On the AJ conjecture of connected sums of knots

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The A-polynomial

The A-polynomial $A_K(M, L)$ of a knot K parametrizes the affine variety of $\mathrm{SL}(2, \mathbb{C})$ representations of the knot complement, viewed from the boundary torus:

- M_K: S^3 minus a tubular neighborhood of K (knot complement)
- character variety: $X_{M_K} = \mathrm{Hom}(\pi_1(M_K), \mathrm{SL}(2, \mathbb{C}))$ (modulo conjugation)
- boundary: $X_{\partial(M_K)} = \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$
- consider the restriction map $\phi : X_{M_K} \to X_{\partial(M_K)}$
- its image is defined by a bivariate polynomial, $A_K(M, L)$
- difficult to compute (e.g., using elimination)
- even unknown for some knots with only 9 crossings.

Example: trefoil

The fundamental group of the trefoil is

$$\pi_1(S^3 \setminus 3_1) = \langle a, b \mid aabbb \rangle$$

$\mathrm{SL}(2, \mathbb{C})$ representations:

$$a \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} =: A \qquad \text{(w.l.o.g.)}$$

$$b \to \begin{pmatrix} v & w \\ x & y \end{pmatrix} =: B \quad \text{with } \det B = 1$$

There are two distinguished elements in $\pi_1(S^3 \setminus K)$, the meridian μ and the longitude λ.
For 3_1 we get:

$$\mu = bab$$
$$\lambda = ba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}ab$$

Example: trefoil

Conditions:

$$\operatorname{tr}\left(\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} - \mathcal{M}\right) = \operatorname{tr}\left(\begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} - \Lambda\right) = 0$$

where

$$\begin{aligned} \mathcal{M} &= BAB \\ \Lambda &= BA^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}AB^{-1}A^{-1}B^{-1}AB \end{aligned}$$

Thus we have to consider the ideal

$$\left\langle vy - wx - 1, AABBB - \mathrm{Id}_2, M + M^{-1} - \operatorname{tr}(\mathcal{M}), L + L^{-1} - \operatorname{tr}(\Lambda) \right\rangle$$

and intersect it with $\mathbb{Q}[M, L]$.

This can be achieved, e.g., by elimination using Gröbner bases.

In this case, we obtain $A_{3_1}(M, L) = L + M^6$.

The colored Jones function

The colored Jones function $J_{K,n}(q)$ of a knot K is a generalization of the classical Jones polynomial. It is a sequence of Laurent polynomials:

$$J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}}.$$

It can be defined using the n-th parallels of K:

$$J_{K,n}(q) = \sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} J_{K^{(k)}}(q)$$

where $J_{K^{(k)}}(q)$ is the Jones polynomial of the k-th parallel of K.

- $J_{K,0}(q) = 1$ for all knots
- $J_{K,1}(q)$ is the Jones polynomial of K

The colored Jones function

Alternative definition via state sums using a diagram of K:

- label the n crossings with variables $k_1, \ldots, k_n$
- label the arcs as follows: at crossing k_i add k_i to the label of the underpass and subtract k_i from the label of the overpass
- associate to crossing k_i a certain proper q-hypergeometric expression R_i, depending on the labels (locally):

$$R_i = \begin{bmatrix} \operatorname{lin}(\boldsymbol{k}) \\ k_i \end{bmatrix}_q \left(q^{n-\operatorname{lin}(\boldsymbol{k})}; q^{-1}\right)_{k_i} (-1)^{\operatorname{lin}(\boldsymbol{k},n)} q^{\operatorname{quad}(\boldsymbol{k},n)}$$

- the colored Jones function is given by an n-fold sum:

$$J_{K,n}(q) = \sum_{0 \leq \boldsymbol{k} \leq n} R_1 \cdots R_n$$

q-calculus

Recall some notation from q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$$

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

$$[n]! = \prod_{k=1}^{n}[k]$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}$$

$\longrightarrow$ all these terms are (proper) q-hypergeometric.

$$f_n(q) \text{ is } q\text{-hg.} \iff \frac{f_{n+1}(q)}{f_n(q)} \in \mathbb{K}(q, q^n)$$

Wilf-Zeilberger

Fundamental theorem of WZ theory: every (multi-) sum over a proper q-hypergeometric term is q-holonomic.

$\longrightarrow$ The colored Jones function is a q-holonomic sequence.

q-holonomic sequences

Notation:

- $\mathbb{K}$: field of characteristic zero
- q: indeterminate, transcendental over $\mathbb{K}$

A univariate sequence $\big(f_n(q)\big)_{n\in\mathbb{N}}$ is called **q-holonomic** if it satisfies a nontrivial linear recurrence with coefficients that are polynomials in q and q^n:

$$\sum_{j=0}^{d} c_j(q, q^n) f_{n+j}(q) = 0 \qquad (n \in \mathbb{N})$$

where d is a nonnegative integer and $c_j(u,v) \in \mathbb{K}[u,v]$ are bivariate polynomials for $j = 0, \dots, d$ with $c_d(u,v) \neq 0$.

The noncommutative A-polynomial

Introduce operator notation:

$$(Lf)_n(q) = f_{n+1}(q), \qquad (Mf)_n(q) = q^n f_n(q)$$

and let

$$\mathbb{O} = \mathbb{K}(q, M)\langle L\rangle/(LM - qML).$$

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The noncommutative A-polynomial $A_K(q, M, L)$ of a knot K is the (homogeneous and content-free) q-holonomic recurrence for $J_{K,n}(q)$ that has minimal order.

The AJ Conjecture

There is a close relation between the A-polynomial $A_K(M, L)$ and the recurrence (given as an operator $A_K(q, M, L) \in \mathbb{O}$) for the colored Jones function:

AJ Conjecture:
For every knot K the following identity holds:

$$A_K(1, M, L) = \mathrm{poly}(M) \cdot A_K(M^{1/2}, L)$$

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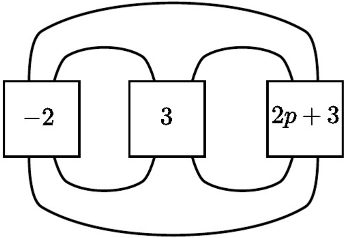
$$A_K(1, M, L) = \mathrm{poly}(M) \cdot A_K(M^{1/2}, L)$$

$\longrightarrow$ The AJ conjecture has been verified (rigorously / non-rigorously) for some knots with few crossings, by explicit computations, as well as for some special families of knots.

Pretzel knots

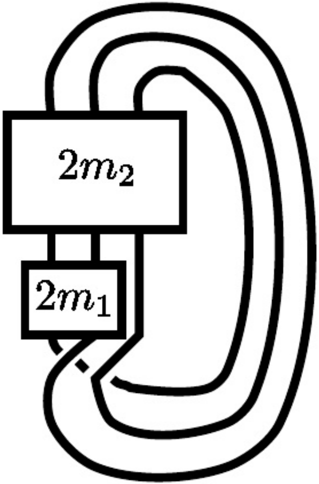
Consider 1-parameter family of pretzel knots $K_p = (-2, 3, 2p+3)$



Pretzel knots

- K_{-1} is the torus knot 5_1
- $K_0 = 8_{19}$ and $K_1 = 10_{124}$ (both torus knots)
- K_p is hyperbolic for $p \neq -1, 0, 1$

The pretzel knots K_p are members of a 2-parameter family of 2-fusion knots $K(m_1, m_2)$ for integers m_1 and m_2:



We have: $K_p = K(p, 1)$.

Formula for the colored Jones polynomial

$$J_{K(m_1,m_2),n+1}(1/q) =$$

$$\frac{\mu(n)^{-w(m_1,m_2)}}{\mathrm{U}(n)} \sum_{(k_1,k_2)\in nP\cap\mathbb{Z}^2} \nu(2k_1,n,n)^{2m_1+2m_2}\nu(n+2k_2,2k_1,n)^{2m_2+1}$$

$$\times \frac{\mathrm{U}(2k_1)\mathrm{U}(n+2k_2)}{\Theta(n,n,2k_1)\Theta(n,2k_1,n+2k_2)}\mathrm{Tet}(n,2k_1,2k_1,n,n,n+2k_2)$$

where

$$\mu(a) = (-1)^a q^{a(a+2)/4}$$

$$w(m_1,m_2) = 2m_1+6m_2+2$$

$$P = \mathrm{Polygon}(\{(0,0),(1/2,-1/2),(1,0),(1,1)\})$$

$$\nu(c,a,b) = (-1)^{(a+b-c)/2}q^{(-a(a+2)-b(b+2)+c(c+2))/8}$$

$$\Theta(a,b,c) = (-1)^{(a+b+c)/2}\left[\frac{a+b+c}{2}+1\right]\begin{bmatrix} \frac{a+b+c}{2} \\ \frac{-a+b+c}{2}, \frac{a-b+c}{2}, \frac{a+b-c}{2}\end{bmatrix}_q$$

$$\mathrm{U}(a) = (-1)^a[a+1]$$

Formula for the colored Jones polynomial (2)

$$\mathrm{Tet}(a,b,c,d,e,f) = \sum_{k=\max T_i}^{\min S_j} (-1)^k [k+1]$$

$$\times \begin{bmatrix} k \\ S_1-k, S_2-k, S_3-k, k-T_1, k-T_2, k-T_3, k-T_4 \end{bmatrix}_q$$

where

$$S_1 = \frac{1}{2}(a+d+b+c), \quad S_2 = \frac{1}{2}(a+d+e+f), \quad S_3 = \frac{1}{2}(b+c+e+f)$$

and

$$T_1 = \frac{1}{2}(a+b+e), \qquad T_2 = \frac{1}{2}(a+c+f),$$
$$T_3 = \frac{1}{2}(c+d+e), \qquad T_4 = \frac{1}{2}(b+d+f).$$

Guessing

A candidate for a q-recurrence of $J_{K,n}(q)$ can be obtained by **guessing**:

1. use the formula to compute the values of $J_{K,n}(q)$ for $1 \leq n \leq N$
2. for the recurrence equation make an ansatz of the form

   $$A(n) = \sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j}(q) q^{jn} J_{K,n+i}(q)$$

   with undetermined coefficients $c_{i,j} \in \mathbb{K}(q)$
3. solve the linear system $A(1) = \cdots = A(N-r) = 0$ for the $c_{i,j}$
4. if there is a solution for $N - r \geq (r+1)(d+1)$, then this is a very plausible candidate

Degree of the colored Jones polynomial

Size of the colored Jones polynomial at $n = 10, 20, 30$ for the pretzel knot family, where $d(p) = d_1 + d_2$ for a Laurent polynomial $\sum_{i=-d_1}^{d_2} c_i q^i$ with $c_{-d_1} \neq 0$ and $c_{d_2} \neq 0$:

p	$d(J_{K_p,10}(q))$	$d(J_{K_p,20}(q))$	$d(J_{K_p,30}(q))$
-5	453	1919	4400
-4	363	1546	3549
-3	282	1197	2735
-2	225	950	2175
-1	225	950	2175
0	265	1130	2595
1	330	1410	3240
2	406	1736	3991
3	491	2098	4821
4	579	2469	5671
5	667	2843	6529

Some tricks

- evaluate $J_{K_p,n}(q)$ for specific integers q and modulo a prime
- guess the recurrence (still for that particular q and modulo prime)
- use interpolation and rational reconstruction (modulo prime), then chinese remaindering, to obtain the desired recurrence equation
- trade order versus degree of the recurrence and compute the (supposedly minimal-order) recurrence by gcrd
- exploit palindromicity

Palindromicity

We say that an operator $P \in \mathbb{K}(q)\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - qML)$ is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

If $P = \sum_{i,j} p_{i,j} M^i L^j$ then this implies that

$$p_{i,j} = (-1)^a q^{b(i - m/2)} p_{m-i, \ell - j} \text{ for all } i, j \in \mathbb{Z}.$$

$\longrightarrow$ All operators here are palindromic!

An Easy Sufficient Criterion for Irreducibility

Consider

$$A(q, M, L) = \sum_{j=0}^{d} a_j(q, M) L^j \in \mathbb{O}$$

with $d > 1$ and assume

- $A(1, M, L) \in \mathbb{K}(M)[L]$ is well-defined,
- irreducible,
- and $a_0(1, M) a_d(1, M) \neq 0$.

Then $A(q, M, L)$ is irreducible in $\mathbb{O}$.

$\longrightarrow$ Most of the guessed operators are irreducible by this criterion and therefore of minimal order.

Some data about the guessed recurrences

p	L-degree	M-degree	q-degree	largest cf.	ByteCount
-5	12	125	946	3.0×10^8	5.7×10^7
-4	9	66	392	12345	1.1×10^7
-3	6	27	85	33	1.1×10^6
-2	3	12	19	4	32032
-1	1	6	3	1	1192
0	2	13	13	2	1616
1	2	16	16	2	1616
2	6	58	233	6	47016
3	9	114	514	118	2.3×10^6
4	12	191	1151	386444	1.9×10^7
5	15	288	2174	2.2×10^{11}	8.6×10^7

Consistency with volume conjecture

The N-th *Kashaev invariant* $\langle K\rangle_N$ of a knot K is defined by

$$\langle K\rangle_N = J_{K,N}(e^{2\pi i/N}).$$

The volume conjecture of Kashaev states that if K is a hyperbolic knot, then

$$\lim_{N\to\infty} \frac{\log|\langle K\rangle_N|}{N} = \frac{\mathrm{vol}(K)}{2\pi}$$

where $\mathrm{vol}(K)$ is the volume of the hyperbolic knot K.

Since we are specializing to a root of unity, we might as well consider the remainder $\tau_{K,N}(q)$ of $J_{K,N}(q)$ by the N-th cyclotomic polynomial $\Phi_N(q)$.

Example

$\tau_{K_2,100}(q) =$

$-1420771679897311607360 - 1402034476570732425908q - 1377764083694494707679q^2 -$
$1348056285420017550322q^3 - 1313028324854995190830q^4 - 1272818441358081463973q^5 -$
$1227585324968178744317q^6 - 1177507490130630983388q^7 - 1122782571182284245313q^8 -$
$1063626542375688303231q^9 + 4204988143666367344 11q^{10} + 4690629079033903065 37q^{11} +$
$5157758244381450144 36q^{12} + 5604532094294288909 01q^{13} + 6029187416487414419 24q^{14} +$
$6430048290431369057 36q^{15} + 6805532701383559215 66q^{16} + 7154158783904514892 64q^{17} +$
$7474550670139139652 48q^{18} + 7765443919677783021 55q^{19} - 6182026289225117431 88q^{20} -$
$5766081399732864303 88q^{21} - 5327380421232863639 77q^{22} - 4867654706066105171 17q^{23} -$
$4388718581582598272 94q^{24} - 3892462189876528123 32q^{25} - 3380844028211724322 80q^{26} -$
$2855883219716462216 47q^{27} - 2319651544885405703 26q^{28} - 1774265265162966208 08q^{29} +$
$1298584002796105745794q^{30} + 1335567867823634101034q^{31} + 1367280856639633305993q^{32} +$
$1393597812566394292363q^{33} + 1414414874600710903331q^{34} + 1429649887309469255114q^{35} +$
$1439242725058651352936q^{36} + 1443155529298983637839q^{37} + 1441372857979981026638q^{38} +$
$1433901746491878528487q^{39}$

$$2\pi\frac{\log|\tau_{K_2,100}(e^{2\pi i/100})|}{N} = 3.22309\ldots$$

But: $\mathrm{vol}(K_2) = 2.828122088330 7827\ldots$

$\longrightarrow$ Compute values for several N and fit a curve:

$$2.82813 + 9.41764\frac{\log(n)}{n} - 3.89193\frac{1}{n}.$$

Excursion: holonomic systems approach

1. Functions and sequences are represented by their annihilating left ideals (and initial values).
2. Holonomic functions are closed under certain operations, e.g., addition, multiplication, but **not** division.
3. An annihilating ideal is given by its Gröbner basis (i.e., a finite set of generators that allows to decide ideal membership and equality of ideals).
4. Integrals and sums are treated by the method of creative telescoping.
5. The output is always given as an annihilating ideal, not as a closed form.

Ore Algebras

$\longrightarrow$ Generalization of the **Weyl algebra** (ring of differential operators with polynomial coefficients)

Let $\mathbb{A}$ be a ring,

- $\sigma\colon \mathbb{A} \to \mathbb{A}$ an automorphism on $\mathbb{A}$, and
- $\delta\colon \mathbb{A} \to \mathbb{A}$ be a σ-derivation, i.e.,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b \quad \text{for all } a, b \in \mathbb{A}.$$

Then the polynomial ring $\mathbb{O} = \mathbb{A}[\partial; \sigma, \delta] = \mathbb{A}\langle\partial\rangle$ whose non-commutative multiplication is defined by

$$\partial a = \sigma(a)\partial + \delta(a)$$

is called an **Ore algebra**.

Example: Let $\mathbb{A} = \mathbb{K}(x)$, $\sigma = \mathrm{id}$, and $\delta = \frac{\mathrm{d}}{\mathrm{d}x}$. In this case we denote $\partial = D_x$ and get $\mathbb{O} = \mathbb{K}(x)\langle D_x\rangle$.

Examples of Ore Algebras

Ore operator	∂	σ	δ
Differential operator	D_x	$\sigma = \mathrm{id}$	$\delta = \frac{\mathrm{d}}{\mathrm{d}x}$
Euler operator	θ_x	$\sigma = \mathrm{id}$	$\delta = x\frac{\mathrm{d}}{\mathrm{d}x}$
Shift operator	S_n	$\sigma(n) = n + 1$	$\delta = 0$
Difference operator	Δ_n	$\sigma(n) = n + 1$	$\delta(n) = 1$
q-Shift operator	$S_{z,q}$	$\sigma(z) = qz$	$\delta = 0$
q-Difference operator	$\Delta_{z,q}$	$\sigma(z) = qz$	$\delta(z) = (q-1)z$

Multivariate Ore Algebras

The construction of Ore algebras can be iterated:

$$\mathbb{A}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_r; \sigma_r, \delta_r] = \mathbb{A}\langle \partial_1, \ldots, \partial_r \rangle$$

In this case, one must ensure that the ∂_i's commute: $\partial_i \partial_j = \partial_j \partial_i$.

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In this talk $\mathbb{A}$ is always a **rational** function field:

$$\mathbb{A} = \mathbb{K}(v_1, \ldots, v_r) = \mathbb{K}(\boldsymbol{v}).$$

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Each ∂_i is related to exactly one variable, say v_i, i.e., $\partial_i v_j = v_j \partial_i$ for $i \neq j$; write ∂_{v_i} for ∂_i.

Summarizing, Ore algebras in this talk are always of the form

$$\mathbb{O} = \mathbb{K}(v_1, \ldots, v_r)\langle \partial_{v_1}, \ldots, \partial_{v_r} \rangle = \mathbb{K}(\boldsymbol{v})\langle \boldsymbol{\partial_v} \rangle.$$

Action!

Define how operators act on functions.

Let $\mathcal{F}$ be an appropriate space of functions / sequences, $f \in \mathcal{F}$.

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Differential operator:	$D_x \bullet f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$
Euler operator:	$\theta_x \bullet f(x) = x \frac{\mathrm{d}}{\mathrm{d}x} f(x)$
Shift operator:	$S_n \bullet f(n) = f(n+1)$
Difference operator:	$\Delta_n \bullet f(n) = f(n+1) - f(n)$
q-Shift operator:	$S_{z,q} \bullet f(z) = f(qz)$
q-Difference operator:	$\Delta_{z,q} \bullet f(z) = f(qz) - f(z)$

$\longrightarrow$ The action $\bullet$: $\mathbb{O} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a left $\mathbb{O}$-module.

Definitions

1. The **annihilator** of a function f w.r.t. an Ore algebra $\mathbb{O}$:

$$\operatorname{ann}_{\mathbb{O}}(f) = \left\{ P \in \mathbb{O} \mid P \bullet f = 0 \right\}$$

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2. A function is called **∂-finite w.r.t. $\mathbb{O}$** ("holonomic") if

$$\dim_{\mathbb{K}(\boldsymbol{v})} \left(\mathbb{O}/\operatorname{ann}_{\mathbb{O}}(f)\right) < \infty$$

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$$\operatorname{ann}_{\mathbb{O}}(f) = \left\{ P \in \mathbb{O} \mid P \bullet f = 0 \right\}$$

2. A function is called **$\boldsymbol{\partial}$-finite w.r.t. $\mathbb{O}$** ("holonomic") if

$$\dim_{\mathbb{K}(\boldsymbol{v})} \left(\mathbb{O} / \operatorname{ann}_{\mathbb{O}}(f)\right) < \infty$$

$\big(\operatorname{ann}_{\mathbb{O}}(f)$ is a zero-dimensional left ideal in $\mathbb{O}\big)$

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4. The definitions **$\boldsymbol{\partial}$-finite** and **holonomic** differ only by some technical conditions.

Example: Legendre Polynomials $P_n(x)$

Important family of orthogonal polynomials $P_0(x), P_1(x), \ldots$:

$$\deg(P_n(x)) = n, \qquad \text{and} \qquad \frac{2n+1}{2}\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \delta_{m,n}.$$

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They are a particular solution of the Legendre differential equation:

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Corresponding operator: $(x^2-1)D_x^2 + 2xD_x - n(n+1)$.

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Corresponding operator: $(x^2 - 1) D_x^2 + 2x D_x - n(n+1)$.

Legendre polynomials also satisfy the three-term recurrence

$$n P_n(x) = (2n-1) x P_{n-1}(x) - (n-1) P_{n-2}(x).$$

Corresponding operator: $(n+2) S_n^2 - (2n+3) x S_n + (n+1)$.

Example: Legendre Polynomials $P_n(x)$

Important family of orthogonal polynomials $P_0(x), P_1(x), \ldots$:

$$\deg(P_n(x)) = n, \qquad \text{and} \qquad \frac{2n+1}{2}\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \delta_{m,n}.$$

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These operators live in the Ore algebra

$$\mathbb{K}(x,n)\langle D_x, S_n\rangle = \mathbb{K}(x,n)[D_x; 1, \tfrac{\mathrm{d}}{\mathrm{d}x}][S_n; \sigma_n, 0].$$

(Incomplete) List of ∂-Finite Functions

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, BesselI, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, BesselJ, StruveL, ArcSec, Factorial2, KelvinBer, BesselK, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin,

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n,k)$

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Creative Telescoping: write

$$c_r(n) f(n+r,k) + \cdots + c_0(n) f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_r(n) F(n+r) + \cdots + c_0(n) F(n) = g(n,b+1) - g(n,a).$$

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Consider the following integration problem: $F(x) = \int_a^b f(x,y)\,\mathrm{d}y$

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Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right) \mathrm{d}y \qquad = g(x,b) - g(x,a)$.

Creative Telescoping: write

$$c_r(x) \tfrac{\mathrm{d}^r}{\mathrm{d}x^r} f(x,y) + \cdots + c_0(x) f(x,y) \;= \tfrac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_r(x) \tfrac{\mathrm{d}^r}{\mathrm{d}x^r} F(x) + \cdots + c_0(x) F(x) \;= g(x,b) - g(x,a)$$

The Right-Hand Side

$$c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) = g(n,k+1) - g(n,k)$$
$$= (S_k - 1) \cdot g(n,k).$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_r(n)f(n+r,i) + \cdots + c_0(n)f(n,i)\big)$$

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A reasonable choice for where to look for g is $\mathbb{O}\cdot f$.

Then the task is to find $P(n,S_n) = c_r(n)S_n^r + \cdots + c_0(n)$ and $Q \in \mathbb{O}$ such that

$$\big(P - (S_k - 1)Q\big)\cdot f = 0 \quad \iff \quad P - (S_k - 1)Q \in \mathrm{Ann}_{\mathbb{O}}(f).$$

$\longrightarrow$ There are algorithms and implementations for that.

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

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```
<< RISC`HolonomicFunctions`
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
$$2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}$$

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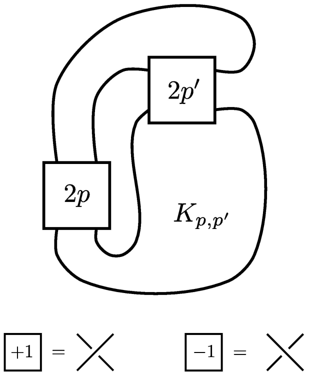
```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*BesselJ[a,2*Sqrt[t*x]
                    Der[t], {S[a], S[n], Der[x]}]
```

$$\{\{-2S_n + 2xD_x + (a+2n+2),\\ 4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),\\ 2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},\\ \{-2t, -4tx, -2tx\}\}$$

$\longrightarrow$ The annihilating ideals agree; check a few initial values.

Double Twist Knots

Consider the family of double twist knots $K_{p,p'}$:



$\longrightarrow$ Interesting family because their A-polynomials are reducible.

Colored Jones Function of $K_{p,p'}$

Using the Habiro theory of the colored Jones function, we get

$$J_{K_{p,p'},n}(q) = \sum_{k=0}^{n-1} (-1)^k c_{p,k}(q) c_{p',k}(q) q^{-kn-\frac{k(k+3)}{2}} (q^{n-1}; q^{-1})_k (q^{n+1}; q)_k$$

where $(a;q)_n$ denotes the q-Pochhammer symbol defined as

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

and where

$$c_{p,n}(q) = \sum_{k=0}^{n} (-1)^{k+n} q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2p} \frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

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$\longrightarrow$ Apply the HolonomicFunctions package.

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ which is 7_4.

Result:

- Recurrence of order 5, with M-degree 24 and q-degree 65
- corresponds to 4 printed pages

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Creative telescoping doesn't necessarily give the minimal-order recurrence.

Strategy:

To prove minimality, we show that the corresponding operator is irreducible.

$\longrightarrow$ Unfortunately, we cannot apply the previous criterion, since $A(1, M, L)$ in our case is reducible (double twist knots!).

Casoratian (shift analogue of the Wronskian):

For k sequences $f_n^{(i)}$, $i = 1, \ldots, k$, it is given by

$$W\big(f^{(1)}, \ldots, f^{(k)}\big)_n = \det_{\substack{0 \le j \le k-1 \\ 1 \le i \le k}} f_{n+j}^{(i)} = \begin{vmatrix} f_n^{(1)} & \cdots & f_n^{(k)} \\ \vdots & & \vdots \\ f_{n+k}^{(1)} & \cdots & f_{n+k}^{(k)} \end{vmatrix}.$$

Exterior Powers

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Exterior Powers:

- $P \in \mathbb{O}$ with $\deg_L(P) = d$
- notation: $\bigwedge^k P$ ("k-th exterior power of P")
- definition: minimal-order operator for $W\left(f^{(1)}, \ldots, f^{(k)}\right)_n$
- where $f^{(1)}, \ldots, f^{(k)}$ are assumed to be linearly independent solutions of $Pf = 0$.

Lemma

Lemma: Let $P = L^d + \sum_{j=0}^{d-1} a_j L^j \in \mathbb{O}$ with $a_0 \neq 0$, let $\left\{ f_n^{(1)}, \ldots, f_n^{(d)} \right\}$ be a fundamental solution set of the equation $Pf = 0$, and let $w = W(f^{(1)}, \ldots, f^{(d)})$. Then $w_{n+1} - (-1)^d a_0 w_n = 0$.

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Proof: This is proven by an elementary calculation

$$w_{n+1} = \begin{vmatrix} f_{n+1}^{(1)} & \cdots & f_{n+1}^{(d)} \\ \vdots & & \vdots \\ f_{n+d}^{(1)} & \cdots & f_{n+d}^{(d)} \end{vmatrix} = \begin{vmatrix} f_{n+1}^{(1)} & \cdots & f_{n+1}^{(d)} \\ \vdots & & \vdots \\ f_{n+d-1}^{(1)} & \cdots & f_{n+d-1}^{(d)} \\ -a_0 f_n^{(1)} & \cdots & -a_0 f_n^{(d)} \end{vmatrix} = (-1)^d a_0 w_n$$

(use $f_{n+d}^{(i)} = -\sum_{j=0}^{d-1} a_j f_{n+j}^{(i)}$ and row operations).

A Necessary and Sufficient Criterion for Irreducibility

Lemma: Let $P, Q, R \in \mathbb{O}$ such that $P = QR$ is a factorization of P, and let k denote the order of R, i.e., $k = \deg_L(R)$. Then $\bigwedge^k P$ has a linear right factor of the form $L - a$ for some $a \in \mathbb{K}(q, M)$.

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Proof:

- Let $F = \{f^{(1)}, \ldots, f^{(k)}\}$ be a fundamental solution set of R.
- By the lemma it follows that $w = W(f^{(1)}, \ldots, f^{(k)})$ satisfies a recurrence of order 1, say $w_{n+1} = a w_n, a \in \mathbb{K}(q, M)$.
- But F is also a set of linearly independent solutions of $Pf = 0$ and therefore w is contained in the solution space of $\bigwedge^k P$.
- It follows that $\bigwedge^k P$ has the right factor $L - a$.

Computation of Exterior Powers

As before let d denote the L-degree of P.

1. Ansatz for $\bigwedge^k P$:

$$c_\ell(q,M)w_{n+\ell} + \cdots + c_1(q,M)w_{n+1} + c_0(q,M)w_n = 0.$$

2. Replace all occurrences of w_{n+j} by the expansion of the Wronskian, e.g., for $k=2$:

$$w_{n+j} = f^{(1)}_{n+j} f^{(2)}_{n+j+1} - f^{(1)}_{n+j+1} f^{(2)}_{n+j}.$$

3. Rewrite each $f^{(i)}_{n+j}$ with $j \geq d$ as a $\mathbb{K}(q,M)$-linear combination of $f^{(i)}_n, \ldots, f^{(i)}_{n+d-1}$, using the equation $Pf^{(i)} = 0$.
4. Coefficient comparison with respect to $f^{(i)}_{n+j}$, $1 \leq i \leq k$, $0 \leq j < d$, yields a linear system for $c_0, \ldots, c_\ell$.

Exterior Powers of P_{7_4}

Some statistics concerning P_{7_4} and its exterior powers, according to the factorization of $P_{7_4}(1, M, L)$:

	L-degree	M-degree	q-degree	`ByteCount`
P_{7_4}	5	24	65	463,544
$\bigwedge^2 P_{7_4}$	10	134	749	37,293,800
$\bigwedge^3 P_{7_4}$	10	183	1108	62,150,408

$\longrightarrow$ We now have to prove that $\bigwedge^2 P_{7_4}$ and $\bigwedge^3 P_{7_4}$ have no linear right factors.

qHyper

Let $P(q, M, L) = p_d(q, M)L^d + \cdots + p_0(q, M)$, $p_i \in \mathbb{K}[q, M]$.

The qHyper algorithm (Abramov+Paule+Petkovšek 1998) attempts to find a right factor $L - r(q, M)$ of P where

$$r(q, M) = z(q)\frac{a(q, M)}{b(q, M)}\frac{c(q, qM)}{c(q, M)}, \quad a, b, c \in \mathbb{K}[q, M]$$

is assumed to be in normal form, defined by the conditions

$$\begin{aligned} \gcd\big(a(q, M), b(q, q^n M)\big) &= 1 \text{ for all } n \in \mathbb{N}, \\ \gcd\big(a(q, M), c(q, M)\big) &= 1, \\ \gcd\big(b(q, M), c(q, qM)\big) &= 1. \end{aligned}$$

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It is not difficult to show that under these assumptions

$$a(q, M) \mid p_0(q, M) \qquad \text{and} \qquad b(q, M) \mid p_d(q, q^{1-d}M).$$

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$\longrightarrow$ qHyper proceeds by testing all admissible choices of a and b.

Application of qHyper

Now let's apply qHyper to $P^{(2)}(q,M,L) := \bigwedge^2 P_{7_4}$ whose trailing and leading coefficients are given by

$$p_0(q,M) = q^{162}M^{44}(M-1)\left(\prod_{i=6}^{9}(q^iM-1)\right) \times \left(\prod_{i=6}^{10}(q^iM+1)(q^{2i+1}M^2-1)\right)F_1(q,M)$$

$$p_{10}(q,q^{-9}M) = q^{-397}(q^2M-1)\left(\prod_{i=4}^{7}(M-q^i)\right) \times \left(\prod_{i=4}^{8}(M+q^i)(M^2-q^{2i+1})\right)F_2(q,M)$$

where F_1 and F_2 are large irreducible polynomials, related by $q^{280}F_1(q,M) = F_2(q,q^{10}M)$.

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$$p_0(q,M) = q^{162}M^{44}(M-1)\left(\prod_{i=6}^{9}(q^iM-1)\right)$$

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$\longrightarrow$ A blind application of qHyper would result in $45 \cdot 2^{16} \cdot 2^{16} = 193\,273\,528\,320$ possible choices for a and b.

Confine the Number of qHyper's Test Cases

We exploit two facts:

Fact 1: Study the image under $q = 1$:

$$P^{(2)}(1, M, L) = R_1(M) \cdot (L - M^4) \cdot Q_1(M, L) \cdot Q_2(M, L)$$

where Q_1 and Q_2 are irreducible of L-degree 3 and 6, respectively. Thus we need only to test pairs (a, b) which satisfy the condition

$$(*) \qquad a(1, M) = M^4 b(1, M).$$

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Fact 2: a and b must fulfill the gcd condition:

$$\gcd(a(q, M), b(q, q^n M)) = 1 \text{ for all } n \in \mathbb{N}.$$

$\longrightarrow$ These two facts allow to exclude most of the admissible choices for a and b.

Structure of Leading and Trailing Coefficient

$$p_0(q,M) = q^{162}M^{44}(M-1)\Big(\prod_{i=6}^{9}(q^iM-1)\Big) \times \Big(\prod_{i=6}^{10}(q^iM+1)(q^{2i+1}M^2-1)\Big)F_1(q,M)$$

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	$p_0(q,M)$	$p_{10}(q,q^{-9}M)$
q^iM-1	0, 6, 7, 8, 9	-7, -6, -5, -4, 2
q^iM+1	6, 7, 8, 9, 10	-8, -7, -6, -5, -4
q^iM^2-1	13, 15, 17, 19, 21	-17, -15, -13, -11, -9

Linear and quadratic factors of the leading and trailing coefficients; each cell contains the values of i of the corresponding factors.

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2. Clearly the factor M^4 in (*) can only come from M^{44} in p_0; thus all other (linear and quadratic) factors in $a(1,M)/b(1,M)$ must cancel completely.
3. The most simple admissible choice is $a(q,M) = M^4$ and $b(q,M) = 1$.
4. Because of the gcd condition, a cancellation can almost never take place among factors which are equivalent under the substitution $q = 1$. This is reflected by the fact that the entries in the first column of the table are (row-wise) larger than those in the second column, e.g., $(q^6 M + 1) \mid a(q,M)$ and $(q^{-4} M + 1) \mid b(q,M)$ violates the gcd condition.

Which Combinations to Test

5. The only exception is that $(M-1) \mid a(q,M)$ cancels with $(q^2M-1) \mid b(q,M)$ in $a(1,M)/b(1,M)$. In that case, the gcd condition excludes further factors of the form q^iM-1, and together with (*) we see that no other factors at all can occur. This gives the choice $a(q,M) = M^4(M-1)$ and $b(q,M) = q^2M-1$.

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$\longrightarrow$ Summing up, we have to test 4504 cases only!

Results for Double Twist Knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order (irreducible!)

$K_{3,3}$:

- rigorous computation of $A(q, M, L)$
- (q, M, L)-degree $= (458, 74, 11)$
- minimality proof out of scope (requires $\bigwedge^5 P$ and $\bigwedge^6 P$)

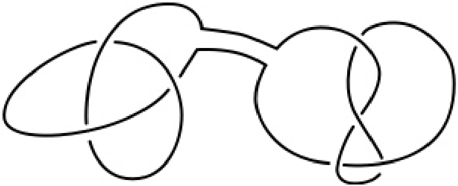
$K_{4,4}$:

- $A(q, M, L)$ guessed
- (q, M, L)-degree $= (2045, 184, 19)$

$K_{5,5}$:

- $A(q, M, L)$ guessed
- (q, M, L)-degree $= (6922, 396, 29)$, ByteCount = 8GB

Colored Jones for connected sum of knots



Fact: Let K_1 and K_2 be two knots in 3-space. Then the colored Jones function of their connected sum is given by

$$J_{K_1\#K_2,n}(q) = J_{K_1,n}(q)\, J_{K_2,n}(q) \quad \text{for all } n \in \mathbb{N}.$$

$\longrightarrow$ Like for the classical Jones polynomial.

Symmetric product

For $P_1, P_2 \in \mathbb{O}$ the **symmetric product** $P_1 \star P_2$ is the operator $P \in \mathbb{O}$ with minimal L-degree such that $P(f \cdot g) = 0$ for all sequences f and g for which $P_1(f) = 0$ and $P_2(g) = 0$.

Remark 1: P is unique up to multiplication by elements from $\mathbb{K}(q, M) \setminus \{0\}$.

Remark 2: The definition does not imply that the symmetric product gives the shortest recurrence for the product of two sequences.

Corollary: Let K_1 and K_2 be two knots and let $P_1, P_2 \in \mathbb{O}$ be annihilating operators of their colored Jones functions, respectively. Then the symmetric product $P_1 \star P_2$ annihilates $J_{K_1 \# K_2, n}(q)$.

Example

Consider the sequence $f(n) = q^n + (-1)^n$ whose minimal-order annihilating operator is $P = L^2 + (1-q)L - q$. As expected, the symmetric product $P \star P$ is of order 3:

$$\begin{aligned} P \star P &= L^3 - (q^2 - q + 1)L^2 - (q^2 - q + 1)L + q^3 \\ &= (L-1)(L+q)(L-q^2). \end{aligned}$$

On the other hand, we have $f(n)^2 = q^{2n} + 1 + 2(-q)^n$ and this expression is annihilated by the second-order operator

$$(qM^2 + 1)L^2 - (q-1)(q^2M^2 - 1)L - q(q^3M^2 + 1).$$

A-polynomial for connected sums

For two bivariate polynomials $A_1(M, L)$ and $A_2(M, L)$ we define the **A-product** $A_1 \diamond A_2$ as follows:

- let $I \subseteq \mathbb{K}(M)[L_1, L_2, L]$ be the ideal

$$\langle A_1(M, L_1), A_2(M, L_2), L - L_1 L_2 \rangle$$

- $A_1 \diamond A_2$ is the generator of the elimination ideal $I \cap \mathbb{K}(M)[L]$
- note that $\mathbb{K}(M)[L]$ is a PID, thence $A_1 \diamond A_2$ is unique up to multiplication by elements from $\mathbb{K}(M) \setminus \{0\}$.

Fact: Let K_1 and K_2 be two knots and $A_1(M, L)$ and $A_2(M, L)$ their respective A-polynomials. Then the A-polynomial of $K_1 \# K_2$ is given by $A_1 \diamond A_2$.

Theorem

We introduce the map ψ by

$$\psi\colon \mathbb{O} \to \mathbb{K}(M)[L],\ P(q, M, L) \mapsto P(1, M, L).$$

Theorem: Let $P_1(q, M, L)$ and $P_2(q, M, L)$ be two operators in the algebra $\mathbb{O}$. Then the following divisibility condition holds:

$$\psi(P_1) \diamond \psi(P_2) \mid \psi(P_1 \star P_2)$$

as polynomials in $\mathbb{K}(M)[L]$, provided that the above quantities are defined.

Proof (1)

Recall the algorithm for computing the symmetric power $P_1 \star P_2$.

- let $f(n)$ and $g(n)$ be generic sequences that are annihilated by P_1 and P_2, respectively
- make an ansatz for the minimal-order q-recurrence for the product $h(n) = f(n)g(n)$:

  $$c_d(q, M)h(n + d) + \cdots + c_0(q, M)h(n) = 0$$

  with undetermined coefficients $c_j \in \mathbb{K}(q, M)$.
- let d_1 and d_2 denote the L-degrees of P_1 and P_2, respectively.
- using the q-recurrence represented by P_1, we can rewrite $f(n + s)$ as a $\mathbb{K}(q, M)$-linear combination of $f(n), \ldots, f(n + d_1 - 1)$ for any $s \in \mathbb{N}$, and similarly for $g(n + s)$
- the ansatz therefore can be reduced to the following form:

  $$\sum_{s=0}^{d_1-1} \sum_{t=0}^{d_2-1} R_{s,t}(q, M, c_0, \ldots, c_d) f(n + s) g(n + t) = 0$$

Proof (2)

$$\sum_{s=0}^{d_1-1} \sum_{t=0}^{d_2-1} R_{s,t}(q, M, c_0, \ldots, c_d) f(n+s) g(n+t) = 0$$

- notation for the 2-tuples corresponding to the summands:

$$\{(s_0, t_0), (s_1, t_1), \ldots\} = \{(s,t) \mid 0 \le s \le d_1 - 1,\ 0 \le t \le d_2 - 1\}$$

- for example, put $s_i = \lfloor i/d_2 \rfloor$ and $t_i = i \bmod d_2$
- equating all $R_{s,t}$ to zero yields a linear system $M\boldsymbol{c} = 0$
- the matrix M is given by

$$M = (m_{i,j})_{0 \le i \le d_1 d_2 - 1, 0 \le j \le d} \quad \text{with} \quad m_{i,j} = \langle c_j \rangle R_{s_i, t_i}$$

- the algorithm proceeds by trying $d = 0$, $d = 1$, etc., until a solution is found; this guarantees minimality.
- if $d \ge d_1 d_2$ the linear system has more unknowns than equations so that a solution must exist; this ensures termination.

Proof (3)

To prove the claim, apply the above algorithm to $\psi(P_1)$ and $\psi(P_2)$.

- rewriting of $f(n+s)$ into $f(n), \ldots, f(n+d_1-1)$ can be rephrased as the (noncommutative) polynomial reduction of the operator L^s with P_1
- if instead $\psi(P_1)$ is used the noncommutativity disappears
- the reduction procedure boils down to a polynomial division with remainder in $\mathbb{K}(M)[L]$
- let $\operatorname{rem}(a, b)$ denote the remainder of dividing the polynomial a by b
- obtain a matrix $\tilde{M}$ with $\tilde{M} = \psi(M)$
- the entries $\psi(m_{i,j})$ of the matrix $\tilde{M}$ are obtained as follows:

$$\begin{aligned}\psi(m_{i,j}) &= \big(\langle L^{s_i}\rangle \operatorname{rem}(L^j, \psi(P_1))\big) \cdot \big(\langle L^{t_i}\rangle \operatorname{rem}(L^j, \psi(P_2))\big) \\ &= \langle L_1^{s_i} L_2^{t_i}\rangle \Big(\operatorname{rem}(L_1^j, P_1(1, M, L_1)) \cdot \operatorname{rem}(L_2^j, P_2(1, M, L_2))\Big)\end{aligned}$$

Proof (4)

- note that the set $G = \{P_1(1, M, L_1), P_2(1, M, L_2)\}$ is a Gröbner basis in $\mathbb{K}(M)[L_1, L_2]$ by Buchberger's product criterion
- can define $\operatorname{red}(P, G)$ for $P \in \mathbb{K}(M)[L_1, L_2]$ as the unique reductum of P with G
- Observe that

$$\operatorname{rem}\big(L_1^j, P_1(1, M, L_1)\big) \cdot \operatorname{rem}\big(L_2^j, P_2(1, M, L_2)\big) = \operatorname{red}\big((L_1 L_2)^j, G\big).$$

- the linear system $\tilde{M}\boldsymbol{c} = 0$ translates to the problem: find $c_0, \ldots, c_d \in \mathbb{K}(M)$ such that

$$\sum_{j=0}^{d} c_j(M) \operatorname{red}\big((L_1 L_2)^j, G\big) = 0.$$

Proof (5)

$$\sum_{j=0}^{d} c_j(M)\,\mathrm{red}\big((L_1L_2)^j, G\big) = 0.$$

- this can be rephrased as an elimination problem
- identify L_1L_2 with a new indeterminate L
- want to find a polynomial in $\mathbb{K}(M)[L]$, free of L_1 and L_2, in the ideal generated by G and $L - L_1L_2$
- this elimination problem is just the definition of $\psi(P_1) \diamond \psi(P_2)$
- Hence we have shown:

$$\psi(P_1) \star \psi(P_2) = \psi(P_1) \diamond \psi(P_2).$$

- we have $\deg_L \big(\psi(P_1 \star P_2)\big) \geq \deg_L \big(\psi(P_1) \star \psi(P_2)\big)$
- moreover: $\psi(P_1 \star P_2)$ is an element of the elimination ideal generated by $\psi(P_1) \diamond \psi(P_2)$
- therefore $\psi(P_1) \diamond \psi(P_2) \mid \psi(P_1 \star P_2)$ as claimed

To do

Problem: We now have established that both $\psi(P_1) \diamond \psi(P_2)$ and $\psi(P)$ divide $\psi(P_1 \star P_2)$, but of course this doesn't tell us anything about divisibility properties between $\psi(P_1) \diamond \psi(P_2)$ and $\psi(P)$. We would be interested exactly in that.

- identify nice conditions under which the symmetric product yields the minimal-order recurrence
- investigate degree drop under ψ

Minimality of inhomogeneous recurrences

Lemma: Let $f = (f_n)_{n\in\mathbb{N}}$ be a q-holonomic sequence and let $R \in \mathbb{O}$ be a minimal-order operator such that $Rf = u$ for some $u \in \mathbb{K}(q, M)$. If $Pf = 1$ for some $P \in \mathbb{O}$ then $u \neq 0$ and $P = QR$ for some $Q \in \mathbb{O}$.

Proof: Using right division with remainder, we can write $P = QR + S$ with $Q, S \in \mathbb{O}$ and $\deg_L(S) < \deg_L(R)$. Applying this operator to f yields

$$1 = Pf = QRf + Sf = Qu + Sf.$$

The remainder S must be zero, since otherwise $Sf = 1 - Qu$ is a contradiction to the minimality assumption on R; note that $Qu \in \mathbb{K}(q, M)$. Hence u must satisfy the equation $Qu = 1$, which implies $u \neq 0$, and $P = QR$ as claimed.

Minimize the order

algorithm for deriving minimal-order inhomogeneous recurrences:

- given: a particular sequence f satisfying $Pf = 1$
- want: the minimal-order operator $R \in \mathbb{O}$ with $Rf = 1$
- compute all possible monic right factors $R_1, \dots, R_s$ of P (including the trivial ones: 1 and P) so that $P = Q_1 R_1 = \cdots = Q_s R_s$
- for each $1 \leq i \leq s$ compute a basis $u_1, \dots, u_r \in \mathbb{K}(q, M)$ of rational solutions of $Q_i u = 1$
- if there is no such solution, then R_i is not a candidate for the minimal-order operator R
- otherwise, use finitely many values of f to decide whether there are $c_1, \dots, c_r \in \mathbb{K}(q)$ such that $R_i f = c_1 u_1 + \cdots + c_r u_r$
- choose the minimal-order R_i for which a $u \in \mathbb{K}(q, M)$ exists such that $R_i f = u$
- return $R = (1/u) R_i$

Example

Consider the connected sum $3_1\#3_1$. Its colored Jones polynomial satisfies $PJ_{3_1\#3_1,n}(q) = b$ with

$$
\begin{aligned}
P = {}& \left(M^4q^5 - 2M^3q^3 - M^2q^4 + M^2q + 2Mq^2 - 1\right) L^2 \\
&+ \left(-M^{10}q^{13} + 2M^9q^{12} + M^8q^{12} - M^8q^{11} - M^7q^{11} - M^6q^{10} + M^5 \right. \\
&+ \left(-M^{13}q^{13} + 2M^{12}q^{13} - M^{11}q^{13} + M^{11}q^{10} - 2M^{10}q^{10} + M^9q^{10}\right) \\
b = {}& M^{11}q^{11} - 2M^9q^{10} - M^9q^8 - M^8q^9 + M^7q^9 + 2M^7q^7 + M^6q^8 \\
&+ 2M^6q^6 - M^5q^6 - 2M^4q^5 - M^4q^3 + M^2q^2
\end{aligned}
$$

The operator P is reducible:

$$
\begin{aligned}
P = {}& \left((M^2q - 1)L + M^5q^9 - M^3q^6\right) \\
& \times \left((M^2q^2 - 2Mq + 1)L - M^8q^4 + 2M^7q^4 - M^6q^4\right)
\end{aligned}
$$

But this factorization doesn't yield a lower order recurrence for $J_{3_1\#3_1,n}(q)$. Hence P is of minimal order.

Some results

Consider connected sums of 3_1 and 4_1:

- $3_1\#3_1$: $\deg_L(P) = 2$, reducible into $1 + 1$
- $3_1\#4_1$: $\deg_L(P) = 5$, reducible into $2 + 1 + 2$ and $1 + 2 + 2$
- $4_1\#4_1$: $\deg_L(P) = 5$, reducible into $2 + 3$

$\longrightarrow$ in all cases the operator is reducible

$\longrightarrow$ nevertheless, in all cases it is already minimal