q-shift operators in knot theory

Christoph Koutschan
(joint work with Stavros Garoufalidis)

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

16 December 2014
Symbolic Analysis Workshop, FoCM, Montevideo

Overview

Knot Theory

- AJ Conjecture
  - **A**-polynomial
  - Colored **J**ones polynomial

Computer Algebra

- Guessing
- Symbolic Summation
  - Holonomic Systems Approach
  - Creative Telescoping
- Factorization of q-shift operators

Computer algebra matters for knot theory!

Basics of knot theory

Knot:

- embedding of the circle S^1 in S^3 (or in Euclidean space $\mathbb{R}^3$)
- "knotted (closed) string"
- oriented or non-oriented

Basics of knot theory

Knot:

- embedding of the circle S^1 in S^3 (or in Euclidean space $\mathbb{R}^3$)
- "knotted (closed) string"
- oriented or non-oriented

Equivalence of knots:

- equivalence relation: ambient isotopy
- "two knots are the same if they can be transformed into each other without cutting the string"

Basics of knot theory

Knot:

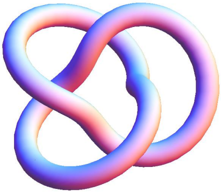
- embedding of the circle S^1 in S^3 (or in Euclidean space $\mathbb{R}^3$)
- "knotted (closed) string"
- oriented or non-oriented

Equivalence of knots:

- equivalence relation: ambient isotopy
- "two knots are the same if they can be transformed into each other without cutting the string"

Examples:

- unknot: $\bigcirc$
- trefoil knot 3_1:



Basics of knot theory

Fundamental problem:

Determine whether two descriptions (e.g., knot diagrams) represent the same knot.

Basics of knot theory

Fundamental problem:

Determine whether two descriptions (e.g., knot diagrams) represent the same knot.

Knot invariants:

- combinatorial invariants
- knot polynomials
- quantum invariants

Basics of knot theory

Fundamental problem:
Determine whether two descriptions (e.g., knot diagrams) represent the same knot.

Knot invariants:

- combinatorial invariants
- knot polynomials
- quantum invariants

Knot polynomials:

- Alexander polynomial (1928)
- Jones polynomial (1984)
- A-polynomial
- HOMFLY polynomial

The A-polynomial

The A-polynomial $A_K(M, L)$ of a knot K parametrizes the affine variety of $\mathrm{SL}(2, \mathbb{C})$ representations of the knot complement, viewed from the boundary torus:

- $M_K := S^3$ minus a tubular neighborhood of K ("knot complement")
- character variety: $X_{M_K} = \mathrm{Hom}(\pi_1(M_K), \mathrm{SL}(2, \mathbb{C}))$ (modulo conjugation)
- boundary: $X_{\partial(M_K)} = \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$
- consider the restriction map $\phi : X_{M_K} \to X_{\partial(M_K)}$
- its image is defined by a bivariate polynomial, $A_K(M, L)$
- difficult to compute (e.g., using elimination)
- even unknown for some knots with only 9 crossings.

Example: trefoil

A finite presentation of the fundamental group of the trefoil knot:

$$\pi_1(S^3 \setminus 3_1) = \langle a, b \mid aabbb \rangle$$

$\mathrm{SL}(2, \mathbb{C})$ representations:

$$a \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} =: A \qquad \text{(w.l.o.g.)}$$

$$b \to \begin{pmatrix} v & w \\ x & y \end{pmatrix} =: B \quad \text{with } \det B = 1$$

There are two distinguished elements in $\pi_1(S^3 \setminus K)$, the meridian μ and the longitude λ, which live on the boundary torus.

$$\mu = bab$$
$$\lambda = ba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}ab$$

Example: trefoil

Impose the following conditions:

$$\mathrm{tr}\left(\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} - \mathcal{M}\right) = \mathrm{tr}\left(\begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} - \Lambda\right) = 0$$

where

$$\mathcal{M} = BAB,$$
$$\Lambda = BA^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}AB^{-1}A^{-1}B^{-1}AB.$$

Example: trefoil

Impose the following conditions:

$$\mathrm{tr}\left(\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} - \mathcal{M}\right) = \mathrm{tr}\left(\begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} - \Lambda\right) = 0$$

where

$$\mathcal{M} = BAB,$$
$$\Lambda = BA^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}AB^{-1}A^{-1}B^{-1}AB.$$

Putting things together, we have to consider the ideal

$$\left\langle vy - wx - 1, AABBB - \mathrm{Id}_2, M + M^{-1} - \mathrm{tr}(\mathcal{M}), L + L^{-1} - \mathrm{tr}(\Lambda) \right\rangle$$

and intersect it with $\mathbb{Q}[M, L]$, e.g., by Gröbner basis elimination.

In this case, we obtain $A_{3_1}(M, L) = L + M^6$.

The Jones polynomial

Skein relation:

- a means to define/compute polynomial invariants
- three-term relation connecting the polynomials of knots which differ only locally:

The Jones polynomial

Skein relation:

- a means to define/compute polynomial invariants
- three-term relation connecting the polynomials of knots which differ only locally:

Definition. The skein relation for the Jones polynomial $J(K)$ is

$$q^{-1}J(L_+) - qJ(L_-) = (q^{1/2} - q^{-1/2})J(L_0)$$

where L_+, L_-, L_0 denote positive, negative, no crossing, resp.
Initial condition: $J(\bigcirc) = 1$.

The colored Jones function

The colored Jones function $J_{K,n}(q)$ of a knot K is a generalization of the classical Jones polynomial. It is a sequence of Laurent polynomials:

$$J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}}.$$

It can be defined using the n-th parallels of K:

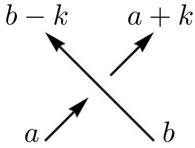
$$J_{K,n}(q) = \sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} J(K^{(k)})$$

where $J(K^{(k)})$ denotes the Jones polynomial of $K^{(k)}$, the k-th parallel of K.

The colored Jones function

Alternative definition via state sums using a diagram of K:

- label the m crossings with variables $\boldsymbol{k} = k_1, \dots, k_m$
- label the arcs: at a left-hand crossing k_i
  - add k_i to the label $a(\boldsymbol{k})$ of the underpass
  - subtract k_i from the label $b(\boldsymbol{k})$ of the overpass



- associate to each crossing k_i a proper q-hypergeometric expression R_i, depending locally on the labels:

$$R_i(n, \boldsymbol{k}) = q^{-n/2 - a(\boldsymbol{k})(n + k_i - b(\boldsymbol{k}))} \left(q^{a(\boldsymbol{k})-n}; q\right)_{k_i} \begin{bmatrix} b(\boldsymbol{k}) \\ k_i \end{bmatrix}_q$$

- the colored Jones function of K is given by an m-fold sum:

$$J_{K,n}(q) = \sum_{0 \le \boldsymbol{k} \le n} R_1 \cdots R_m$$

q-calculus

Recall some notation from q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

$$[n]! = \prod_{k=1}^{n} [k]$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}$$

q-calculus

Recall some notation from q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

$$[n] = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$$

$$[n]! = \prod_{k=1}^{n}[k]$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}$$

$\longrightarrow$ All these terms are (proper) q-hypergeometric:

$$f_n(q) \text{ is } q\text{-hg.} \iff \frac{f_{n+1}(q)}{f_n(q)} \in \mathbb{K}(q,q^n)$$

Wilf-Zeilberger theory

Theorem. ("fundamental theorem of WZ theory")
Every (multi-) sum over a proper q-hypergeometric term is q-holonomic.

Wilf-Zeilberger theory

Theorem. ("fundamental theorem of WZ theory")
Every (multi-) sum over a proper q-hypergeometric term is q-holonomic.

$\longrightarrow$ The colored Jones function is a q-holonomic sequence.

q-holonomic sequences

Notation.

- $\mathbb{K}$: field of characteristic zero
- q: indeterminate, transcendental over $\mathbb{K}$

Definition.
A univariate sequence $\big(f_n(q)\big)_{n\in\mathbb{N}}$ is called q-holonomic if it satisfies a nontrivial linear recurrence with coefficients that are polynomials in q and q^n:

$$\sum_{j=0}^{d} c_j(q, q^n) f_{n+j}(q) = 0 \qquad (n \in \mathbb{N})$$

where d is a nonnegative integer and $c_j(x, y) \in \mathbb{K}[x, y]$ are bivariate polynomials for $j = 0, \ldots, d$ with $c_d(x, y) \neq 0$.

The noncommutative A-polynomial

Notation.

Introduce operator notation:

$$(Lf)_n(q) = f_{n+1}(q), \qquad (Mf)_n(q) = q^n f_n(q)$$

and let

$$\mathbb{O} = \mathbb{K}(q, M)\langle L\rangle/(LM - qML).$$

The noncommutative A-polynomial

Notation.

Introduce operator notation:

$$(Lf)_n(q) = f_{n+1}(q), \qquad (Mf)_n(q) = q^n f_n(q)$$

and let

$$\mathbb{O} = \mathbb{K}(q, M)\langle L\rangle/(LM - qML).$$

Definition.

The noncommutative A-polynomial $A_K(q, M, L) \in \mathbb{O}$ of a knot K is the minimal-order operator (denominator- and content-free) that annihilates $J_{K,n}(q)$.

The AJ conjecture

There is a close relation between the A-polynomial $A_K(M, L)$ and the annihilator $A_K(q, M, L)$ of the colored Jones function:

AJ Conjecture:
For every knot K the following identity holds:

$$A_K(1, M, L) = \mathrm{poly}(M) \cdot A_K(M^{1/2}, L).$$

The AJ conjecture

There is a close relation between the A-polynomial $A_K(M, L)$ and the annihilator $A_K(q, M, L)$ of the colored Jones function:

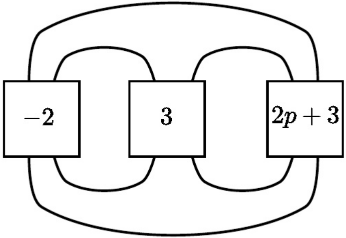
AJ Conjecture:
For every knot K the following identity holds:

$$A_K(1, M, L) = \mathrm{poly}(M) \cdot A_K(M^{1/2}, L).$$

The AJ conjecture has been verified (rigorously / non-rigorously) for some knots with few crossings, by explicit computations, as well as for some special families of knots.

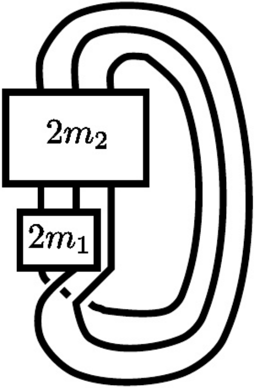
Pretzel knots

Consider 1-parameter family of pretzel knots $K_p = (-2, 3, 2p+3)$:



2-fusion knots

The pretzel knots K_p are members of a 2-parameter family of 2-fusion knots $K(m_1, m_2)$ for integers m_1 and m_2:



We have: $K_p = K(p, 1)$.

Formula for the colored Jones polynomial

$$J_{K(m_1,m_2),n+1}(1/q) =$$

$$\frac{\mu(n)^{-w(m_1,m_2)}}{\mathrm{U}(n)} \sum_{(k_1,k_2)\in nP\cap\mathbb{Z}^2} \nu(2k_1,n,n)^{2m_1+2m_2}\nu(n+2k_2,2k_1,n)^{2m_2+1}$$

$$\times \frac{\mathrm{U}(2k_1)\mathrm{U}(n+2k_2)}{\Theta(n,n,2k_1)\Theta(n,2k_1,n+2k_2)}\mathrm{Tet}(n,2k_1,2k_1,n,n,n+2k_2)$$

where

$$\mu(a) = (-1)^a q^{a(a+2)/4}$$

$$w(m_1,m_2) = 2m_1+6m_2+2$$

$$P = \mathrm{Polygon}(\{(0,0),(1/2,-1/2),(1,0),(1,1)\})$$

$$\nu(c,a,b) = (-1)^{(a+b-c)/2} q^{(-a(a+2)-b(b+2)+c(c+2))/8}$$

$$\Theta(a,b,c) = (-1)^{(a+b+c)/2}\left[\frac{a+b+c}{2}+1\right]\begin{bmatrix} \frac{a+b+c}{2} \\ \frac{-a+b+c}{2}, \frac{a-b+c}{2}, \frac{a+b-c}{2}\end{bmatrix}_q$$

$$\mathrm{U}(a) = (-1)^a[a+1]$$

Formula for the colored Jones polynomial

$$\mathrm{Tet}(a,b,c,d,e,f) = \sum_{k=\max T_i}^{\min S_j} (-1)^k [k+1]$$

$$\times \begin{bmatrix} k \\ S_1-k, S_2-k, S_3-k, k-T_1, k-T_2, k-T_3, k-T_4 \end{bmatrix}_q$$

where

$$S_1 = \frac{1}{2}(a+d+b+c), \quad S_2 = \frac{1}{2}(a+d+e+f), \quad S_3 = \frac{1}{2}(b+c+e+f)$$

and

$$T_1 = \frac{1}{2}(a+b+e), \qquad T_2 = \frac{1}{2}(a+c+f),$$
$$T_3 = \frac{1}{2}(c+d+e), \qquad T_4 = \frac{1}{2}(b+d+f).$$

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.
2. For the recurrence equation make an ansatz of the form

$$A(n) = \sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j}(q) q^{jn} J_{K,n+i}(q)$$

   with undetermined coefficients $c_{i,j} \in \mathbb{K}(q)$.

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.
2. For the recurrence equation make an ansatz of the form

$$A(n) = \sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j}(q) q^{jn} J_{K,n+i}(q)$$

   with undetermined coefficients $c_{i,j} \in \mathbb{K}(q)$.
3. Solve the linear system $A(1) = \cdots = A(N-r) = 0$ for the $c_{i,j}$.

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by "guessing":

1. Use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$.
2. For the recurrence equation make an ansatz of the form

$$A(n) = \sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j}(q) q^{jn} J_{K,n+i}(q)$$

   with undetermined coefficients $c_{i,j} \in \mathbb{K}(q)$.
3. Solve the linear system $A(1) = \cdots = A(N-r) = 0$ for the $c_{i,j}$.
4. If there is a solution for $N - r \geq (r+1)(d+1)$, then this is a very plausible candidate.

Degree of the colored Jones polynomial

Size of the colored Jones polynomial at $n = 10, 20, 30$ for the pretzel knot family, where $d(p) = d_1 + d_2$ for a Laurent polynomial $\sum_{i=-d_1}^{d_2} c_i q^i$ with $c_{-d_1} \neq 0$ and $c_{d_2} \neq 0$:

p	$d(J_{K_p,10}(q))$	$d(J_{K_p,20}(q))$	$d(J_{K_p,30}(q))$
-5	453	1919	4400
-4	363	1546	3549
-3	282	1197	2735
-2	225	950	2175
-1	225	950	2175
0	265	1130	2595
1	330	1410	3240
2	406	1736	3991
3	491	2098	4821
4	579	2469	5671
5	667	2843	6529

Some tricks

1. Use modular computations (evaluation – interpolation)
   - evaluate $J_{K_p,n}(q)$ for specific integers q and modulo a prime
   - guess the recurrence (for that particular q and modulo prime)
   - do this for many q and many primes
   - use interpolation and rational reconstruction (modulo prime), then chinese remaindering, to obtain the desired recurrence equation
2. Trade order versus degree of the recurrence and compute the (supposedly minimal-order) recurrence by gcrd.
3. Use information about the Newton polygon known from the A-polynomial.
4. Exploit palindromicity to halve the number of unknowns.

Palindromicity

We say that an operator $P \in \mathbb{K}(q)\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - qML)$ is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

Palindromicity

We say that an operator $P \in \mathbb{K}(q)\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - qML)$ is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \operatorname{ldeg}_M(P)$ and $\ell = \deg_L(P) + \operatorname{ldeg}_L(P)$.

If $P = \sum_{i,j} p_{i,j} M^i L^j$ then this implies that

$$p_{i,j} = (-1)^a q^{b(i-m/2)} p_{m-i,\ell-j} \text{ for all } i, j \in \mathbb{Z}.$$

Palindromicity

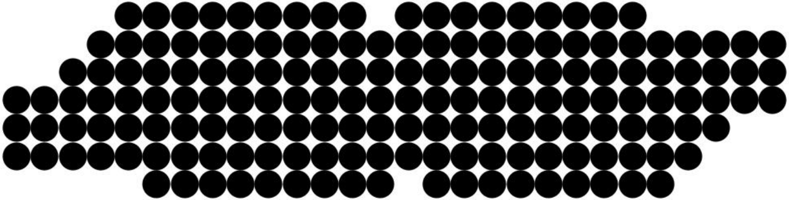
We say that an operator $P \in \mathbb{K}(q)\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - qML)$ is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

If $P = \sum_{i,j} p_{i,j} M^i L^j$ then this implies that

$$p_{i,j} = (-1)^a q^{b(i-m/2)} p_{m-i,\ell-j} \text{ for all } i, j \in \mathbb{Z}.$$



Palindromicity

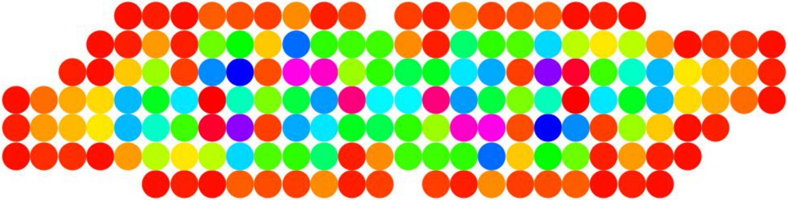
We say that an operator $P \in \mathbb{K}(q)\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - qML)$ is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \operatorname{ldeg}_M(P)$ and $\ell = \deg_L(P) + \operatorname{ldeg}_L(P)$.

If $P = \sum_{i,j} p_{i,j} M^i L^j$ then this implies that

$$p_{i,j} = (-1)^a q^{b(i-m/2)} p_{m-i,\ell-j} \text{ for all } i, j \in \mathbb{Z}.$$



Verification of AJ conjecture

1. The A-polynomials of $K_{-5}, \ldots, K_5$ were known.
2. Compute the $q = 1$ images of the guessed recurrence operators.
3. The results are in accordance with the AJ conjecture.
4. Assuming that the guessed operators are correct, how can we know that they are of minimal order?

Verification of AJ conjecture

1. The A-polynomials of $K_{-5}, \ldots, K_5$ were known.
2. Compute the $q = 1$ images of the guessed recurrence operators.
3. The results are in accordance with the AJ conjecture.
4. Assuming that the guessed operators are correct, how can we know that they are of minimal order?
5. Try to show irreducibility, which implies minimality.

An easy sufficient criterion for irreducibility

Consider

$$A(q, M, L) = \sum_{j=0}^{d} a_j(q, M) L^j \in \mathbb{O}$$

with $d > 1$ and assume

- $A(1, M, L) \in \mathbb{K}(M)[L]$ is well-defined,
- irreducible,
- and $a_0(1, M) a_d(1, M) \neq 0$.

Then $A(q, M, L)$ is irreducible in $\mathbb{O}$.

An easy sufficient criterion for irreducibility

Consider

$$A(q, M, L) = \sum_{j=0}^{d} a_j(q, M) L^j \in \mathbb{O}$$

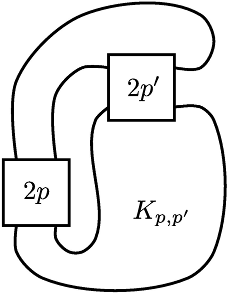
with $d > 1$ and assume

- $A(1, M, L) \in \mathbb{K}(M)[L]$ is well-defined,
- irreducible,
- and $a_0(1, M) a_d(1, M) \neq 0$.

Then $A(q, M, L)$ is irreducible in $\mathbb{O}$.

$\longrightarrow$ Most of the guessed operators are irreducible by this criterion and therefore of minimal order.

Consider the family of double twist knots $K_{p,p'}$:



$\longrightarrow$ Interesting family because their A-polynomials are reducible.

Colored Jones function of $K_{p,p'}$

Using the Habiro theory of the colored Jones function, we get

$$J_{K_{p,p'},n}(q) = \sum_{k=0}^{n-1} (-1)^k c_{p,k}(q) c_{p',k}(q) q^{-kn-\frac{k(k+3)}{2}} (q^{n-1}; q^{-1})_k (q^{n+1}; q)_k$$

where the sequence $c_{p,n}(q)$ is defined by

$$c_{p,n}(q) = \sum_{k=0}^{n} (-1)^{k+n} q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2p} \frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

Colored Jones function of $K_{p,p'}$

Using the Habiro theory of the colored Jones function, we get

$$J_{K_{p,p'},n}(q) = \sum_{k=0}^{n-1} (-1)^k c_{p,k}(q) c_{p',k}(q) q^{-kn-\frac{k(k+3)}{2}} (q^{n-1}; q^{-1})_k (q^{n+1}; q)_k$$

where the sequence $c_{p,n}(q)$ is defined by

$$c_{p,n}(q) = \sum_{k=0}^{n} (-1)^{k+n} q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2 p} \frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

$\longrightarrow$ Apply CK's HolonomicFunctions package.
www.risc.jku.at/research/combinat/software/HolonomicFunctions/

- symbolic summation via creative telescoping
- closure properties
- delivers a q-holonomic recurrence for the sum

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ (which is 7_4).

Result:

- (inhomogeneous) recurrence of order 5
- M-degree 24 and q-degree 65
- corresponds to 4 printed pages

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ (which is 7_4).

Result:

- (inhomogeneous) recurrence of order 5
- M-degree 24 and q-degree 65
- corresponds to 4 printed pages

Problem:

Creative telescoping doesn't necessarily give the minimal-order recurrence, but at least it certifies that it is correct.

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ (which is 7_4).

Result:

- (inhomogeneous) recurrence of order 5
- M-degree 24 and q-degree 65
- corresponds to 4 printed pages

Problem:

Creative telescoping doesn't necessarily give the minimal-order recurrence, but at least it certifies that it is correct.

Strategy:

Again, we try to show that the corresponding operator is irreducible.

How to show irreducibility?

Unfortunately, we cannot apply the previous criterion, since $A(1, M, L)$ in our case is reducible (double twist knots!).

For example, for $K_{2,2}$ one gets

$$\Big(L^3 + (M^7 - 2M^6 + 3M^5 + 2M^4 - 7M^3 + 2M^2 + 6M - 2)L^2 + (2M^7 - 6M^6 - 2M^5 + 7M^4 - 2M^3 - 3M^2 + 2M - 1)L + M^7\Big)$$
$$\times\Big(L^2 - (M^4 - M^3 - 2M^2 - M + 1)L + M^4\Big)$$

How to show irreducibility?

Unfortunately, we cannot apply the previous criterion, since $A(1, M, L)$ in our case is reducible (double twist knots!).

For example, for $K_{2,2}$ one gets

$$\Big(L^3 + (M^7 - 2M^6 + 3M^5 + 2M^4 - 7M^3 + 2M^2 + 6M - 2)L^2 + (2M^7 - 6M^6 - 2M^5 + 7M^4 - 2M^3 - 3M^2 + 2M - 1)L + M^7\Big)$$
$$\times\Big(L^2 - (M^4 - M^3 - 2M^2 - M + 1)L + M^4\Big)$$

This means, if a factorization exists then it must be of the form

- (irreducible of order 2) $\cdot$ (irreducible of order 3)
- (irreducible of order 3) $\cdot$ (irreducible of order 2)

Necessary and sufficient criterion for irreducibility

Lemma: Let $P, Q, R \in \mathbb{O}$ such that $P = QR$ is a factorization of P, and let k denote the order of R, i.e., $k = \deg_L(R)$. Then $\bigwedge^k P$ has a linear right factor $L - a$ for some $a \in \mathbb{K}(q, M)$.

Necessary and sufficient criterion for irreducibility

Lemma: Let $P, Q, R \in \mathbb{O}$ such that $P = QR$ is a factorization of P, and let k denote the order of R, i.e., $k = \deg_L(R)$. Then $\bigwedge^k P$ has a linear right factor $L - a$ for some $a \in \mathbb{K}(q, M)$.

Proof:

- Let $F = \{f^{(1)}, \ldots, f^{(k)}\}$ be a fundamental solution set of R.
- By the lemma it follows that $w = W(f^{(1)}, \ldots, f^{(k)})$ satisfies a recurrence of order 1, say $w_{n+1} = a w_n, a \in \mathbb{K}(q, M)$.
- But F is also a set of linearly independent solutions of $Pf = 0$ and therefore w is contained in the solution space of $\bigwedge^k P$.
- It follows that $\bigwedge^k P$ has the right factor $L - a$.

Exterior powers of P_{7_4}

Some statistics concerning P_{7_4} and its exterior powers:

	L-degree	M-degree	q-degree	ByteCount
P_{7_4}	5	24	65	463,544
$\bigwedge^2 P_{7_4}$	10	134	749	37,293,800
$\bigwedge^3 P_{7_4}$	10	183	1108	62,150,408

$\longrightarrow$ We now have to prove that $\bigwedge^2 P_{7_4}$ and $\bigwedge^3 P_{7_4}$ have no linear right factors.

Exterior powers of P_{7_4}

Some statistics concerning P_{7_4} and its exterior powers:

	L-degree	M-degree	q-degree	ByteCount
P_{7_4}	5	24	65	463,544
$\bigwedge^2 P_{7_4}$	10	134	749	37,293,800
$\bigwedge^3 P_{7_4}$	10	183	1108	62,150,408

$\longrightarrow$ We now have to prove that $\bigwedge^2 P_{7_4}$ and $\bigwedge^3 P_{7_4}$ have no linear right factors.

This can be achieved by an optimized version of the qHyper algorithm (Abramov+Paule+Petkovšek 1998).

Results for double twist knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order (irreducible!)

Results for double twist knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order (irreducible!)

$K_{3,3}$:

- rigorous computation of $A(q, M, L)$
- (q, M, L)-degree $= (458, 74, 11)$
- minimality proof out of scope (requires $\bigwedge^5 P$ and $\bigwedge^6 P$)

Results for double twist knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order (irreducible!)

$K_{3,3}$:

- rigorous computation of $A(q, M, L)$
- (q, M, L)-degree $= (458, 74, 11)$
- minimality proof out of scope (requires $\bigwedge^5 P$ and $\bigwedge^6 P$)

$K_{4,4}$:

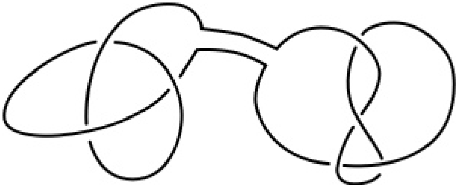
- $A(q, M, L)$ guessed
- (q, M, L)-degree $= (2045, 184, 19)$

$K_{5,5}$:

- $A(q, M, L)$ guessed
- (q, M, L)-degree $= (6922, 396, 29)$, ByteCount = 8GB

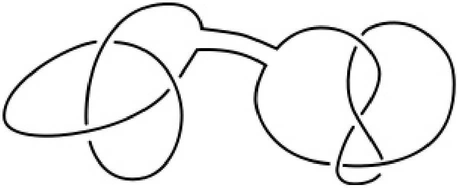
Colored Jones for connected sum of knots

Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



Colored Jones for connected sum of knots

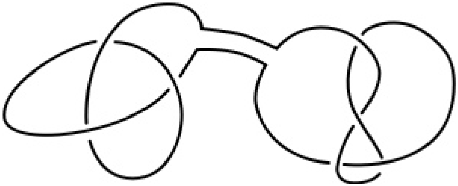
Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



- A knot is irreducible if it cannot be written as connected sum of two nontrivial knots.

Colored Jones for connected sum of knots

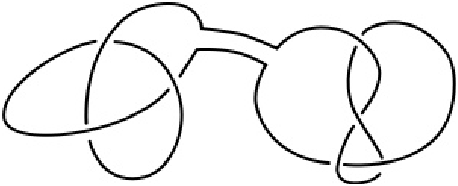
Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



- A knot is irreducible if it cannot be written as connected sum of two nontrivial knots.
- Each knot has a "unique factorization".

Colored Jones for connected sum of knots

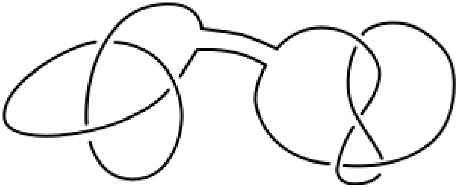
Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



- A knot is irreducible if it cannot be written as connected sum of two nontrivial knots.
- Each knot has a "unique factorization".
- Rolfsen's table contains only irreducible knots.

Colored Jones for connected sum of knots

Connected sum $K_1 \# K_2$ of two knots K_1 and K_2:



- A knot is irreducible if it cannot be written as connected sum of two nontrivial knots.
- Each knot has a "unique factorization".
- Rolfsen's table contains only irreducible knots.

Fact: Let K_1 and K_2 be two knots in 3-space. Then the colored Jones function of their connected sum is given by

$$J_{K_1 \# K_2, n}(q) = J_{K_1,n}(q)\, J_{K_2,n}(q) \quad \text{for all } n \in \mathbb{N}.$$

$\longrightarrow$ Like for the classical Jones polynomial.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the symmetric product $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the symmetric product $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Remark 1: P is unique up to multiplication by elements from $\mathbb{K}(q, M) \setminus \{0\}$.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the symmetric product $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Remark 1: P is unique up to multiplication by elements from $\mathbb{K}(q, M) \setminus \{0\}$.

Remark 2: The definition does not imply that the symmetric product gives the shortest recurrence for the product of two sequences.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the symmetric product $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Remark 1: P is unique up to multiplication by elements from $\mathbb{K}(q, M) \setminus \{0\}$.

Remark 2: The definition does not imply that the symmetric product gives the shortest recurrence for the product of two sequences.

Corollary: Let K_1 and K_2 be two knots and let $P_1, P_2 \in \mathbb{O}$ be annihilating operators of their colored Jones functions, respectively. Then the symmetric product $P_1 \star P_2$ annihilates $J_{K_1 \# K_2, n}(q)$.

Example.

Consider the sequence $f(n) = q^n + (-1)^n$ whose minimal-order annihilating operator is $P = L^2 + (1-q)L - q$. As expected, the symmetric product $P \star P$ is of order 3:

$$\begin{aligned} P \star P &= L^3 - (q^2 - q + 1)L^2 - (q^2 - q + 1)L + q^3 \\ &= (L-1)(L+q)(L-q^2). \end{aligned}$$

On the other hand, we have $f(n)^2 = q^{2n} + 1 + 2(-q)^n$ and this expression is annihilated by the second-order operator

$$(qM^2 + 1)L^2 - (q-1)(q^2M^2 - 1)L - q(q^3M^2 + 1).$$

A-polynomial for connected sums

Definition.

For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the “A-product” $A_1 \diamond A_2$ as follows:

A-polynomial for connected sums

Definition.

For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the "A-product" $A_1 \diamond A_2$ as follows:

- let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal

$$\langle A_1(M, L_1), A_2(M, L_2), L - L_1 L_2 \rangle$$

A-polynomial for connected sums

Definition.
For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the "A-product" $A_1 \diamond A_2$ as follows:

- let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal

$$\langle A_1(M, L_1), A_2(M, L_2), L - L_1 L_2 \rangle$$

- $A_1 \diamond A_2$ is the generator of the elimination ideal $I \cap \mathbb{K}(M)[L]$

A-polynomial for connected sums

Definition.
For two bivariate polynomials $A_1(M,L)$ and $A_2(M,L)$ we define the “A-product” $A_1 \diamond A_2$ as follows:

- let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal

$$\langle A_1(M,L_1), A_2(M,L_2), L - L_1 L_2 \rangle$$

- $A_1 \diamond A_2$ is the generator of the elimination ideal $I \cap \mathbb{K}(M)[L]$
- Note that $\mathbb{K}(M)[L]$ is a PID, thence $A_1 \diamond A_2$ is unique up to multiplication by elements from $\mathbb{K}(M) \setminus \{0\}$.

A-polynomial for connected sums

Definition.
For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the "A-product" $A_1 \diamond A_2$ as follows:

- let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal

$$\langle A_1(M, L_1), A_2(M, L_2), L - L_1 L_2 \rangle$$

- $A_1 \diamond A_2$ is the generator of the elimination ideal $I \cap \mathbb{K}(M)[L]$
- Note that $\mathbb{K}(M)[L]$ is a PID, thence $A_1 \diamond A_2$ is unique up to multiplication by elements from $\mathbb{K}(M) \setminus \{0\}$.

Fact: Let K_1 and K_2 be two knots and $A_1(M, L)$ and $A_2(M, L)$ their respective A-polynomials. Then the A-polynomial of $K_1 \# K_2$ is given by $A_1 \diamond A_2$.

Theorem

Notation: We introduce the map ψ by

$$\psi\colon \mathbb{O} \to \mathbb{K}(M)[L],\ P(q,M,L) \mapsto P(1,M,L).$$

Theorem

Notation: We introduce the map ψ by

$$\psi\colon \mathbb{O} \to \mathbb{K}(M)[L],\; P(q,M,L) \mapsto P(1,M,L).$$

Theorem.
Let $P_1(q,M,L)$ and $P_2(q,M,L)$ be two operators in the algebra $\mathbb{O}$. Then the following divisibility condition holds:

$$\psi(P_1) \diamond \psi(P_2) \mid \psi(P_1 \star P_2)$$

as polynomials in $\mathbb{K}(M)[L]$, provided that the above quantities are defined.

Theorem

Notation: We introduce the map ψ by

$$\psi \colon \mathbb{O} \to \mathbb{K}(M)[L],\ P(q, M, L) \mapsto P(1, M, L).$$

Theorem.
Let $P_1(q, M, L)$ and $P_2(q, M, L)$ be two operators in the algebra $\mathbb{O}$. Then the following divisibility condition holds:

$$\psi(P_1) \diamond \psi(P_2) \mid \psi(P_1 \star P_2)$$

as polynomials in $\mathbb{K}(M)[L]$, provided that the above quantities are defined.

Proof (1)

Recall the algorithm for computing the symmetric power $P_1 \star P_2$.

- let $f(n)$ and $g(n)$ be generic sequences that are annihilated by P_1 and P_2, respectively
- make an ansatz for the minimal-order q-recurrence for the product $h(n) = f(n)g(n)$:

$$c_d(q, M)h(n+d) + \cdots + c_0(q, M)h(n) = 0$$

  with undetermined coefficients $c_j \in \mathbb{K}(q, M)$.
- let d_1 and d_2 denote the L-degrees of P_1 and P_2, respectively.
- using the q-recurrence represented by P_1, we can rewrite $f(n+s)$ as a $\mathbb{K}(q, M)$-linear combination of $f(n), \ldots, f(n+d_1-1)$ for any $s \in \mathbb{N}$, and similarly for $g(n+s)$
- the ansatz therefore can be reduced to the following form:

$$\sum_{s=0}^{d_1-1} \sum_{t=0}^{d_2-1} R_{s,t}(q, M, c_0, \ldots, c_d) f(n+s) g(n+t) = 0$$

Proof (2)

$$\sum_{s=0}^{d_1-1} \sum_{t=0}^{d_2-1} R_{s,t}(q, M, c_0, \ldots, c_d) f(n+s) g(n+t) = 0$$

- notation for the 2-tuples corresponding to the summands:

  $$\{(s_0, t_0), (s_1, t_1), \ldots\} = \{(s,t) \mid 0 \leq s \leq d_1{-}1,\, 0 \leq t \leq d_2{-}1\}$$
- for example, put $s_i = \lfloor i/d_2 \rfloor$ and $t_i = i \bmod d_2$
- equating all $R_{s,t}$ to zero yields a linear system $M\boldsymbol{c} = 0$
- the matrix M is given by

  $$M = (m_{i,j})_{0 \leq i \leq d_1 d_2 - 1, 0 \leq j \leq d} \quad \text{with} \quad m_{i,j} = \langle c_j \rangle R_{s_i, t_i}$$
- the algorithm proceeds by trying $d = 0$, $d = 1$, etc., until a solution is found; this guarantees minimality.
- if $d \geq d_1 d_2$ the linear system has more unknowns than equations so that a solution must exist; this ensures termination.

Proof (3)

To prove the claim, apply the above algorithm to $\psi(P_1)$ and $\psi(P_2)$.

- rewriting of $f(n+s)$ into $f(n), \dots, f(n+d_1-1)$ can be rephrased as the (noncommutative) polynomial reduction of the operator L^s with P_1
- if instead $\psi(P_1)$ is used the noncommutativity disappears
- the reduction procedure boils down to a polynomial division with remainder in $\mathbb{K}(M)[L]$
- let $\operatorname{rem}(a,b)$ denote the remainder of dividing the polynomial a by b
- obtain a matrix $\tilde{M}$ with $\tilde{M} = \psi(M)$
- the entries $\psi(m_{i,j})$ of the matrix $\tilde{M}$ are obtained as follows:

$$\begin{aligned}\psi(m_{i,j}) &= \big(\langle L^{s_i}\rangle \operatorname{rem}(L^j, \psi(P_1))\big) \cdot \big(\langle L^{t_i}\rangle \operatorname{rem}(L^j, \psi(P_2))\big) \\ &= \langle L_1^{s_i} L_2^{t_i}\rangle \Big(\operatorname{rem}(L_1^j, P_1(1, M, L_1)) \cdot \operatorname{rem}(L_2^j, P_2(1, M, L_2))\Big)\end{aligned}$$

Proof (4)

- note that the set $G = \{P_1(1, M, L_1), P_2(1, M, L_2)\}$ is a Gröbner basis in $\mathbb{K}(M)[L_1, L_2]$ by Buchberger's product criterion
- can define $\operatorname{red}(P, G)$ for $P \in \mathbb{K}(M)[L_1, L_2]$ as the unique reductum of P with G
- Observe that

$$\operatorname{rem}\big(L_1^j, P_1(1, M, L_1)\big) \cdot \operatorname{rem}\big(L_2^j, P_2(1, M, L_2)\big) = \operatorname{red}\big((L_1 L_2)^j, G\big).$$

- the linear system $\tilde{M}\boldsymbol{c} = 0$ translates to the problem: find $c_0, \ldots, c_d \in \mathbb{K}(M)$ such that

$$\sum_{j=0}^{d} c_j(M) \operatorname{red}\big((L_1 L_2)^j, G\big) = 0.$$

Proof (5)

$$\sum_{j=0}^{d} c_j(M)\,\mathrm{red}\big((L_1L_2)^j, G\big) = 0.$$

- this can be rephrased as an elimination problem
- identify L_1L_2 with a new indeterminate L
- want to find a polynomial in $\mathbb{K}(M)[L]$, free of L_1 and L_2, in the ideal generated by G and $L - L_1L_2$
- this elimination problem is just the definition of $\psi(P_1) \diamond \psi(P_2)$
- Hence we have shown:

$$\psi(P_1) \star \psi(P_2) = \psi(P_1) \diamond \psi(P_2).$$

- we have $\deg_L \big(\psi(P_1 \star P_2)\big) \geq \deg_L \big(\psi(P_1) \star \psi(P_2)\big)$
- moreover: $\psi(P_1 \star P_2)$ is an element of the elimination ideal generated by $\psi(P_1) \diamond \psi(P_2)$
- therefore $\psi(P_1) \diamond \psi(P_2) \mid \psi(P_1 \star P_2)$ as claimed

Example

Consider the connected sum $3_1\#3_1$. Its colored Jones polynomial satisfies $PJ_{3_1\#3_1,n}(q) = b$ with

$$
\begin{aligned}
P = &\big(M^4q^5 - 2M^3q^3 - M^2q^4 + M^2q + 2Mq^2 - 1\big)L^2 \\
&+ \big(- M^{10}q^{13} + 2M^9q^{12} + M^8q^{12} - M^8q^{11} - M^7q^{11} - M^6q^{10} \\
&\quad + M^5q^9 - M^5q^8 + 2M^4q^7 - M^3q^6\big)L \\
&- M^{13}q^{13} + 2M^{12}q^{13} - M^{11}q^{13} + M^{11}q^{10} - 2M^{10}q^{10} + M^9q^{10} \\
b = &M^{11}q^{11} - 2M^9q^{10} - M^9q^8 - M^8q^9 + M^7q^9 + 2M^7q^7 + M^6q^8 \\
&+ 2M^6q^6 - M^5q^6 - 2M^4q^5 - M^4q^3 + M^2q^2
\end{aligned}
$$

The operator P is reducible:

$$
\begin{aligned}
P = &\big((M^2q - 1)L + M^5q^9 - M^3q^6\big) \\
&\times \big((M^2q^2 - 2Mq + 1)L - M^8q^4 + 2M^7q^4 - M^6q^4\big)
\end{aligned}
$$

But this factorization doesn't yield a lower order recurrence for $J_{3_1\#3_1,n}(q)$. Hence P is of minimal order.

Some results

Consider connected sums of 3_1 and 4_1:

- $3_1\#3_1$: $\deg_L(P) = 2$, reducible into $1+1$
- $3_1\#4_1$: $\deg_L(P) = 5$, reducible into $2+1+2$ and $1+2+2$
- $4_1\#4_1$: $\deg_L(P) = 5$, reducible into $2+3$

$\longrightarrow$ In all cases the operators are reducible.

$\longrightarrow$ Nevertheless, in all cases they are already minimal.