Creative Telescoping

Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

RICAM, November 17, 2014
Special Semester on New Trends in Calculus of Variations
Geometric control and related fields

ÖAW
Austrian Academy of Sciences

RICAM
JOHANN · RADON · INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{Bad: no parameter!}$$

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Bad: no parameter!

$$\sum_{k=1}^{\infty} \frac{1}{k(k+n)}$$

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{Bad: no parameter!}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{\gamma + \psi(n)}{n}$$

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{Bad: no parameter!}$$

$$f_n := \sum_{k=1}^{\infty} \frac{1}{k(k+n)} \rightsquigarrow (n+2)^2 f_{n+2} = (n+1)(2n+3) f_{n+1} - n(n+1) f_n$$

What is Creative Telescoping?

Creative telescoping is a methodology

- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for such expressions
- that became popular in computer algebra in the past 25 years

Creative telescoping can be performed in different settings:

- difference fields (Carsten Schneider, RISC)
- differential fields (Clemens Raab, DESY $\to$ RICAM)
- holonomic functions (here)

Motivating Example

T. Combot, C. K.: Third order integrability conditions for homogeneous potentials of degree -1. Define:

$$P_n(t) = \frac{1}{t^2-1}\frac{\partial^{n-1}}{\partial t^{n-1}}(t^2-1)^n$$

$$Q_n(t) = \epsilon_n P_n(t)\,\mathrm{arctanh}\left(\frac{1}{t}\right) + \frac{W_n(t)}{t^2-1}$$

where W_n are polynomials and ϵ_n is a real sequence, given by

$$W_{2k}(t) = \frac{(-1)^k(t^2-1)}{2^{4k}}\left(\frac{\pi\ {}_2F_1\left(\frac{1}{2}-k, k+1, \frac{1}{2}, t^2\right)}{\Gamma\left(k+\frac{1}{2}\right)^2} + \frac{2kt(2k+1)\,\mathrm{arctanh}(t)\ {}_2F_1\left(1-k, k+\frac{3}{2}, \frac{3}{2}, t^2\right)}{(k!)^2}\right),$$

$$W_{2k+1}(t) = \ldots$$

$$\epsilon_n = \frac{n(n+1)}{4^n(n!)^2}.$$

Motivating Example

Task: Compute recurrences for the following quantities:

$$f_1(n) = \langle\alpha^3\rangle\, 2\epsilon_n^{-2}\,\mathrm{Res}_{t=\infty}\Bigg((t^2-1)^2(Q_n+\epsilon_n\alpha P_n)^2(Q_2+\epsilon_2\alpha P_2) \times \int (t^2-1)^2(Q_n+\epsilon_n\alpha P_n)^2 P_2\,\mathrm{d}t\Bigg),$$

$$f_2(n) = \langle\alpha^3\rangle\, 2\epsilon_n^{-2}\,\mathrm{Res}_{t=\infty}\Bigg((t^2-1)^2(Q_n+\epsilon_n\alpha P_n)^3 \times \int (t^2-1)^2(Q_n+\epsilon_n\alpha P_n)^2 P_n\,\mathrm{d}t\Bigg),$$

$$f_3(n) = \langle\alpha^3\rangle\, \frac{1}{6}\epsilon_n^{-2}\,\mathrm{Res}_{t=\infty}\left((t^2-1)^3(Q_n+\epsilon_n\alpha P_n)^4\right).$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) := \sum_{k=a}^{b} f(n,k)$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) := \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$.

Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) := \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$.

Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

Creative Telescoping: write

$$c_d(n)f(n+d,k) + \cdots + c_0(n)f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_d(n)F(n+d) + \cdots + c_0(n)F(n) = g(n,b+1) - g(n,a).$$

Creative Telescoping

Method for doing integrals and sums (aka Feynman's differentiating under the integral sign)

Consider the following integration problem: $F(x) := \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: write $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right)\mathrm{d}y \qquad = g(x,b) - g(x,a)$.

Creative Telescoping: write

$$c_d(x)\frac{\mathrm{d}^d}{\mathrm{d}x^d} f(x,y) + \cdots + c_0(x) f(x,y) \;= \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_d(x)\frac{\mathrm{d}^d}{\mathrm{d}x^d} F(x) + \cdots + c_0(x) F(x) \;= g(x,b) - g(x,a)$$

D-finite and P-recursive

Definition: A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[x]$ not all zero.

D-finite and P-recursive

Definition: A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[x]$ not all zero.

Definition: A sequence $f(n)$ is called **P-recursive** (or **P-finite**) if it satisfies a linear recurrence equation with polynomial coefficients:

$$p_d(n)f(n+d) + \cdots + p_1(n)f(n+1) + p_0(n)f(n) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[n]$ not all zero.

D-finite and P-recursive

Definition: A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[x]$ not all zero.

Definition: A sequence $f(n)$ is called **P-recursive** (or **P-finite**) if it satisfies a linear recurrence equation with polynomial coefficients:

$$p_d(n)f(n+d) + \cdots + p_1(n)f(n+1) + p_0(n)f(n) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[n]$ not all zero.

$\longrightarrow$ In both cases, only finitely many initial conditions!

D-finite and P-recursive

Definition: A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[x]$ not all zero.

Definition: A sequence $f(n)$ is called **P-recursive** (or **P-finite**) if it satisfies a linear recurrence equation with polynomial coefficients:

$$p_d(n)f(n+d) + \cdots + p_1(n)f(n+1) + p_0(n)f(n) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[n]$ not all zero.

$\longrightarrow$ In both cases, only finitely many initial conditions!
$\longrightarrow$ Also called **holonomic function** resp. **holonomic sequence**.

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geq 2)$$

with initial conditions $H_0 = 0$ and $H_1 = 1$.

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geq 2)$$

with initial conditions $H_0 = 0$ and $H_1 = 1$.

Can express any shift as $\mathbb{K}(n)$-linear combination of H_n and H_{n+1}:

$$H_{n+4}$$

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad\qquad (n \geq 2)$$

with initial conditions $H_0 = 0$ and $H_1 = 1$.

Can express any shift as $\mathbb{K}(n)$-linear combination of H_n and H_{n+1}:

$$H_{n+4} = \tfrac{2n+7}{n+4} H_{n+3} - \tfrac{n+3}{n+4} H_{n+2}$$

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geq 2)$$

with initial conditions $H_0 = 0$ and $H_1 = 1$.

Can express any shift as $\mathbb{K}(n)$-linear combination of H_n and H_{n+1}:

$$\begin{aligned} H_{n+4} &= \tfrac{2n+7}{n+4} H_{n+3} - \tfrac{n+3}{n+4} H_{n+2} \\ &= \tfrac{2n+7}{n+4}\left(\tfrac{2n+5}{n+3} H_{n+2} - \tfrac{n+2}{n+3} H_{n+1}\right) - \tfrac{n+3}{n+4} H_{n+2} \end{aligned}$$

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geq 2)$$

with initial conditions $H_0 = 0$ and $H_1 = 1$.

Can express any shift as $\mathbb{K}(n)$-linear combination of H_n and H_{n+1}:

$$\begin{aligned} H_{n+4} &= \tfrac{2n+7}{n+4} H_{n+3} - \tfrac{n+3}{n+4} H_{n+2} \\ &= \tfrac{2n+7}{n+4}\left(\tfrac{2n+5}{n+3} H_{n+2} - \tfrac{n+2}{n+3} H_{n+1}\right) - \tfrac{n+3}{n+4} H_{n+2} \\ &= \tfrac{3n^2+18n+26}{(n+3)(n+4)} H_{n+2} - \tfrac{(2n+7)(n+2)}{(n+3)(n+4)} H_{n+1} \end{aligned}$$

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geq 2)$$

with initial conditions $H_0 = 0$ and $H_1 = 1$.

Can express any shift as $\mathbb{K}(n)$-linear combination of H_n and H_{n+1}:

$$\begin{aligned}
H_{n+4} &= \tfrac{2n+7}{n+4} H_{n+3} - \tfrac{n+3}{n+4} H_{n+2} \\
&= \tfrac{2n+7}{n+4} \left(\tfrac{2n+5}{n+3} H_{n+2} - \tfrac{n+2}{n+3} H_{n+1} \right) - \tfrac{n+3}{n+4} H_{n+2} \\
&= \tfrac{3n^2+18n+26}{(n+3)(n+4)} H_{n+2} - \tfrac{(2n+7)(n+2)}{(n+3)(n+4)} H_{n+1} \\
&= \tfrac{3n^2+18n+26}{(n+3)(n+4)} \left(\tfrac{2n+3}{n+2} H_{n+1} - \tfrac{n+1}{n+2} H_n \right) - \tfrac{(2n+7)(n+2)}{(n+3)(n+4)} H_{n+1}
\end{aligned}$$

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geq 2)$$

with initial conditions $H_0 = 0$ and $H_1 = 1$.

Can express any shift as $\mathbb{K}(n)$-linear combination of H_n and H_{n+1}:

$$\begin{aligned}
H_{n+4} &= \tfrac{2n+7}{n+4} H_{n+3} - \tfrac{n+3}{n+4} H_{n+2} \\
&= \tfrac{2n+7}{n+4} \left(\tfrac{2n+5}{n+3} H_{n+2} - \tfrac{n+2}{n+3} H_{n+1} \right) - \tfrac{n+3}{n+4} H_{n+2} \\
&= \tfrac{3n^2+18n+26}{(n+3)(n+4)} H_{n+2} - \tfrac{(2n+7)(n+2)}{(n+3)(n+4)} H_{n+1} \\
&= \tfrac{3n^2+18n+26}{(n+3)(n+4)} \left(\tfrac{2n+3}{n+2} H_{n+1} - \tfrac{n+1}{n+2} H_n \right) - \tfrac{(2n+7)(n+2)}{(n+3)(n+4)} H_{n+1} \\
&= \tfrac{2(2n+5)(n^2+5n+5)}{(n+2)(n+3)(n+4)} H_{n+1} - \tfrac{(n+1)(3n^2+18n+26)}{(n+2)(n+3)(n+4)} H_n
\end{aligned}$$

Closure Properties

If $f(x)$ and $g(x)$ are D-finite then also the following are D-finite

- $f(x) + g(x)$
- $f(x) \cdot g(x)$
- $f(a(x))$ if $a(x)$ is algebraic

Closure Properties

If $f(x)$ and $g(x)$ are D-finite then also the following are D-finite

- $f(x) + g(x)$
- $f(x) \cdot g(x)$
- $f(a(x))$ if $a(x)$ is algebraic

If $f(n)$ and $g(n)$ are P-recursive then also the following are P-recursive

- $f(n) + g(n)$
- $f(n) \cdot g(n)$
- $f(an + b)$ for integers a and b

Closure Properties

If $f(x)$ and $g(x)$ are D-finite then also the following are D-finite

- $f(x) + g(x)$
- $f(x) \cdot g(x)$
- $f(a(x))$ if $a(x)$ is algebraic

If $f(n)$ and $g(n)$ are P-recursive then also the following are P-recursive

- $f(n) + g(n)$
- $f(n) \cdot g(n)$
- $f(an + b)$ for integers a and b

A sequence is P-recursive iff its generating function is D-finite.

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.
Ansatz: want to find $c_0, \ldots, c_d \in \mathbb{K}[n]$ such that

$$0 = c_d(n)h(n+d) + \cdots + c_0(n)h(n)$$

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.
Ansatz: want to find $c_0, \ldots, c_d \in \mathbb{K}[n]$ such that

$$\begin{aligned} 0 &= c_d(n)h(n+d) + \cdots + c_0(n)h(n) \\ &= c_d(n)f(n+d)g(n+d) + \cdots + c_0(n)f(n)g(n) \end{aligned}$$

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.
Ansatz: want to find $c_0, \dots, c_d \in \mathbb{K}[n]$ such that

$$
\begin{aligned}
0 &= c_d(n)h(n+d) + \cdots + c_0(n)h(n) \\
&= c_d(n)f(n+d)g(n+d) + \cdots + c_0(n)f(n)g(n) \\
&= c_d(n)\big(f_{d,d_1-1}f(n+d_1-1) + \cdots + f_{d,0}f(n)\big) \\
&\quad \times \big(g_{d,d_2-1}g(n+d_2-1) + \cdots + g_{d,0}g(n)\big) + \cdots + c_0(n)f(n)g(n)
\end{aligned}
$$

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.
Ansatz: want to find $c_0, \ldots, c_d \in \mathbb{K}[n]$ such that

$$\begin{aligned}
0 &= c_d(n)h(n+d) + \cdots + c_0(n)h(n) \\
&= c_d(n)f(n+d)g(n+d) + \cdots + c_0(n)f(n)g(n) \\
&= c_d(n)\big(f_{d,d_1-1}f(n+d_1-1) + \cdots + f_{d,0}f(n)\big) \\
&\quad \times \big(g_{d,d_2-1}g(n+d_2-1) + \cdots + g_{d,0}g(n)\big) + \cdots + c_0(n)f(n)g(n) \\
&= \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} r_{i,j}(c_0, \ldots, c_d, n) f(n+i)g(n+j)
\end{aligned}$$

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.
Ansatz: want to find $c_0, \ldots, c_d \in \mathbb{K}[n]$ such that

$$\begin{aligned}
0 &= c_d(n)h(n+d) + \cdots + c_0(n)h(n) \\
&= c_d(n)f(n+d)g(n+d) + \cdots + c_0(n)f(n)g(n) \\
&= c_d(n)\big(f_{d,d_1-1}f(n+d_1-1) + \cdots + f_{d,0}f(n)\big) \\
&\quad \times \big(g_{d,d_2-1}g(n+d_2-1) + \cdots + g_{d,0}g(n)\big) + \cdots + c_0(n)f(n)g(n) \\
&= \sum_{i=0}^{d_1-1}\sum_{j=0}^{d_2-1} r_{i,j}(c_0, \ldots, c_d, n)f(n+i)g(n+j)
\end{aligned}$$

All coefficients $r_{i,j}$ must vanish: this yields $d_1 d_2$ equations for the unknowns $c_0, \ldots, c_d$. The choice $d = d_1 d_2$ ensures a solution.

Multivariate Generalization

Try to generalize the finiteness property to

- multivariate functions $f(x_1, \ldots, x_s)$
  (the x_i are called **continuous variables**)

Multivariate Generalization

Try to generalize the finiteness property to

- multivariate functions $f(x_1, \ldots, x_s)$
  (the x_i are called **continuous variables**)
- multidimensional sequences $f(n_1, \ldots, n_s)$
  (the n_i are called **discrete variables**)

Multivariate Generalization

Try to generalize the finiteness property to

- multivariate functions $f(x_1, \ldots, x_s)$
  (the x_i are called **continuous variables**)
- multidimensional sequences $f(n_1, \ldots, n_s)$
  (the n_i are called **discrete variables**)
- mixed setting: functions in several continuous and discrete variables $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2 - 1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$

Example: Legendre Polynomials $P_n(x)$

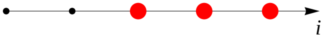
This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$

$$-\frac{6x}{x^2-1}P_n^{(3)}(x) + \frac{(n-2)(n+3)}{x^2-1}P_n''(x)$$



$$(x^2-1)P_n^{(4)}(x) + 6xP_n^{(3)}(x) - (n-2)(n+3)P_n''(x) = 0$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2 - 1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$

$$-\tfrac{6x}{x^2-1}P_n^{(3)}(x) + \tfrac{(n-2)(n+3)}{x^2-1}P_n''(x)$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$
$$-\frac{6x}{x^2-1}P_n^{(3)}(x) + \frac{(n-2)(n+3)}{x^2-1}P_n''(x)$$

$$(x^2-1)P_n^{(3)}(x) + 4xP_n''(x) - (n-1)(n+2)P_n'(x) = 0$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) = \frac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x) - \frac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$

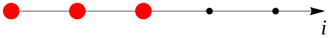
Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) =$$
$$\frac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x)$$
$$-\frac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$



$$(x^2-1)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0$$

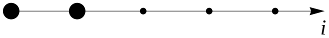
Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0.$$

Consider the set $\left\{P_n^{(i)}(x) : i \geq 0\right\}$.

$$P_n^{(4)}(x) = -\frac{8x(n^2x^2-n^2+nx^2-n+3x^2+3)}{(x^2-1)^3}P_n'(x) + \frac{n(n+1)(n^2x^2-n^2+nx^2-n+18x^2+6)}{(x^2-1)^3}P_n(x)$$



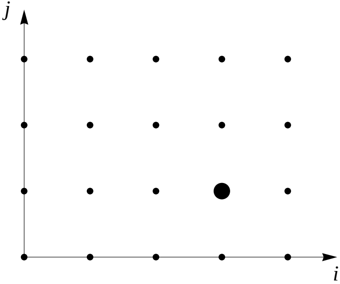
$\longrightarrow$ $P_n(x)$ is D-finite w.r.t. x.

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2 - 1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



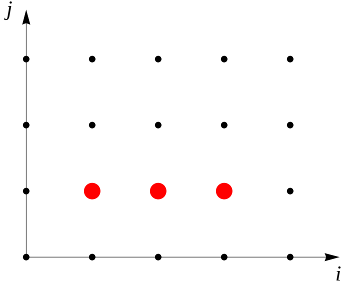
$$P_{n+1}^{(3)}(x) =$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+1}^{(3)}(x) =$$

$$-\tfrac{4x}{x^2-1}P_{n+1}''(x) + \tfrac{n(n+3)}{x^2-1}P_{n+1}'(x)$$

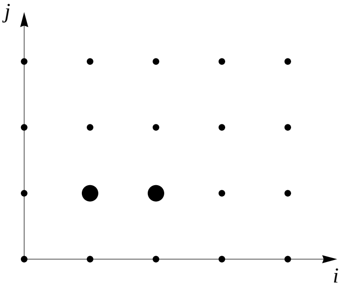
$$(x^2-1)P_{n+1}^{(3)}(x) + 4xP_{n+1}''(x) - n(n+3)P_{n+1}'(x) = 0$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+1}^{(3)}(x) =$$

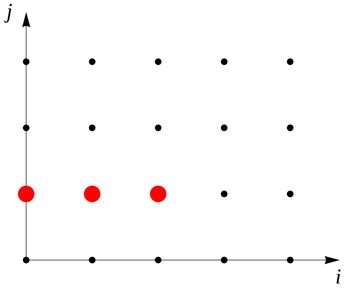
$$-\tfrac{4x}{x^2-1}P_{n+1}''(x) + \tfrac{n(n+3)}{x^2-1}P_{n+1}'(x)$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+1}^{(3)}(x) =$$

$$-\frac{4x}{x^2-1}P_{n+1}''(x) + \frac{n(n+3)}{x^2-1}P_{n+1}'(x)$$

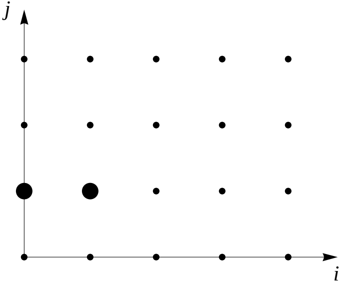
$$(x^2-1)P_{n+1}''(x) + 2xP_{n+1}'(x) - (n+1)(n+2)P_{n+1}(x) = 0$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



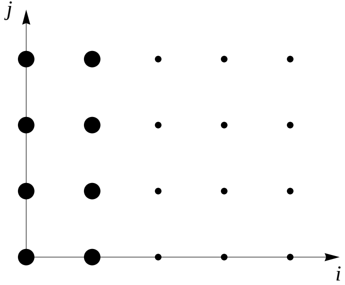
$$P_{n+1}^{(3)}(x) = \frac{(n^2x^2-n^2+3nx^2-3n+8x^2)}{(x^2-1)^2}P_{n+1}'(x) - \frac{4(n^2x+3nx+2x)}{(x^2-1)^2}P_{n+1}(x)$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.

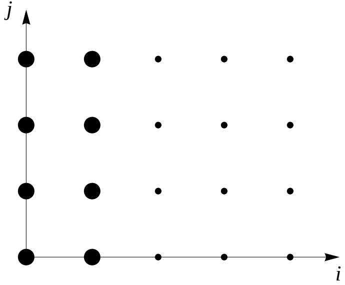


Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.

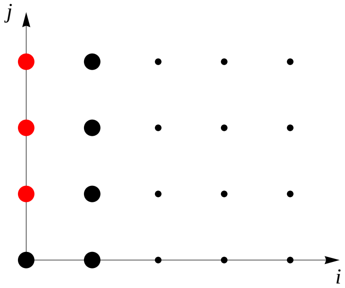


Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$P_{n+3}(x) =$

$\frac{(2n+5)x}{n+3} P_{n+2}(x) - \frac{n+2}{n+3} P_{n+1}(x)$

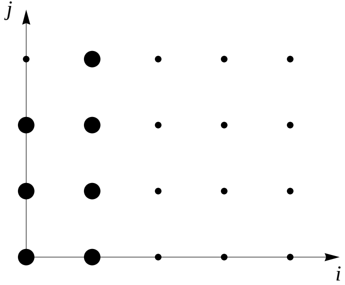
$$(n+3)P_{n+3}(x) - (2n+5)xP_{n+2}(x) + (n+2)P_{n+1}(x) = 0$$

Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P^{(i)}_{n+j}(x) : i, j \geq 0\right\}$.



$P_{n+3}(x) =$

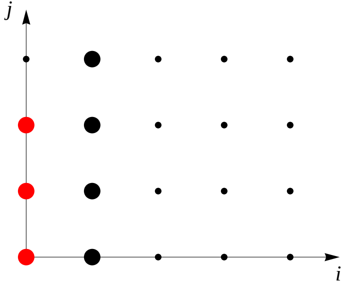
$\frac{(2n+5)x}{n+3} P_{n+2}(x) - \frac{n+2}{n+3} P_{n+1}(x)$

Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$$P_{n+3}(x) = \frac{4n^2x^2 - n^2 + 16nx^2 - 4n + 15x^2 - 4}{(n+2)(n+3)} P_{n+1}(x) - \frac{2n^2x + 7nx + 5x}{(n+2)(n+3)} P_n(x)$$

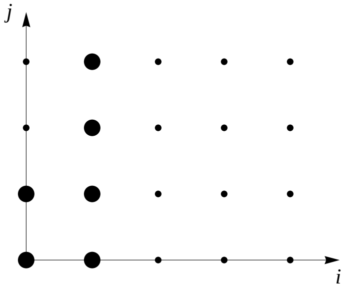
$$(n+2)P_{n+2}(x) - (2n+3)xP_{n+1}(x) + (n+1)P_n(x) = 0$$

Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P^{(i)}_{n+j}(x) : i, j \geq 0\right\}$.



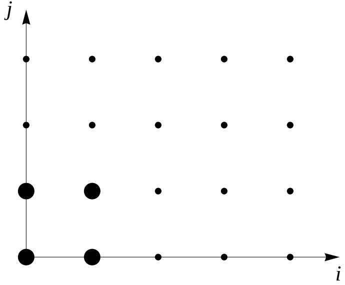
$$P_{n+3}(x) = \frac{4n^2x^2 - n^2 + 16nx^2 - 4n + 15x^2 - 4}{(n+2)(n+3)} P_{n+1}(x) - \frac{2n^2x + 7nx + 5x}{(n+2)(n+3)} P_n(x)$$

Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.

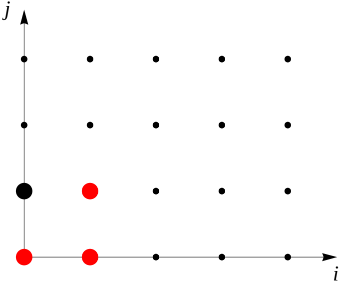


Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P^{(i)}_{n+j}(x) : i, j \geq 0\right\}$.



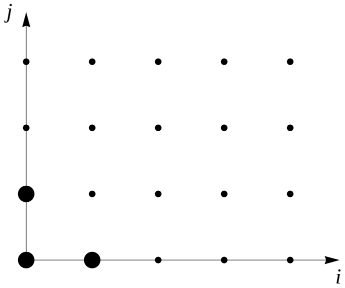
$P'_{n+1}(x) - xP'_n(x) - (n+1)P_n(x) = 0$

Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.

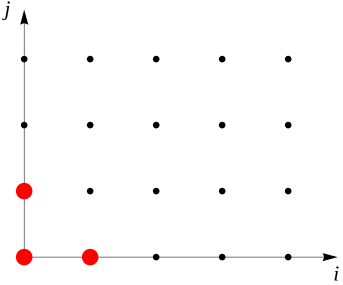


Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



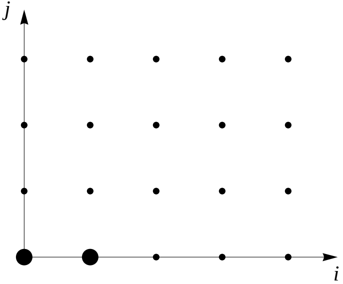
$(n+1)P_{n+1}(x) + (1-x^2)P_n'(x) - (n+1)xP_n(x) = 0$

Example: Legendre Polynomials $P_n(x)$

The Legendre polynomials can be defined recursively:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \end{aligned}$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geq 0\right\}$.



$\longrightarrow$ $P_n(x)$ is ∂-finite w.r.t. n and x (of rank 2).

∂-Finiteness

First attempt at a definition:

Let $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$ be a function in the continuous variables $x_1, \ldots, x_s$ and in the discrete variables $n_1, \ldots, n_r$. If there is a finite set of basis functions of the form

$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \ldots, x_s, n_1 + j_1, \ldots, n_r + j_r)$$

with $i_1, \ldots, i_s, j_1, \ldots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \ldots, x_s, n_1, \ldots, n_r)$-linear combination of the basis functions.

∂-Finiteness

First attempt at a definition:

Let $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$ be a function in the continuous variables $x_1, \ldots, x_s$ and in the discrete variables $n_1, \ldots, n_r$. If there is a finite set of basis functions of the form

$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \ldots, x_s, n_1 + j_1, \ldots, n_r + j_r)$$

with $i_1, \ldots, i_s, j_1, \ldots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \ldots, x_s, n_1, \ldots, n_r)$-linear combination of the basis functions.

Again, finitely many (modulo details) initial conditions suffice to fix f.

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
- q-shift operator $S_{v,q}$: $S_{v,q} f(v) = f(qv)$
- arbitrary operator ∂_v: any of the above

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
- q-shift operator $S_{v,q}$: $S_{v,q} f(v) = f(qv)$
- arbitrary operator ∂_v: any of the above

Example 1: The Legendre differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0$$

translates to the operator

$$(x^2-1)D_x^2 + 2xD_x - n(n+1).$$

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
- q-shift operator $S_{v,q}$: $S_{v,q} f(v) = f(qv)$
- arbitrary operator ∂_v: any of the above

Example 1: The Legendre differential equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0$$

translates to the operator

$$(x^2-1)D_x^2 + 2xD_x - n(n+1).$$

Example 2: The three-term recurrence

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

translates to the operator

$$(n+2)S_n^2 - (2n+3)xS_n + (n+1).$$

Operator Algebra

Differential equations and recurrences are translated to skew polynomials.

Noncommutative multiplication:

$$D_x x = x D_x + 1, \quad S_n n = n S_n + S_n, \quad \text{etc.}$$

Operator Algebra

Differential equations and recurrences are translated to skew polynomials.

Noncommutative multiplication:

$$D_x x = x D_x + 1, \quad S_n n = n S_n + S_n, \quad \text{etc.}$$

More general:

$$D_x a(x) = a(x) D_x + a'(x) \quad S_n a(n) = a(n+1) S_n, \quad \text{etc.}$$

Operator Algebra

Differential equations and recurrences are translated to skew polynomials.

Noncommutative multiplication:

$$D_x x = x D_x + 1, \quad S_n n = n S_n + S_n, \quad \text{etc.}$$

More general:

$$D_x a(x) = a(x) D_x + a'(x) \quad S_n a(n) = a(n+1) S_n, \quad \text{etc.}$$

Even more general:

$$\partial_v a = \sigma(a) \partial_v + \delta(a)$$

where σ is an automorphism and δ a σ-derivation, i.e.,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

Operator Algebra

Such operators form an **Ore algebra**

$$\mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle,$$

i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $v, w, \dots$, where $\mathbb{K}$ is a computable field of characteristic 0 (i.e., containing $\mathbb{Q}$).

Operator Algebra

Such operators form an **Ore algebra**

$$\mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle,$$

i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $v, w, \dots$, where $\mathbb{K}$ is a computable field of characteristic 0 (i.e., containing $\mathbb{Q}$).

In fact, the above notation is a shortcut for

$$\mathbb{K}(v, w, \dots)[\partial_v; \sigma_v, \delta_v][\partial_w; \sigma_w, \delta_w] \cdots$$

Operator Algebra

Such operators form an **Ore algebra**

$$\mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle,$$

i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $v, w, \dots$, where $\mathbb{K}$ is a computable field of characteristic 0 (i.e., containing $\mathbb{Q}$).

In fact, the above notation is a shortcut for

$$\mathbb{K}(v, w, \dots)[\partial_v; \sigma_v, \delta_v][\partial_w; \sigma_w, \delta_w] \cdots$$

Example: The operators that we encountered with the Legendre polynomials live in the Ore algebra

$$\mathbb{K}(x, n)\langle D_x, S_n \rangle = \mathbb{K}(x, n)[D_x; 1, \tfrac{\mathrm{d}}{\mathrm{d}x}][S_n; \sigma_n, 0].$$

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P \cdot f = 0\}$$

is called the **annihilator** of the function f.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P \cdot f = 0\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two linear equations for a function f, given by the operators P and Q, i.e., $P \cdot f = Q \cdot f = 0$.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \left\{ P \in \mathbb{O} : P \cdot f = 0 \right\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two linear equations for a function f, given by the operators P and Q, i.e., $P \cdot f = Q \cdot f = 0$. Then

- $\mathrm{Eq}_1 + \mathrm{Eq}_2$
- $r \cdot \mathrm{Eq}_1$
- $\frac{\mathrm{d}}{\mathrm{d}v}\mathrm{Eq}_1$
- $\mathrm{Eq}_1|_{v \to v+1}$

are also valid equations for f.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \left\{P \in \mathbb{O} : P \cdot f = 0\right\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two linear equations for a function f, given by the operators P and Q, i.e., $P \cdot f = Q \cdot f = 0$. Then

- $\mathrm{Eq}_1 + \mathrm{Eq}_2$ corresponds to $P + Q$
- $r \cdot \mathrm{Eq}_1$ corresponds to $r \cdot P$
- $\frac{\mathrm{d}}{\mathrm{d}v}\mathrm{Eq}_1$ corresponds to $D_v \cdot P$
- $\mathrm{Eq}_1|_{v \to v+1}$ corresponds to $S_v \cdot P$

are also valid equations for f.

Annihilating Ideals

Let now $\mathbb{O}$ be such an Ore algebra. The set

$$\operatorname{Ann}_{\mathbb{O}} f := \{P \in \mathbb{O} : P \cdot f = 0\}$$

is called the **annihilator** of the function f.

Let Eq_1 and Eq_2 be two linear equations for a function f, given by the operators P and Q, i.e., $P \cdot f = Q \cdot f = 0$. Then

- $\mathrm{Eq}_1 + \mathrm{Eq}_2$ corresponds to $P + Q$
- $r \cdot \mathrm{Eq}_1$ corresponds to $r \cdot P$
- $\frac{\mathrm{d}}{\mathrm{d}v}\mathrm{Eq}_1$ corresponds to $D_v \cdot P$
- $\mathrm{Eq}_1|_{v \to v+1}$ corresponds to $S_v \cdot P$

are also valid equations for f. More generally,

$$P, Q \in \operatorname{Ann}_{\mathbb{O}} f \implies P + Q \in \operatorname{Ann}_{\mathbb{O}} f$$
$$L \in \mathbb{O}, P \in \operatorname{Ann}_{\mathbb{O}} f \implies L\,P \in \operatorname{Ann}_{\mathbb{O}} f$$

which means that $\operatorname{Ann}_{\mathbb{O}} f$ is a **left ideal** in $\mathbb{O}$.

Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle\partial_v, \partial_w, \dots\rangle$ be an Ore algebra.

A function $f(v, w, \dots)$ is ∂-finite w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space:

$$\dim_{\mathbb{K}(v,w,\dots)} \left(\mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)\right) < \infty.$$

Definition: ∂-Finite Function

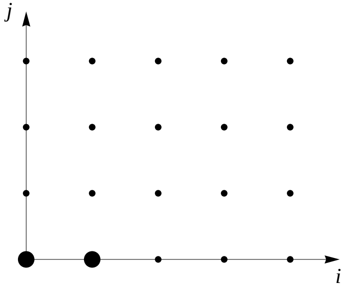
Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function $f(v, w, \dots)$ is ∂-finite w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space:

$$\dim_{\mathbb{K}(v,w,\dots)} \left(\mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)\right) < \infty.$$



Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function $f(v, w, \dots)$ is ∂-finite w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space:

$$\dim_{\mathbb{K}(v,w,\dots)} \left(\mathbb{O} / \operatorname{Ann}_{\mathbb{O}}(f)\right) < \infty.$$

In other words, if the left ideal of annihilating operators of f

$$\operatorname{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} : P \cdot f = 0\}$$

is a zero-dimensional ideal.

Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function $f(v, w, \dots)$ is ∂-finite w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space:

$$\dim_{\mathbb{K}(v,w,\dots)} \left(\mathbb{O} / \operatorname{Ann}_{\mathbb{O}}(f)\right) < \infty.$$

In other words, if the left ideal of annihilating operators of f

$$\operatorname{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} : P \cdot f = 0\}$$

is a zero-dimensional ideal.

We use (noncommutative) **Gröbner Bases** for certain operations on left ideals, e.g., for normal form computation modulo an ideal.

Why ∂-Finite Functions?

1. Definition gives rise to a finite data structure.
    - annihilating ideal of operators (Gröbner basis is finite)
    - finitely many initial values

Why ∂-Finite Functions?

1. Definition gives rise to a finite data structure.
   - annihilating ideal of operators (Gröbner basis is finite)
   - finitely many initial values
2. This set of functions is closed under many operations.
   - addition, e.g., $x^n + P_n(x)$
   - multiplication, e.g., $P_n(x)P_{n+1}(x)$
   - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
   - operator application, e.g., $P'_{n+2}(x)$

Why ∂-Finite Functions?

1. Definition gives rise to a finite data structure.
   - annihilating ideal of operators (Gröbner basis is finite)
   - finitely many initial values
2. This set of functions is closed under many operations.
   - addition, e.g., $x^n + P_n(x)$
   - multiplication, e.g., $P_n(x)P_{n+1}(x)$
   - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
   - operator application, e.g., $P'_{n+2}(x)$
3. These operations (closure properties) can be executed algorithmically.

Why ∂-Finite Functions?

1. Definition gives rise to a finite data structure.
   - annihilating ideal of operators (Gröbner basis is finite)
   - finitely many initial values
2. This set of functions is closed under many operations.
   - addition, e.g., $x^n + P_n(x)$
   - multiplication, e.g., $P_n(x)P_{n+1}(x)$
   - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
   - operator application, e.g., $P'_{n+2}(x)$
3. These operations (closure properties) can be executed algorithmically.
4. Many elementary and special functions are covered.

(Incomplete) List of ∂-Finite Functions

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, BesselI, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, BesselJ, StruveL, ArcSec, Factorial2, KelvinBer, BesselK, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin, SphericalBesselY, WhittakerW, Pochhammer

That is nice, but we want (and can) do more...

What about integrals

$$\int_a^b f(x, \dots)\,\mathrm{d}x$$

and sums

$$\sum_{n=a}^{b} f(n, \dots)$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$.

Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k) \big) = g(n,b+1) - g(n,a)$.

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$.

Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k) \big) = g(n,b+1) - g(n,a)$.

Creative Telescoping: write

$$c_d(n) f(n+d,k) + \cdots + c_0(n) f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_d(n) F(n+d) + \cdots + c_0(n) F(n) = g(n,b+1) - g(n,a).$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following integration problem: $F(x) = \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: write $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right)\mathrm{d}y \qquad = g(x,b) - g(x,a)$.

Creative Telescoping: write

$$c_d(x)\frac{\mathrm{d}^d}{\mathrm{d}x^d} f(x,y) + \cdots + c_0(x) f(x,y) \;= \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_d(x)\frac{\mathrm{d}^d}{\mathrm{d}x^d} F(x) + \cdots + c_0(x) F(x) \;= g(x,b) - g(x,a)$$

Creative Telescoping, $\mathbb{O} = \mathbb{K}(n,k)\langle S_n, S_k\rangle$

$$\begin{aligned} c_d(n)f(n+d,k) + \cdots + c_0(n)f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1)\cdot g(n,k). \end{aligned}$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_d(n)f(n+d,i) + \cdots + c_0(n)f(n,i)\big)$$

Creative Telescoping, $\mathbb{O} = \mathbb{K}(n,k)\langle S_n, S_k \rangle$

$$\begin{aligned} c_d(n) f(n+d,k) + \cdots + c_0(n) f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1) \cdot g(n,k). \end{aligned}$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_d(n) f(n+d,i) + \cdots + c_0(n) f(n,i) \big)$$

A reasonable choice for where to look for g is $\mathbb{O} \cdot f$.

Then the task is to find $P(n, S_n) = c_d(n) S_n^d + \cdots + c_0(n)$ and $Q \in \mathbb{O}$ such that

$$\big(P - (S_k - 1)Q\big) \cdot f = 0 \quad \iff \quad P - (S_k - 1)Q \in \operatorname{Ann}_{\mathbb{O}}(f).$$

Creative Telescoping (Example 1)

Let $F(n)$ denote the double sum over the trinomial coefficients

$$F(n) = \sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i, j, n-i-j} = \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{n!}{i!j!(n-i-j)!}.$$

Creative Telescoping (Example 1)

Let $F(n)$ denote the double sum over the trinomial coefficients

$$F(n) = \sum_{j=0}^{n}\sum_{i=0}^{n}\binom{n}{i, j, n-i-j} = \sum_{j=0}^{n}\sum_{i=0}^{n}\frac{n!}{i!j!(n-i-j)!}.$$

Then the creative telescoping operator

$$CT = S_n - 3 + (S_i - 1)\frac{i}{n-i-j+1} + (S_j - 1)\frac{j}{n-i-j+1}$$

with $CT\left(\binom{n}{i,j,n-i-j}\right) = 0$ implies that

$$F(n+1) = 3F(n).$$

Creative Telescoping (Example 2)

The lattice Green's function of the square lattice is given by

$$P(z) = \int_0^1 \int_0^1 \frac{1}{(1-xyz)\sqrt{1-x^2}\sqrt{1-y^2}}\,\mathrm{d}x\,\mathrm{d}y.$$

Creative Telescoping (Example 2)

The lattice Green's function of the square lattice is given by

$$P(z) = \int_0^1 \int_0^1 \frac{1}{(1 - xyz)\sqrt{1 - x^2}\sqrt{1 - y^2}} \,\mathrm{d}x\,\mathrm{d}y.$$

The creative telescoping operator

$$(z^3 - z)D_z^2 + (3z^2 - 1)D_z + z + D_x \frac{y(1 - x^2)}{xyz - 1} + D_y \frac{yz(1 - y^2)}{xyz - 1}$$

that annihilates the integrand, certifies that $P(z)$ satisfies the differential equation

$$(z^3 - z)P''(z) + (3z^2 - 1)P'(z) + zP(z) = 0.$$

Creative Telescoping in Several Variables

In general, a creative telescoping operator has the form

$$P(\boldsymbol{v}, \boldsymbol{\partial_v}) + \Delta_1 Q_1(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w}) + \cdots + \Delta_m Q_m(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$$

where $\Delta_i = S_{w_i} - 1$ or $\Delta_i = D_{w_i}$ (depending on the problem).

Creative Telescoping in Several Variables

In general, a creative telescoping operator has the form

$$P(\boldsymbol{v}, \boldsymbol{\partial_v}) + \Delta_1 Q_1(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w}) + \cdots + \Delta_m Q_m(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$$

where $\Delta_i = S_{w_i} - 1$ or $\Delta_i = D_{w_i}$ (depending on the problem).

- Corresponds to an m-fold summation/integration problem.
- $\boldsymbol{w} = w_1, \ldots, w_m$ are the summation/integration variables.
- $\boldsymbol{v} = v_1, v_2, \ldots$ are the surviving parameters.
- $P(\boldsymbol{v}, \boldsymbol{\partial_v})$ is called the **telescoper**.
- The $Q_i(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$ are called the **certificates**: they certify the correctness of the telescoper.

Holonomy

Question: Does there always exist such a telescoping operator?

Holonomy

Question: Does there always exist such a telescoping operator?

Answer 1: No!

Holonomy

Question: Does there always exist such a telescoping operator?

Answer 1: No!

Answer 2: Yes!

Holonomy

Question: Does there always exist such a telescoping operator?

Answer 1: No! In general not for ∂-finite functions.

Answer 2: Yes!

Holonomy

Question: Does there always exist such a telescoping operator?

Answer 1: No! In general not for ∂-finite functions.

Answer 2: Yes! If additionally the function is holonomic.

Holonomy

Question: Does there always exist such a telescoping operator?

Answer 1: No! In general not for ∂-finite functions.

Answer 2: Yes! If additionally the function is holonomic.

Combine the two notions:

- Use ∂-finiteness for computations.
- Use holonomy for justifications.

Holonomic Functions

Assume that $f(x_1, \ldots, x_s)$ depends only on continuous variables. Consider the **Weyl algebra**

$$\mathbb{W} = \mathbb{K}[x_1, \ldots, x_s]\langle D_{x_1}, \ldots, D_{x_s}\rangle.$$

Then f is holonomic if the left ideal $\operatorname{Ann}_{\mathbb{W}}(f)$ has dimension s (which, by Bernstein's inequality, is the minimum possible).

Holonomic Functions

Assume that $f(x_1, \ldots, x_s)$ depends only on continuous variables. Consider the **Weyl algebra**

$$\mathbb{W} = \mathbb{K}[x_1, \ldots, x_s]\langle D_{x_1}, \ldots, D_{x_s} \rangle.$$

Then f is holonomic if the left ideal $\operatorname{Ann}_{\mathbb{W}}(f)$ has dimension s (which, by Bernstein's inequality, is the minimum possible).

Differently stated: f is holonomic if for any $(s-1)$-subset

$$E \subset \{x_1, \ldots, x_s, D_{x_1}, \ldots, D_{x_s}\}, \quad |E| = s - 1,$$

there exists a nonzero element in $\operatorname{Ann}_{\mathbb{W}}(f)$ that is free of all generators in E.

Holonomic Functions

Assume that $f(x_1, \ldots, x_s)$ depends only on continuous variables. Consider the **Weyl algebra**

$$\mathbb{W} = \mathbb{K}[x_1, \ldots, x_s]\langle D_{x_1}, \ldots, D_{x_s} \rangle.$$

Then f is holonomic if the left ideal $\operatorname{Ann}_{\mathbb{W}}(f)$ has dimension s (which, by Bernstein's inequality, is the minimum possible).

Differently stated: f is holonomic if for any $(s-1)$-subset

$$E \subset \big\{x_1, \ldots, x_s, D_{x_1}, \ldots, D_{x_s}\big\}, \quad |E| = s - 1,$$

there exists a nonzero element in $\operatorname{Ann}_{\mathbb{W}}(f)$ that is free of all generators in E.

$\longrightarrow$ This is why a creative telescoping operator always exists.

∂-Finite and Holonomic Functions

Theorem: The function $f(x_1, \ldots, x_s)$ is holonomic if and only if it is ∂-finite.

∂-Finite and Holonomic Functions

Theorem: The function $f(x_1, \ldots, x_s)$ is holonomic if and only if it is ∂-finite.

$\longrightarrow$ This equivalence holds only in the continuous case!

∂-Finite and Holonomic Functions

Theorem: The function $f(x_1, \ldots, x_s)$ is holonomic if and only if it is ∂-finite.

$\longrightarrow$ This equivalence holds only in the continuous case!

A sequence is defined to be holonomic if its (multivariate) generating function is holonomic.

∂-Finite and Holonomic Functions

Theorem: The function $f(x_1, \ldots, x_s)$ is holonomic if and only if it is ∂-finite.

$\longrightarrow$ This equivalence holds only in the continuous case!

A sequence is defined to be holonomic if its (multivariate) generating function is holonomic.

Example: The sequence $\frac{1}{n^2+k^2}$ is ∂-finite but not holonomic.

Creative Telescoping Algorithms

- **Zeilberger's slow algorithm:**
  Eliminate all summation/integration variables from the annihilator ideal. Can be done, e.g., with Gröbner bases.

Creative Telescoping Algorithms

- **Zeilberger's slow algorithm:**
  Eliminate all summation/integration variables from the annihilator ideal. Can be done, e.g., with Gröbner bases.
- **Takayama's algorithm:**
  Assuming natural boundaries. Mod out the certificate parts and then eliminate the variables using module Gröbner bases.

Creative Telescoping Algorithms

- **Zeilberger's slow algorithm:**
  Eliminate all summation/integration variables from the annihilator ideal. Can be done, e.g., with Gröbner bases.
- **Takayama's algorithm:**
  Assuming natural boundaries. Mod out the certificate parts and then eliminate the variables using module Gröbner bases.
- **Chyzak's algorithm:**
  Ansatz with undetermined coefficients of the telescoper and the certificate. No Gröbner bases, but solving a coupled first-order linear difference/differential system is required.

Creative Telescoping Algorithms

- **Zeilberger's slow algorithm:**
  Eliminate all summation/integration variables from the annihilator ideal. Can be done, e.g., with Gröbner bases.
- **Takayama's algorithm:**
  Assuming natural boundaries. Mod out the certificate parts and then eliminate the variables using module Gröbner bases.
- **Chyzak's algorithm:**
  Ansatz with undetermined coefficients of the telescoper and the certificate. No Gröbner bases, but solving a coupled first-order linear difference/differential system is required.
- **C.K.'s heuristic method:**
  Refined Chyzak-type ansatz. Yields a linear system for the undetermined coefficients.

Creative Telescoping Algorithms

- **Zeilberger's slow algorithm:**
  Eliminate all summation/integration variables from the annihilator ideal. Can be done, e.g., with Gröbner bases.
- **Takayama's algorithm:**
  Assuming natural boundaries. Mod out the certificate parts and then eliminate the variables using module Gröbner bases.
- **Chyzak's algorithm:**
  Ansatz with undetermined coefficients of the telescoper and the certificate. No Gröbner bases, but solving a coupled first-order linear difference/differential system is required.
- **C.K.'s heuristic method:**
  Refined Chyzak-type ansatz. Yields a linear system for the undetermined coefficients.
- **Algorithms using Hermite reduction:**
  Allow to compute the telescoper without the certificates. So far, only available for special cases, e.g., hyperexponential functions.

Principia Holonomica

1. Functions and sequences are represented by their annihilating left ideals (and initial values).

Principia Holonomica

1. Functions and sequences are represented by their annihilating left ideals (and initial values).

2. An annihilating ideal is given by its Gröbner basis (i.e., a finite set of generators that allows to decide ideal membership and equality of ideals).

Principia Holonomica

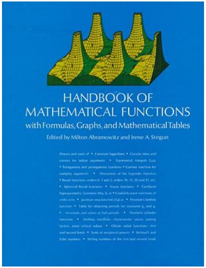
1. Functions and sequences are represented by their annihilating left ideals (and initial values).
2. An annihilating ideal is given by its Gröbner basis (i.e., a finite set of generators that allows to decide ideal membership and equality of ideals).
3. Integrals and sums are treated by the method of creative telescoping.

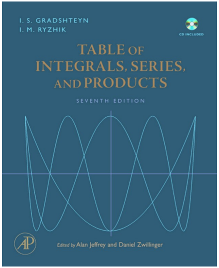
Principia Holonomica

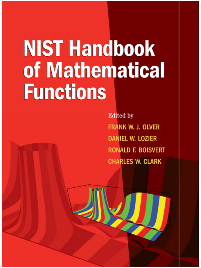
1. Functions and sequences are represented by their annihilating left ideals (and initial values).
2. An annihilating ideal is given by its Gröbner basis (i.e., a finite set of generators that allows to decide ideal membership and equality of ideals).
3. Integrals and sums are treated by the method of creative telescoping.
4. The output is always given as an annihilating ideal, not as a closed form.

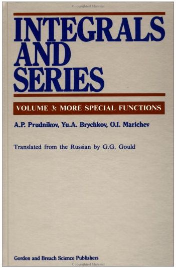
Standard Application

Special Function Identities









Some Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{1}$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x = \frac{\pi P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \tag{2}$$

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\mathrm{d}t \tag{3}$$

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!n!}\,\mathrm{d}x = \sqrt{\pi}e^{2rs} \tag{4}$$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu) J_{n+\nu}(a)}{2^{\nu-1}a^\nu n!\Gamma(\nu)} \tag{5}$$

$$\frac{\sin\left(\sqrt{z^2+2tz}\right)}{z} = \sum_{n=0}^{\infty} \frac{(-t)^n y_{n-1}(z)}{n!} \tag{6}$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

```
<< HolonomicFunctions.m
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\,\mathrm{d}t.$$

```
<< HolonomicFunctions.m
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),\\ 4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),\\ 2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*BesselJ[a,2*Sqrt[t*x]
                    Der[t], {S[a], S[n], Der[x]}]
```

$$\{\{-2S_n + 2xD_x + (a+2n+2),\\ 4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),\\ 2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},\\ \{-2t, -4tx, -2tx\}\}$$

$\longrightarrow$ The annihilating ideals agree; check a few initial values.