Reduction-Based Creative Telescoping for Fuchsian D-finite Functions[1]

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Abstract

Continuing a series of articles in the past few years on creative telescoping using reductions, we adapt Trager's Hermite reduction for algebraic functions to fuchsian D-finite functions whose singularities have real exponents. We develop a reduction-based creative telescoping algorithm for this class of functions, thereby generalizing our recent reduction-based algorithm for algebraic functions, presented at ISSAC 2016.

Keywords: D-finite function, Integral basis, Trager's reduction, Telescoper

1. Introduction

The classical question in symbolic integration is whether the integral of a given function can be written in "closed form". In its most restricted form, the question is whether for a given function f belonging to some domain D there exists another function g, also belonging to D, such that $f = g'$. For example, if D is the field of rational functions, then for $f = 1/x^2$ we can find $g = -1/x$, while for $f = 1/x$ no suitable g exists. When no g exists in D, there are several other questions we may ask. One possibility is to ask whether there is some extension E of D

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such that in E there exists some g with $f = g'$. For example, in the case of elementary functions, Liouville's principle restricts the possible extensions E, and there are algorithms which construct such extensions whenever possible. Another possibility is to ask whether for some modification $\tilde{f} \in D$ of f there exists a $g \in D$ such that $\tilde{f} = g'$. Creative telescoping leads to a question of this type. Here we are dealing with domains D containing functions in several variables, say x and t, and the question is whether there is a linear differential operator P, nonzero and free of x, such that there exists a $g \in D$ with $P \cdot f = g'$, where g' denotes the derivative of g with respect to x. Typically, g itself has the form $Q \cdot f$ for some operator Q (which may be zero and need not be free of x). In this case, we call P a telescoper for f, and Q a certificate for P.

Creative telescoping is the backbone of definite integration, because $P \cdot f = (Q \cdot f)'$ implies, for instance, $P \cdot \int_0^1 f(x,t)dx = (Q \cdot f)(1) - (Q \cdot f)(0)$. A telescoper P for f thus gives rise to an annihilating operator for the definite integral $F(t) = \int_0^1 f(x,t)dx$.

Example 1 ((Manin, 1958))**.** *The algebraic function*

$$f(x,t) = \frac{1}{\sqrt{x(x-1)(x-t)}}$$

does not admit an elementary integral with respect to x. However, we have $P \cdot f = (Q \cdot f)'$ for

$$P = 4(t-1)t\,D_t^2 + 4(2t-1)D_t + 1, \quad Q = \frac{2x(x-1)}{t-x}.$$

This implies

$$P \cdot \int_0^1 f(x,t)dx = \Big[\frac{2x(x-1)}{t-x} f(x,t)\Big]_{x=0}^{x=1},$$

so the integral $F(t) = \int_0^1 \frac{1}{\sqrt{x(x-1)(x-t)}}dx$ satisfies the differential equation

$$4(t-1)t\,F''(t) + 4(2t-1)F'(t) + F(t) = 0.$$

In the common case when the right-hand side collapses to zero, we say that the integral has "natural boundaries". Readers not familiar with creative telescoping are referred to the literature (Petkovšek et al., 1996; Zeilberger, 1990a, 1991, 1990b; Koepf, 1998; Kauers and Paule, 2011) for additional motivation, theory, algorithms, implementations, and applications. There are several ways to find telescopers for a given $f \in D$. In recent years, an approach has become popular which has the feature that it can find a telescoper without also constructing the corresponding certificate. This is interesting because certificates tend to be much larger than telescopers, and in some applications, for instance when an integral has natural boundaries, only the telescoper is of interest. This approach was first formulated for rational functions $f \in C(t,x)$ by Bostan et al. (2010) and later generalized to rational functions in several variables (Bostan et al., 2013b; Lairez, 2016), to hyperexponential functions (Bostan et al., 2013a) and, for the shift case, to hypergeometric terms (Chen et al., 2015; Huang, 2016) and binomial sums (Bostan et al., 2016). At ISSAC'16, three of the present authors have given a version for algebraic functions (Chen et al., 2016). In the present article, we extend this algorithm to fuchsian D-finite functions.

The basic principle of the general approach is as follows. Assume that the x-constants $\mathrm{Const}_x(D) = \{c \in D : c' = 0\}$ form a field and that D is a vector space over the field of x-constants. Assume further that there is some $\mathrm{Const}_x(D)$-linear map $[\cdot]\colon D \to D$ such that for every $f \in D$ there exists a $g \in D$ with $f - [f] = g'$. Such a map is called a *reduction.* For example, in $D = C(t, x)$ Hermite reduction (Ostrogradskiĭ, 1845; Hermite, 1872) decomposes any $f \in D$ into $f = g' + h$ with $g, h \in D$ such that h is a proper rational function (i.e., the numerator degree is smaller than the denominator degree) with a squarefree denominator. In this case, we can take $[f] = h$. In order to find a telescoper, we can compute $[f]$, $[\partial_t \cdot f]$, $[\partial_t^2 \cdot f]$, $\ldots$, until we find that they are linearly dependent over $\mathrm{Const}_x(D)$. Once we find a relation $p_0[f] + \cdots + p_r[\partial_t^r \cdot f] = 0$, then, by linearity, $[p_0 f + \cdots + p_r\partial_t^r \cdot f] = 0$, and then, by definition of $[\cdot]$, there exists a $g \in D$ such that $(p_0 + \cdots + p_r\partial_t^r) \cdot f = g'$. In other words, $P = p_0 + \cdots + p_r\partial_t^r$ is a telescoper.

There are two ways to guarantee that this method terminates. The first requires that we already know for other reasons that a telescoper exists. The idea is then to show that the reduction $[\cdot]$ has the property that when $f \in D$ is such that there exists a $g \in D$ with $g' = f$, then $[f] = 0$. If this is the case and $P = p_0 + \cdots + p_r\partial_t^r$ is a telescoper for f, then $P \cdot f$ is integrable in D, so $[P \cdot f] = 0$, and by linearity $[f], \ldots, [\partial_t^r \cdot f]$ are linearly dependent over $\mathrm{Const}_x(D)$. This means that the method won't miss any telescoper. In particular, this argument has the nice feature that we are guaranteed to find a telescoper of smallest possible order r. This approach was taken by Chen et al. (2015). The second way consists in showing that the $\mathrm{Const}_x(D)$-vector space generated by $\{ [\partial_t^i \cdot f] : i \in \mathbb{N} \}$ has finite dimension. This approach was taken by Bostan et al. (2010, 2013a). It has the nice additional feature that every bound for the dimension of this vector space gives rise to a bound for the order of the telescoper. In particular, it implies the existence of a telescoper.

In this paper, we generalize Trager's Hermite reduction for algebraic functions to fuchsian D-finite functions (Section 4). We turn this in two ways into reduction-based creative telescoping algorithms. In Section 5 we continue to follow Trager's reasoning and exploit the property of the integrand having a double root at infinity. As an alternative, we generalize the concept of polynomial reduction Bostan et al. (2013a); Chen et al. (2015, 2016) in Section 6. The termination of both algorithms will be shown by both approaches sketched above. In particular, they are guaranteed to find the minimal-order telescoper and they give rise to bounds on the order of the telescoper.

2. Fuchsian D-finite Functions

Throughout the paper, let C be a field of characteristic zero. We consider linear differential operators $L = \ell_0 + \cdots + \ell_n\partial_x^n$ with $\ell_0, \ldots, \ell_n$ belonging to some ring R containing C. Typical choices for R will be $C[x]$ or $C(x)$. When $\ell_n \neq 0$, we say that $\mathrm{ord}(L) := n$ is the order of L.

Let R be a differential ring and write $'$ for its derivation. We write $R[\partial_x]$ for the algebra consisting of all linear differential operators, together with the usual addition and the unique non-commutative multiplication satisfying $\partial_x a = a\partial_x + a'$ for all $a \in R$. We shall assume throughout that $C \subseteq \mathrm{Const}_x(R)$. The algebra $R[\partial_x]$ acts on a differential R-module F via

$$(\ell_0 + \ell_1\partial_x + \cdots + \ell_n\partial_x^n) \cdot f = \ell_0 f + \ell_1 f' + \cdots + \ell_n f^{(n)}.$$

An element $y \in F$ is called a solution of an operator $L \in R[\partial_x]$ if $L \cdot y = 0$.

By $\bar{C}$ we denote some algebraically closed field containing C (not necessarily the smallest). An operator L of order n is called fuchsian at a point $a \in \bar{C}$ if it admits n linearly independent solutions in

$$\bar{C}[[[x-a]]] := \bigcup_{\nu \in C} (x-a)^\nu \bar{C}[[x-a]][\log(x-a)].$$

It is called fuchsian at ∞ if it admits n linearly independent solutions in

$$\bar{C}[[[x^{-1}]]] := \bigcup_{\nu \in C} x^{-\nu} \bar{C}[[x^{-1}]][\log(x)].$$

It is simply called *fuchsian* if it is fuchsian at all $a \in \bar{C} \cup \{\infty\}$. Note that the exponents ν are restricted to C, not to the larger field $\bar{C}$. For simplicity, the dependence on C is not reflected in the notation.

Examples for fuchsian operators are operators that have a basis of algebraic function solutions, the Gauss hypergeometric differential operator, or the operator $L = x\partial_x^2 - \partial_x$, whose solutions are 1 and $\log(x)$. However, the class of fuchsian D-finite functions considered in this paper is not as rich as it may seem at first glance, because we require that the operators under consideration should be fuchsian at all points *including infinity*. Functions such as $\exp(x)$, $\sin(x)$, $\cos(x)$, Bessel functions, etc. are only fuchsian at all finite points but not at infinity.

For a fixed fuchsian operator L, we will consider the left $R[\partial_x]$-module $A = R[\partial_x]/\langle L \rangle$, where $\langle L \rangle$ denotes the left ideal generated by L in $R[\partial_x]$. Then $1 \in A$ is a solution of L, because we have $L \cdot 1 = L = 0$ in A. We can say that A consists of all the "functions" f which can be obtained from a "generic" solution y of L by applying some operator $P \in R[\partial_x]$ to it. When R is a field, then A is an R-vector space of dimension $n = \operatorname{ord}(L)$, generated by $1, \partial_x, \dots, \partial_x^{n-1}$.

It is instructive to compare this setup to the situation for algebraic functions. Comparing $A = R[\partial_x]/\langle L \rangle$ to an algebraic function field $R[Y]/\langle M \rangle$ (when R is a field), our operator L plays the role of the minimal polynomial M. In the algebraic case, Y is a formal solution of the equation $M = 0$, similar as $1 \in A$ is a formal solution of L. Besides these formal solutions there are, for each fixed $a \in \bar{C}$, exactly $\deg_Y(M)$ different Puiseux series solutions of $M = 0$ at places above a. They correspond in the differential setting to the series solutions of L in $\bar{C}[[[x - a]]]$, which generate a $\bar{C}$-vector space of dimension $\operatorname{ord}(L)$.

The exponents of an element $f \in A = R[\partial_x]/\langle L \rangle$ at a point $a \in \bar{C} \cup \{\infty\}$ are the values α such that one of the series in $\bar{C}[[[x - a]]]$ (or $\bar{C}[[[x^{-1}]]]$, respectively) associated to f has $(x - a)^\alpha \log(x - a)^\beta$ (or $(\frac{1}{x})^\alpha \log(x)^\beta$) as initial term. For an element $f \in A$, let n_f be the minimal order of an operator $\tilde{L} \in R[\partial_x] \setminus \{0\}$ with $\tilde{L} \cdot f = 0$. We say that a is an ordinary point of f if the set of exponents of f at a is $\{0, 1, \dots, n_f - 1\}$ and the solutions at a do not involve logarithms. There can be at most finitely many non-ordinary points; these are called the singular points. The defect of $f \in A$ at $a \in \bar{C} \cup \{\infty\}$, denoted $\operatorname{defect}_a(f)$, is defined as the sum of the exponents of f at a minus $\sum_{k=0}^{n_f-1} k = \frac{1}{2} n_f(n_f - 1)$. Then the Fuchs relation (Schlesinger, 1895; Ince, 1926) says that we have

$$\sum_{a \in \bar{C} \cup \{\infty\}} \operatorname{defect}_a(f) = n_f(1 - n_f)$$

for all $f \in A$. This relation is the counterpart of the Bézout relation in the algebraic case. Note that when a is an ordinary point, then $\operatorname{defect}_a(f) = 0$, but $\operatorname{defect}_a(f) = 0$ does in general not imply that a is an ordinary point.

In the context of creative telescoping, we let $\bar{C}$ be some algebraically closed field containing the rational function field $K = C(t)$, and we use $R = K(x)$ instead of $C(x)$. Integration will always be with respect to x, but besides the derivation ∂_x there is now also the derivation with respect to t. The notation f' will always refer to the derivative of f with respect to x, not with respect to t. In addition to the operator algebra $R[\partial_x]$, we consider the operator algebra $R[\partial_x, \partial_t]$, in which ∂_x, ∂_t commute with each other (although they need not commute with elements of R).

The action of $R[\partial_x]$ on $\bar{C}[[[x-a]]]$ or $\bar{C}[[[x^{-1}]]]$ is extended to $R[\partial_x, \partial_t]$ by letting ∂_t act coefficient-wise. We further assume that the action of $R[\partial_x]$ on $A = R[\partial_x]/\langle L\rangle$ is extended to an action of $R[\partial_x, \partial_t]$ on A in a way that is compatible with the action of $R[\partial_x, \partial_t]$ on series domains. This means that when $y \in \bar{C}[[[x-a]]]$ is a solution of L and f is an element of A, so that $f \cdot y$ is an element of $\bar{C}[[[x-a]]]$, then we want to have $(\partial_t \cdot f) \cdot y = \partial_t \cdot (f \cdot y)$, where the $\cdot$ in $(\partial_t \cdot f)$ refers to the action of $R[\partial_x, \partial_t]$ on A and the three other dots refer to the action on $\bar{C}[[[x-a]]]$.

If $u \in A$ is such that $\partial_t \cdot 1 = u$ and $U \in R[\partial_x]$ is such that $u = U + \langle L\rangle$, then the annihilator $\mathfrak{a} \subseteq R[\partial_x, \partial_t]$ of $1 \in A$ in $R[\partial_x, \partial_t]$ contains L and $\partial_t - U$. We therefore have $\dim \mathfrak{a} = 0$, which is the usual definition of D-finiteness in the case of several variables (Zeilberger, 1990b; Chyzak and Salvy, 1998; Koutschan, 2009; Kauers, 2015). Degenerate situations, where we also have $\dim \mathfrak{a} = 0$ but $\mathfrak{a}$ does not have a basis of the form $\langle L, \partial_t - U\rangle$, are not considered in this paper.

3. Integral Bases

Trager's Hermite reduction for algebraic functions rests on the notion of integral bases. The notion of integral bases has been generalized to D-finite functions (Kauers and Koutschan, 2015), and an algorithm was also given there for computing such bases. We recall here the relevant definitions and properties. We will use below the integral bases of Kauers and Koutschan (2015) in very much the same way as Trager used integral bases in his integration algorithm. For another recent application of integral bases for D-finite functions, see (Imamoglu and van Hoeij, to appear).

Although the elements of a generalized series ring $\bar{C}[[[x-a]]]$ are formal objects, the series notation suggests certain analogies with complex functions. For simplicity, let us assume throughout that $C \subseteq \mathbb{R}$. Terms $(x-a)^\alpha \log(x-a)^\beta$ or $(\frac{1}{x})^\alpha \log(x)^\beta$ are called *integral* if $\alpha \geq 0$. A series in $\bar{C}[[[x-a]]]$ or $\bar{C}[[[x^{-1}]]]$ is called integral if it only contains integral terms. A non-integral series is said to have a *pole* at the reference point. Note that in this terminology also $1/\sqrt{x}$ has a pole at 0, while $\log(x)$ does not; this convention differs slightly from the default setting of (Kauers and Koutschan, 2015, Ex. 2), but can be achieved by defining the function ι (Kauers and Koutschan, 2015, Def. 1) accordingly. Note also that the terminology only refers to x but not to t.

Integrality at $a \in \bar{C}$ is not preserved by differentiation, but if f is integral at a, then so is $(x-a)f'$. On the other hand, integrality at infinity is preserved by differentiation; we even have the stronger property that when f is integral at infinity, then not only f' but also $xf' = (x^{-1})^{-1}f'$ is integral at infinity.

Let K be some field with $C \subseteq K \subseteq \bar{C}$. Let $L \in K(x)[\partial_x]$ be a fuchsian operator. An element $f \in A = K(x)[\partial_x]/\langle L\rangle$ is called (locally) integral at $a \in \bar{C} \cup \{\infty\}$ if for every solution y of L in $\bar{C}[[[x-a]]]$ or $\bar{C}[[[x^{-1}]]]$, respectively, the series $f \cdot y$ is integral. f is called (globally) integral if it is locally integral at every $a \in \bar{C}$ ("at all finite places").

For an element $f \in A$ to have a "pole" at $a \in \bar{C} \cup \{\infty\}$ means that f is not locally integral at a; to have a "double pole" at a means that $(x-a)f$ (or $\frac{1}{x}f$ if $a = \infty$) is not integral; to have a "double root" at a means that $f/(x-a)^2$ (or $f/(\frac{1}{x})^2 = x^2 f$ if $a = \infty$) is integral, and so on.

The set of all globally integral elements $f \in A$ forms a $K[x]$-submodule of A. A basis $\{\omega_1, \ldots, \omega_n\}$ of this module is called an *integral basis* for A. Kauers and Koutschan (2015) proposed an algorithm which computes an integral basis for a given A. This algorithm is a generalization of van Hoeij's algorithm (van Hoeij, 1994) for computing integral bases of algebraic function fields (Trager, 1984; Rybowicz, 1991).

For a fixed $a \in \bar{C}$, let $\bar{C}(x)_a$ be the ring of rational functions p/q with $q(a) \neq 0$, and write $\bar{C}(x)_\infty$ for the ring of all rational functions p/q with $\deg_x(p) \leq \deg_x(q)$. Then the set of all $f \in A$ which are locally integral at some fixed $a \in \bar{C} \cup \{\infty\}$ forms a $\bar{C}(x)_a$-module. A basis of this module is called a *local integral basis* at a for A. The algorithm given by Kauers and Koutschan (2015) for computing (global) integral bases computes local integral bases at finite points as an intermediate step. By an analogous algorithm, it is also possible to compute a local integral basis at infinity.

An integral basis $\{\omega_1, \dots, \omega_n\}$ is always also a $K(x)$-vector space basis of A. A key feature of integral bases is that they make poles explicit. Writing an element $f \in A$ as a linear combination $f = \sum_{i=1}^n f_i \omega_i$ for some $f_i \in K(x)$, we have that f has a pole at $a \in \bar{C}$ if and only if at least one of the f_i has a pole there.

Lemma 2. *Let L be a fuchsian operator and let $\{\omega_1, \dots, \omega_n\}$ be a local integral basis of $A = K(x)[\partial_x]/\langle L\rangle$ at $a \in \bar{C} \cup \{\infty\}$. Let $f \in A$ and $f_1, \dots, f_n \in K(x)$ be such that $f = \sum_{i=1}^n f_i \omega_i$. Then f is integral at a if and only if each $f_i \omega_i$ is integral at a.*

Proof. The direction "$\Leftarrow$" is obvious. To show "$\Rightarrow$", suppose that f is integral at a. Then there exist $\tilde{f}_1, \dots, \tilde{f}_n \in \bar{C}(x)_a$ such that $f = \sum_{i=1}^n \tilde{f}_i \omega_i$. Thus $\sum_{i=1}^n (\tilde{f}_i - f_i)\omega_i = 0$, and then $\tilde{f}_i = f_i$ for all i, because $\{\omega_1, \dots, \omega_n\}$ is a vector space basis of A. As elements of $\bar{C}(x)_a$, the f_i are integral at a, and hence also all the $f_i \omega_i$ are integral at a. ■

The lemma says in particular that poles of the f_i in a linear combination $\sum_{i=1}^n f_i \omega_i$ cannot cancel each other.

Lemma 3. *Let L be a fuchsian operator and let $\{\omega_1, \dots, \omega_n\}$ be an integral basis of $A = K(x)[\partial_x]/\langle L\rangle$. Let $e \in K[x]$ and $M = ((m_{i,j}))_{i,j=1}^n \in K[x]^{n\times n}$ be such that*

$$e\,\omega_i' = \sum_{j=1}^n m_{i,j}\omega_j$$

for $i = 1, \dots, n$ and $\gcd(e, m_{1,1}, \dots, m_{n,n}) = 1$. Then e is squarefree.

Proof. Let $a \in \bar{C}$ be a root of e. We show that a is not a multiple root. Since ω_i is integral, it is in particular locally integral at a. Therefore $(x-a)\omega_i'$ is locally integral at a. Since $\{\omega_1, \dots, \omega_n\}$ is an integral basis, it follows that $(x-a)m_{i,j}/e \in \bar{C}(x)_a$ for all i, j. Because of $\gcd(e, m_{1,1}, \dots, m_{n,n}) = 1$, no factor $x - a$ of e can be canceled by all the $m_{i,j}$. Therefore the factor $x - a$ can appear in e only once. ■

Lemma 4. *Let L be a fuchsian operator and let $\{\omega_1, \dots, \omega_n\}$ be a local integral basis at infinity of $A = K(x)[\partial_x]/\langle L\rangle$. Let $e \in K[x]$ and $M = ((m_{i,j}))_{i,j=1}^n \in K[x]^{n\times n}$ be defined as in Lemma 3. Then $\deg_x(m_{i,j}) < \deg_x(e)$ for all i, j.*

Proof. Since every ω_i is locally integral at infinity, so is every $x\,\omega_i'$. Since $\{\omega_1, \dots, \omega_n\}$ is an integral basis at infinity, it follows that $xm_{i,j}/e \in \bar{C}(x)_\infty$ for all i, j. This means that $1 + \deg_x(m_{i,j}) \leq \deg_x(e)$ for all i, j, and therefore $\deg_x(m_{i,j}) < \deg_x(e)$, as claimed. ■

A $K(x)$-vector space basis $\{\omega_1, \dots, \omega_n\}$ of $A = K(x)[\partial_x]/\langle L\rangle$ is called *normal* at $a \in \bar{C} \cup \{\infty\}$ if there exist $r_1, \dots, r_n \in K(x)$ such that $\{r_1\omega_1, \dots, r_n\omega_n\}$ is a local integral basis at a. Trager (1984) shows for the case of algebraic function fields how to construct an integral basis which is normal at infinity from a given integral basis and a given local integral basis at infinity. The same procedure also applies in the present situation. It works as follows.

Let $\{\omega_1, \ldots, \omega_n\}$ be a global integral basis and $\{\nu_1, \ldots, \nu_n\}$ be a local integral basis at infinity. Let $m_{i,j} \in K(x)$ be such that

$$\omega_i = \sum_{j=1}^{n} m_{i,j} \nu_j.$$

For each i, let $\tau_i \in \mathbb{Z}$ be the largest integer such that $x^{\tau_i} m_{i,j}$ has no pole at infinity for any j. Then each $x^{\tau_i}\omega_i$ is locally integral at infinity. Let $B \in K^{n\times n}$ be the matrix obtained by evaluating $((x^{\tau_i} m_{i,j}))_{i,j=1}^{n}$ at infinity (this is possible by the choice of τ_i). If B is invertible, then the $x^{\tau_i}\omega_i$ form a local integral basis at infinity and we are done. Otherwise, there exists a nonzero vector $a = (a_1, \ldots, a_n) \in K^n$ with $aB = 0$. Among the indices ℓ with $a_\ell \neq 0$ choose one where τ_ℓ is minimal, and then replace ω_ℓ by $\sum_{i=1}^{n} a_i x^{\tau_i - \tau_\ell} \omega_i$. Note that the resulting basis is still global integral. Repeating the process, it can be checked that the value of $\tau_1 + \cdots + \tau_n$ strictly increases in each iteration. According to the following lemma, the sum is bounded, so the procedure must terminate after a finite number of iterations.

Lemma 5. *Let $\{\omega_1, \ldots, \omega_n\}$, $\{\nu_1, \ldots, \nu_n\}$, and $\tau_1, \ldots, \tau_n$ be as above. Let N be the number of points $a \in \bar{C}$ where at least one of the ω_i does not have n distinct exponents in $\mathbb{N}$ (i.e., N counts the finite singular points of L that are not "apparent" singularities). Then*

$$\tau_1 + \cdots + \tau_n \leq \tfrac{1}{2} n(n-1)(N-1).$$

Proof. We show that when $\tau_i \in \mathbb{Z}$ is such that $x^{\tau_i}\omega_i$ is locally integral at infinity, then $\tau_i \leq \frac{1}{2}(n-1)(N-1)$, for every i. Let n_i be the minimal order of an operator $L \in R[\partial_x] \setminus \{0\}$ with $L \cdot \omega_i = 0$. By the Fuchs relation we have

$$\sum_{a\in\bar{C}\cup\{\infty\}} \operatorname{defect}_a(\omega_i) = n_i(1-n_i),$$

hence

$$\operatorname{defect}_\infty(\omega_i) = n_i(1-n_i) - \sum_{a\in\bar{C}} \operatorname{defect}_a(\omega_i).$$

When all exponents of the series associated to ω_i at $a \in \bar{C}$ form a subset of $\mathbb{N}$ of size n_i, then $\operatorname{defect}_a(\omega_i) \geq 0$. At all other points a, of which there are at most N by assumption, we still have the estimate $\operatorname{defect}_a(\omega_i) \geq -\frac{1}{2}n_i(n_i-1)$, because ω_i is integral at all finite places. It follows that

$$\operatorname{defect}_\infty(\omega_i) \leq n_i(1-n_i) + \tfrac{1}{2}n_i(n_i-1)N = \tfrac{1}{2}n_i(n_i-1)(N-2).$$

Next, for every $r \in \mathbb{Z}$ we have $\operatorname{defect}_\infty(x^{-r}\omega_i) = rn_i + \operatorname{defect}_\infty(\omega_i)$. Moreover, if $\tau \in \mathbb{Z}$ is such that $x^\tau\omega_i$ is integral at infinity, then we must have $\operatorname{defect}_\infty(x^\tau\omega_i) \geq -\frac{1}{2}n_i(n_i-1)$, i.e.,

$$\operatorname{defect}_\infty(\omega_i) - \tau n_i \geq -\tfrac{1}{2}n_i(n_i-1),$$

and hence,

$$\begin{aligned}\tau &\leq \frac{1}{n_i}\Big(\frac{1}{2}n_i(n_i-1) + \operatorname{defect}_\infty(\omega_i)\Big)\\ &\leq \tfrac{1}{2}(n_i-1) + \tfrac{1}{2}(n_i-1)(N-2) = \tfrac{1}{2}(n_i-1)(N-1) \leq \tfrac{1}{2}(n-1)(N-1)\end{aligned}$$

as claimed. ■

Although normality is a somewhat weaker condition on a basis than integrality, it also excludes the possibility that poles in the terms of a linear combination of basis elements can cancel:

Lemma 6. *Let L be a fuchsian operator and let $\{\omega_1, \ldots, \omega_n\}$ be a basis of $A = K(x)[\partial_x]/\langle L\rangle$ which is normal at some $a \in \bar{C} \cup \{\infty\}$. Let $f = \sum_{i=1}^n f_i\omega_i$ for some $f_1, \ldots, f_n \in K(x)$. Then f has a pole at a if and only if there is some i such that $f_i\omega_i$ has a pole at a.*

Proof. Let $r_1, \ldots, r_n \in K(x)$ be such that $\{r_1\omega_1, \ldots, r_n\omega_n\}$ is a local integral basis at a. By $f = \sum_{i=1}^n (f_i r_i^{-1})(r_i\omega_i)$ and by Lemma 2, f is integral at a iff all $f_i r_i^{-1} r_i\omega_i = f_i\omega_i$ are integral at a. ■

We will mostly be using bases that are integral at every point in $\bar{C} \cup \{\infty\}$ except one. For the case of algebraic functions, the reason is that the only algebraic functions which are integral at all finite places and also at infinity are the constant functions (Chevalley's theorem (Chevalley, 1951, p. 9, Cor. 3)). The results of Chen et al. (2016) depend heavily on this fact. There is no analogous result for fuchsian D-finite functions: such functions may be integral at all finite places and also at infinity without being constant. It is easy to construct examples using the Papperitz symbol.

Example 7. *The operator $L = 3x(x^2-1)D_x^2 + 2(3x^2-1)D_x \in \mathbb{Q}(x)[\partial_x]$ has three singular points $0, +1, -1$. Infinity is an ordinary point of L. At all three singularities, there is one local solution starting with exponent 0 and another starting with exponent $1/3$, so all the solutions are integral everywhere according to our standard definition of integrality of generalized series.*

Fortunately, we can still be sure that there are not too many such functions.

Lemma 8. *Let $A = K(x)[\partial_x]/\langle L\rangle$ for some fuchsian operator L, let $\{\omega_1, \ldots, \omega_n\}$ be a global integral basis of A which is normal at infinity, and let $\tau_1, \ldots, \tau_n \in \mathbb{Z}$ be such that $\{x^{\tau_1}\omega_1, \ldots, x^{\tau_n}\omega_n\}$ is a local integral basis at infinity. Denote by V the set of all $f \in A$ which are integral at all finite places and at infinity. Then V is a K-vector space of finite dimension, and $\{\, x^j\omega_i : i = 1, \ldots, n; j = 0, \ldots, \tau_i \,\}$ is a basis of V.*

Proof. It is clear that V is closed under taking K-linear combinations, so it is clearly a vector space. We show that $B = \{\, x^j\omega_i : i = 1, \ldots, n; j = 0, \ldots, \tau_i \,\}$ is a basis.

Every $x^j\omega_i \in B$ is by definition integral at all finite places and because of $0 \leq j \leq \tau_i$ also integral at infinity. Therefore $x^j\omega_i \in V$, and therefore B generates a subspace of V.

Conversely, let $f \in V$ be arbitrary. Then f is in particular integral at all finite places, and since $\{\omega_1, \ldots, \omega_n\}$ is a global integral basis we can write $f = \sum_{i=1}^n p_i\omega_i$ for some polynomials $p_1, \ldots, p_n$. If we had $\deg(p_i) > \tau_i$ for some i, then $p_i\omega_i$ would not be integral at infinity (by definition of the numbers τ_i), and then, because $\{\omega_1, \ldots, \omega_n\}$ is normal at infinity, Lemma 6 implies that f would not be integral at infinity. But f is in V and therefore integral at all points, including infinity. It follows that $\deg(p_i) \leq \tau_i$ for all i, and therefore f is a K-linear combination of elements of B. ■

Corollary 9. *With the notation of Lemmas 5 and 8, we have*

$$\dim_K(V) \leq n(\tfrac{1}{2}(n-1)(N-1)+1).$$

Proof. Note that the exponents τ_i in Lemma 8 are the same as in Lemma 5; in the proof of the latter it was shown that $\tau_i \leq \frac{1}{2}(n-1)(N-1)$. The estimate on the dimension of V then follows immediately. ■

4. Hermite Reduction

Hermite reduction for rational functions was invented at least twice during the 19th century (Ostrogradskiĭ, 1845; Hermite, 1872). This reduction was later extended to elementary functions by Risch (Risch, 1969, 1970; Geddes et al., 1992; Bronstein, 1998b, 2005), to algebraic functions by Trager (Trager, 1984; Geddes et al., 1992; Bronstein, 1998b), and to hyperexponential functions (Bostan et al., 2013a). For the case of summation, it was first formulated for rational functions by Abramov (1975) and later generalized to hypergeometric terms by Abramov and Petkovšek (2002a,b). These generalizations are the key step in many integration and summation algorithms, including the creative telescoping algorithm presented in this paper. It turns out that the Hermite reduction for fuchsian D-finite functions is literally the same as Trager's reduction for algebraic functions.

We start with a technical lemma, which is needed later to ensure that the Hermite reduction always works. The analogous statement for algebraic functions and its proof can be found in (Trager, 1984, pp. 46–47); Trager's proof for the algebraic case directly carries over and is reproduced here only for the convenience of the reader.

Throughout this section, let $L \in K(x)[\partial_x]$ be a fuchsian operator of order n and let $A = K(x)[\partial_x]/\langle L\rangle$.

Lemma 10. *Let $v \in K[x]$ be a squarefree polynomial and let $\{\omega_1, \dots, \omega_n\}$ be a basis of A that is a locally integral basis at all roots of v. For some integer $\mu > 1$ we define $\psi_i := v^\mu \left(v^{1-\mu}\omega_i\right)'$; then $\{\psi_1, \dots, \psi_n\}$ is a local integral basis at each root of v.*

Proof. By expanding $\psi_i = v\omega_i' - (\mu-1)v'\omega_i$ one sees that the ψ_i themselves are integral at all roots of v. Let now $a \in \bar{C}$ be an arbitrary but fixed root of v. We have to show that each $f \in A$ that is integral at a can be expressed as a linear combination of the ψ_i with coefficients in $\bar{C}(x)_a$. To the contrary, assume that there exists an integral element f that requires $x-a$ in the denominator of some coefficient, i.e.,

$$f = \frac{1}{v}\sum_{i=1}^{n} c_i\psi_i \quad \text{with } c_i \in \bar{C}(x)_a \text{ and } c_i(a) \neq 0 \text{ for some } i$$

(here we use the fact that v is squarefree). Further let $g = \sum_{i=1}^n c_i'\omega_i$, which is obviously integral. Then also their sum

$$f + g = v^{\mu-1}\sum_{i=1}^{n}\left(c_i(v^{1-\mu}\omega_i)' + c_i'v^{1-\mu}\omega_i\right) = v^{\mu-1}\sum_{i=1}^{n}(c_iv^{1-\mu}\omega_i)'$$

must be integral. Since $\{\omega_1, \dots, \omega_n\}$ is an integral basis at a, there exists for each $i = 1, \dots, n$ a series solution $y_i \in \bar{C}[[[x-a]]]$ of L such that $\omega_i \cdot y_i$ involves a term $T = (x-a)^\alpha \log(x-a)^\beta$ with $0 \le \alpha < 1$ and $\beta \in \mathbb{N}$. Let now i be an index such that $c_i(a) \neq 0$; this implies that T appears in $(c_i\omega_i)\cdot y_i$. Using the fact that the ω_i form a local integral basis, it follows by Lemma 2 that T is also present in $h \cdot y_i$ where $h = \sum_{i=1}^n c_i\omega_i$. Let now T be the dominant term of $h \cdot y_i$, i.e., among all terms with minimal α the one with the largest exponent β. From

$$((x-a)^{\mu-1}\partial_x(x-a)^{1-\mu})\cdot T = (1-\mu+\alpha)(x-a)^{\alpha-1}\log(x-a)^\beta + \beta(x-a)^{\alpha-1}\log(x-a)^{\beta-1} \quad (1)$$

it follows that $(x-a)^{\alpha-1}\log(x-a)^\beta$ is the dominant term of $(v^{\mu-1}\partial_x v^{1-\mu})\cdot(h\cdot y_i)$; here we use the assumption that $\mu > 1$, because for $\mu = 1$ and $\alpha = 0$ the coefficient $(1-\mu+\alpha)$ in (1) is zero. This calculation reveals that $v^{\mu-1}(v^{1-\mu}h)' = f + g$ is not integral at a, which contradicts our assumption on the integrality of f. Hence $\{\psi_1, \dots, \psi_n\}$ is a local integral basis at a. ■

Let $\{\omega_1,\dots,\omega_n\}$ be an integral basis for A. Further let $e, m_{i,j} \in K[x]$ $(1 \le i, j \le n)$ be such that $e\omega_i' = \sum_{j=1}^n m_{i,j}\omega_i$ and $\gcd(e, m_{1,1}, m_{1,2}, \dots, m_{n,n}) = 1$ as in Lemma 3. For describing the Hermite reduction we fix an integrand $f \in A$ and represent it in the integral basis, i.e., $f = \sum_{i=1}^n (f_i/D)\,\omega_i$ with $D, f_1, \dots, f_n \in K[x]$. The purpose is to find $g, h \in A$ such that $f = g' + h$ and $h = \sum_{i=1}^n (h_i/D^*)\,\omega_i$ with $h_1, \dots, h_n \in K[x]$ and D^* denoting the squarefree part of D. As differentiating the ω_i can introduce denominators, namely the factors of e, it is convenient to consider those denominators from the very beginning on, which means that we shall assume $e \mid D$. Note that $\gcd(D, f_1, \dots, f_n)$ can then be nontrivial.

We now execute one step of the Hermite reduction, where the multiplicity $\mu > 1$ of some nontrivial squarefree factor $v \in K[x]$ of D is reduced. Let $u \in K[x]$ be such that $D = uv^\mu$; it follows that $\gcd(u, v) = 1$ and $\gcd(v, v') = 1$. We want to find $g_1, \dots, g_n, h_1, \dots, h_n \in K[x]$ such that

$$\sum_{i=1}^n \frac{f_i}{uv^\mu}\omega_i = \Bigl(\sum_{i=1}^n \frac{g_i}{v^{\mu-1}}\omega_i\Bigr)' + \sum_{i=1}^n \frac{h_i}{uv^{\mu-1}}\omega_i. \tag{2}$$

By a repeated application of such reduction steps one can decompose any $f \in A$ as $f = g' + h$ where the denominators of the coefficients of h are squarefree and the coefficients of g are proper rational functions.

In order to determine the unknown polynomials $g_1, \dots, g_n$ in (2), clearing the denominator uv^μ yields

$$\sum_{i=1}^n f_i\omega_i = \sum_{i=1}^n \Bigl(uvg_i'\omega_i + uv^\mu g_i \left(v^{1-\mu}\omega_i\right)' + vh_i\omega_i\Bigr), \tag{3}$$

and then this equation is reduced modulo v:

$$\sum_{i=1}^n f_i\omega_i = \sum_{i=1}^n g_i uv^\mu \left(v^{1-\mu}\omega_i\right)' \mod v. \tag{4}$$

By Lemma 10 and from $\gcd(u, v) = 1$ it follows that the elements $uv^\mu \left(v^{1-\mu}\omega_i\right)'$ form a local integral basis at each root of v, which implies that the coefficients g_i are uniquely determined modulo v.

By Lemma 3 the polynomial e is squarefree and therefore $e \mid uv$; hence we can write $uv = ew$ for some $w \in K[x]$. By rewriting the derivatives of the ω_i in terms of the integral basis, Equation (4) turns into

$$\begin{aligned}
\sum_{i=1}^n f_i\omega_i &= \sum_{i=1}^n g_i(uv\omega_i' - (\mu-1)uv'\omega_i) \mod v \\
&= \sum_{i=1}^n g_i\Bigl(w\sum_{j=1}^n m_{i,j}\omega_j - (\mu-1)uv'\omega_i\Bigr) \mod v.
\end{aligned}$$

Comparing coefficients w.r.t. $\omega_1, \dots, \omega_n$ yields a system of linear equations over $K[x]/\langle v\rangle$ for the unknown functions $g_1, \dots, g_n$. This system has a unique solution. For the coordinates of the solution vector $(g_1, \dots, g_n)$, we can choose representatives in $K[x]$ whose degrees are less than $\deg_x(v)$.

The remaining unknowns $h_1, \dots, h_n$ are obtained by plugging $g_1, \dots, g_n$ into Equation (3).

Example 11. *We consider the fuchsian D-finite function*

$$\frac{1}{x^2}\log\Big(\frac{1}{x^2}-1\Big)\sqrt{\frac{1+x}{1-x}}.$$

This function is annihilated by the second-order differential operator

$$L = (x^2-1)^2x^2\partial_x^2 + (x^2-1)(x+1)(7x-5)x\partial_x + 8x^4 + 5x^3 - 11x^2 - 5x + 4.$$

Using the algorithm described of Kauers and Koutschan (2015) we compute the following integral basis $\{\omega_1,\omega_2\}$ for $A = \mathbb{C}(x)[\partial_x]/\langle L\rangle$:

$$\{(x-1)x^2,\ (x^2-1)(x-1)x^3\partial_x + 2(x-1)x^4\}$$

For the differentiation matrix, a simple calculation yields

$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = \frac{1}{e}\begin{pmatrix}(x-1)(x+2) & 1\\ -x^3-x^2+5x-4 & x^2-x+2\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \tag{5}$$

with $e = (x^2-1)x$. As the integrand corresponds to $1 \in A$, its representation in the integral basis is

$$f = \frac{1}{(x-1)x^2}\omega_1 = \frac{x+1}{ex}\omega_1;$$

using the notation employed above, we have $D = (x^2-1)x^2$, $f_1 = x+1$, and $f_2 = 0$. Here we can only reduce the power of x in the denominator, so we start with $u = x^2-1$, $v = x$, and $\mu = 2$. Then Equation (4) leads to the following linear system for the unknowns g_1 and g_2:

$$\begin{pmatrix}x-1 & -x^3-x^2+5x-4\\ 1 & 3-x\end{pmatrix}\begin{pmatrix}g_1\\ g_2\end{pmatrix} = \begin{pmatrix}x+1\\ 0\end{pmatrix} \mod x.$$

Thus we get $g_1 = 3$ and $g_2 = -1$ and the final result of the Hermite reduction is

$$f = \Big(\underbrace{\frac{3}{x}\omega_1 - \frac{1}{x}\omega_2}_{=g}\Big)' + \underbrace{\frac{-x^2-x+3}{(x^2-1)x}\omega_1 - \frac{1}{(x^2-1)x}\omega_2}_{=h}.$$

For the integration of algebraic functions, it is known that Hermite reduction itself often takes less time than the construction of an integral basis. If Hermite reduction is applied to some other basis, for instance the standard basis $\{1, y, \dots, y^{n-1}\}$, it either succeeds or it runs into a division by zero. Bronstein (1998a) noticed that when a division by zero occurs, then the basis can be replaced by some other basis that is a little closer to an integral basis, just as much as is needed to avoid this particular division by zero. After finitely many such basis changes, the Hermite reduction will come to an end and produce a correct output. This variant is known as lazy Hermite reduction. The same technique also applies in the present situation.

5. The Canonical Form Property

Recall from the introduction that reduction-based creative telescoping requires some K-linear map $[\cdot]\colon A \to A$ with the property that $f - [f]$ is integrable in A for every $f \in A$. This is sufficient for the correctness of the method, but additional properties are needed in order to ensure that the method terminates.

As also explained already in the introduction, one possibility consists in showing that $[f] = 0$ whenever f is integrable. For the special case of algebraic functions, Trager showed that his Hermite reduction has this property (Trager, 1984, p. 50, Thm. 1). Essentially the same argument works for the fuchsian case, as we will show next. The main difference is that in the case of algebraic functions, we can exploit that all the functions which have no poles at either a finite place or at infinity are the constant functions. As we have pointed out above, this is no longer true in the fuchsian D-finite case, but in this case, we can still exploit that the space of all the functions which have no poles at all (neither at finite points nor at infinity) form a finite-dimensional vector space over K.

Lemma 12. *Let $\{\omega_1, \dots, \omega_n\}$ be an integral basis for A that is normal at infinity. Let $g = \sum_{i=1}^n g_i\omega_i \in A$ be such that all its coefficients $g_i \in K(x)$ are proper rational functions. If an integral element $f \in A$ has a pole at infinity, then also $f + g$ has a pole at infinity.*

Proof. Since f is integral we can write it as $f = f_1\omega_1 + \cdots + f_n\omega_n$ with $f_i \in K[x]$. If f has a pole at infinity, there is at least one index i such that $f_i\omega_i$ has a pole at infinity. Since f_i is a polynomial and g_i is a proper rational function, f_i and $f_i + g_i$ have the same exponent at infinity. Therefore, if $f_i\omega_i$ has a pole at infinity, so does $(f_i + g_i)\omega_i$. But then, by Lemma 6, also $f + g = \sum_{i=1}^n (f_i + g_i)\omega_i$ has a pole at infinity. ■

Theorem 13. *Suppose that $f \in A$ has at least a double root at infinity (i.e., every series in $\bar{C}[[[x^{-1}]]]$ associated to f only contains monomials $(1/x)^\alpha \log(x)^\beta$ with $\alpha \geq 2$). Let $W = \{\omega_1, \dots, \omega_n\}$ be an integral basis for A that is normal at infinity, and let $f = g' + h$ be the result of the Hermite reduction with respect to W. Let $V \subseteq A$ be the K-vector space of all elements that are integral at all places, including infinity, and let $U = \{v' : v \in V\}$ be the space of all elements of A that are integrable in V. Then f is integrable in A if and only if $h \in U$.*

Proof. The direction "$\Leftarrow$" is trivial. To show the implication "$\Rightarrow$" assume that f is integrable in A. From $f = g' + h$ it follows that then also h is integrable in A; let $H \in A$ be such that $H' = h$. In order to show that $h \in U$, we show that $H \in V$, i.e., we show that H has no finite poles and no poles at infinity.

It is clear that H has no finite poles because h has at most simple poles (i.e., all series associated to h have only exponents $\alpha \geq -1$). This follows from the facts that the ω_i are integral and that the coefficients of h have squarefree denominators.

If H has a pole at infinity, then by Lemma 12 also $g + H$ must have a pole at infinity, because Hermite reduction produces $g = \sum_i g_i\omega_i$ with proper rational functions g_i. On the other hand, since $f = g' + h = (g + H)'$ has at least a double root at infinity by assumption, $g + H$ must have at least a single root at infinity. This is a contradiction. ■

By Lemma 8, the vector space V has finite dimension and computing a vector space basis of it is not harder than computing an integral basis. Once a basis $\{b_1, \dots, b_d\}$ of V is known, it is also easy to obtain a basis of U, as this space is generated by $\{b_1', \dots, b_d'\}$. Therefore, we can decide whether a given $f \in A$ with a double root at infinity is integrable in A by first executing Hermite reduction, and then checking whether the Hermite remainder h belongs to U. More generally, by performing a reduction modulo U as a post-processing step after the Hermite reduction, we can even ensure that f is integrable if and only if $h = 0$.

Example 14. *Let $L = 3(x^3 - x)D_x^2 + 2(3x^2 - 1)D_x$ be the same operator as in Example 7; its solution space is spanned by $y_1(x) = 1$ and $y_2(x) = x^{1/3}\,{}_2F_1(\frac{1}{6}, \frac{2}{3}; \frac{7}{6}; x^2)$. An integral basis for $A = \mathbb{Q}(x)[\partial_x]/\langle L\rangle$ that is also normal at infinity is given by $\omega_1 = 1$ and $\omega_2 = (x^3 - x)\partial_x$. Recall that both solutions are integral everywhere, and hence $\omega_1 \in V$. Actually, the K-vector space V is spanned by ω_1, as can be seen from the fact that $\tau_1 = 0$ and $\tau_2 = -1$. The straightforward calculation*

$$W' = \frac{1}{e}MW = \frac{1}{x^3 - x}\begin{pmatrix} 0 & 1 \\ 0 & x^2 - \frac{1}{3} \end{pmatrix} W \quad \textit{for } W = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

exhibits that $e = x^3 - x$. Consider now

$$f = \frac{3}{x^2}\omega_1 + \frac{2(2x+1)}{(x^3 - x)^2}\omega_2,$$

which has a double root at infinity. The result of the Hermite reduction is

$$f = \left(-\frac{3}{x}\omega_1 - \frac{3(2x+1)}{2(x^3 - x)}\omega_2\right)' - \frac{3}{x^3 - x}\omega_2,$$

which has a nonzero remainder. According to Theorem 13, f is integrable if this remainder lies in the subspace $U = \{v' : v \in V\}$. Using the matrix M above, we find that $\omega_1' = \frac{1}{x^3 - x}\omega_2$, which is indeed a scalar multiple of our remainder. Hence, f is integrable:

$$f = \left(-\frac{3(x+1)}{x}\omega_1 - \frac{3(2x+1)}{2(x^3 - x)}\omega_2\right)'.$$

Remark 15.

1. *The condition in Theorem 13 that f has a double root at infinity is not a restriction, as it can always be achieved by a suitable change of variables. Let $a \in C$ be an ordinary point of L; by the substitution $x \to a + 1/x$ the ordinary point a is moved to infinity. From*

   $$\int f(x)\,\mathrm{d}x = \int f\left(\frac{1}{x} + a\right)\left(-\frac{1}{x^2}\right)\mathrm{d}x$$

   we see that the new integrand has a double root at infinity.
2. *Since the action of ∂_t on series domains is defined coefficient-wise, it follows that when f has at least a double root at infinity (with respect to x), this is also true for $\partial_t \cdot f, \partial_t^2 \cdot f, \partial_t^3 \cdot f, \ldots$, and then also for every K-linear combination $p_0 f + p_1\partial_t \cdot f + \cdots + p_r\partial_t^r \cdot f$. Thus Theorem 13 implies that $p_0 + p_1\partial_t + \cdots + p_r\partial_t^r$ is a telescoper for f if* and only if $[p_0 f + p_1\partial_t \cdot f + \cdots + p_r\partial_t^r \cdot f] = 0$.
3. *We already know for other reasons (Zeilberger, 1990b; Chyzak, 2000) that telescopers for D-finite functions exist, and therefore the reduction-based creative telescoping procedure with Hermite reduction with respect to an integral basis that is normal at infinity plus reduction modulo U as reduction function* $[\cdot]$ *succeeds when applied to an integrand $f \in A$ that has a double root at infinity. In particular, the method finds a telescoper of smallest possible order. Again, if f has no double root at infinity, we can produce one by a change of variables. Note that a change of variables $x \to a + 1/x$ with $a \in C$ has no effect on the telescoper.*

One of the anonymous referees has made us aware that it can further be shown that the vector space of Hermite remainders has finite dimension. This in turn gives rise to a bound on the order of the telescoper. The referee's argument is formulated in the following proposition.

Proposition 16. *Let $W = \{\omega_1, \ldots, \omega_n\}$ be an integral basis for A that is normal at infinity, and let $\tau_1, \ldots, \tau_n \in \mathbb{Z}$ be such that $\{x^{\tau_1}\omega_1, \ldots, x^{\tau_n}\omega_n\}$ is a local integral basis at infinity.*

Let $f \in A$ have a double root at infinity, write $f = \sum_{i=1}^n (f_i/D)\omega_i \in A$ for some $D, f_1, \ldots, f_n \in K[x]$. Let $e \in K[x]$ be as in Lemma 3, and let $E = \operatorname{lcm}(e, D^*)$, where $D^* \in K[x]$ denotes the squarefree part of D.

Then f admits a telescoper of order at most $n(\deg_x(E) - 1 - \min(0, \min_j(\tau_j))) + \sum_{i=1}^n \tau_i$.

Proof. By Hermite reduction, we have $f = g' + h$ for some $g, h \in A$ which we may write as $g = \sum_{i=1}^n g_i\omega_i$ and $h = \sum_{i=1}^n (h_i/E)\omega_i$ with proper rational functions g_i and polynomials h_i. We seek to bound the degrees of the polynomials h_i.

Let the polynomials $m_{i,j}$ be as in Lemma 3; then we have

$$g' = \sum_{i=1}^n \left(g_i'\omega_i + g_i \sum_{j=1}^n \frac{m_{i,j}}{e}\omega_j \right) = \sum_{i=1}^n \left(g_i' + \sum_{j=1}^n g_j \frac{m_{j,i}}{e} \right)\omega_i.$$

Since W is also a $K(x)$-vector space basis of A, it follows from $f - g' - h = 0$ that also the coefficients $f_i/D - g_i' - \sum_{j=1}^n g_j(m_{j,i}/e) - h_i/E$ must be zero for all i.

The function $x^2 f$ is integral at infinity, because f has a double root at infinity, and hence we can write it in the local integral basis as $x^2 f = \sum_{i=1}^n x^{2-\tau_i}(f_i/D)(x^{\tau_i}\omega_i)$. Then also all the coefficients $x^{2-\tau_i}(f_i/D)$ are integral at infinity, which implies that the series expansion of any f_i/D involves only monomials $(1/x)^\alpha$ with $\alpha \geq 2 - \tau_i$. Since the g_i are proper rational functions, the exponents α in their expansions at infinity satisfy $\alpha \geq 1$, and similarly $\alpha \geq 2$ in the expansions of the g_i'. By Lemma 4 applied to the local integral basis $\{x^{\tau_1}\omega_1, \ldots, x^{\tau_n}\omega_n\}$, we have $\deg_x(m_{j,i}) < \deg_x(e) - \tau_j + \tau_i$ for all i, j, so $m_{j,i}/e$ only contains terms $(1/x)^\alpha$ with $\alpha \geq \tau_j - \tau_i + 1$. Therefore, for the exponents α in the expansion of $g_i' + \sum_{j=1}^n g_j(m_{j,i}/e)$ we have the estimate $\alpha \geq 2 + \min(0, \min_j(\tau_j) - \tau_i)$.

It now follows that the expansion of any h_i/E at infinity cannot have any exponents α smaller than

$$\begin{aligned} b_i &:= 2 + \min(-\tau_i, 0, \min_j(\tau_j) - \tau_i) = 2 - \max(0, \underbrace{\tau_i, \underbrace{\tau_i - \min_j(\tau_j)}_{\leq \tau_i}}_{\geq 0}) \\ &= 2 - \max(\tau_i, \tau_i - \min_j(\tau_j)) = 2 - \tau_i - \max(0, -\min_j(\tau_j)) \\ &= 2 - \tau_i + \min(0, \min_j(\tau_j)), \end{aligned}$$

because such terms could not be canceled by f_i/D or g_i.

We have thus shown that $\deg_x(h_i) \leq \deg_x(E) - b_i$. In other words, the Hermite remainder $h = \sum_{i=1}^n (h_i/E)\omega_i$ is constrained to belong to the K-vector space generated by $(x^j/E)\omega_i$ for $i = 1, \ldots, n$ and $j = 0, \ldots, \deg_x(E) - b_i$. The dimension of this space is

$$\sum_{i=1}^n (\deg_x(E) - b_i + 1) = n(\deg_x(E) - 1 - \min(0, \min_j(\tau_j))) + \sum_{i=1}^n \tau_i.$$

As the Hermite remainders of $\partial_t \cdot f, \partial_t^2 \cdot f, \ldots$ also belong to this K-vector space, the claimed bound on the order of the telescoper follows. ∎

6. Polynomial Reduction

Under the assumption that the input function $f \in A$ has a double root at infinity, the success of reduction-based creative telescoping can be ensured in two ways:

1. The canonical form property enables us to find the minimal telescoper for f; together with the known existence of telescopers for D-finite functions it ensures termination.
2. The finite-dimensionality property of the K-vector space generated by $\{[\partial_t^i f] : i \in \mathbb{N}\}$ reproves the existence of telescopers and leads to a bound on their orders.

As an alternative, we now introduce an additional reduction called *polynomial reduction,* which we apply after Hermite reduction. We then show that the combined reduction (Hermite reduction followed by polynomial reduction) has the same canonical form property and the same finite-dimensionality property for the space of remainders, but without using the double-root assumption on the inputs. As a result, we obtain another bound on the order of the telescoper, which is similar to that of Chen et al. (2014) and that of Proposition 16.

In this approach, we use two integral bases. First we use a global integral basis in order to perform Hermite reduction. Then we write the remainder h with respect to some local integral basis at infinity and perform the polynomial reduction on this representation.

Throughout this section let $W = (\omega_1, \dots, \omega_n)^T \in A^n$ be such that $\{\omega_1, \dots, \omega_n\}$ is a global integral basis of A, and let $e \in K[x]$ and $M = ((m_{i,j})) \in K[x]^{n\times n}$ be such that $eW' = MW$ and $\gcd(e, m_{1,1}, m_{1,2}, \dots, m_{n,n}) = 1$. The Hermite reduction described in Section 4 decomposes an element $f \in A$ into the form

$$f = g' + h = g' + \sum_{i=1}^{n} \frac{h_i}{de}\omega_i, \qquad g, h \in A,$$

with $h_i, d \in K[x]$ such that $\gcd(d, e) = \gcd(d, h_1, \dots, h_n) = 1$ and d is squarefree.

Lemma 17. *Let h be as above. If h is integrable in A, then its integral is integral, and we have $d \in K$.*

Proof. Suppose that h is integrable in A, i.e., there exist $b_i \in K(x)$ such that $h = (\sum_{i=1}^n b_i\omega_i)'$. If one of the b_i had a pole at a finite place, then $\sum_{i=1}^n b_i\omega_i$ would have a pole at a finite place, because $\{\omega_1, \dots, \omega_n\}$ is an integral basis. But then h would have a pole of order greater than 1 there, which is impossible because $\gcd(d, e) = 1$ and d is squarefree and $\{\omega_1, \dots, \omega_n\}$ is an integral basis. Therefore, $b_1, \dots, b_n \in K[x]$ and we have shown that the integral of h is integral. The claim on d then follows directly from the definition of e. ■

Note that the lemma continues to hold when $\{\omega_1, \dots, \omega_n\}$ is a local integral basis at a finite place $a \in \bar{C}$. In this case, we can conclude that the integral is locally integral at a and $d \in K(x)_a$.

By the extended Euclidean algorithm, we compute $r_i, s_i \in K[x]$ such that $h_i = r_i e + s_i d$ and $\deg_x(r_i) < \deg_x(d)$. Then the Hermite remainder h decomposes as

$$\sum_{i=1}^{n} \frac{h_i}{de}\omega_i = \sum_{i=1}^{n} \frac{r_i}{d}\omega_i + \sum_{i=1}^{n} \frac{s_i}{e}\omega_i. \tag{6}$$

We now introduce the *polynomial reduction* whose goal is to confine the s_i to a finite-dimensional vector space over K. Similar reductions have been introduced and used in creative telescoping for hyperexponential functions (Bostan et al., 2013a) and hypergeometric terms (Chen et al., 2015). Our version below is slightly different from these, and also from the reduction given by Chen et al. (2016) for the algebraic case, because we will be considering Laurent polynomials instead of polynomials. Note that the same idea can be applied in the polynomial reduction for q-hypergeometric terms Du et al. (2016).

Throughout the rest of the section, let $V = (v_1, \ldots, v_n)^T \in A^n$ be such that its entries form a local integral basis at ∞ as well as a local integral basis at every $a \in \bar{C} \setminus \{0\}$, and which is normal at 0. The existence of such a basis follows from the existence of global integral bases that are normal at infinity, as follows.

Lemma 18. *Let $W = \{\omega_1, \ldots, \omega_n\}$ be an integral basis of A that is normal at infinity. Then there exist integers $\tau_1, \ldots, \tau_n \in \mathbb{Z}$ such that $V := \{v_1, \ldots, v_n\}$ with $v_i = x^{\tau_i}\omega_i$ ($i = 1, \ldots, n$) is a basis of A which is normal at 0 and integral at all other points (including infinity).*

Proof. It is clear that such a basis V will be normal at zero, because multiplying the generators by the rational functions $x^{-\tau_i}$ brings it back to a global integral basis, which is in particular a local integral basis at zero. It is also clear that such a basis will be integral at every other point $a \in \bar{C} \setminus \{0\}$, because the multipliers x^{τ_i} are locally units at such a. Finally, since the original basis is normal at infinity, there exist rational functions $u_1, \ldots, u_n$ such that $\{u_1\omega_1, \ldots, u_n\omega_n\}$ is a local integral basis at infinity. Since u_i can be written as $u_i = x^{\tau_i}\tilde{u}_i$ with $\tau_i \in \mathbb{Z}$ and $\tilde{u}_i$ being a unit in $\bar{C}(x)_\infty$, we see that also V is a local integral basis at infinity. ∎

In the case of algebraic functions (Chen et al., 2016), it is a consequence of Chevalley's theorem that the exponents τ_i can never be positive, and that they can be zero only if the function is constant. In the more general fuchsian situation, this is no longer the case. Let $a \in K[x]$ and $B = ((b_{i,j})) \in K[x]^{n\times n}$ be such that $aV' = BV$ and $\gcd(a, b_{1,1}, b_{1,2}, \ldots, b_{n,n}) = 1$. By Lemma 17, we may assume that $a = x^\lambda e$ for some $\lambda \in \mathbb{N}$, and we will do so. Writing the expression $\sum_{i=1}^n \frac{s_i}{e}\omega_i$ from equation (6) in terms of the basis V, we obtain $\sum_{i=1}^n \frac{x^{\lambda-\tau_i}s_i}{x^\lambda e}v_i$, where now the numerators of the coefficients are Laurent polynomials. Note however that since $s_1, \ldots, s_n$ are in $K[x]$, the new numerators cannot have arbitrarily negative exponents. In fact, they will live in $x^{-\tau}K[x]$ where $\tau = \max(\tau_1, \ldots, \tau_n)$. An a priori bound for τ follows from Lemma 5.

The purpose of polynomial reduction is to write

$$\sum_{i=1}^n \frac{x^{\lambda-\tau_i}s_i}{x^\lambda e}v_i = \tilde{g}' + \sum_{i=1}^n \frac{\tilde{s}_i}{x^\lambda e}v_i,$$

where the $\tilde{s}_i$ belong to a finite-dimensional subspace of $x^{-\tau}K[x]$. Let us write $K[x]_{\eta,\mu}$ for the subspace of $K[x, x^{-1}]$ consisting of all Laurent polynomials whose exponents are at least η and at most μ. Then $x^{-\tau}K[x] = K[x]_{-\tau,\infty}$, and while $x^{-\tau_1}s_1, \ldots, x^{-\tau_n}s_n$ belong to this space, we will show that it is possible to choose $\tilde{s}_1, \ldots, \tilde{s}_n$ that belong to $K[x]_{-\tau,\delta}$ for some finite $\delta \in \mathbb{Z}$. To this end, note that for any $P = (p_1, \ldots, p_n) \in K[x, x^{-1}]^n$ we have

$$(PV)' = \sum_{i=1}^n (p_i v_i)' = \frac{x^\lambda eP' + PB}{x^\lambda e}V. \tag{7}$$

This motivates the following definition.

Definition 19. *Let*

$$\phi_V : K[x]^n_{-\tau+1,\infty} \to K[x]^n_{-\tau,\infty}, \quad \phi_V(P) = x^\lambda eP' + PB.$$

We call ϕ_V the map for polynomial reduction *with respect to V, and call the subspace*

$$\mathrm{im}(\phi_V) = \{\phi_V(P) \mid P \in K[x]^n_{-\tau+1,\infty}\} \subseteq K[x]^n_{-\tau,\infty}$$

the subspace for polynomial reduction *with respect to V.*

Note that, by construction and because of Lemma 17, $Q \in K[x]^n_{-\tau,\infty}$ belongs to $\mathrm{im}(\phi_V)$ if and only if $\frac{1}{x^\lambda e} QV$ is integrable in A.

We can always view an element of $K[x]^n_{\eta,\mu}$ (resp. $K[x]^{n\times n}_{\eta,\mu}$) as a Laurent polynomial in x with coefficients in K^n (resp. $K^{n\times n}$). In this sense we use the notation $\mathrm{lc}(\cdot)$ for the leading coefficient and $\mathrm{lt}(\cdot)$ for the leading term of a vector (resp. matrix). For example, if $P \in K[x]^n_{\eta,\mu}$ is of the form

$$P = p^{(i_1)}x^{i_1} + \cdots + p^{(i_m)}x^{i_m}, \quad p^{(i)} \in K^n, \quad i_1 \leq \cdots \leq i_m, \quad p^{(i_m)} \neq 0$$

then $\deg_x(P) = i_m$, $\mathrm{lc}(P) = p^{(i_m)}$, and $\mathrm{lt}(P) = p^{(i_m)}x^{i_m}$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of K^n. Then the K-vector space $K[x]^n_{\eta,\mu}$ is generated by

$$\mathcal{X}_{\eta,\mu} := \{e_i x^j \mid 1 \leq i \leq n,\, \eta \leq j \leq \mu\}.$$

Definition 20. *Let N_V be the K-subspace of $K[x]^n_{-\tau,\infty}$ generated by*

$$\{t \in \mathcal{X}_{-\tau,\infty} \mid t \neq \mathrm{lt}(P) \textit{ for all } P \in \mathrm{im}(\phi_V)\}.$$

Then $K[x]^n_{-\tau,\infty} = \mathrm{im}(\phi_V) \oplus N_V$. We call N_V the standard complement *of* $\mathrm{im}(\phi_V)$*. For any $P \in K[x]^n_{-\tau,\infty}$, there exist $P_1 \in K[x]^n_{-\tau+1,\infty}$ and $P_2 \in N_V$ such that $P = \phi_V(P_1) + P_2$ and*

$$\frac{1}{x^\lambda e} PV = (P_1 V)' + \frac{1}{x^\lambda e} P_2 V.$$

This decomposition is called the polynomial reduction *of P with respect to V.*

Lemma 21. *Let $H \in K^{n\times n}$ and $\lambda_1, \ldots, \lambda_s \in \bar{K}$ be all of the distinct eigenvalues of H. Then*

$$\sum_{i=1}^{s} \Big(n - \dim(\mathrm{im}(H - \lambda_i I_n))\Big) \leq n,$$

where $\mathrm{im}(T) := \{w \in \bar{K}^n \mid \exists v \in \bar{K}^n \text{ s.t. } Tv = w\}$ for $T \in \bar{K}^{n\times n}$.

Proof. Note that $\dim(\mathrm{im}(T)) = \mathrm{rank}(T)$ and $\dim(\ker(T)) = n - \mathrm{rank}(T)$. Then it suffices to show that $\sum_{i=1}^s \dim(\ker(H - \lambda_i I_n)) \leq n$. Let $p(x) = (x-\lambda_1)^{m_1} \cdots (x - \lambda_s)^{m_s} \in K[x]$ be the characteristic polynomial of H. Since the geometric multiplicity $\dim(\ker(H - \lambda_i I_n))$ is at most the algebraic multiplicity m_i for each eigenvalue λ, we have $\sum_{i=1}^s \dim(\ker(H - \lambda_i I_n)) \leq \sum_{i=1}^s m_i = n$. ∎

Proposition 22. *Let $\lambda \in \mathbb{N}$, $e \in K[x]$ and $B \in K[x]^{n\times n}$ be such that $x^\lambda e V' = BV$, as before. If $\deg_x(B) \leq \lambda + \deg_x(e) - 1$, then N_V is a K-vector space of dimension at most $n(\tau + \lambda + \deg_x(e) + 1)$.*

Proof. For brevity, let $\delta := \lambda + \deg_x(e) - 1$. We distinguish two cases.

Case 1. Assume that $\deg_x(B) < \delta$. For any $P \in K[x]^n_{-\tau+1,\infty}$ of degree μ, we have

$$\mathrm{lt}(\phi_V(P)) = \mu\, \mathrm{lc}(e)\, \mathrm{lc}(P) x^{\mu+\delta}.$$

Thus all monomials $e_i x^j$ with $1 \le i \le n$ and $j \ge \delta + 1$ are not in N_V, and thus $\dim N_V \le n(\tau + \delta + 1) < \infty$.

Case 2. Assume that $\deg_x(B) = \delta$. For any $P \in K[x]^n_{-\tau+1,\infty}$ of degree μ, we have

$$\mathrm{lt}(\phi_V(P)) = \mathrm{lc}(P)(\mu\,\mathrm{lc}(e)I_n + \mathrm{lc}(B))x^{\mu+\delta}.$$

Let $\mathcal{L} := \{\ell \in \mathbb{N} \mid -\ell\,\mathrm{lc}(e) \text{ is an eigenvalue of } \mathrm{lc}(B) \in K^{n\times n}\}$. Then for any $\mu \in \mathbb{N} \setminus \mathcal{L}$, the matrix $\mu\,\mathrm{lc}(e)I_n + \mathrm{lc}(B) \in K^{n\times n}$ is invertible. So any monomial $e_i x^{j+\delta}$ with $j \notin \mathcal{L}$ or $e_i \in \mathrm{im}(j\,\mathrm{lc}(e)I_n + \mathrm{lc}(B))$ for $j \in \mathcal{L}$ is not in N_V for any $i = 1, \ldots, n$, and thus $\dim N_V \le n(\tau + \delta + 1) + \sum_{\ell\in\mathcal{L}}(n - \dim(\mathrm{im}(\ell\,\mathrm{lc}(e)I_n + \mathrm{lc}(B))))$. By Lemma 21, we have $\dim N_V \le n(\tau+\delta+2) = n(\tau+\lambda+\deg_x(e)+1)$. ■

It follows from our general assumptions on V that the condition $\deg_x(B) \le \lambda + \deg_x(e) - 1$ is always satisfied. Therefore, by the combination of Hermite reduction described in Section 4 with polynomial reduction, we get the following theorem.

Theorem 23. *Let $W \in A^n$ be an integral basis of A that is normal at infinity. Let*

$$T = \mathrm{diag}(x^{\tau_1}, \ldots, x^{\tau_n}) \in K(x)^{n\times n}$$

be such that $V := TW$ is a local integral basis at infinity. Let $e \in K[x]$, $\lambda \in \mathbb{N}$, and $B, M \in K[x]^{n\times n}$ be such that $eW' = MW$ and $x^\lambda eV' = BV$. Then any element $f \in A$ can be decomposed into

$$f = g' + \frac{1}{d}RW + \frac{1}{x^\lambda e}QV, \tag{8}$$

where $g \in A$, $d \in K[x]$ is squarefree and $\gcd(d, e) = 1$, $R, Q \in K[x]^n$ with $\deg_x(R) < \deg_x(d)$ and $Q \in N_V$, which is a finite-dimensional K-vector space. Moreover, $R = Q = 0$ if and only if f is integrable in A.

Proof. After performing the Hermite reduction on f, we get

$$f = \tilde{g}' + \frac{1}{d}RW + \frac{1}{e}SW, \tag{9}$$

where $R = (r_1, \ldots, r_n) \in K[x]^n$ and $S = (s_1, \ldots, s_n) \in K[x]^n$ with r_i, s_i introduced in (6). By Lemma 18, there exists $T = \mathrm{diag}(x^{\tau_1}, \ldots, x^{\tau_n}) \in K(x)^{n\times n}$ such that $V = TW$ is a local integral basis at infinity. By the same lemma it follows that V is also normal at 0 and integral at all other points. The derivative of V can be reexpressed in the same basis as

$$V' = (TW)' = \left(T' + \frac{1}{e}TM\right)T^{-1}V = \frac{1}{x^\lambda e}BV,$$

Since V is a local integral basis at infinity, it follows by Lemma 4 that $\deg_x(B) \le \lambda + \deg_x(e) - 1$, which is a prerequisite for applying Proposition 22; hence the corresponding space N_V is finite-dimensional, as claimed. We rewrite the last summand in (9) w.r.t. the new basis V:

$$\frac{1}{e}SW = \frac{1}{x^\lambda e}\tilde{S}V,$$

where $\tilde{S} = x^\lambda S T^{-1} \in K[x, x^{-1}]^n$. Note that the entries of $\tilde{S}$ are not necessarily polynomials, but Laurent polynomials in x. Indeed, because of Lemma 8, some of the τ_i may be positive, and actually, $\tilde{S} \in K[x]^n_{-\tau,\infty}$ where $\tau := \max\{\tau_1, \ldots, \tau_n\}$, as before. Next, using the polynomial reduction, we decompose $\tilde{S}$ into $\tilde{S} = \phi_V(S_1) + S_2$ with $S_1 \in K[x]^n_{-\tau+1,\infty}$ and $S_2 \in N_V$, which means that

$$\frac{1}{e} S W = (S_1 V)' + \frac{1}{x^\lambda e} S_2 V.$$

We finally obtain the decomposition (8) by setting $g = \tilde{g} + S_1 V$ and $Q = S_2$.

For the last assertion, assume that f is integrable (the other direction of the equivalence holds trivially). Then Lemma 17 implies that $d \in K$, and therefore R must be zero because $\deg_x(R) < \deg_x(d)$. Hence the last summand in (8) is also integrable, i.e., there exist $c_i \in K(x)$ such that

$$\frac{1}{x^\lambda e} Q V = \left(\sum_{i=1}^{n} c_i v_i\right)'.$$

Note that the expression on the left-hand side has only simple poles at finite points except 0. Therefore, by Lemma 17, its integral is integral at all nonzero finite points. In other words, the coefficients c_i are actually Laurent polynomials in $K[x, x^{-1}]$, which implies that $Q \in \operatorname{im}(\phi_V)$. Since $\operatorname{im}(\phi_V) \cap N_V = \{0\}$, it follows that $Q = 0$. ■

The decomposition in (8) is called an *additive decomposition* of f with respect to x. We now discuss how to compute telescopers for elements of A via Hermite reduction and polynomial reduction.

We first consider the additive decompositions of the successive derivatives $\partial_t^i \cdot f$ for $i \in \mathbb{N}$. Assume that

$$\partial_t \cdot W = \frac{1}{\tilde{e}} \tilde{M} W \quad \text{and} \quad \partial_t \cdot V = \frac{1}{\tilde{a}} \tilde{B} V, \tag{10}$$

for some polynomials $\tilde{e}, \tilde{a} \in K[x]$ and matrices $\tilde{M}, \tilde{B} \in K[x]^{n\times n}$ such that $\tilde{e}$ is coprime with $\tilde{M}$ and $\tilde{a}$ is coprime with $\tilde{B}$. By (Chen et al., 2014, Prop. 7), we have that $\tilde{e} \mid e$ and $\tilde{a} \mid x^\lambda e$. Hence, we can take $\tilde{e} = e$ and $\tilde{a} = x^\lambda e$ in (10), by multiplying the matrices $\tilde{M}$ and $\tilde{B}$ by some factors of $x^\lambda e$. Now we differentiate (8) with respect to t, and obtain, after a direct calculation, $\partial_t \cdot f = (\partial_t \cdot g)' + h$, where

$$h = \left(\partial_t \cdot \left(\frac{1}{d} R\right) + \frac{1}{de} R \tilde{M}\right) W + \left(\partial_t \cdot \left(\frac{1}{x^\lambda e} Q\right) + \frac{1}{x^{2\lambda} e^2} Q \tilde{B}\right) V.$$

Obviously the squarefree part of the denominator of h divides xde. Applying Hermite reduction and polynomial reduction to h then yields

$$h = \tilde{g}_1' + \frac{1}{d} R_1 W + \frac{1}{x^\lambda e} Q_1 V,$$

where $R_1, Q_1 \in K[x]^n$ with $\deg_x(R_1) < \deg_x(d)$ and $Q_1 \in N_V$. Repeating this discussion, we get the following lemma.

Lemma 24. *For any $i \in \mathbb{N}$, the derivative $\partial_t^i \cdot f$ has an additive decomposition of the form*

$$\partial_t^i \cdot f = g_i' + \frac{1}{d} R_i W + \frac{1}{x^\lambda e} Q_i V,$$

where $g_i \in A$, $R_i, Q_i \in K[x]^n$ with $\deg_x(R_i) < \deg_x(d)$ and $Q_i \in N_V$.

As application of the above lemma, we can compute the minimal telescoper for f by finding the first linear dependence among the (R_i, Q_i) over K. We also obtain an upper bound for the order of telescopers.

Corollary 25. *Every $f \in A$ has a telescoper of order at most $n \deg_x(d) + \dim_K(N_V)$, which is bounded by $n(\deg_x(d) + \deg_x(e) + \tau + \lambda + 1)$.*

Remark 26. *Note that the order bound as above is obtained without the double-root assumption on the input, which however generically matches the bound given in Proposition 16, since the set $\mathcal{L}$ in the proof of Proposition 22 is generically empty. Both bounds are tight since they can be reached at least for rational inputs. Indeed, let us consider rational functions of the form $f(t, x) = P(t, x)/Q(t, x)$, where $P, Q \in \mathbb{C}[t, x]$ with $\deg_x(P) = \deg_x(Q) - 1$ and Q being irreducible. To turn f into a rational function having a double root at infinity, we perform the substitution $x \to a + 1/x$ for some $a \in \mathbb{C}$ such that $Q(t, a) \neq 0$. Then the telescoping problem for f is translated to that for $\tilde{f} = \tilde{P}/\tilde{Q} = -f(t, a + 1/x)/x^2$. If no cancellation happens, we have $\deg_x(\tilde{P}) = \deg_x(P)$ and $\deg_x(\tilde{Q}) = \deg_x(Q)+1$. By substituting the order bound $n(\deg_x(E)-1-\min(0, \min_j(\tau_j)))+\sum_{i=1}^n \tau_i$ in Proposition 16 with $n = 1$, $\deg_x(E) = \deg_x(\tilde{Q})$ and $\tau_i = 0$ for all $i \in \{1, 2, \ldots, n\}$, we get the order bound $\deg_x(Q)$ for the input $f(t, x)$. By Corollary 25, we also get the order bound $\deg_x(Q)$, since $n = 1$, $\dim_K(N_v) = 0$, and $\deg_x(d) = \deg_x(Q)$. So the two bounds match each other for such rational inputs generically.*

Example 27. *We compute a minimal telescoper for the function*

$$F := \frac{1}{x^2} \log\Big(\frac{1}{x^2} - t^2\Big) \sqrt{\frac{1 + tx}{1 - tx}}.$$

Note that for $t = 1$ we obtain the function from Example 11. The operator L with $LF = 0$ and the integral basis $\{\omega_1, \omega_2\}$ for $A = \mathbb{C}(t, x)[\partial_x]/\langle L\rangle$ are very similar to those in Example 11. As before, F is represented by $f = 1 \in A$. Also the computation of its Hermite reduction is analogous, yielding

$$h = \frac{-t^3x^2 - t^2x + 3t}{(t^2x^2 - 1)x}\omega_1 - \frac{t}{(t^2x^2 - 1)x}\omega_2$$

as the Hermite remainder h. The matrix M representing the differentiation of the ω_i does not satisfy the degree condition of Proposition 22; this fact is already visible in (5). Hence we perform a change of basis to $v_1 = x^{-1}\omega_1$, $v_2 = x^{-2}\omega_2$, which is an integral basis at infinity. We have $xeV' = BV$ for $V = (v_1, v_2)^T$, $e = x(t^2x^2 - 1)$, and some matrix $B \in K[x]^{2\times 2}$ with $\deg_x(B) = 3$; since $\delta = \lambda + \deg_x(e) - 1 = 3$ we are in Case 2 of Proposition 22. By investigating the eigenvalues of $\operatorname{lc}(B)$ we find that $\ell = 1$. In order to determine a basis for N_V and to execute the polynomial reduction conveniently, we consider the matrix whose rows are constituted by $\phi_V(t)$ for all $t \in \mathcal{X}_{0,\ell}$, written in the basis $\mathcal{X}_{0,\ell+\delta}$. The echelon form of this 4×10 matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2/t^3 & 0 & 4/t^4 & -4/t^4 & 0 \\ 0 & 0 & 1 & 1/t & 0 & 0 & 0 & -4/t^3 & 4/t^3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Writing the Hermite remainder h in terms of the v_i, we see that all denominators divide the polynomial xe. Thus we write $h = \frac{1}{xe}(h_1v_1 + h_2v_2)$ and in (6) we get $s_1 = h_1 = -t^3x^4 - t^2x^3 + 3tx^2$, $s_2 = h_2 = -tx^3$, and $r_1 = r_2 = 0$. In the basis $\mathcal{X}_{0,4}$ the vector (s_1, s_2) reads

$$(-t^3, 0, -t^2, -t, 3t, 0, 0, 0, 0, 0).$$

The polynomial reduction now corresponds to reducing this vector with the rows of (11), yielding the final result $[f] = -\frac{x^2}{xe}\nu_2$*. Next, we consider the derivative* $\partial_t \cdot f = tx^3\partial_x + 2tx^2 + x \in A$ *whose Hermite remainder is*

$$\frac{1}{(tx+1)x}\omega_1 + \frac{1}{(t^2x^2-1)x}\omega_2.$$

(Note that we could as well take $\partial_t \cdot [f]$ *instead of* $\partial_t \cdot f$*, which in general should result in a faster algorithm.) After polynomial reduction we obtain*

$$\frac{1}{xe}\Big(-\frac{4}{t^2}\nu_1 + \frac{(tx+4)x}{t^2}\nu_2\Big).$$

Since there is no linear dependence over $\mathbb{C}(t)$ *yet, we continue with*

$$\partial_t^2 \cdot f = \frac{(tx+1)x^3}{1-tx}\partial_x - \frac{x^2(2t^4x^4+5t^3x^3+2t^2x^2-5tx-3)}{(t^2x^2-1)^2}.$$

Writing $\partial_t^2 \cdot f$ *in terms of the integral basis produces the denominator* $d = (tx-1)^3(tx+1)^2x$*, which means that the Hermite reduction consists of three reduction steps. As a Hermite remainder we obtain*

$$\frac{t^2x^2+2tx-4}{(t^2x^2-1)tx}\omega_1 + \frac{2}{(t^2x^2-1)tx}\omega_2$$

which by polynomial reduction is converted into

$$\frac{1}{xe}\Big(-\frac{4}{t^3}\nu_1 + \frac{2(tx+2)x}{t^3}\nu_2\Big).$$

Now we can find a linear dependence that gives rise to the telescoper $t^2\partial_t^2 - t\partial_t + 1$*, which is indeed the minimal one for this example.*

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