Holonomic Integration

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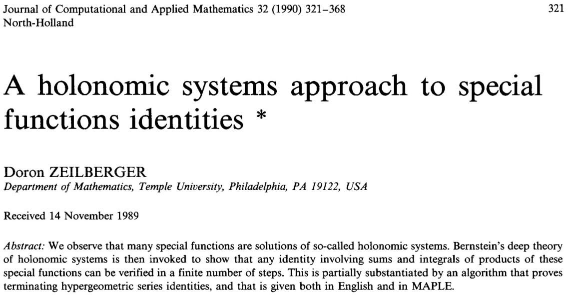
Austrian Academy of Sciences

Antidifferentiation and the Calculation of Feynman Amplitudes
6 October 2020, DESY Zeuthen

ÖAW RICAM

The Holonomic Systems Approach



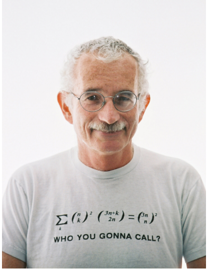
Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Received 14 November 1989

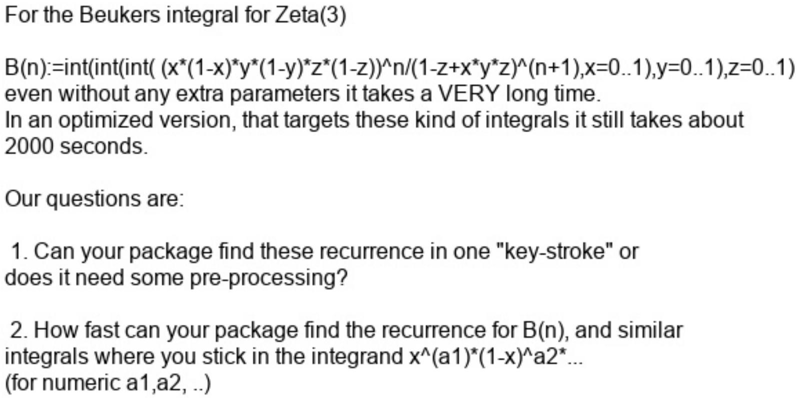
Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.



- seminal paper by Doron Zeilberger in 1990
- created a huge research area
- many applications in mathematics and elsewhere
  (maybe also in the Calculation of Feynman Amplitudes?)

Motivating Example

An e-mail from Doron Zeilberger (dated 28.08.2020):



For the Beukers integral for Zeta(3)

B(n):=int(int(int((x*(1-x)*y*(1-y)*z*(1-z))^n/(1-z+x*y*z)^(n+1),x=0..1),y=0..1),z=0..1)
even without any extra parameters it takes a VERY long time.
In an optimized version, that targets these kind of integrals it still takes about
2000 seconds.

Our questions are:

1. Can your package find these recurrence in one "key-stroke" or
does it need some pre-processing?

2. How fast can your package find the recurrence for B(n), and similar
integrals where you stick in the integrand x^(a1)*(1-x)^a2*...
(for numeric a1,a2, ..)

Beukers Integral

The Beukers integral for $\zeta(3)$

$$F(n) = \int_0^1 \int_0^1 \int_0^1 \frac{\bigl(x(1-x)y(1-y)z(1-z)\bigr)^n}{(1-z+xyz)^{n+1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

satisfies the second-order (Apéry) recurrence:

$$(n+2)^3F(n+2) = (2n+3)(17n^2+51n+39)F(n+1)-(n+1)^3F(n).$$

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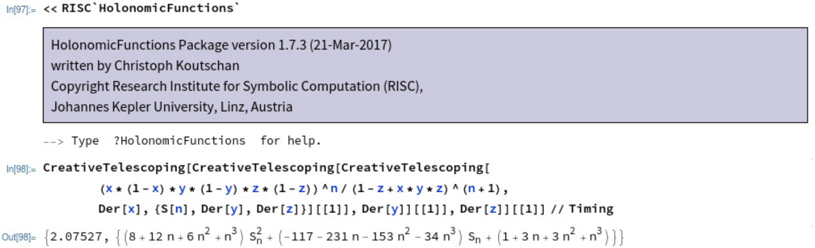
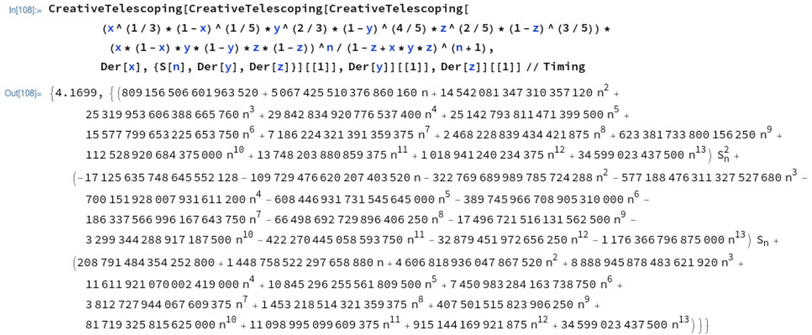


```
In[97]:= << RISC`HolonomicFunctions`

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

--> Type  ?HolonomicFunctions  for help.

In[98]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
           (x * (1 - x) * y * (1 - y) * z * (1 - z)) ^ n / (1 - z + x * y * z) ^ (n + 1),
           Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing

Out[98]= {2.07527, {(8 + 12 n + 6 n^2 + n^3) S_n^2 + (-117 - 231 n - 153 n^2 - 34 n^3) S_n + (1 + 3 n + 3 n^2 + n^3)}}
```

Beukers Integral with Numeric Parameters

$$\int_0^1\int_0^1\int_0^1 x^{1/3}(1-x)^{1/5}y^{2/3}(1-y)^{4/5}z^{2/5}(1-z)^{3/5} \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$



```
In[108]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
             (x ^ (1 / 3) * (1 - x) ^ (1 / 5) * y ^ (2 / 3) * (1 - y) ^ (4 / 5) * z ^ (2 / 5) * (1 - z) ^ (3 / 5)) *
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Out[108]= {4.1699, {(809 156 506 601 963 520 + 5 067 425 510 376 860 160 n + 14 542 081 347 310 357 120 n^2 +
          25 319 953 606 388 665 760 n^3 + 29 842 834 920 776 537 400 n^4 + 25 142 793 811 471 399 500 n^5 +
          15 577 799 653 225 653 750 n^6 + 7 186 224 321 391 359 375 n^7 + 2 468 228 839 434 421 875 n^8 + 623 381 733 800 156 250 n^9 +
          112 528 920 684 375 000 n^10 + 13 748 203 880 859 375 n^11 + 1 018 941 240 234 375 n^12 + 34 599 023 437 500 n^13) S_n^2 +
        (-17 125 635 748 645 552 128 - 109 729 476 620 207 403 520 n - 322 769 689 989 785 724 288 n^2 - 577 188 476 311 327 527 680 n^3 -
          700 151 928 007 931 611 200 n^4 - 608 446 931 731 545 645 000 n^5 - 389 745 966 708 905 310 000 n^6 -
          186 337 566 996 167 643 750 n^7 - 66 498 692 729 896 406 250 n^8 - 17 496 721 516 131 562 500 n^9 -
          3 299 344 288 917 187 500 n^10 - 422 270 445 058 593 750 n^11 - 32 879 451 972 656 250 n^12 - 1 176 366 796 875 000 n^13) S_n +
        (208 791 484 354 252 800 + 1 448 758 522 297 658 880 n + 4 606 818 936 047 867 520 n^2 + 8 888 945 878 483 621 920 n^3 +
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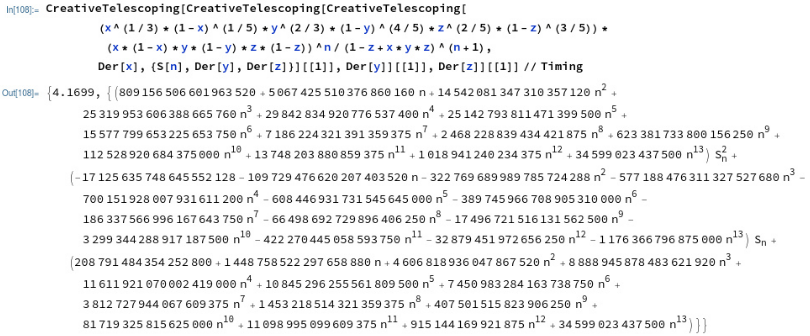

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```

$\longrightarrow$ Wow, we are really impressed!

We will rave about your package in our forthcoming paper...

Beukers Integral with Six Symbolic Parameters

$$\int_0^1\int_0^1\int_0^1 x^b(1-x)^{c-f}y^e(1-y)^{a+f}z^a(1-z)^c \times \frac{\bigl(x(1-x)y(1-y)z(1-z)\bigr)^n}{(1-z+xyz)^{n+d+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

D-finite and P-recursive

A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_d(x)f^{(d)}(x) \;+\; \dots \;+\; p_1(x)f'(x) \;+\; p_0(x)f(x) \;=\; 0,$$

$p_0, \dots, p_d \in \mathbb{K}[x]$ not all zero.

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$\longrightarrow$ In both cases, only **finitely** many initial conditions are needed!

$\longrightarrow$ Also called **holonomic function** resp. **holonomic sequence**.

Example: Harmonic Numbers

Example: The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the recurrence

$$nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2} \qquad (n \geqslant 2)$$

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Generalize the finiteness property to

- multivariate functions $f(x_1, \ldots, x_s)$
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- multivariate functions $f(x_1, \ldots, x_s)$ (the x_i are **continuous variables**)
- multidimensional sequences $f(n_1, \ldots, n_r)$ (the n_i are **discrete variables**)
- mixed setting: functions in several continuous and discrete variables $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2-1)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0.$$

Consider the set $\left\{P_n^{(i)}(x) \mid i \geqslant 0\right\}$.

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$$-\tfrac{6x}{x^2-1}P_n^{(3)}(x)+\tfrac{(n-2)(n+3)}{x^2-1}P_n''(x)$$

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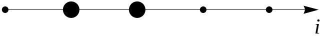
$$P_n^{(4)}(x) =$$

$$\tfrac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x)$$

$$-\tfrac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$



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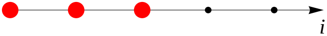
$$P_n^{(4)}(x) = \\ \tfrac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x) \\ -\tfrac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$



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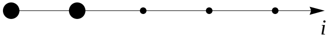
Consider the set $\left\{P_n^{(i)}(x) \mid i \geqslant 0\right\}$.

$$P_n^{(4)}(x) =$$

$$-\frac{8x(n^2x^2-n^2+nx^2-n+3x^2+3)}{(x^2-1)^3}P_n'(x)$$

$$+\frac{n(n+1)(n^2x^2-n^2+nx^2-n+18x^2+6)}{(x^2-1)^3}P_n(x)$$



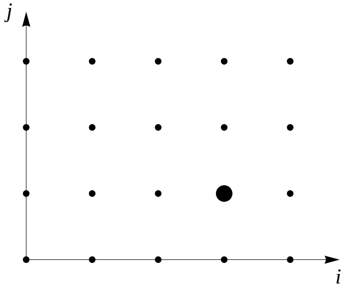
$\longrightarrow$ $P_n(x)$ is D-finite w.r.t. x.

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2 - 1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

Consider the set $\left\{P_{n+j}^{(i)}(x) \mid i, j \geqslant 0\right\}$.



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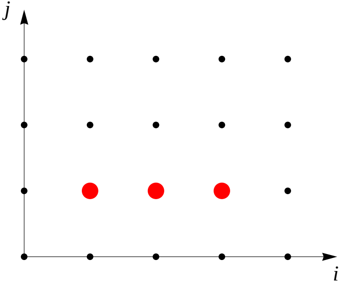

$$P_{n+1}^{(3)}(x) =$$

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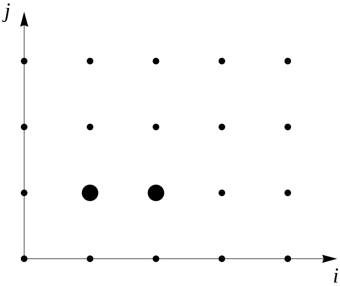

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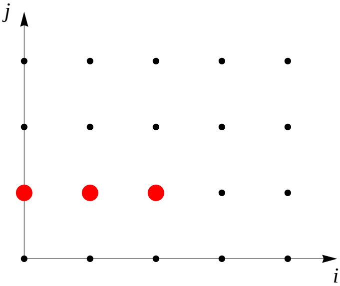

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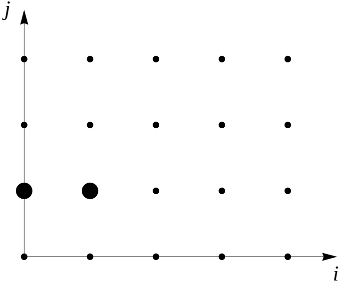

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$$\frac{(n^2x^2-n^2+3nx^2-3n+8x^2)}{(x^2-1)^2}P_{n+1}'(x)$$
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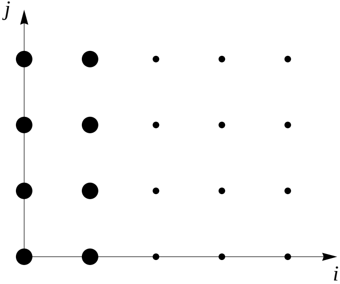

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The Legendre polynomials can be defined recursively:

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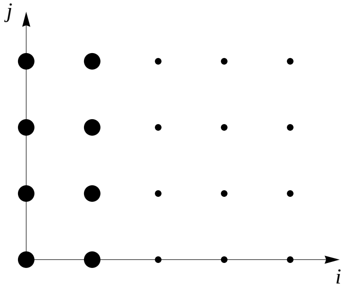

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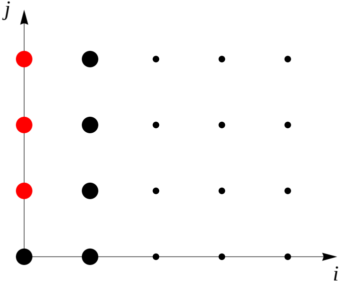

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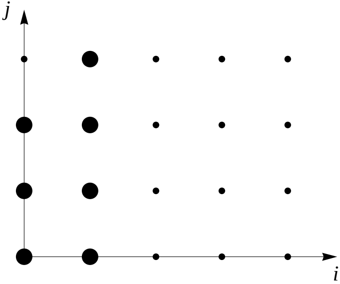

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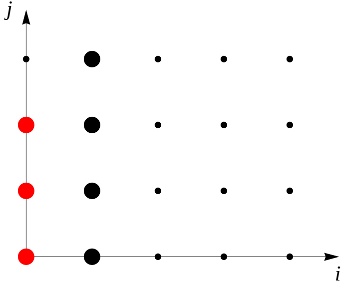

$$P_{n+3}(x) = \frac{4n^2x^2-n^2+16nx^2-4n+15x^2-4}{(n+2)(n+3)}P_{n+1}(x) - \frac{2n^2x+7nx+5x}{(n+2)(n+3)}P_n(x)$$

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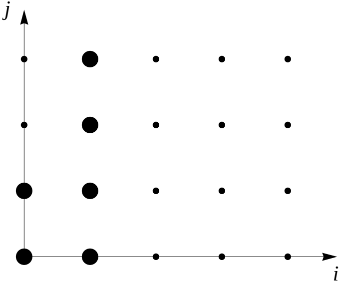

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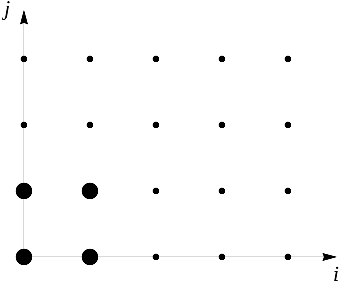

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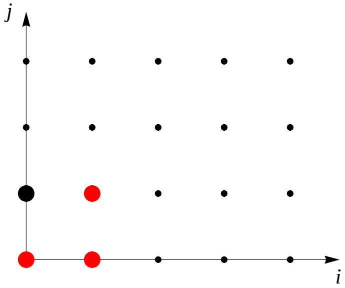

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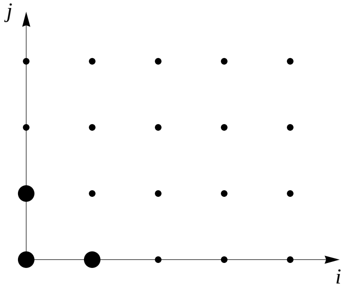

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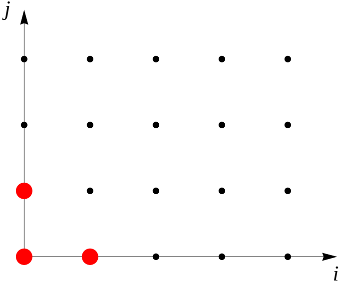

$$(n+1)P_{n+1}(x) + (1-x^2)P_n'(x) - (n+1)xP_n(x) = 0$$

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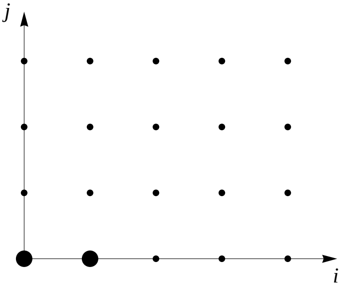

$\longrightarrow P_n(x)$ is ∂-finite w.r.t. n and x (of holonomic rank 2).

∂-Finiteness

Let $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$ be a function in the continuous variables $x_1, \ldots, x_s$ and in the discrete variables $n_1, \ldots, n_r$.

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Definition: f is called ∂-finite (or D-finite) if there is a finite set of basis functions of the form

$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \ldots, x_s, n_1 + j_1, \ldots, n_r + j_r)$$

with $i_1, \ldots, i_s, j_1, \ldots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \ldots, x_s, n_1, \ldots, n_r)$-linear combination of the basis functions.

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Again, finitely many initial conditions suffice to specify / fix f.

Algebraic Setting: Operators

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
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Example 2: The three-term recurrence

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

translates to the operator

$$(n+2)S_n^2 - (2n+3)xS_n + (n+1).$$

Operator Algebra

Differential equations/recurrences are translated to skew polynomials.

Noncommutative multiplication:

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Even more general:

$$\partial_v \cdot a = \sigma(a) \cdot \partial_v + \delta(a)$$

where σ is an automorphism and δ is a σ-derivation, i.e.,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

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Definition: Such operators form an **Ore algebra**

$$\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle,$$

i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $v, w, \dots$, where $\mathbb{K}$ is a field $(\mathrm{char}(\mathbb{K}) = 0)$.

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Definition: We define the **annihilator** of a function f to be the set

$$\mathrm{Ann}_{\mathbb{O}}\, f := \left\{ P \in \mathbb{O} \;\middle|\; P \cdot f = 0 \right\}$$

(it is a **left ideal** in the ring $\mathbb{O}$).

∂-Finite Functions

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

Definition: A function $f(v, w, \dots)$ is ***∂*-finite** w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f \mid P \in \mathbb{O}\}$$

form a **finite-dimensional** $\mathbb{K}(v, w, \dots)$-vector space:

$$\dim_{\mathbb{K}(v,w,\dots)} \big(\mathbb{O} / \operatorname{Ann}_{\mathbb{O}}(f)\big) < \infty.$$

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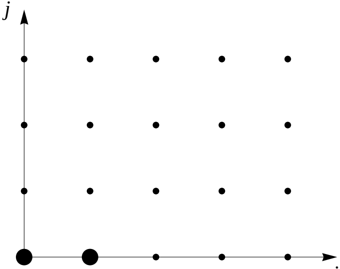

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In other words, if the left ideal of annihilating operators of f

$$\operatorname{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} \mid P \cdot f = 0\}$$

is a **zero-dimensional ideal**.

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   - multiplication, e.g., $P_n(x)P_{n+1}(x)$
   - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
   - operator application, e.g., $P'_{n+2}(x)$

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   - multiplication, e.g., $P_n(x)P_{n+1}(x)$
   - certain substitutions, e.g., $P_{2n+3}\big(\sqrt{x^2+1}\big)$
   - operator application, e.g., $P'_{n+2}(x)$
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Why ∂-Finite Functions?

1. Definition gives rise to a **finite data structure**.
    - annihilating ideal of operators (finite set of generators)
    - finitely many initial values
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3. These operations (closure properties) can be executed **algorithmically**.
4. Many elementary and **special functions** are covered.

Holonomic Functions

Definition: Let $f(x_1, \ldots, x_s)$ depend only on continuous variables. Consider the **Weyl algebra**

$$\mathbb{W} = \mathbb{K}[x_1, \ldots, x_s]\langle D_{x_1}, \ldots, D_{x_s} \rangle.$$

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Differently stated: f is holonomic if for any $(s-1)$-subset

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there exists a nonzero element in $\operatorname{Ann}_{\mathbb{W}}(f)$ that is free of all generators in E, we call this the **elimination property**.

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Sequences: $f(n_1, \ldots, n_s)$ is holonomic if its generating function

$$F(x_1, \ldots, x_s) := \sum_{n_1=0}^{\infty} \cdots \sum_{n_s=0}^{\infty} f(n_1, \ldots, n_s) x_1^{n_1} \cdots x_s^{n_s}$$

is holonomic in the above sense.

∂-Finite and Holonomic Functions

Theorem: The function $f(x_1, \ldots, x_s)$ is holonomic if and only if it is ∂-finite.

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Application: Combine the two notions:

- Use ∂-finiteness for computations.
- Use holonomy for justifications (existence, termination).

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3. Integrals and sums are treated by the method of **creative telescoping**.
4. The output is always given as an annihilating ideal, **not as a closed form**.

List of ∂-Finite and Holonomic Functions

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, BesselI, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, BesselJ, StruveL, ArcSec, Factorial2, KelvinBer, BesselK, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin,

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Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

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$$c_d(n)f(n+d,k) + \ldots + c_0(n)f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

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Method for doing integrals and sums

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Consider the following integration problem: $F(x) = \displaystyle\int_a^b f(x,y)\,\mathrm{d}y$

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$$c_d(x) \tfrac{\mathrm{d}^d}{\mathrm{d}x^d} f(x,y) + \ldots + c_0(x) f(x,y) \;\; = \tfrac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_d(x) \tfrac{\mathrm{d}^d}{\mathrm{d}x^d} F(x) + \ldots + c_0(x) F(x) \;\; = g(x,b) - g(x,a)$$

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We aim at computing a creative telescoping relation of the form:

$$\begin{aligned} c_d(n)f(n+d,k) + \ldots + c_0(n)f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1) \cdot g(n,k). \end{aligned}$$

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Task: find $P(n,S_n)=c_d(n)S_n^d+\ldots+c_0(n)$ and $Q\in\mathbb{O}$ such that

$$\big(P-(S_k-1)Q\big)\cdot f=0 \quad\iff\quad P-(S_k-1)Q\in\operatorname{Ann}_{\mathbb{O}}(f).$$

Creative Telescoping Example

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The function f is ∂-finite with holonomic rank 2 (Basis: f, $\frac{\mathrm{d}}{\mathrm{d}y} f$)

$$\begin{aligned} P &= x^2 D_x^2 + x D_x - x^2 - \nu^2 \\ Q &= (y^2+1) D_y + \frac{y^2 - 2\nu y^2 - 2\nu - 1}{y} \end{aligned}$$

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Integrating $(P - D_y Q) \cdot f = 0$, i.e., $P \cdot f = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$, yields

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Indeed, we have $F(x) = K_\nu(x)$.

Computing CT Relations

Idea: Make an ansatz for the telescoper P and the certificate Q.

Telescoper: Fix an integer r and set

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Certificate:

Let $\mathfrak{U}$ denote the set of monomials under the stairs of a Gröbner basis for $\operatorname{Ann}_{\mathbb{O}}(f)$, or any other vector space basis of $\mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)$.

Since $Q \in \mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)$, we can set

$$Q = \sum_{u \in \mathfrak{U}} q_u(x,y)\, u \qquad \text{with unknowns } q_u \in \mathbb{K}(x,y).$$

Chyzak's Algorithm

Putting things together:

$$P - D_y Q = \sum_{i=0}^{r} p_i(x) D_x^i - D_y \sum_{u \in \mathfrak{U}} q_u(x,y)\, u$$

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Since we want $P - D_y Q \in \mathrm{Ann}_{\mathbb{O}}(f)$ we **reduce** the above expression with a Gröbner basis of $\mathrm{Ann}_{\mathbb{O}}(f)$ and equate the (D_x, D_y)-coefficients to zero.

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Finally: loop over the (a priori) unknown order r of the telescoper.
$\longrightarrow$ This is Chyzak's algorithm (analogously in other Ore algebras).

Creative Telescoping in Full Generality

In general, a creative telescoping operator has the form

$$P(\boldsymbol{x}, \boldsymbol{\partial_x}) + \Delta_1 Q_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\partial_x}, \boldsymbol{\partial_y}) + \ldots + \Delta_m Q_m(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\partial_x}, \boldsymbol{\partial_y})$$

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- The certificates **certify** the correctness of the telescoper.
- Research topic: develop fast algorithms to compute it!

Ansatz with Specific Denominators

For finding CT operators, we proposed an ansatz of the form

$$\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{\alpha}} + \sum_{i=1}^{m} \Delta_i \sum_{u\in\mathfrak{U}} \frac{\sum_{\boldsymbol{\beta}} q_{i,j,\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{y}^{\boldsymbol{\beta}}}{d_{i,j}(\boldsymbol{x},\boldsymbol{y})}\, u$$

with unknowns $p_{\boldsymbol{\alpha}}$ and $q_{i,j,\boldsymbol{\beta}}$, and with specific denominators $d_{i,j}$.

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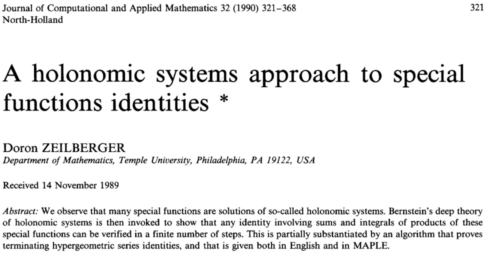
$\longrightarrow$ Hence, the desired telescoper is $p_0 + p_1 D_x + \ldots + p_r D_x^r$.

Application: Special Function Identities



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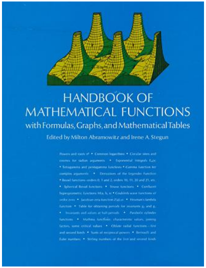
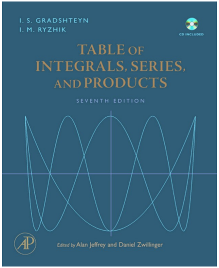
A holonomic systems approach to special functions identities *

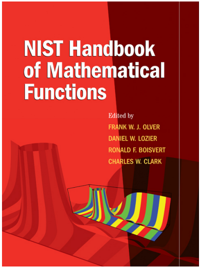
Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

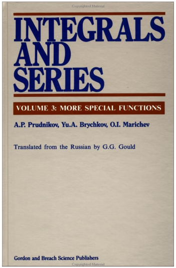
Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.









Examples of Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{1}$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x = \frac{\pi P_m^{(m+\frac{1}{2},-m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \tag{2}$$

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\,\mathrm{d}t \tag{3}$$

$$\int_{-\infty}^\infty \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!n!}\,\mathrm{d}x = \sqrt{\pi}e^{2rs} \tag{4}$$

$$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu)J_{n+\nu}(a)}{2^{\nu-1}a^\nu n!\Gamma(\nu)} \tag{5}$$

$$\frac{\sin\left(\sqrt{z^2+2tz}\right)}{z} = \sum_{n=0}^\infty \frac{(-t)^n y_{n-1}(z)}{n!} \tag{6}$$

Computer Proof of a Special Function Identity

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```
<< RISC`HolonomicFunctions`
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\begin{aligned}&\{2S_n - 2xD_x + (-a-2n-2),\\ &\ 4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),\\ &\ 2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\}\end{aligned}$$

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```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*BesselJ[a,2*Sqrt[t*x]],
                    Der[t], {S[a], S[n], Der[x]}]
```

$$\begin{aligned}\{\{&-2S_n + 2xD_x + (a+2n+2),\\ &4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),\\ &2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},\\ \{&-2t, -4tx, -2tx\}\}\end{aligned}$$

⟶ The annihilating ideals agree; check a few initial values.

Application in Theoretical Physics

Joint work with

Youssef Abdelaziz, Salah Boukraa, Jean-Marie Maillard

- Diagonals of rational functions, pullbacked ${}_2F_1$ hypergeometric functions and modular forms (JPA 51(45), 455201, 2018)
- Heun functions and diagonals of rational functions (JPA 53(7), 075206, 2020)
- On Christol's conjecture (JPA 53(20), 205201, 2020)

Diagonals of Rational Functions

Given a rational function in n variables

$$R(x_1, \ldots, x_n) = \frac{A(x_1, \ldots, x_n)}{B(x_1, \ldots, x_n)},$$

where $A, B \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $B(0, \ldots, 0) \neq 0$.

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Definition: The diagonal of R is defined through its multi-Taylor expansion around $(0, \ldots, 0)$:

$$R(x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} r_{m_1,\ldots,m_n} \cdot x_1^{m_1} \cdots x_n^{m_n},$$

as the power series in one variable:

$$\mathrm{Diag}\big(R(x_1, \ldots, x_n)\big) := \sum_{m=0}^{\infty} r_{m,m,\ldots,m} \cdot x^m.$$

Example of a Diagonal

Consider the Taylor expansion of the bivariate rational function

$$\begin{aligned} f(x,y) &= \frac{1}{1-x-y-2xy} \\ &= 1+x+y+x^2+4xy+y^2+x^3+7x^2y+7xy^2+\dots \end{aligned}$$

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Then the diagonal of f is

$$\mathrm{Diag}(f) = 1 + 4x + 22x^2 + 136x^3 + 886x^4 + 5944x^5 + \dots$$

Properties of Diagonals

Theorem: The diagonal $f(x)$ of every rational function is

- **globally bounded:** there exist integers $c, d \in \mathbb{N}^*$, such that $d \cdot f(cx) \in \mathbb{Z}[[x]]$ and $f(x)$ has nonzero radius of convergence.

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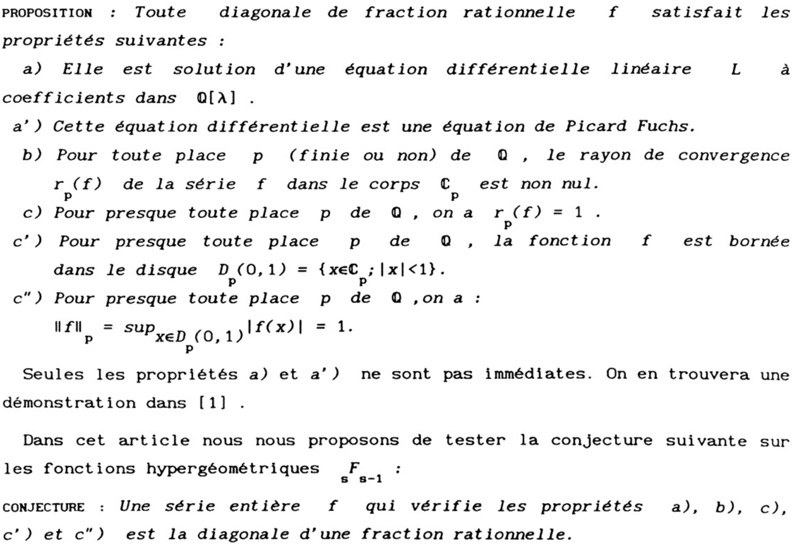
- It doesn't say anything about the number of variables in the rational function.
- One needs at least three variables, but no explicit example requiring more than three variables is known.

Christol's Conjecture



PROPOSITION : *Toute diagonale de fraction rationnelle f satisfait les propriétés suivantes :*

a) Elle est solution d'une équation différentielle linéaire L à coefficients dans $\mathbb{Q}[\lambda]$.

a') Cette équation différentielle est une équation de Picard Fuchs.

b) Pour toute place p (finie ou non) de $\mathbb{Q}$, le rayon de convergence $r_p(f)$ de la série f dans le corps $\mathbb{C}_p$ est non nul.

c) Pour presque toute place p de $\mathbb{Q}$, on a $r_p(f) = 1$.

c') Pour presque toute place p de $\mathbb{Q}$, la fonction f est bornée dans le disque $D_p(0,1) = \{x\in\mathbb{C}_p ; |x|<1\}$.

c") Pour presque toute place p de $\mathbb{Q}$,on a :

$$\|f\|_p = \sup_{x\in D_p(0,1)}|f(x)| = 1.$$

Seules les propriétés *a)* et *a')* ne sont pas immédiates. On en trouvera une démonstration dans [1] .

Dans cet article nous nous proposons de tester la conjecture suivante sur les fonctions hypergéométriques ${}_sF_{s-1}$:

CONJECTURE : *Une série entière f qui vérifie les propriétés a), b), c), c') et c") est la diagonale d'une fraction rationnelle.*

Hadamard Product

Definition: The Hadamard product of two series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n$$

is given by

$$f(x) \star g(x) = \sum_{n=0}^{\infty} \alpha_n \cdot \beta_n \cdot x^n.$$

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Example:

Let $f(x) = \mathrm{Diag}\big(F(x_1, \ldots, x_r)\big)$ and $g(x) = \mathrm{Diag}\big(G(x_1, \ldots, x_s)\big)$.
Then:

$$f(x) \star g(x) = \mathrm{Diag}\Big(F(x_1, \ldots, x_r) \cdot G(x_{r+1}, \ldots, x_{r+s})\Big).$$

Hypergeometric Functions

Definition: Let $(a)_k := a \cdot (a+1) \cdots (a+k-1)$. Then

$${}_pF_q\big([a_1, \ldots, a_p], [b_1, \ldots, b_q]\,, x\big) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \cdot \frac{x^k}{k!}$$

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Note: Any such hypergeometric function is D-finite, for example: the classical Gauß hypergeometric ${}_2F_1([a,b],[c],x)$ function satisfies Euler's differential equation:

$$ x(x-1)\,y''(x) + ((a+b+1)x-c)\,y'(x) + ab\,y(x) = 0. $$

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Hypergeometric functions of the form ${}_pF_{p-1}$ provide a natural testing ground for Christol's conjecture:

- If $q < p-1$ then the ${}_pF_q$ series has zero radius of convergence.

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- If $q < p-1$ then the ${}_pF_q$ series has zero radius of convergence.
- If $q > p-1$ then the ${}_pF_q$ series cannot be globally bounded.

Height

Definition: Let $f(x)$ be a hypergeometric function of the form

$$f(x) = {}_pF_{p-1}\big([a_1, \ldots, a_p], [b_1 \ldots, b_{p-1}], x\big).$$

Setting $b_p = 1$, the height of f is

$$h = \big|\{1 \leqslant j \leqslant p \mid b_j \in \mathbb{Z}\}\big| - \big|\{1 \leqslant j \leqslant p \mid a_j \in \mathbb{Z}\}\big|.$$

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Theorem (Christol): A ${}_pF_{p-1}$ hypergeometric function of height 1 is globally bounded if and only if it is algebraic.

Rational vs. Algebraic

Example: The globally bounded hypergeometric function

$$f(x) = {}_3F_2\big(\big[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\big], [1,1], x\big) = 1 + \tfrac{1}{27}x + \tfrac{8}{729}x^2 + \ldots$$

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By noting that ${}_1F_0([\frac{1}{3}], [\,], x) = (1 - x)^{-1/3}$, we see that $f(x)$ is the diagonal of an **algebraic function** in three variables:

$$f(x) = \operatorname{Diag}\Big((1 - x)^{-1/3} \cdot (1 - y)^{-1/3} \cdot (1 - z)^{-1/3}\Big).$$

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Theorem (Denef, Lipshitz): Any power series in $\mathbb{Q}\,[[x_1, \dots, x_n]]$, algebraic over $\mathbb{Q}(x_1, \dots, x_n)$, is the diagonal of a rational function in $2n$ variables.

Situation for 2F1 Functions

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For example, let $a, b \in \mathbb{Q} \setminus \mathbb{Z}$.

- $c \in \mathbb{N}$: in this case the ${}_2F_1$ function is automatically globally bounded and can be written as the Hadamard product of two (algebraic) ${}_1F_0$ functions.

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Hence, this situation is not particularly interesting for our purposes.

Situation for 3F2 Functions

When is it easy to see that a globally bounded hypergeometric function ${}_3F_2([a, b, c], [d, e], x)$ with $a, b, c \in \mathbb{Q} \setminus \mathbb{Z}$ is the diagonal of a rational function?

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Even in this case, a lot of the ${}_3F_2$ functions are easily seen to be diagonals of rational functions...

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Suppose now that $f(x) = {}_3F_2([a,b,c],[d,1],x)$ is globally bounded, with the parameters $a,b,c,d \in \mathbb{Q} \setminus \mathbb{Z}$.

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Then there are six ways to write this function as a Hadamard product of hypergeometric functions:

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Thus if one of ${}_2F_1([a,b],[d],x)$, ${}_2F_1([b,c],[d],x)$, ${}_2F_1([a,c],[d],x)$ is algebraic, then $f(x)$ is the diagonal of a rational function.

Potential Counterexamples

Potential counterexamples to Christol's conjecture were constructed in a way that avoids them being written as "simple" Hadamard products of algebraic functions.

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Christol came up with an unresolved example to his conjecture

- G. Christol, *Fonctions hypergéométriques bornées*, Groupe d'Etude d'Analyse ultramétrique, vol. 14 (1986–1987), Exposé N$^\circ$ 8, p. 1–16.

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A longer list was generated by Christol and his co-authors in 2012.

- A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard *Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity.* Journal of Physics A: Mathematical and Theoretical **46**(18)

Potential Counterexamples

For example, these two hypergeometric functions are globally bounded, as they can be recast into series with integer coefficients:

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], 3^6 x\right) = 1 + 120x + 47124x^2 + 23483460x^3 + \dots $$

$$ {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}, 1\right], 3^6 x\right) = 1 + 84x + 32760x^2 + 16302000x^3 + \dots $$

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But they cannot be obtained as diagonals through Hadamard products, since the following series are not globally bounded:

$$ {}_2F_1\left(\left[\tfrac{2}{9}, \tfrac{5}{9}\right], \left[\tfrac{2}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{2}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}\right], x\right), $$

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$$ = 1 + \frac{2/9 \cdot 5/9}{2/3 \cdot 1} \cdot x + \frac{(2/9 \cdot 11/9) \cdot (5/9 \cdot 14/9)}{(2/3 \cdot 5/3) \cdot (1 \cdot 2)} \cdot x^2 + \dots $$

$$ \dots + \frac{2 \cdot 11 \cdot 20 \cdots (9k-7) \cdot 5 \cdot 14 \cdot 23 \cdots (9k-4)}{2 \cdot 5 \cdot 8 \cdots (3k-1) \cdot 1 \cdot 2 \cdot 3 \cdots k} \cdot \left(\frac{x}{27}\right)^k + \dots $$

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- If $p \equiv 2 \mod 9$ or if $p \equiv 5 \mod 9$ then it gets cancelled in the k-th term.

Not Globally Bounded

$$ {}_2F_1\left(\left[\tfrac{2}{9}, \tfrac{5}{9}\right], \left[\tfrac{2}{3}\right], x\right) = $$

$$ = 1 + \frac{2/9 \cdot 5/9}{2/3 \cdot 1} \cdot x + \frac{(2/9 \cdot 11/9) \cdot (5/9 \cdot 14/9)}{(2/3 \cdot 5/3) \cdot (1 \cdot 2)} \cdot x^2 + \dots $$

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There are infinitely many prime factors in the Taylor expansion, and therefore the function is not globally bounded.

Towards Christol

Theorem: The hypergeometric functions

$${}_3F_2\big(\big[\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}\big],\big[\tfrac{2}{3},1\big],27x\big) \quad \text{and} \quad {}_3F_2\big(\big[\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9}\big],\big[\tfrac{1}{3},1\big],27x\big)$$

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More precisely, we have:

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More generally, $\operatorname{Diag}\left(\frac{(1-x-y)^{a/b}}{1-x-y-z}\right)$ is shown to evaluate to

$$ {}_3F_2\left(\left[\frac{3a-b}{3a},\frac{2a-b}{3a},\frac{a-b}{3a}\right],\left[\frac{a-b}{a},1\right],27x\right). $$

Proof

The denominator of the algebraic function $\dfrac{(1-x-y)^{a/b}}{(1-x-y-z)}$ is expanded as a geometric series:

$$(1-x-y-z)^{-1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\binom{n}{m}\binom{m}{l}\cdot x^{l}y^{m-l}z^{n-m},$$

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while the numerator can be expanded as

$$\sum_{k=0}^{\infty}\frac{(-a/b)_k}{k!}\cdot(x+y)^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-a/b)_k}{k!}\cdot\binom{k}{j}x^j y^{k-j}.$$

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Multiplying these two sums and re-indexing, we obtain:

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}x^{s}y^{t}z^{u}\sum_{j=0}^{s}\sum_{k=0}^{\infty}\frac{(-a/b)_k}{k!}\binom{k}{j}\binom{s+t+u-k}{s+t-k}\binom{s+t-k}{s-j}.$$

Proof

Hence the diagonal coefficient of $x^n y^n z^n$ is given by

$$\sum_{j=0}^{n} \sum_{k=0}^{\infty} \frac{(-a/b)_k}{k!} \cdot \binom{k}{j} \binom{3n-k}{2n-k} \binom{2n-k}{n-j},$$

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which by the Chu-Vandermonde identity

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Now use a computer algebra tool like Mathematica or Maple to simplify this sum further into a closed form. . .

Proof

More precisely, we employ creative telescoping to find that

$$\binom{2n}{n} \cdot \sum_{k=0}^{2n} \frac{(-a/b)_k}{k!} \cdot \binom{3n-k}{2n-k} \quad =: S(n)$$

satisfies the first-order recurrence

$$\begin{aligned}&(a-3b-3bn)\cdot(a-2b-3bn)\cdot(a-b-3bn)\cdot S(n)\\ &= b^2\cdot(n+1)^2\cdot(a-b-bn)\cdot S(n+1).\end{aligned}$$

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Together with the initial value $S(0)=1$, we get the closed form

$$S(n) = \frac{3^{3n}\cdot\left(\frac{b-a}{3b}\right)_n\cdot\left(\frac{2b-a}{3b}\right)_n\cdot\left(\frac{3b-a}{3b}\right)_n}{\left(\frac{b-a}{b}\right)_n\cdot\left(n!\right)^2},$$

yielding the hypergeom. function representation of the diagonal.

Diagonals as Integrals

Note that a diagonal $\operatorname{Diag}\bigl(R(x,y,z)\bigr)$ can also be expressed as

$$\langle y^0 z^0\rangle\, R\Bigl(\frac{x}{y},\frac{y}{z},z\Bigr) = \operatorname{res}_{y,z}\frac{1}{yz}R\Bigl(\frac{x}{y},\frac{y}{z},z\Bigr) = \oint \frac{1}{yz}R\Bigl(\frac{x}{y},\frac{y}{z},z\Bigr)\mathrm{d}y\,\mathrm{d}z.$$

where $\langle y^0 z^0\rangle$ denotes the constant coefficient w.r.t. y and z.

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Indeed, writing

$$R(x,y,z) = \sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{n\geqslant 0} r_{l,m,n}\, x^l y^m z^n$$

one obtains

$$R\Big(\frac{x}{y},\frac{y}{z},z\Big) = \sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{n\geqslant 0} a_{l,m,n}\, x^l y^{m-l} z^{n-m}.$$

Proof by Creative Telescoping

Compute a linear differential operator that annihilates the diagonal of our algebraic function, by applying creative telescoping to

$$\oint \frac{1}{yz} R\Big(\frac{x}{y}, \frac{y}{z}, z\Big)\,\mathrm{d}y\,\mathrm{d}z = \oint \frac{(1 - x/y - y/z)^{a/b}}{yz - xz - y^2 - yz^2}\,\mathrm{d}y\,\mathrm{d}z$$

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We obtain the following telescoper of order three:

$$\begin{aligned} & b^3 x^2 (1 - 27x) \cdot D_x^3 + b^2 x((27a - 135b) \cdot x - a + 3b) \cdot D_x^2 \\ & - b \cdot ((9a^2 - 63ab + 114b^2) \cdot x + ab - b^2) \cdot D_x \\ & + (a - 3b) \cdot (a - 2b) \cdot (a - b). \end{aligned}$$

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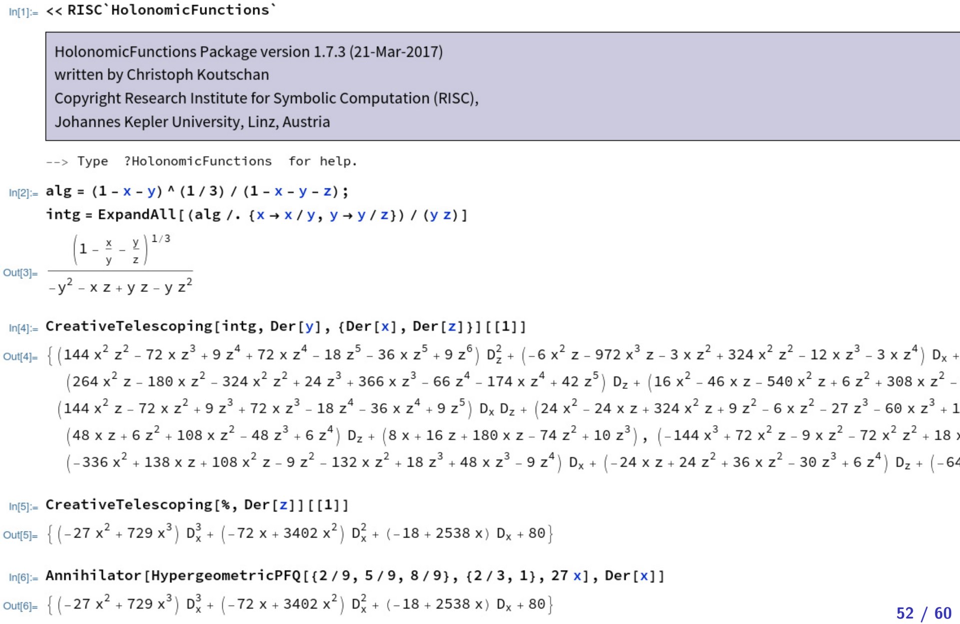
One of its solutions is the claimed ${}_3F_2$ hypergeometric function

$${}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big).$$

Software Demo



```
In[1]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

```
In[2]:= alg = (1 - x - y) ^ (1 / 3) / (1 - x - y - z);
        intg = ExpandAll[(alg /. {x → x / y, y → y / z}) / (y z)]
```

Out[3]=
$$\frac{\left(1-\frac{x}{y}-\frac{y}{z}\right)^{1/3}}{-y^2-x\,z+y\,z-y\,z^2}$$

```
In[4]:= CreativeTelescoping[intg, Der[y], {Der[x], Der[z]}][[1]]
```

Out[4]=
$$\{(144 x^2 z^2 - 72 x z^3 + 9 z^4 + 72 x z^4 - 18 z^5 - 36 x z^5 + 9 z^6)\, D_z^2 + (-6 x^2 z - 972 x^3 z - 3 x z^2 + 324 x^2 z^2 - 12 x z^3 - 3 x z^4)\, D_x +$$
$$(264 x^2 z - 180 x z^2 - 324 x^2 z^2 + 24 z^3 + 366 x z^3 - 66 z^4 - 174 x z^4 + 42 z^5)\, D_z + (16 x^2 - 46 x z - 540 x^2 z + 6 z^2 + 308 x z^2 -$$
$$(144 x^2 z - 72 x z^2 + 9 z^3 + 72 x z^3 - 18 z^4 - 36 x z^4 + 9 z^5)\, D_x D_z + (24 x^2 - 24 x z + 324 x^2 z + 9 z^2 - 6 x z^2 - 27 z^3 - 60 x z^3 + 1$$
$$(48 x z + 6 z^2 + 108 x z^2 - 48 z^3 + 6 z^4)\, D_z + (8 x + 16 z + 180 x z - 74 z^2 + 10 z^3),\ (-144 x^3 + 72 x^2 z - 9 x z^2 - 72 x^2 z^2 + 18$$
$$(-336 x^2 + 138 x z + 108 x^2 z - 9 z^2 - 132 x z^2 + 18 z^3 + 48 x z^3 - 9 z^4)\, D_x + (-24 x z + 24 z^2 + 36 x z^2 - 30 z^3 + 6 z^4)\, D_z + (-64$$

```
In[5]:= CreativeTelescoping[%, Der[z]][[1]]
```

Out[5]=
$$\{(-27 x^2 + 729 x^3)\, D_x^3 + (-72 x + 3402 x^2)\, D_x^2 + (-18 + 2538 x)\, D_x + 80\}$$

```
In[6]:= Annihilator[HypergeometricPFQ[{2 / 9, 5 / 9, 8 / 9}, {2 / 3, 1}, 27 x], Der[x]]
```

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$$\{(-27 x^2 + 729 x^3)\, D_x^3 + (-72 x + 3402 x^2)\, D_x^2 + (-18 + 2538 x)\, D_x + 80\}$$

From Algebraic to Rational

Denef and Lipshitz: For a given algebraic power series $f(x_1, \ldots, x_n)$ in n variables, construct a rational function $R(x_1, \ldots, x_{2n})$ in $2n$ variables such that

$$\mathrm{Diag}\big(R(x_1, \ldots, x_{2n})\big) = \mathrm{Diag}\big(f(x_1, \ldots, x_n)\big).$$

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Moreover, the "partial diagonal" of R, w.r.t. the pairs of variables

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Example: We use the three-variable algebraic function

$$\begin{aligned} f(x, y, z) &= \frac{(1 - x - y)^{1/3}}{1 - x - y - z} \\ &= 1 + \tfrac{2}{3}x + \tfrac{2}{3}y + z + \tfrac{10}{9}xy + \tfrac{5}{3}xz + \tfrac{5}{3}yz + \tfrac{40}{9}xyz + \ldots \end{aligned}$$

Etale Extensions

The minimal polynomial of $f = \frac{(1-x-y)^{1/3}}{1-x-y-z}$ is given by

$$p(x, y, z, f) = \big((x + y + z - 1) \cdot f\big)^3 + 1 - x - y.$$

Denef and Lipshitz's theorem is formulated for étale extensions, which basically means that $\frac{\partial p}{\partial f}$ has a nonzero constant coefficient.

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By considering $\tilde{f} = f - 1$, i.e. by removing the constant term of f, we can achieve an étale extension. The minimal polynomial then reads

$$\tilde{p}(x,y,z,f) = \big((x+y+z-1)\cdot(f+1)\big)^3 + 1 - x - y.$$

and indeed, $\frac{\partial \tilde{p}}{\partial f}(0,0,0,0) = -3 \neq 0$.

Special Diagonal

Now, the rational function

$$\tilde{r}(x,y,z,f) = f^2 \cdot \frac{\frac{\partial \tilde{p}}{\partial f}(xf,yf,zf,f)}{\tilde{p}(xf,yf,zf,f)}$$

has the property that $\mathcal{D}\big(\tilde{r}(x,y,z,f)\big) = \tilde{f}(x,y,z)$, where the operator $\mathcal{D}$ denotes a special kind of "diagonalization" with respect to the last variable:

$$\mathcal{D}\Big(\sum a_{i_1,\dots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n} y^j\Big) = \sum_{j=i_1+\cdots+i_n} a_{i_1,\dots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n}.$$

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Hence $\mathcal{D}\big(r(x,y,z,f)\big) = f(x,y,z)$ for $r(x,y,z,f) = \tilde{r}(x,y,z,f) + 1$.

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In our example we obtain:

$$r(x,y,z,f) = \frac{3f^2 \cdot (f+1)^2 \cdot (xf+yf+zf-1)^3}{(f+1)^3 \cdot (xf+yf+zf-1)^3 - xf - yf + 1} + 1.$$

Rational Function

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This process consists of a sequence of $n-1$ elementary steps, each of which is adding one more variable:

$$r_1(x, y, z, u_1, v_1) = \frac{u_1 \cdot r(x, y, z, u_1) - v_1 \cdot r(x, y, z, v_1)}{u_1 - v_1}$$

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Then r_2 is the desired rational function in six variables.

Final Result

The hypergeometric function

$$ {}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big). $$

is the diagonal of the following rational function in the six variables x, y, z, u, v, w:

$$ 1+\frac{au^3v\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-v)(v-w)} $$

$$ -\frac{av^4\,(1-vx-vy-vz)\,(1+v)^{a-1}(1-vx-vy-vz)^{a-1}}{(1+v)^a(1-vx-vy-vz)^a-(1-vx-vy)^b(u-v)(v-w)} $$

$$ -\frac{au^3w\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-w)(v-w)} $$

$$ -\frac{aw^4\,(1-wx-wy-wz)\,(1+w)^{a-1}(1-wx-wy-wz)^{a-1}}{(1+w)^a(1-wx-wy-wz)^a-(1-wx-wy)^b(u-w)(v-w)} $$

Other Potential Counterexamples

Christol's original example:

$${}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], \left[\frac{1}{3}, 1\right], 27x\right)$$

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Note that our examples,

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], x\right) \quad \text{and} \quad {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}, 1\right], x\right), $$

have an arithmetic progression in the top parameters.

Integral Representation

Recalling the integral representation of the hypergeometric function

$$
{}_3F_2([a,b,c],[d,e],x) = \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(a)\,\Gamma(b)\,\Gamma(d-a)\,\Gamma(e-b)} \times
$$

$$
\times \int_0^1 \int_0^1 y^{a-1} z^{b-1} (1-y)^{-a+d-1} (1-z)^{-b+e-1} (1-xyz)^{-c} \, \mathrm{d}y \, \mathrm{d}z
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For example, let

$$
A(x,y,z) = (1-y)^{d-b-1}\, y^b \,(1-xy^2)^{-a}\,(1-z)^{-c}
$$

then the telescoper of

$$
\frac{1}{yz} A\Big(\frac{x}{y}, \frac{y}{z}, z\Big)
$$

gives precisely the differential equation of ${}_3F_2([a,b,c],[d,1],x)$.

The End

Taking the parameter values $a = \frac{1}{9}, b = \frac{4}{9}, c = \frac{5}{9}, d = \frac{1}{3}$, one could hope that the diagonal of the algebraic function

$$\frac{y^{4/9}}{(1-y)^{10/9}\,(1-xy^2)^{1/9}\,(1-z)^{5/9}}$$

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Note: The diagonal of a rational function and a solution of the corresponding telescoper are close, yet distinct notions: the telescoper annihilates the n-fold integral over all integration cycles.

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Outlook: Meanwhile, Alin Bostan and Sergey Yurkevich came up with a generalization of our result, but Christol's original example still **resists**!