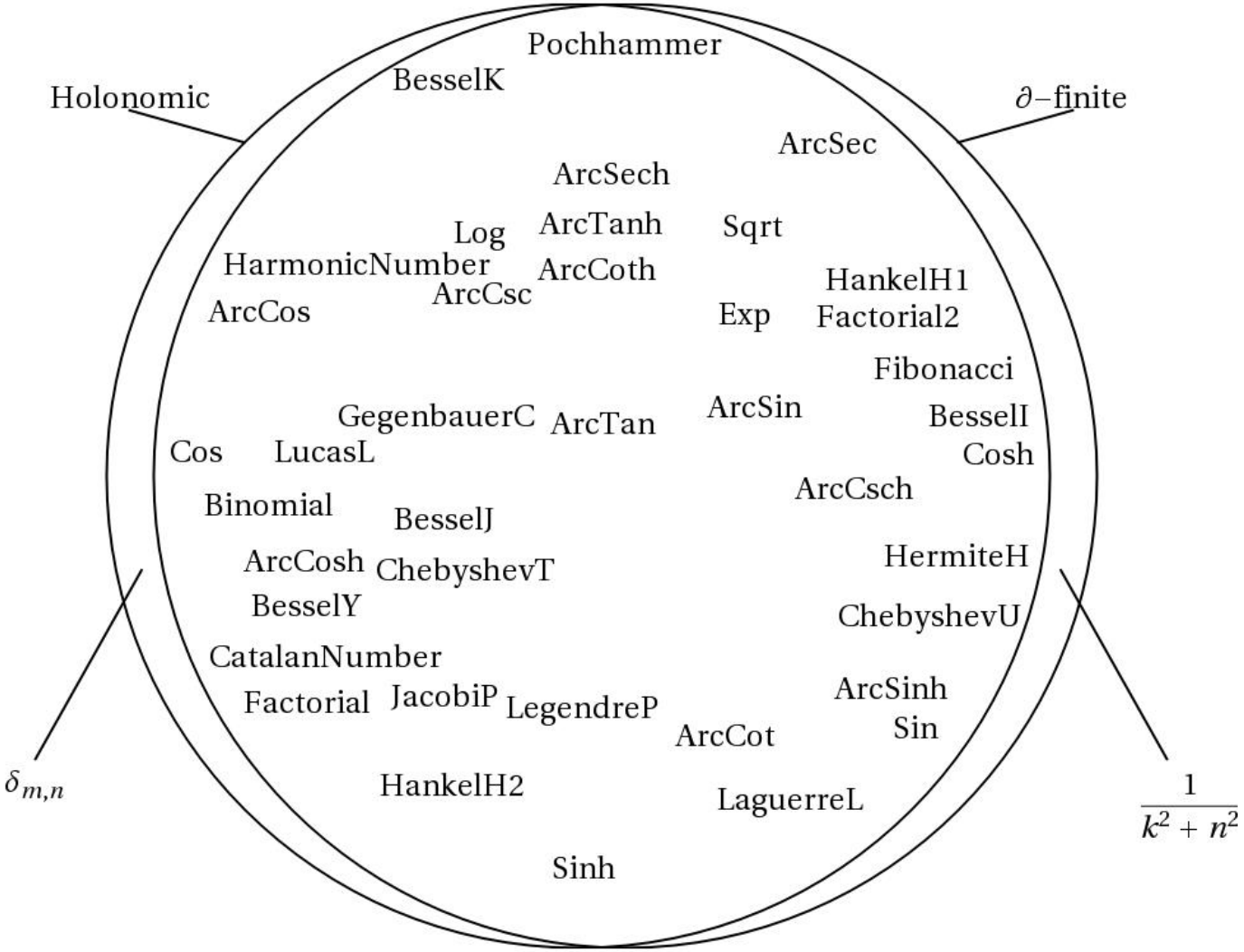

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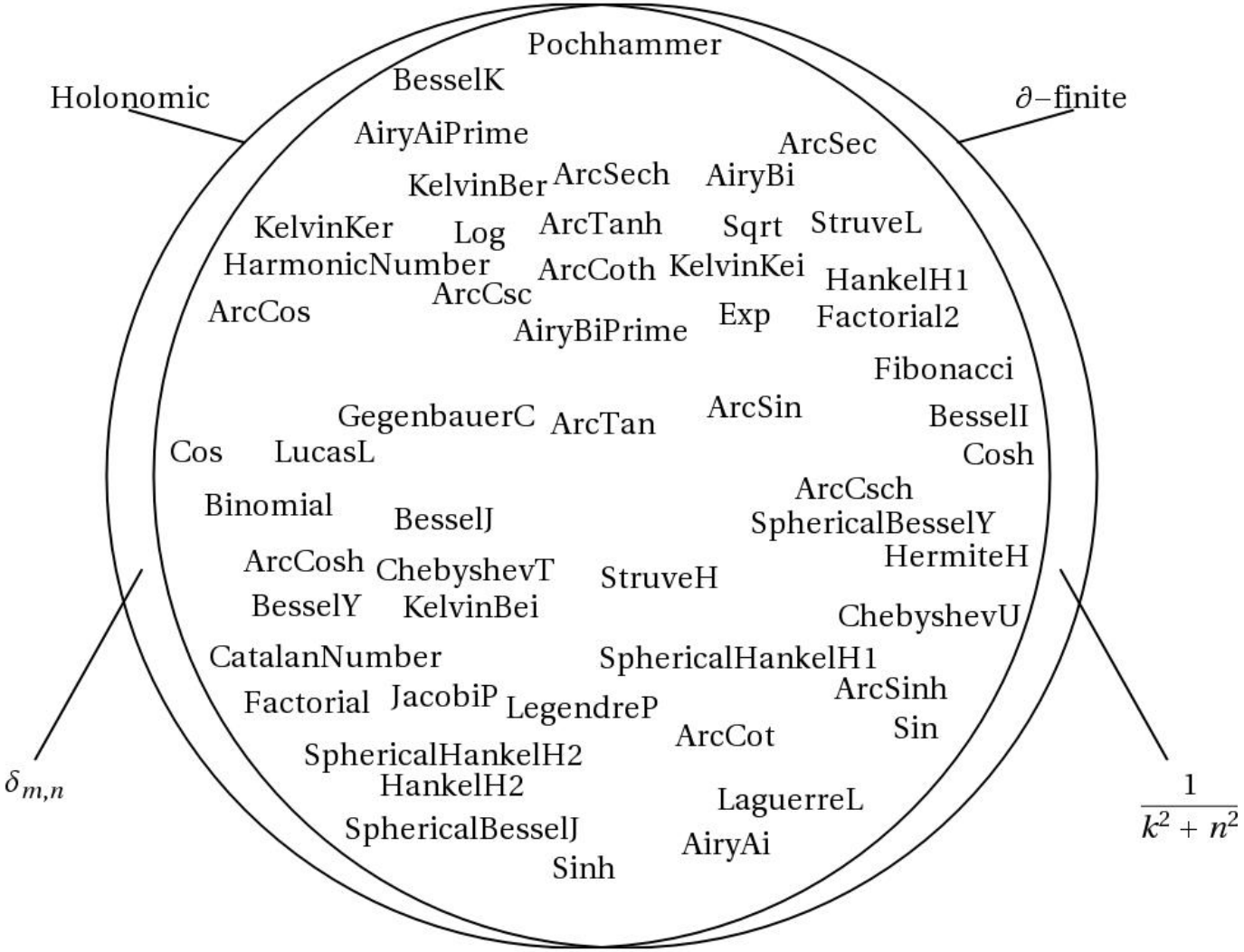

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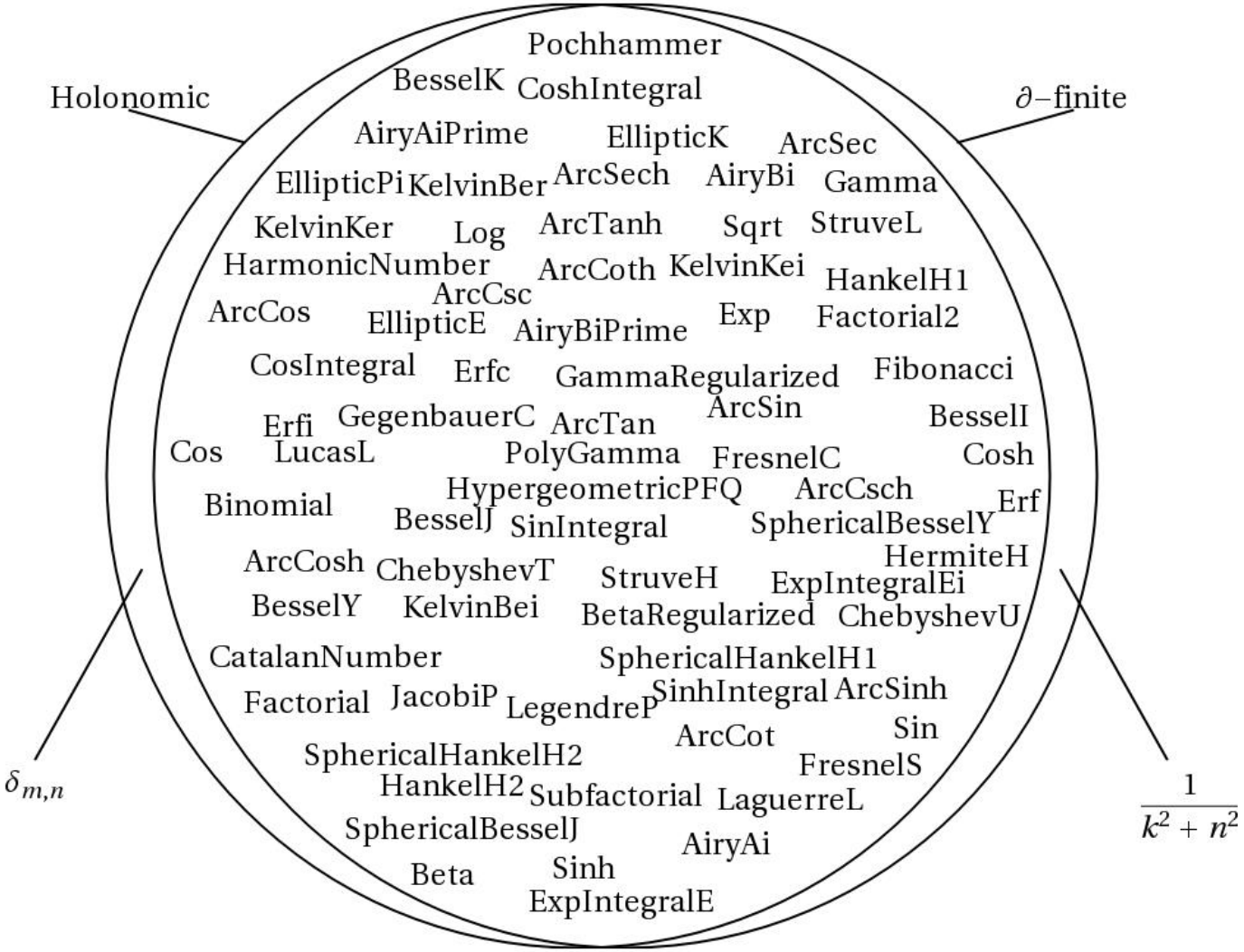

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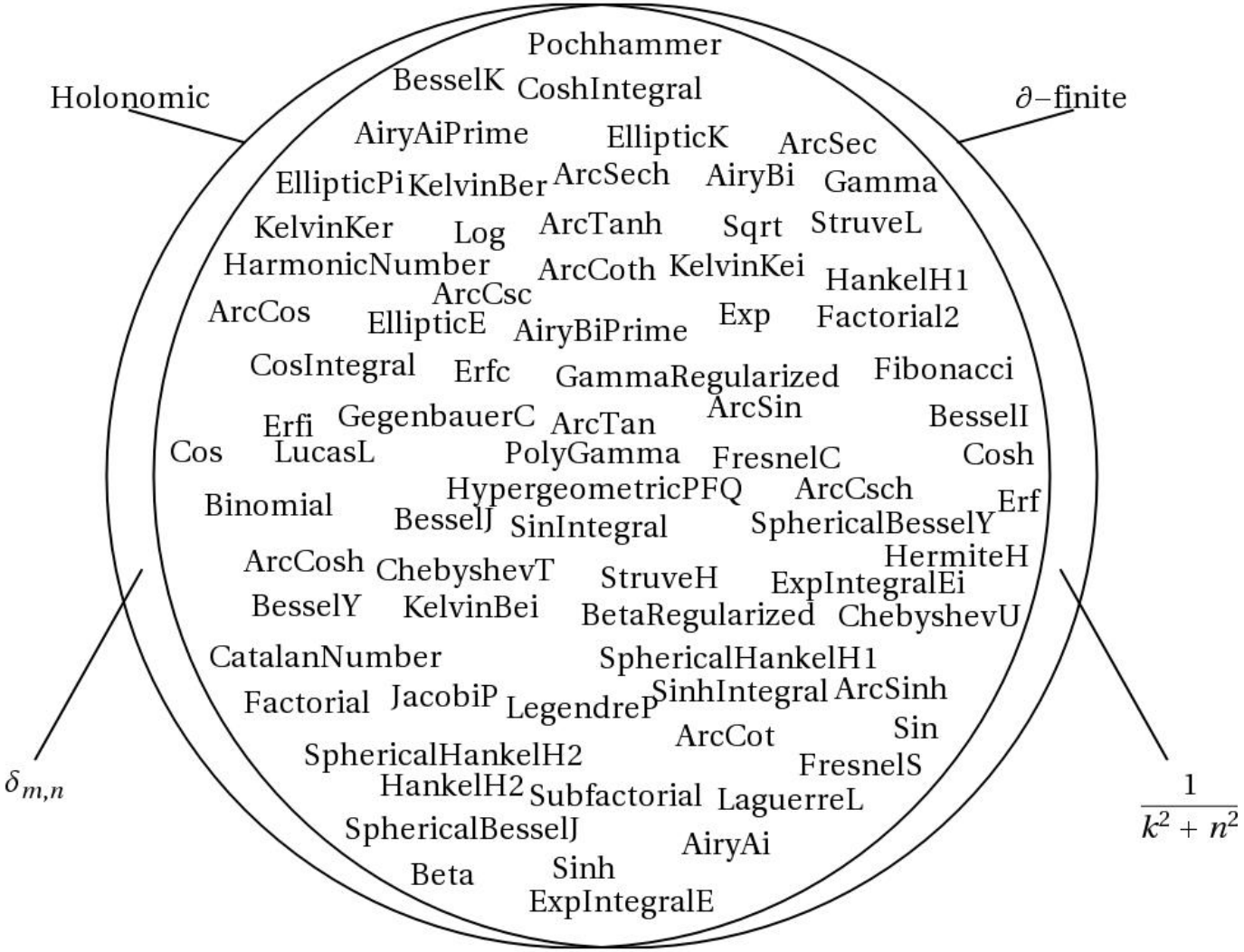

Tan, StirlingS1, BernoulliB, LerchiPhi, ...

Closure Properties

The class of ∂-finite/holonomic functions is closed under

- addition: $f + g$
- multiplication: $f \cdot g$
- certain substitutions
  - algebraic expressions for continuous variables
  - integer-linear expressions for discrete variables
- application of an operator, e.g., $D_x^3 S_n\big(f(n,x)\big)$
- definite integration, e.g., $\int_0^1 f(x,y)\,\mathrm{d}x$
- definite summation, e.g., $\sum_{k=0}^{n} f(n,k)$

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$\longrightarrow$ All these operations can be executed algorithmically.

Principia Holonomica

1. Functions and sequences are represented by their annihilating left ideals (and initial values).

Principia Holonomica

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3. Integrals and sums are treated by the method of creative telescoping.
4. The output is always given as an annihilating ideal, not as a closed form.

How to Solve It

Various problems over the class of ∂-finite/holonomic functions:

- compute a Gröbner basis of annihilating operators for a ∂-finite expression
- prove identities (possibly involving integrals and summations)
- help to evaluate sums and integrals
- find certain identities (e.g. contiguous relations)
- determinant evaluations

Creative Telescoping

Let $F(n)$ denote the double sum over the trinomial coefficients

$$F(n) = \sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i, j, n-i-j} = \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{n!}{i!j!(n-i-j)!}.$$

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Then the creative telescoping operator

$$CT = S_n - 3 + (S_i - 1)\frac{i}{n-i-j+1} + (S_j - 1)\frac{j}{n-i-j+1}$$

with $CT\left(\binom{n}{i,j,n-i-j}\right) = 0$ implies that

$$F(n+1) = 3F(n).$$

Creative Telescoping

The lattice Green's function of the square lattice is given by

$$P(z) = \int_0^1 \int_0^1 \frac{1}{(1-xyz)\sqrt{1-x^2}\sqrt{1-y^2}} \, \mathrm{d}x \, \mathrm{d}y.$$

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The creative telescoping operator

$$(z^3 - z)D_z^2 + (3z^2 - 1)D_z + z + D_x \frac{y(1-x^2)}{xyz-1} + D_y \frac{yz(1-y^2)}{xyz-1}$$

that annihilates the integrand, certifies that $P(z)$ satisfies the differential equation

$$(z^3 - z)P''(z) + (3z^2 - 1)P'(z) + zP(z) = 0.$$

Creative Telescoping

In general, a creative telescoping operator has the form

$$P(\boldsymbol{v}, \boldsymbol{\partial_v}) + \Delta_1 Q_1(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w}) + \cdots + \Delta_m Q_m(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$$

where $\Delta_i = S_{w_i} - 1$ or $\Delta_i = D_{w_i}$ (depending on the problem).

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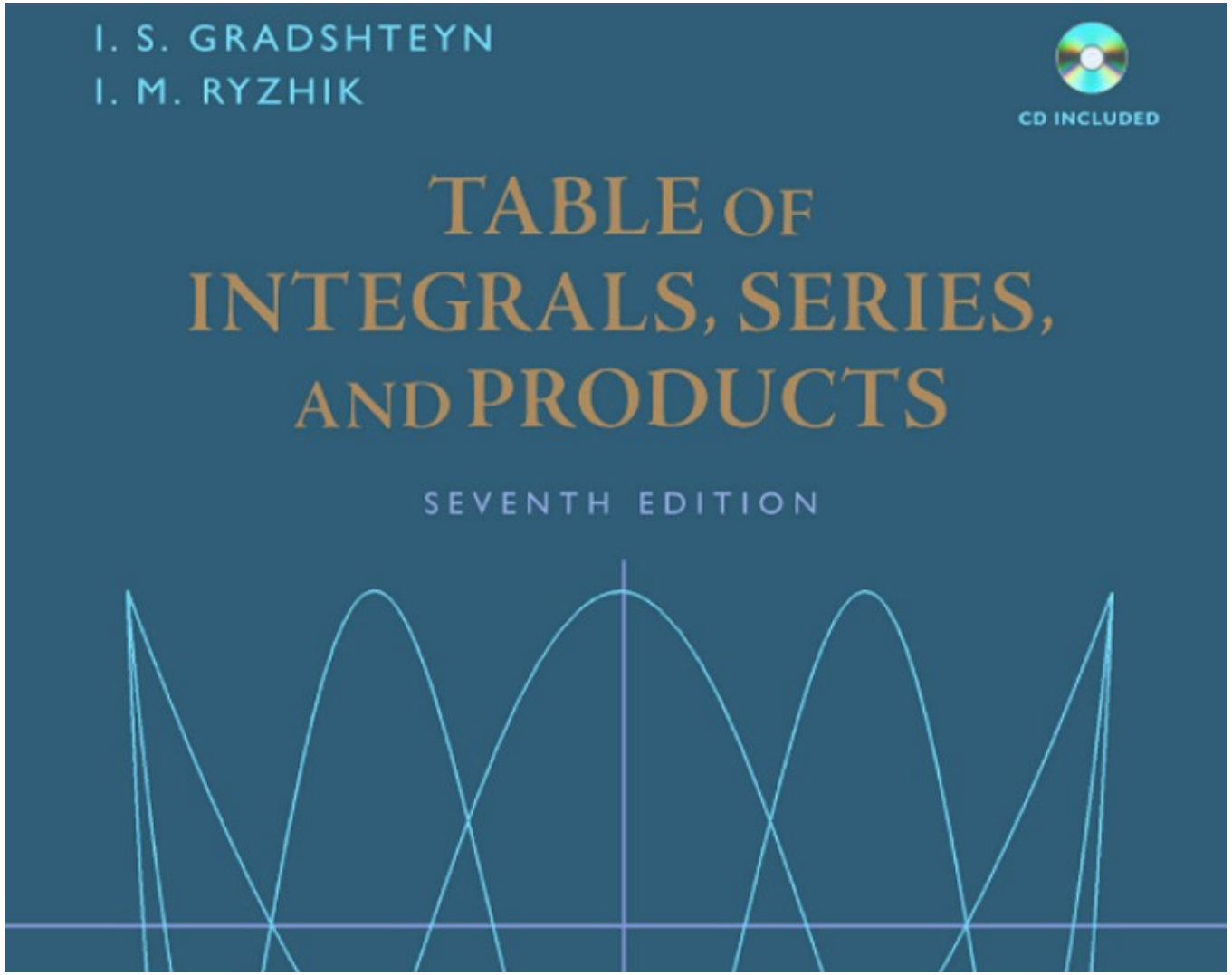
- corresponds to an m-fold summation/integration problem
- $\boldsymbol{w} = w_1, \ldots, w_m$ are the summation/integration variables
- $\boldsymbol{v} = v_1, v_2, \ldots$ are the surviving parameters
- $P(\boldsymbol{v}, \boldsymbol{\partial_v})$ is called the *principal part* or the *telescoper*
- the $Q_i(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$ are called the *delta parts*
- they can be viewed as certificates for the correctness of the principal part

Application 1: Special Function Identities



Some special function identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{k+n}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{k+n}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \tag{1}$$

$$\int_0^\infty \frac{1}{\left(x^4+2ax^2+1\right)^{m+1}}\,\mathrm{d}x=\frac{\pi P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \tag{2}$$

$$e^{-x}x^{a/2}n!L_n^a(x)=\int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\,\mathrm{d}t \tag{3}$$

$$\int_{-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!n!}\,\mathrm{d}x=\sqrt{\pi}e^{2rs} \tag{4}$$

$$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}}e^{iax}C_n^{(\nu)}(x)\,\mathrm{d}x=\frac{\pi 2^{1-\nu}i^n\Gamma(n+2\nu)a^{-\nu}J_{n+\nu}(a)}{n!\Gamma(\nu)} \tag{5}$$

$$\frac{\sin\left(\sqrt{z^2+2tz}\right)}{z}=\sum_{n=0}^{\infty}\frac{(-t)^n y_{n-1}(z)}{n!} \tag{6}$$

Computer proof of a special function identity

We consider identity (3):

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\,\mathrm{d}t.$$

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```
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
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```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*BesselJ[a,2*Sqrt[t*x]
                    Der[t], {S[a], S[n], Der[x]}]
```

$$\{\{-2S_n + 2xD_x + (a+2n+2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},$$
$$\{-2t, -4tx, -2tx\}\}$$

The GradshteynRyzhik Package

GradshteynRyzhik package by Christoph Koutschan, Version 1.1 (30.06.2010)
⟶ Type ?GradshteynRyzhik for help

```
SearchGR[BesselJ[__]^2 + BesselY[__]^2]
```

{6.542, 6.552.3, 6.679.7, 6.785, 6.794.6, 7.184.7, 7.662.3}

```
ShowGR[%]
```

6.542 $\int_0^\infty \frac{J_\nu(ax)\, Y_\nu(bx) - J_\nu(bx)\, Y_\nu(ax)}{x\,\{[J_\nu(bx)]^2 + [Y_\nu(bx)]^2\}}\, dx = -\frac{\pi}{2}\left(\frac{b}{a}\right)^\nu \quad [0 < b < a]$ ET II 352 (16)

6.552.3 $\int_0^\infty K_\nu(xy)\, \frac{dx}{(x^2+a^2)^{1/2}} = \frac{\pi^2}{8} \sec\left(\frac{1}{2}\nu\pi\right) \left\{\left[J_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2 + \left[Y_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2\right\}$ [

$\mathrm{Re}\, a > 0, \quad \mathrm{Re}\, y > 0, \quad |\mathrm{Re}\, \nu| < 1]$ ET II 128 (6)

6.679.7 $\int_0^\infty K_0\left[2\, a \sinh\left(\frac{x}{2}\right)\right] \cos(bx)\, dx = \frac{\pi^2}{4} \left\{[J_{ib}(a)]^2 + [Y_{ib}(a)]^2\right\} \quad [\mathrm{Re}\, a > 0, \quad b > 0]$ ET I 59 (66)

6.785 $\int_0^\infty \frac{\exp\left(\frac{a^2}{2x} - x\right)}{x} \left[1 - \Phi\left(\frac{a}{\sqrt{2x}}\right)\right] K_\nu(x)\, dx = \frac{\pi^{5/2}}{4} \sec(\nu\pi) \left\{[J_\nu(a)]^2 + [Y_\nu(a)]^2\right\}$ [

$\mathrm{Re}\, a > 0, \quad |\mathrm{Re}\, \nu| < \frac{1}{2}]$ ET II 370 (46)

6.794.6 $\int_0^\infty \mathrm{sech}(\pi x) \left\{[J_{ix}(a)]^2 + [Y_{ix}(a)]^2\right\} dx = -Y_0(2a) - E_0(2a) \quad [a > 0]$ ET II 380 (12)

7.184.7 $\int_0^\infty x^{-\nu} \left(x^2+a^2\right)^{\frac{1}{4}-\frac{1}{2}\nu} Q_{-\frac{1}{2}}^{\frac{1}{2}-\nu}\left(1 + 2a^2 x^{-2}\right) K_\nu(xy)\, dx$

$= i\, e^{-i\pi\nu}\, \pi^{3/2}\, 2^{-\nu-3}\, a^{\frac{1}{2}-\nu}\, y^{\nu-1}\, [\Gamma(1-\nu)]^2 \times \left\{\left[J_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)\right]^2 + \left[Y_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)\right]^2\right\}$ [

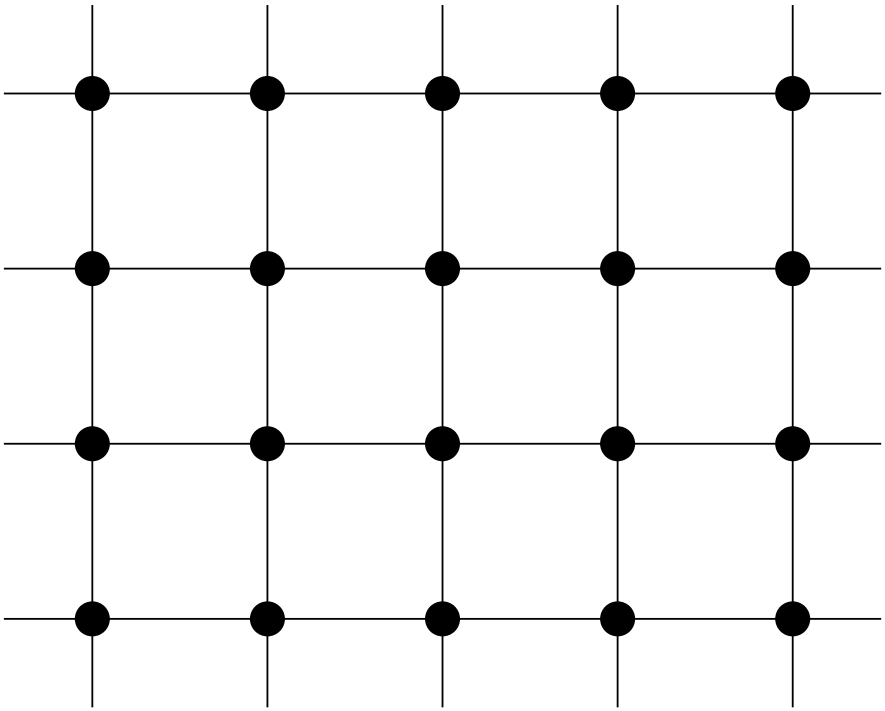
$\mathrm{Re}\, a > 0, \quad \mathrm{Re}\, y > 0, \quad \mathrm{Re}\, \nu < 1]$ ET II 136 (12)

7.662.3 $\int_0^\infty x^{-1} M_{k,0}\left(i\, a\, x^2\right) M_{k,0}\left(-i\, a\, x^2\right) K_0(xy)\, dx = \frac{\pi}{16} \left\{\left[J_k\left(\frac{y^2}{8a}\right)\right]^2 + \left[Y_k\left(\frac{y^2}{8a}\right)\right]^2\right\} \quad [a > 0]$ ET II 152 (83)

Application 2: Lattice Walks

The FCC Lattice

square lattice (= integer lattice $\mathbb{Z}^2$)



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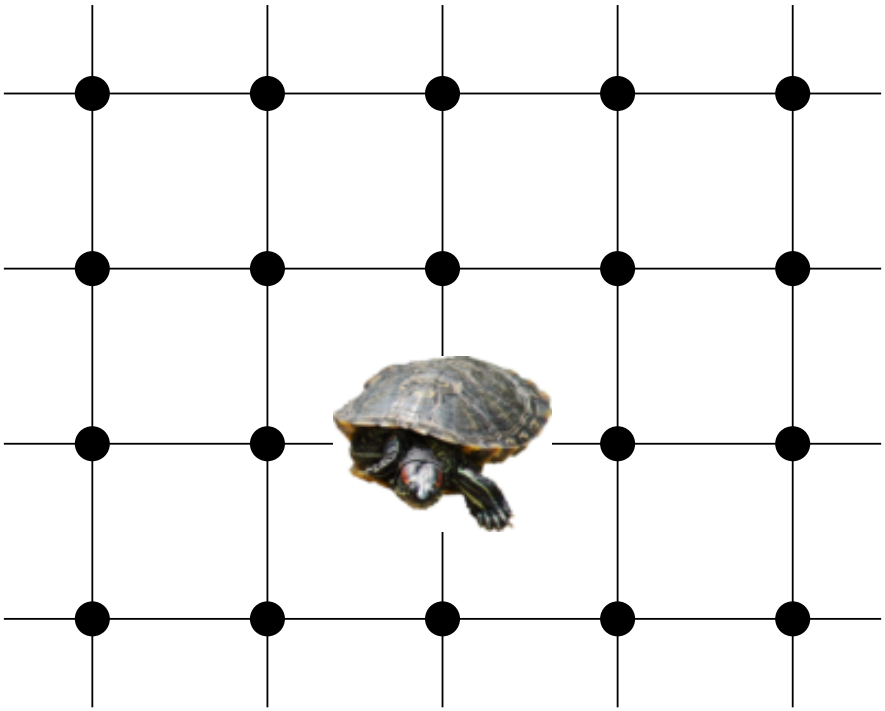

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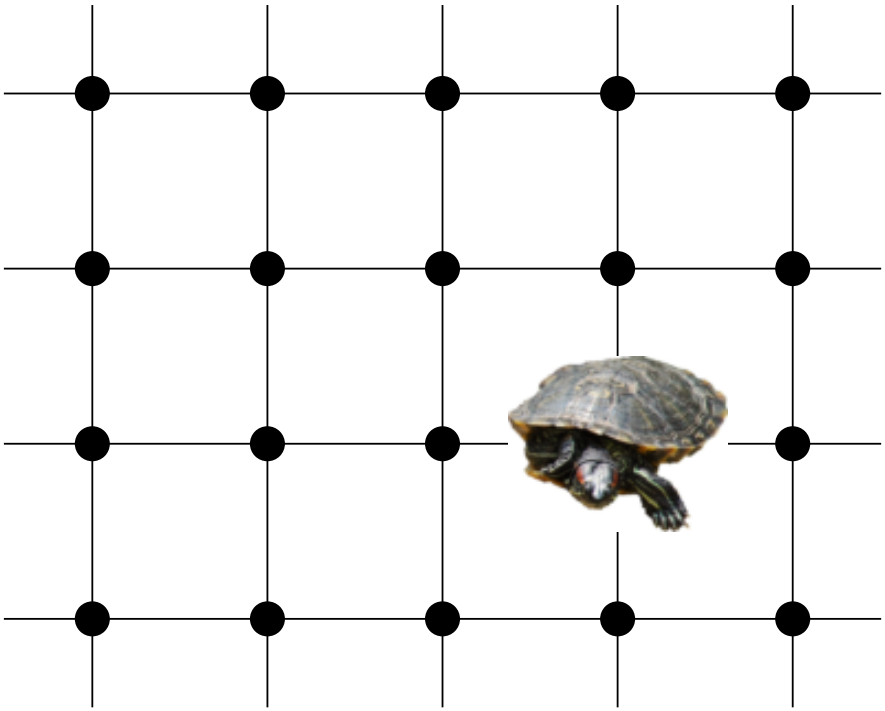

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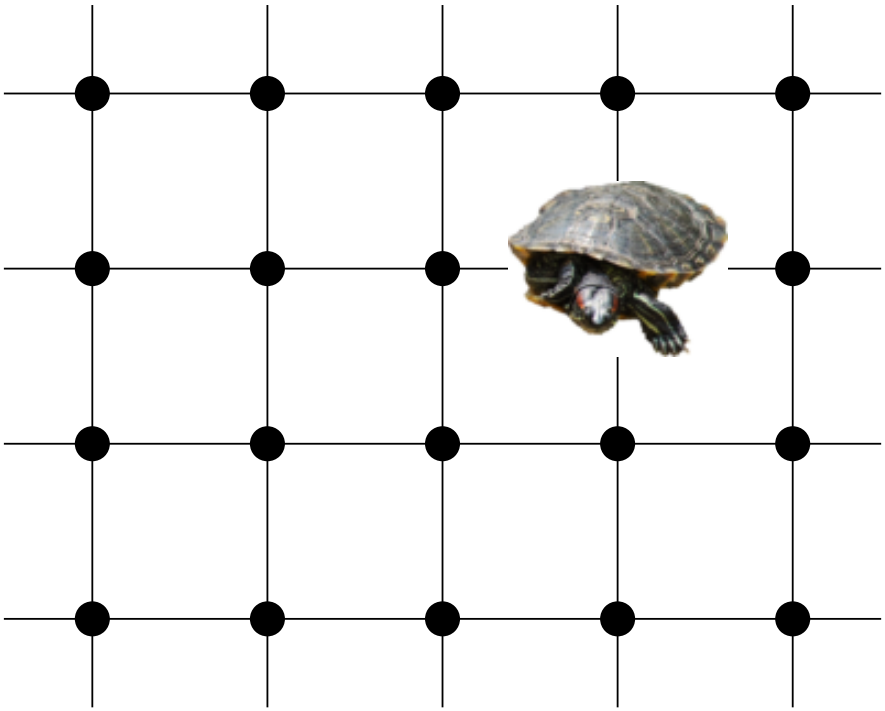

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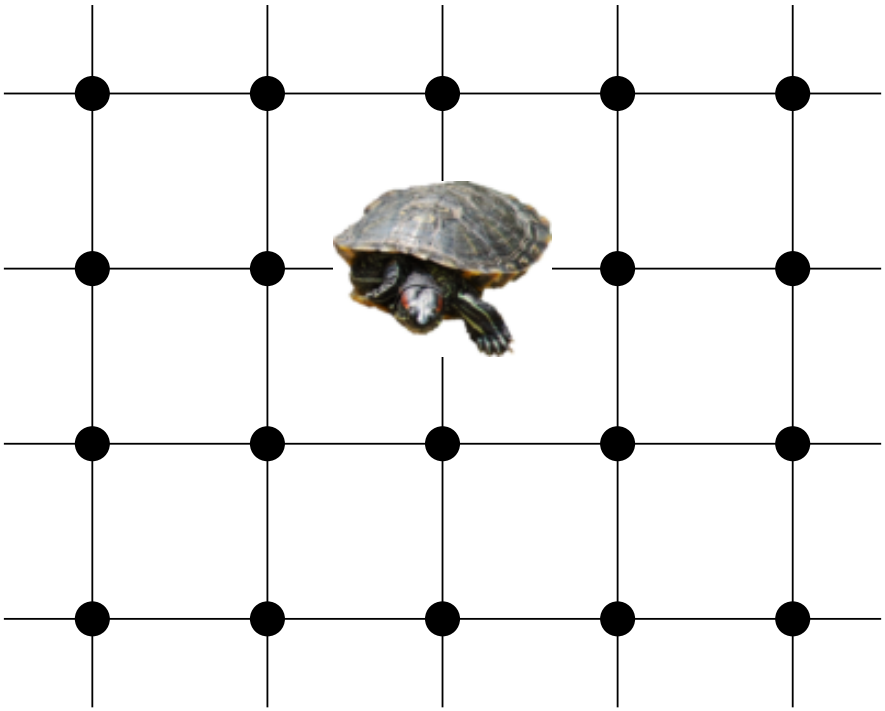

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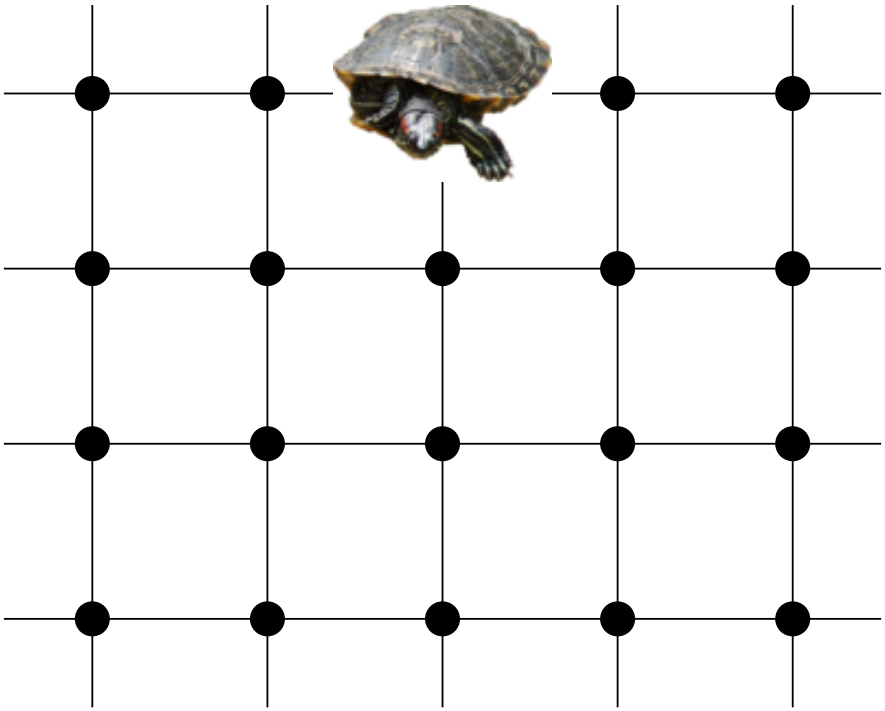

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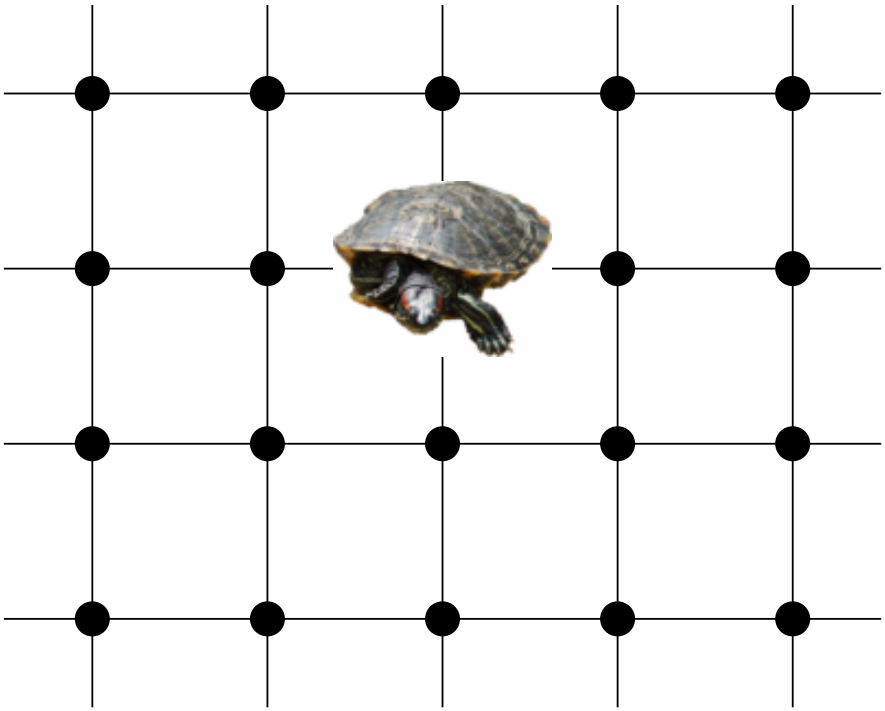

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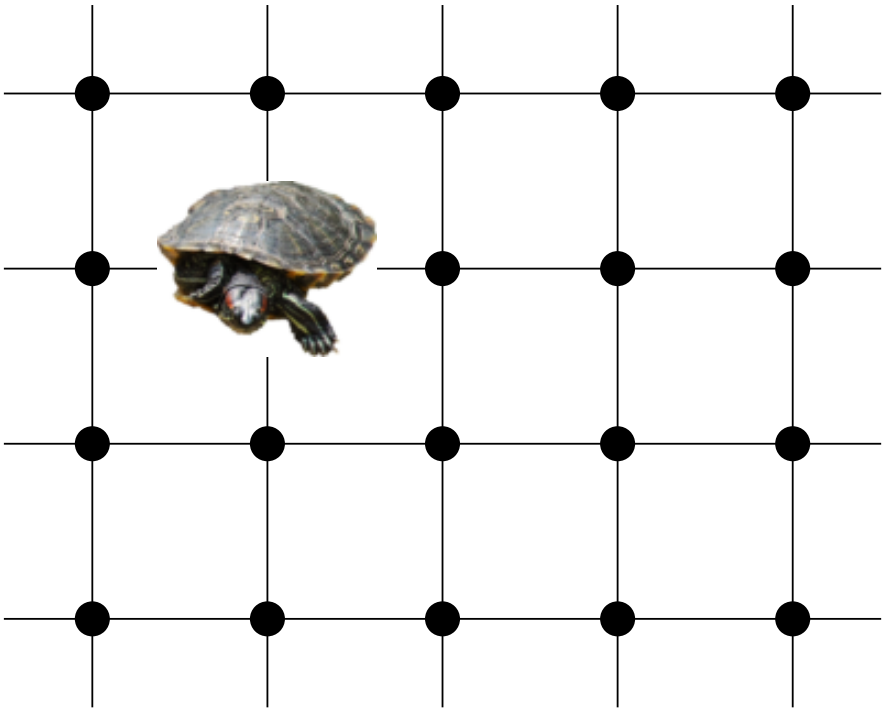

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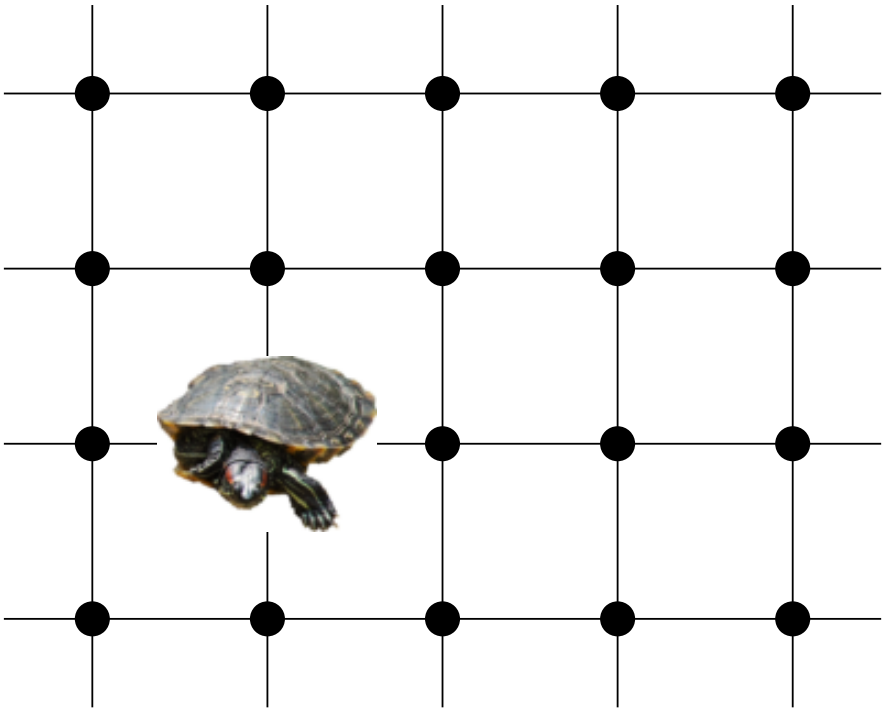

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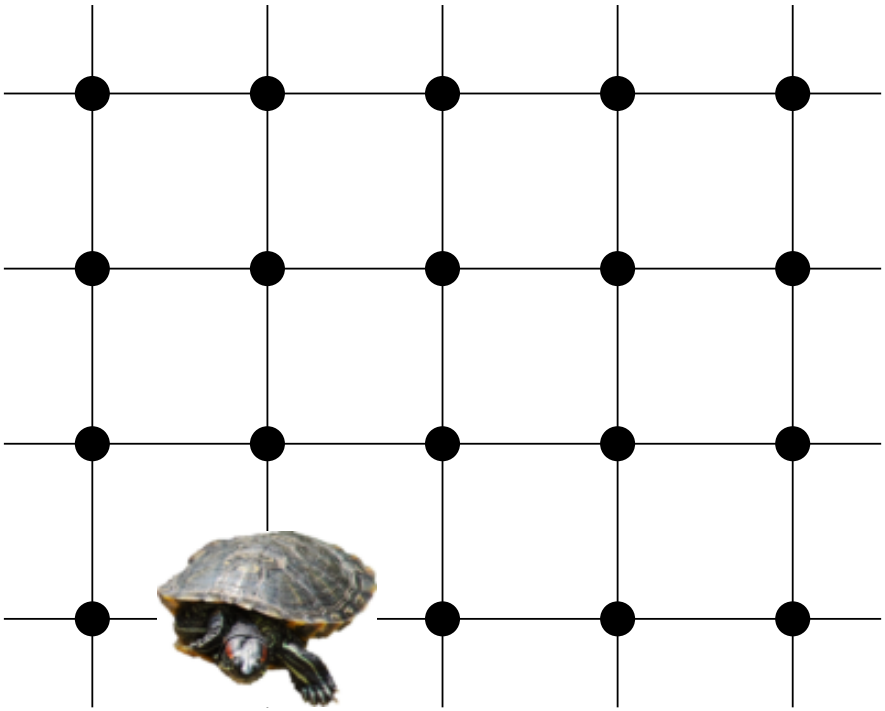

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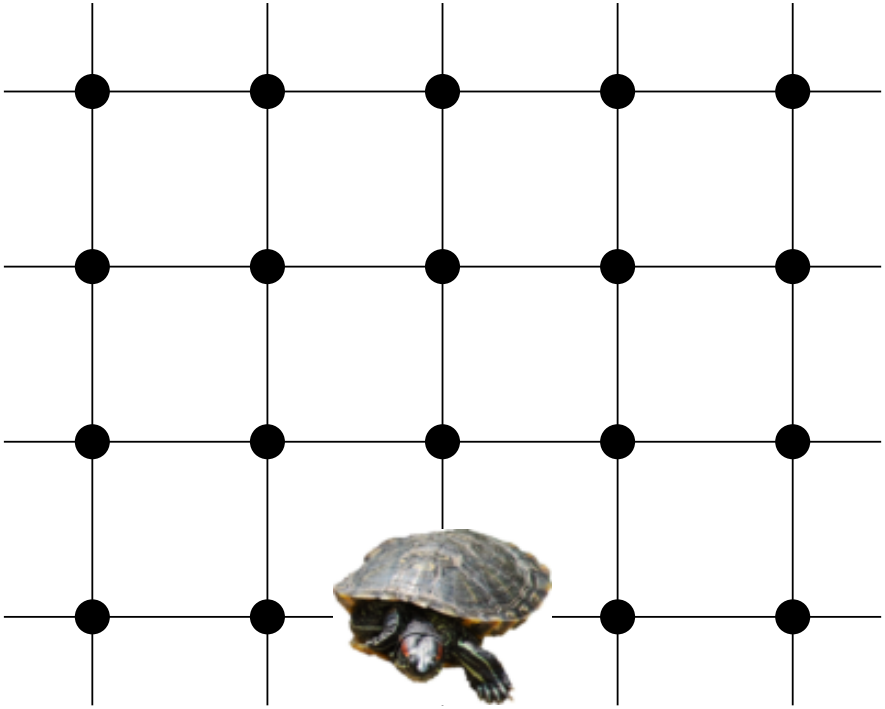

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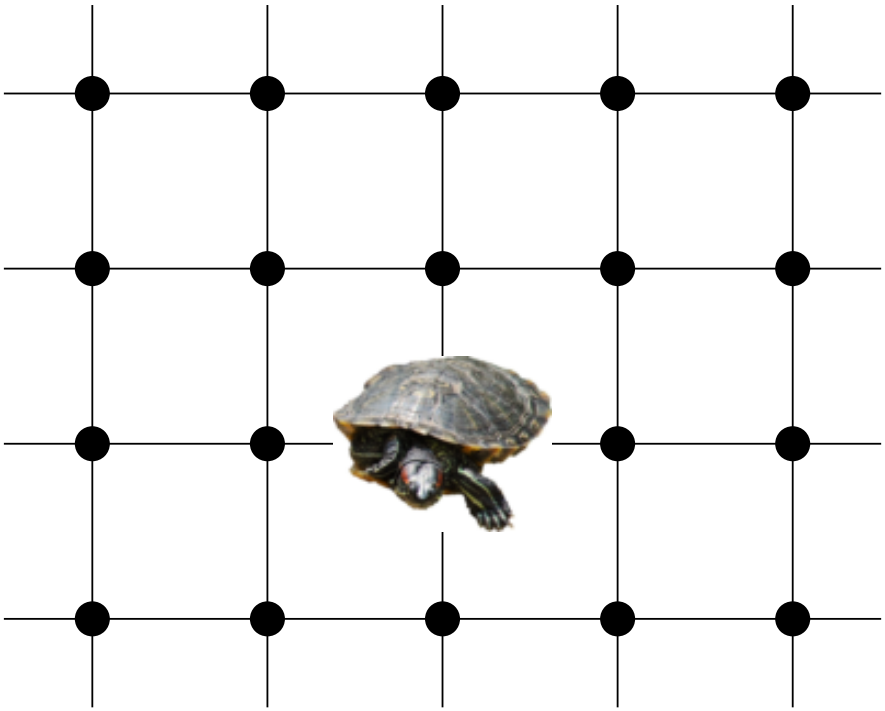

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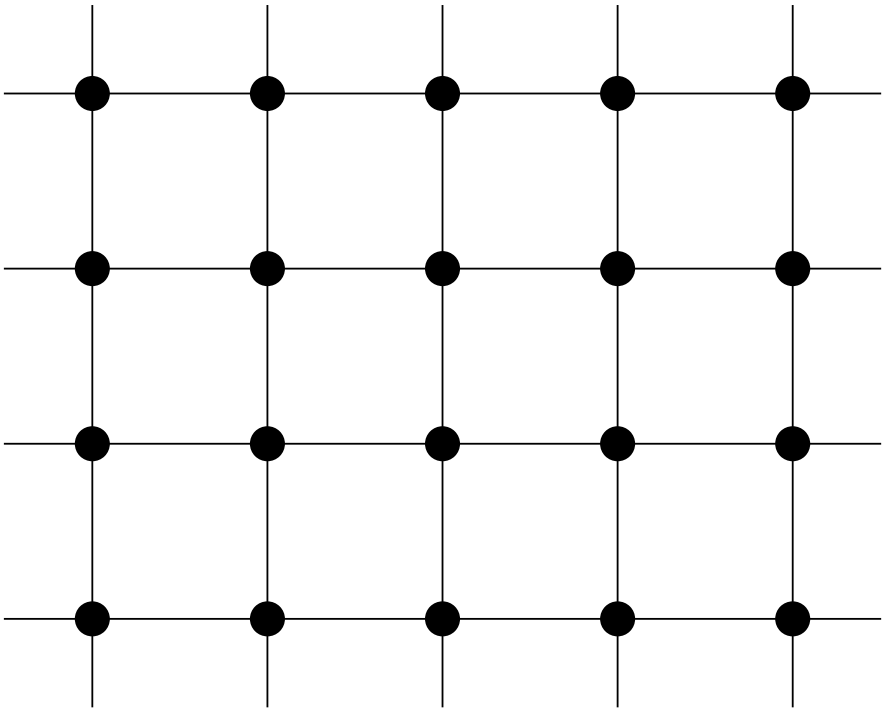
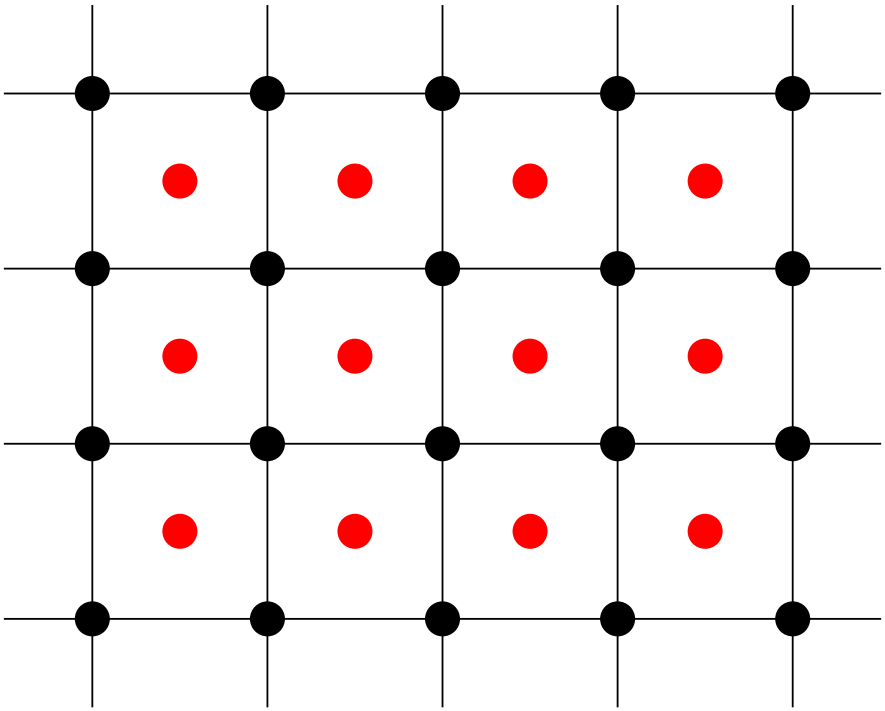


The FCC Lattice

face-centered square lattice



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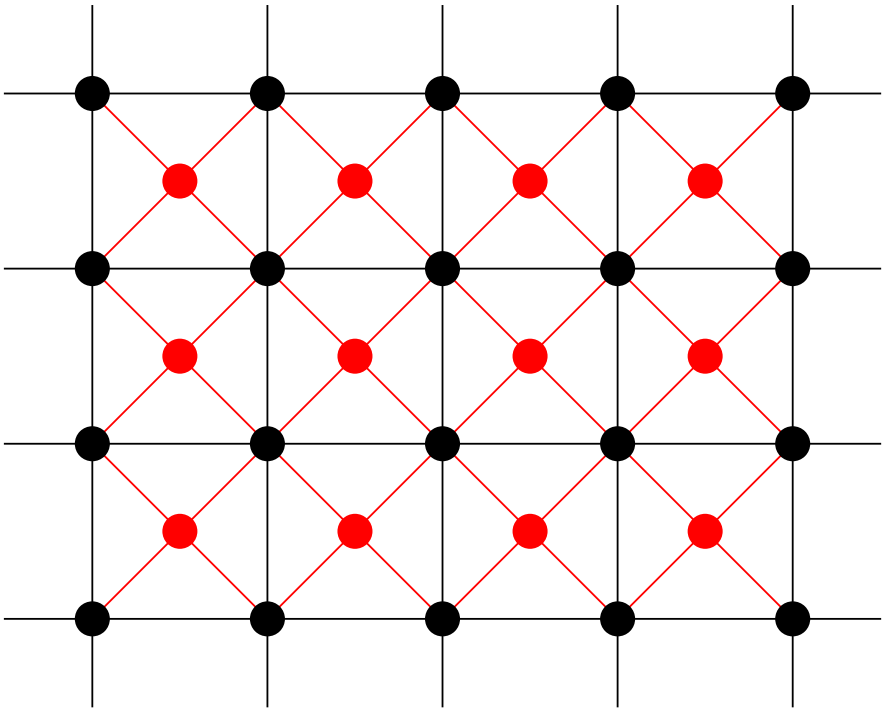

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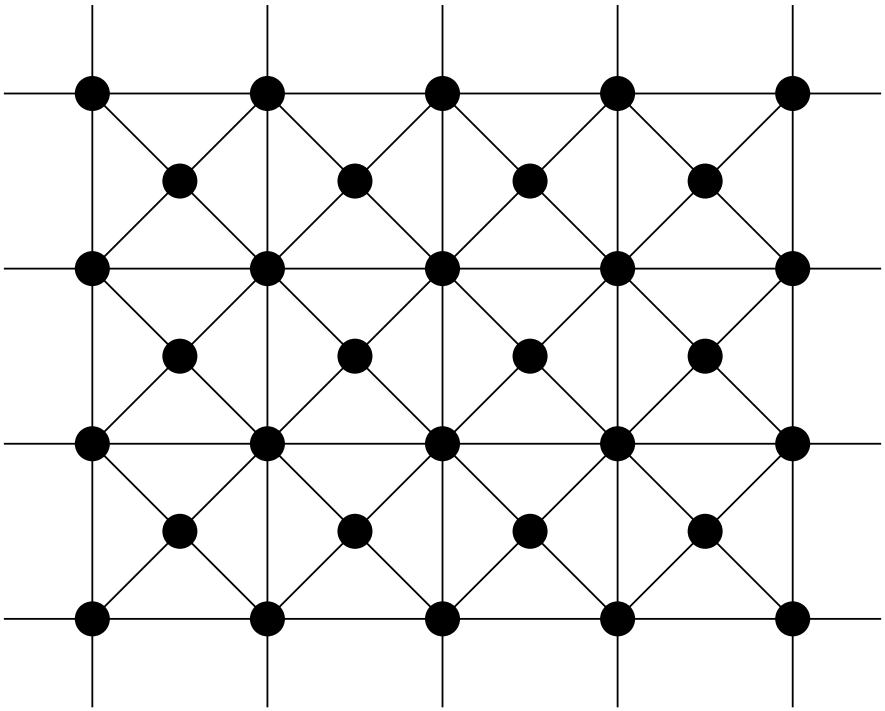

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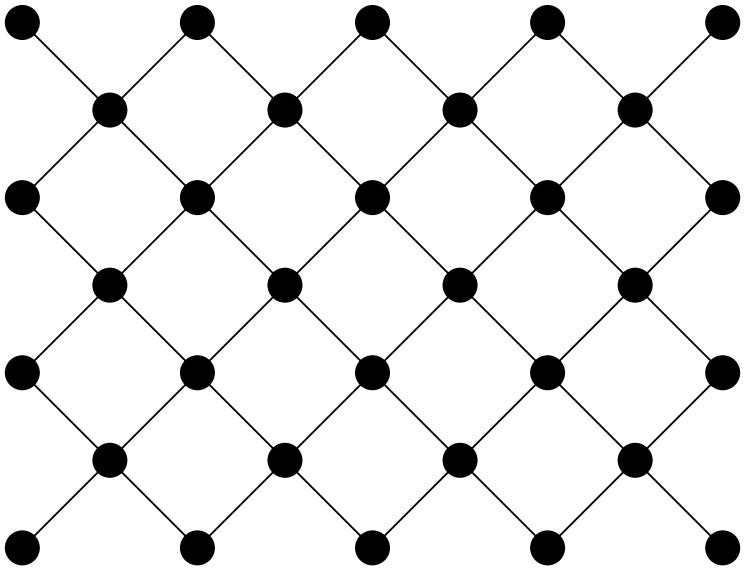
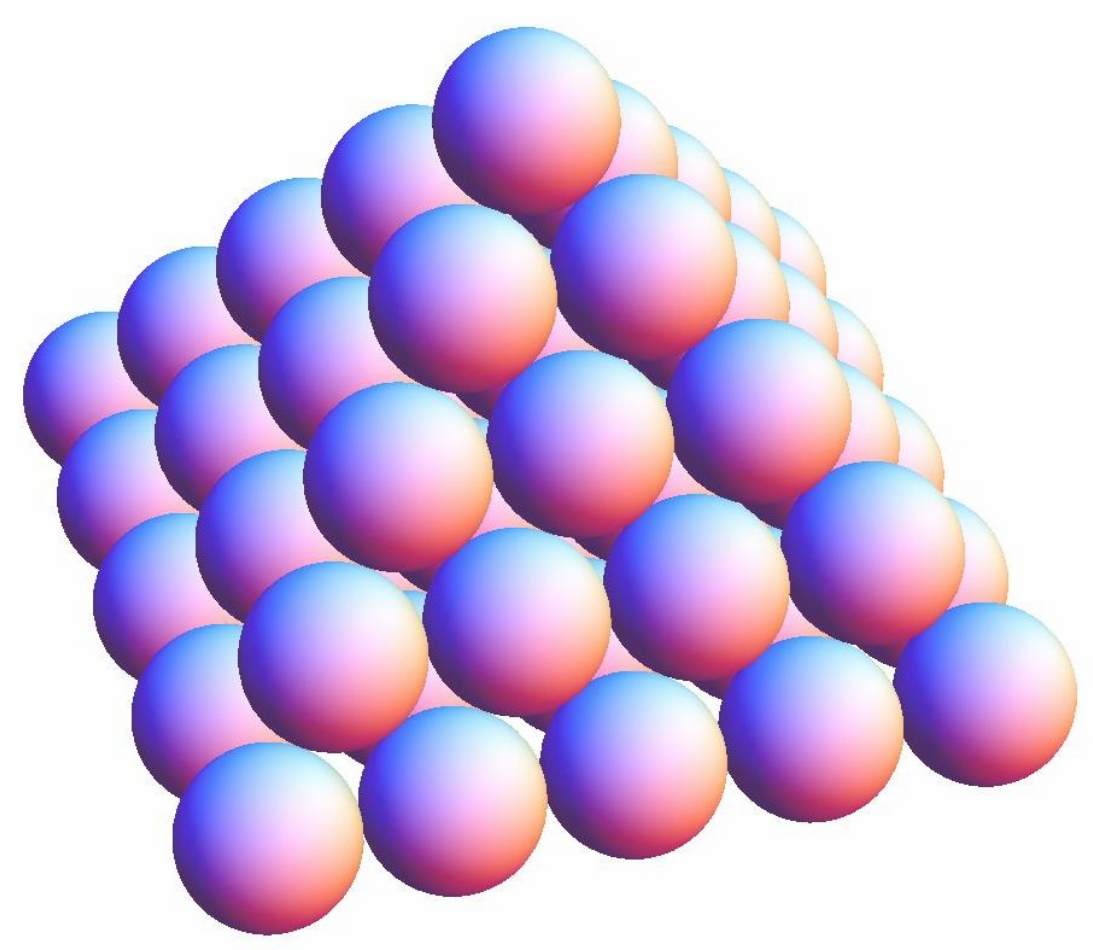


The FCC Lattice

face-centered cubic lattice in 3D



Lattice Green's Functions

We study the fcc lattice in several dimensions: $2 \leq d \leq 6$.

The lattice Green's function is the probability generating function

$$P(\boldsymbol{x}; z) = \sum_{n=0}^{\infty} p_n(\boldsymbol{x}) z^n.$$

Lattice Green's Functions

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Of particular interest is

$$P(\mathbf{0}; z) = \sum_{n=0}^{\infty} p_n(\mathbf{0}) z^n = \frac{1}{\pi^d} \int_0^{\pi} \cdots \int_0^{\pi} \frac{\mathrm{d}k_1 \ldots \mathrm{d}k_d}{1 - z\lambda(\boldsymbol{k})}.$$

that gives the return probabilities.

Here $\lambda(\boldsymbol{k})$ is the structure function that is given by the discrete Fourier transform of the step probabilities.

Lattice Green's Functions

Thus, for the d-dimensional face-centered cubic lattice, we have to compute a d-fold integral of $\frac{1}{1-z\lambda(\boldsymbol{k})}$ where the structure function is

$$\lambda(\boldsymbol{k}) = \binom{d}{2}^{-1} \Big(\cos(k_1)\cos(k_2) + \cdots + \cos(k_{d-1})\cos(k_d) \Big).$$

Note 1: A substitution is required (for holonomizing).
Note 2: For $d = 6$ this is a real challenge!

Lattice Green's Functions

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Timings with our new approach to creative telescoping:

- for $d = 3$: $\sim$ 2 seconds
- for $d = 4$: $\sim$ 3 minutes
- for $d = 5$: $\sim$ 4 hours
- for $d = 6$: $\sim$ 5 days

Results for FCC Lattices

In each case, the result is a linear ODE in z, which gives rise to recurrences for the series coefficients and their partial sums.

From this we can compute the return probability

$$r = 1 - \frac{1}{\sum_{n=0}^{\infty} p_n(\mathbf{0})}$$

to very high accuracy using the asymptotic behaviour of the solutions.

Some results for return probabilities:

- $d = 3$: $r_3 = 1 - \frac{16\sqrt[3]{4}\pi^4}{9(\Gamma(\frac{1}{3}))^6} = 0.2563182365...$
- $d = 4$: $r_4 = 0.09571315417...$
- $d = 5$: $r_5 = 0.04657695746...$
- $d = 6$: $r_6 = 0.02699987828...$

Application 3: Finite Elements

Simulation of Electromagnetic Waves

- joint work with Joachim Schöberl (TU Wien) and Peter Paule (RISC)
- wide range of applications in constructing antennas, mobile phones, etc.
- merchandised by the company CST (Computer Simulation Technology)

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- simulation using the finite element method
- symbolically derived formulae allow a considerable speed-up
- method is part of a registered patent

Problem Setting

Simulate the propagation of electromagnetic waves using the Maxwell equations

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively. Define basis functions (in 2D) in order to approximate the solution:

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\tfrac{2y}{1-x} - 1\right)$$

Problem: need to represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself)

Make an Ansatz!

The numerists need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

1. work in the Ore algebra $\mathbb{Q}(i,j,x,y)\langle D_x, S_i, S_j\rangle$
2. Gröbner basis G of an annihilating ideal for $\varphi_{i,j}(x,y)$
3. choose index sets A and B
4. reduce the above ansatz with G
5. do coefficient comparison with respect to x and y
6. solve the resulting linear system for $a_{k,l}, b_{m,n} \in \mathbb{Q}(i,j)$

Make an Ansatz!

The numerists need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

1. work in the Ore algebra $\mathbb{Q}(i,j,x,y)\langle D_x, S_i, S_j\rangle$
2. Gröbner basis G of an annihilating ideal for $\varphi_{i,j}(x,y)$
3. choose index sets A and B
4. reduce the above ansatz with G
5. do coefficient comparison with respect to x and y
6. solve the resulting linear system for $a_{k,l}, b_{m,n} \in \mathbb{Q}(i,j)$

$\longrightarrow$ can find the "smallest" relation

Note: many optimizations (e.g., homomorphic images) are implemented.

Result

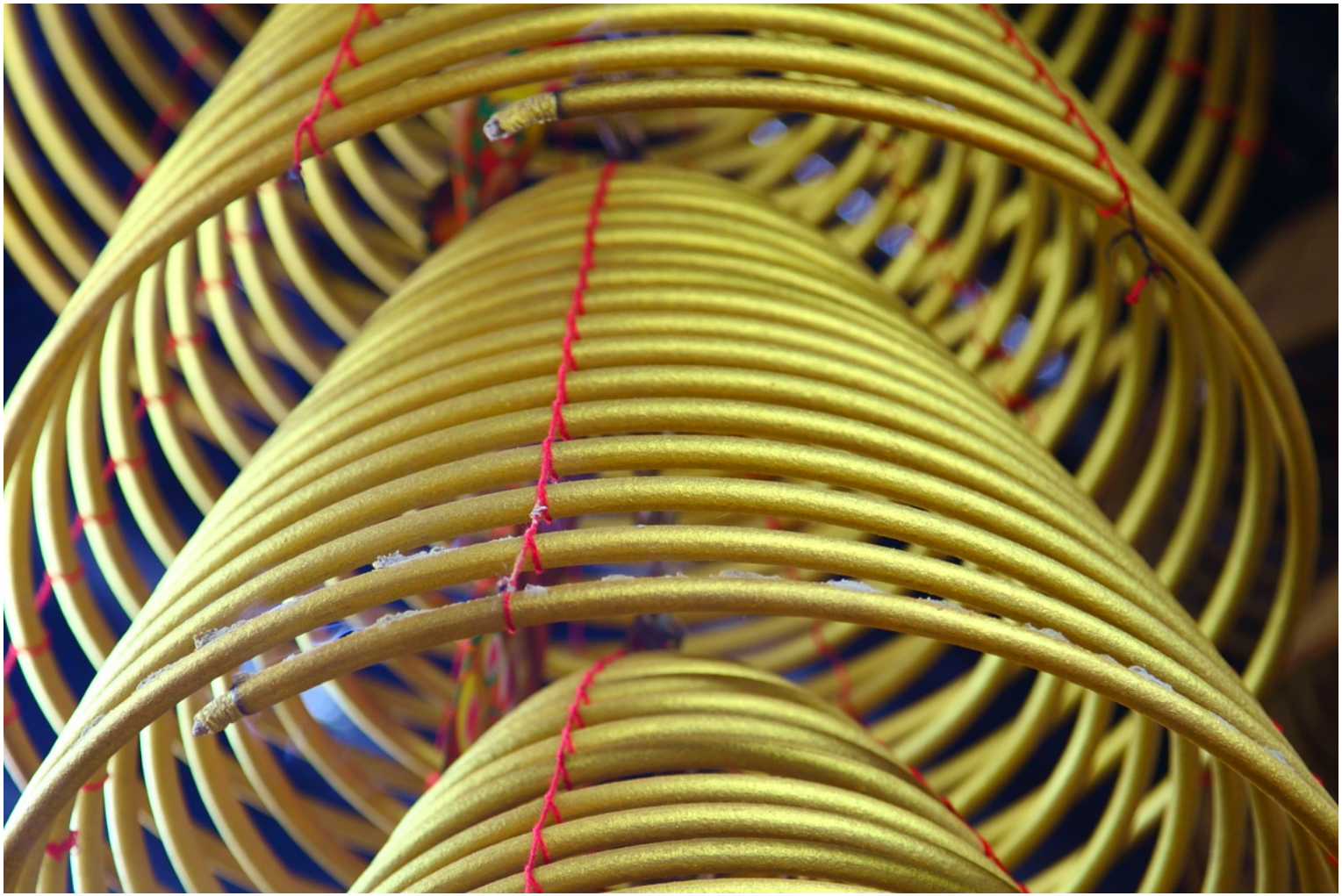
With this method, we find the relation

$$
\begin{aligned}
&(2i+j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)+\\
&2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)-\\
&(j+3)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)+\\
&(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)-\\
&2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)-\\
&(2i+j+5)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)=0
\end{aligned}
$$

and a similar one for $\frac{\mathrm{d}}{\mathrm{d}y}\varphi_{i,j}(x,y)$ in just a few seconds.

$\longrightarrow$ These formulae caused a speed-up of 20 percent (!) in the numerical simulations.

Application 4: Hamiltonian Systems



Problem 1

This is joint work with Thierry Combot (Paris). While studying the integrability of Hamiltonian systems, he encountered the integral

$$\int_{-1}^{1} \tilde{P}_n(x)^{k+1}(x^2-1)^k \,\mathrm{d}x$$

where

$$\tilde{P}_n(x) = \frac{1}{x^2-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n.$$

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Our results (using our new approach to creative telescoping):

- found the recurrence in n for $k = 2, \ldots, 7$
- the recurrence for $k = 6$ has order 6 (with even exponents only) and degree 92 (computation time: 3 hours)
- the recurrence for $k = 7$ has order 4 and degree 70 (computation time: 38 min)
- the asymptotic behaviour of these recurrences can be studied

Problem 2

Consider the integral

$$\int_C (x^2-1)^2 \tilde{P}_n(x) \tilde{Q}_n(x)^2 \left(\int (x^2-1)^2 \tilde{P}_n(x)^3 \, \mathrm{d}x \right) \, \mathrm{d}x$$

where $\tilde{Q}_n(x)$ is annihilated by the same operators as $\tilde{P}_n(x)$.

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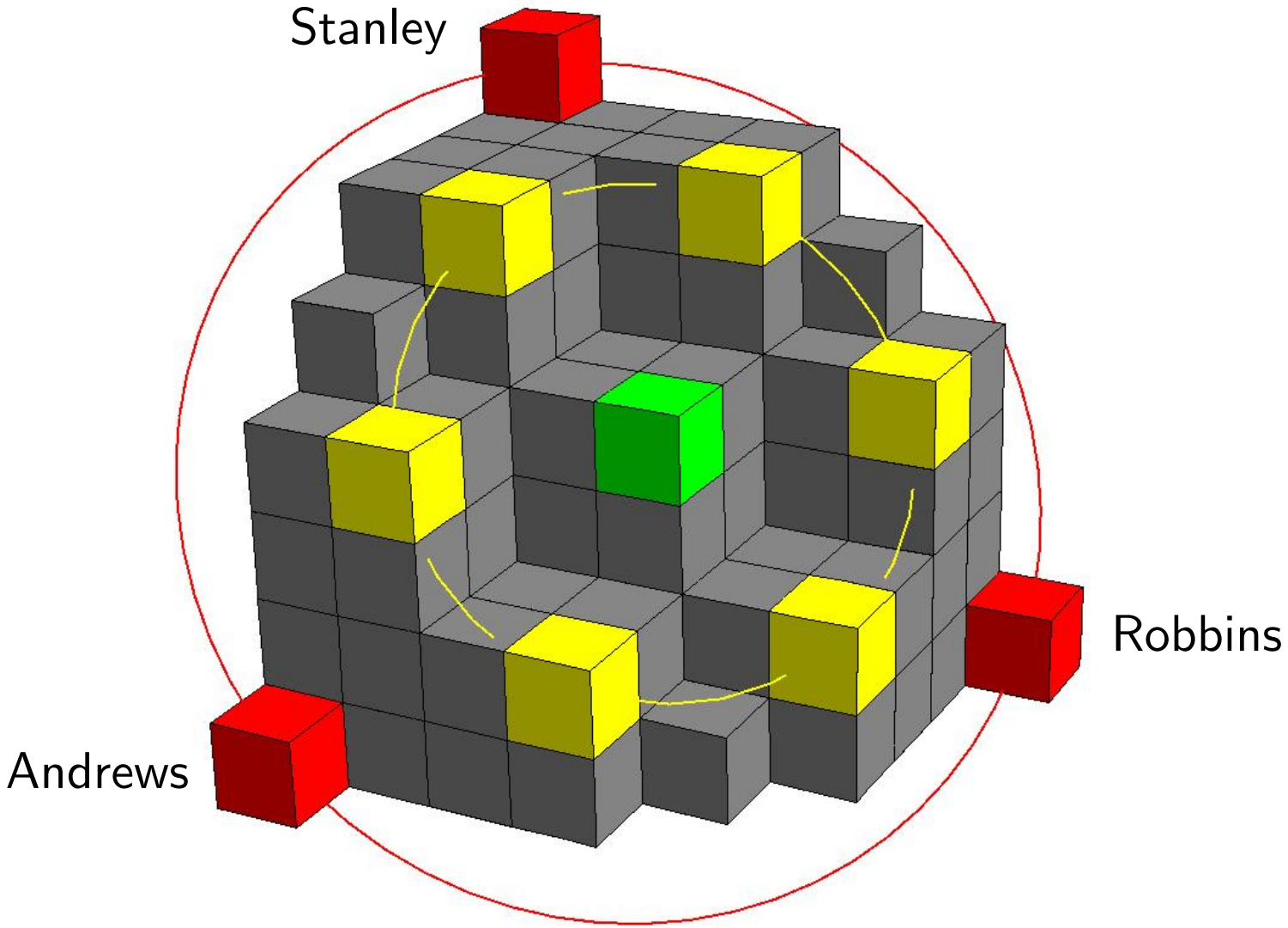
where $\tilde{Q}_n(x)$ is annihilated by the same operators as $\tilde{P}_n(x)$.

We computed the desired recurrence in n:

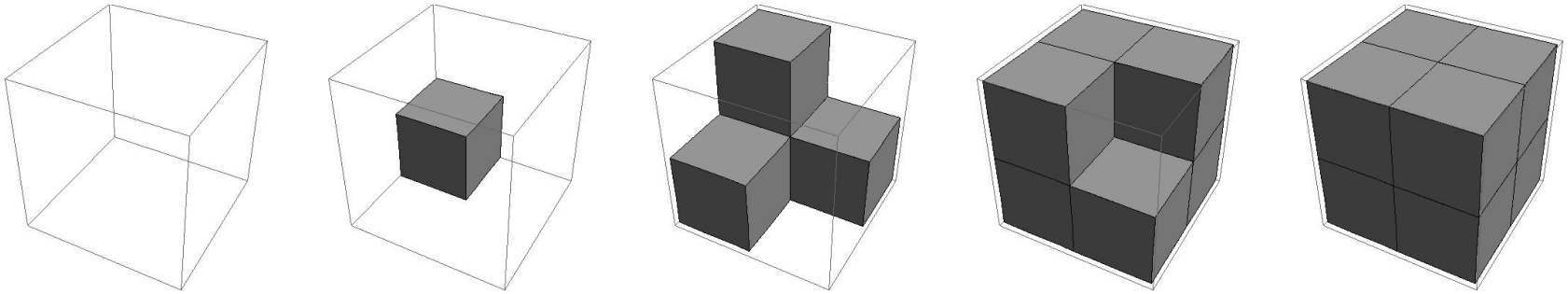
- 24 monomials under the stairs
- ansatz with 1310 unknowns
- total timing is about 50 hours

Application 5: Symmetric Towers

q-TSPP



Let $T(n)$ denote set of TSPPs with largest part at most n.



q^0 q^1 q^2 q^3 q^4

Andrews-Robbins q-TSPP conjecture:

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \le i \le j \le k \le n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

For $q = 1$:

$$|T(n)| = \prod_{1 \le i \le j \le k \le n} \frac{i+j+k-1}{i+j+k-2} \qquad \text{(Stembridge)}$$

The Determinant

Reduction by Soichi Okada:

The q-TSPP conjecture is true if

$$\det(a_{i,j})_{1\le i,j\le n} = \prod_{1\le i\le j\le k\le n} \left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 =: b_n$$

where

$$a_{i,j} := q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q\right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

The Holonomic Ansatz

Second reduction by Doron Zeilberger:

"Pull out of the hat" a discrete function $c_{n,j}$ and prove

$$\begin{aligned} c_{n,n} &= 1 && (n \geq 1), \\ \sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leq i < n), \\ \sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \geq 1). \end{aligned}$$

Then $\det(a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds.

The Result

. . . is about 7GB large (corresponding to some million printed pages).

A short version of this appeared in PNAS (Proceedings of the National Academy of Sciences of the USA):

Christoph Koutschan, Manuel Kauers, Doron Zeilberger:
A proof of George Andrews' and David Robbins' q-TSPP conjecture