

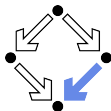
# Computing the Number of Realizations of Laman Graphs

Christoph Koutschan

(joint work with Jose Capco, Matteo Gallet, Georg Grasegger,  
Niels Lubbes, Josef Schicho)

Johann Radon Institute for Computational and Applied Mathematics  
Austrian Academy of Sciences

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CMAPT-Workshop, Linz



## Rigid and Non-Rigid Graphs

**Notation:** Let  $G = (V, E)$  be a graph, and let  $\lambda: E \rightarrow \mathbb{R}_{>0}$  be a labeling of its edges, that is **realizable** (as lengths in  $\mathbb{R}^2$ ).

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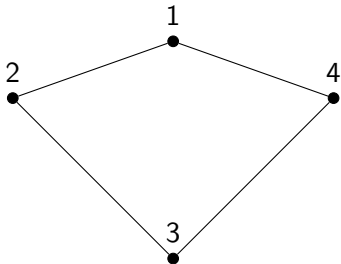
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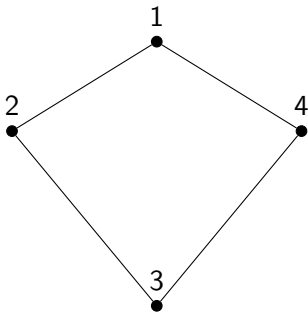
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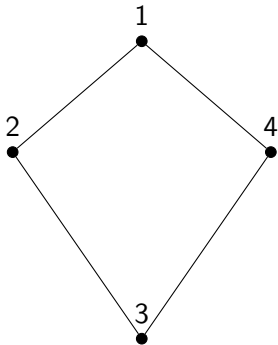
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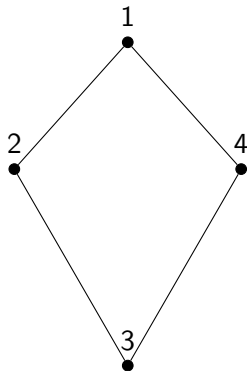
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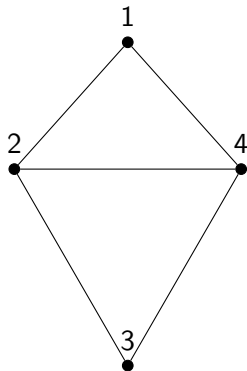
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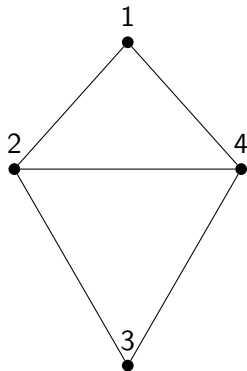
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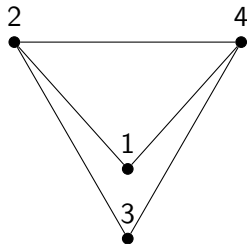
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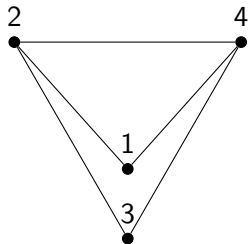
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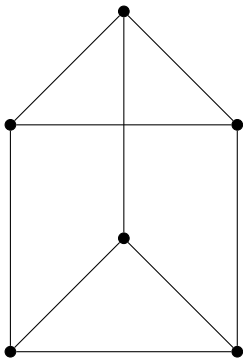
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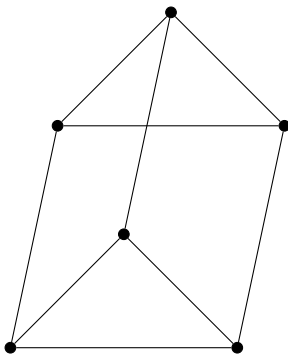
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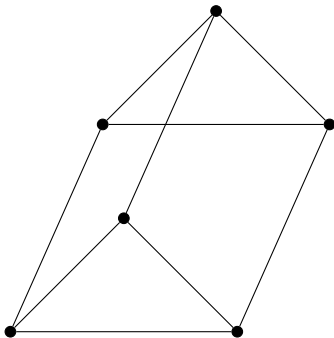
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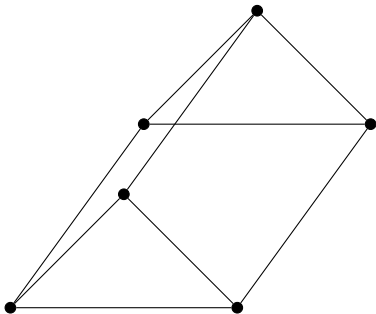
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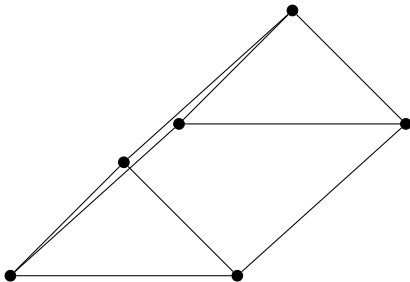
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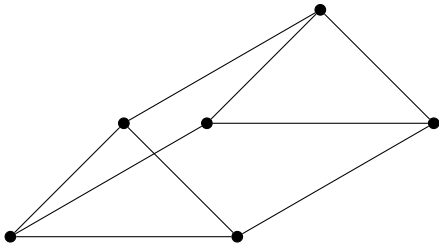
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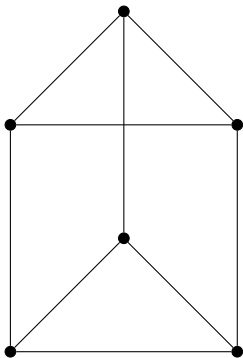
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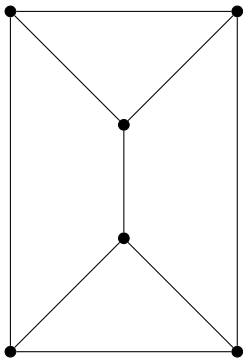
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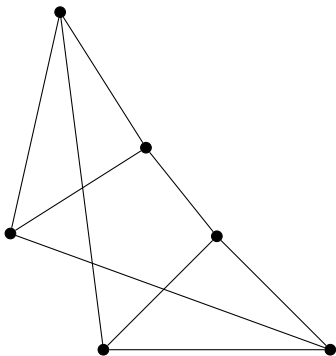
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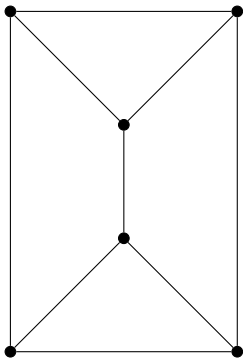
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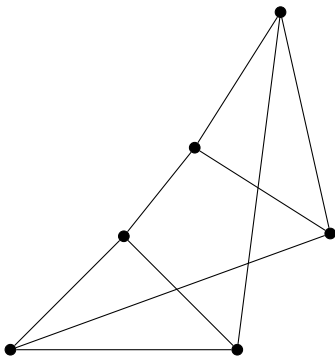
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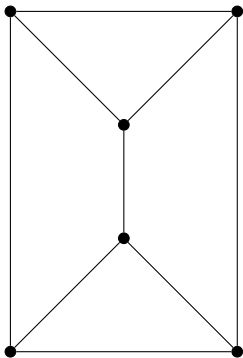
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**Theorem.** (Laman, 1970)

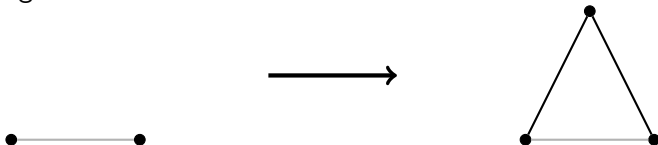
A graph  $G = (V, E)$  is minimally rigid if and only if

1.  $|E| = 2|V| - 3$ ,
2.  $|E'| \leq 2|V'| - 3$  for each subgraph  $G' = (V', E')$  of  $G$ .

## Henneberg Steps

**Theorem.** (Henneberg, 1911) The following two simple rules allow to construct any Laman graph, starting from a single edge.

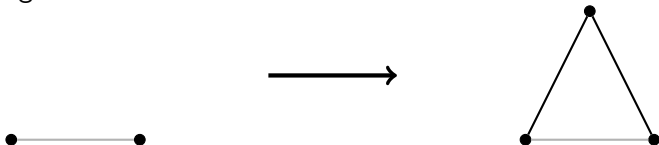
**Henneberg step (type I):** add a new vertex and connect it to two existing vertices.



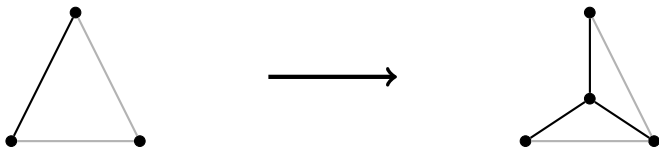
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**Henneberg step (type II):** select three vertices of the graph, at least two of which are connected by an edge  $e$ ; delete the edge  $e$ ; add a new vertex and connect it to the three chosen ones.



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Construct the three-prism graph by Henneberg steps:



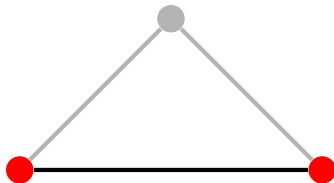
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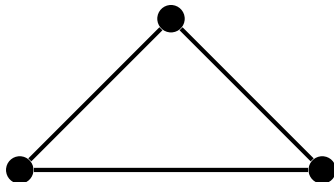
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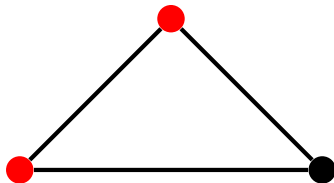
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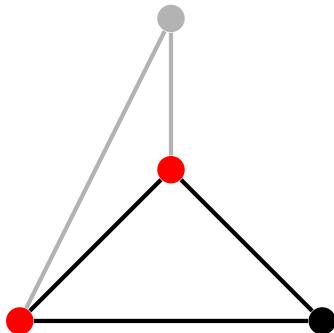
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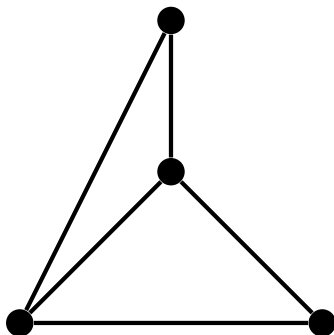
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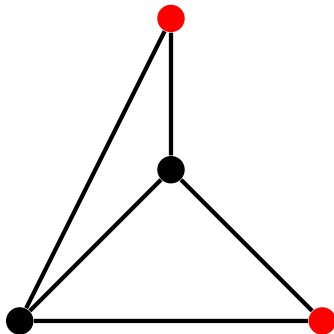
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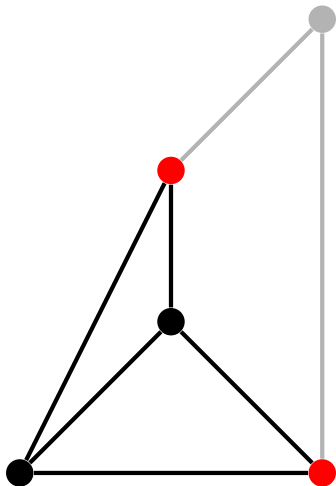
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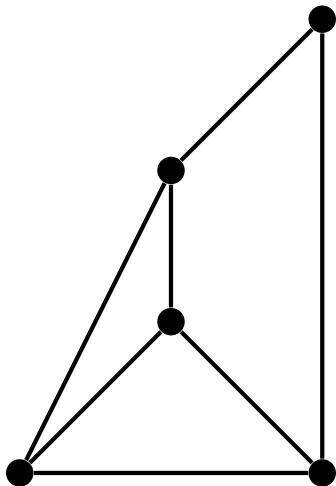
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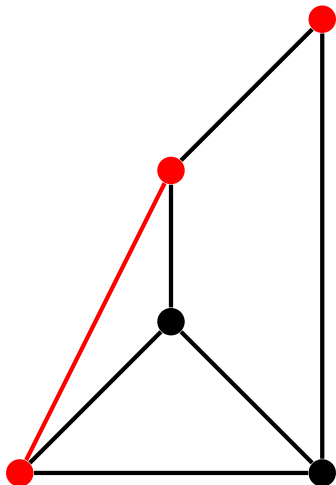
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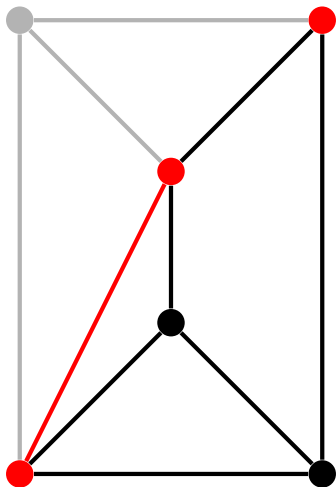
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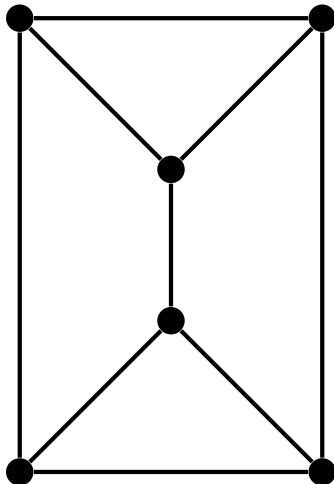
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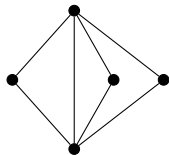
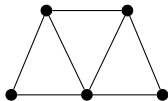
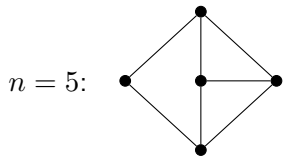
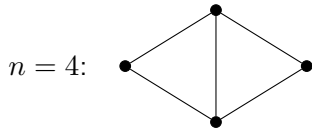
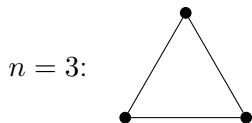
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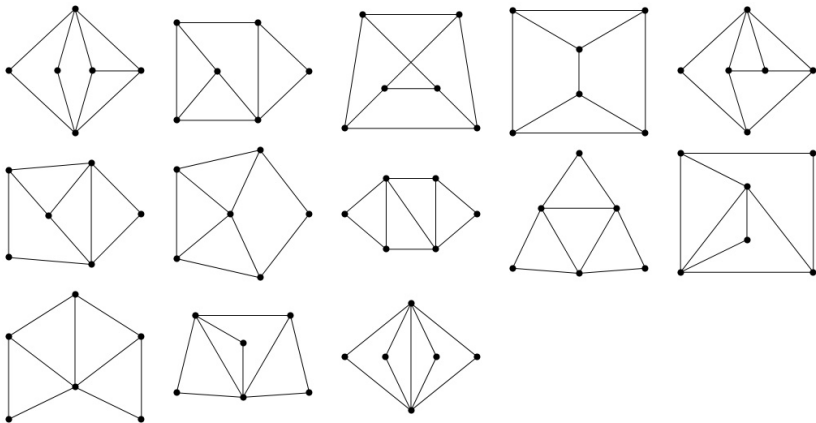
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All Laman graphs with  $2 \leq n \leq 5$  vertices:



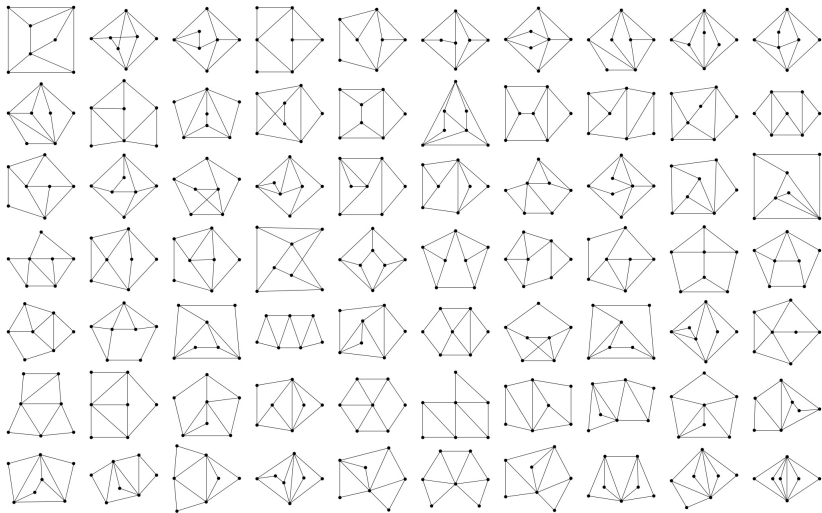
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All Laman graphs with 6 vertices:



## Some Laman Graphs

There are 70 Laman graphs with 7 vertices:



## Enumeration of Laman graphs

Number of Laman graphs with  $n$  vertices:

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```
A227117      Number of minimally rigid graphs on n vertices.
1, 1, 1, 1, 1, 3, 13, 70, 609 (list; graph; refs; listen; history; text; int
OFFSET      1,5
COMMENTS    All the minimally rigid graphs on n vertices
             graphs on n-1 vertices by use of two types
             constructions. In the first type a new ve
             edges are added connecting the new vertex
             of the graph. In the second type of const
             which are connected by an edge are selecte
             edge between v_1 and v_2 is deleted. A new
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- ▶ Define a **normal form** for graphs.
- ▶ Encode graphs as integers.

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Number of Laman graphs with  $n$  vertices:

$n$	#
2	1
3	1
4	1
5	3
6	13
7	70
8	608
9	7222
10	110132
11	2039273
12	44176717

**Question:** How to compute these numbers?

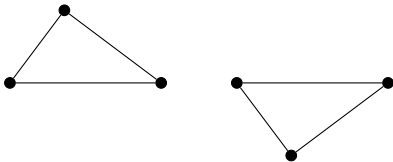
- ▶ Start with all Laman graphs on  $n$  vertices.
- ▶ Apply a single Henneberg step (type I or type II) in **all** possible ways on **all** graphs.
- ▶ Remove duplicates (graphs that are **isomorphic**).
- ▶ Define a **normal form** for graphs.
- ▶ Encode graphs as integers.

## Number of Realizations

Laman graph with 3 vertices: ?

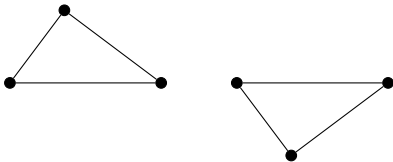
## Number of Realizations

Laman graph with 3 vertices: 2 realizations



## Number of Realizations

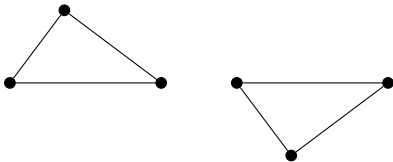
Laman graph with 3 vertices: 2 realizations



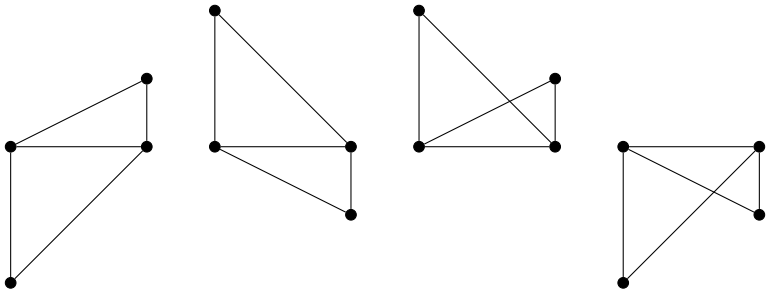
Laman graph with 4 vertices: ?

## Number of Realizations

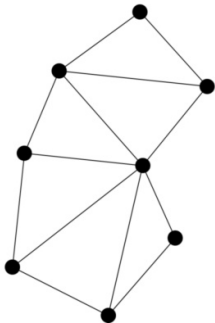
Laman graph with 3 vertices: 2 realizations



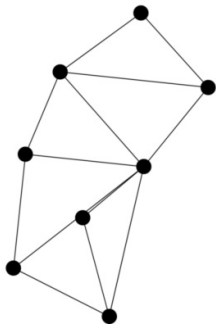
Laman graph with 4 vertices: 4 realizations



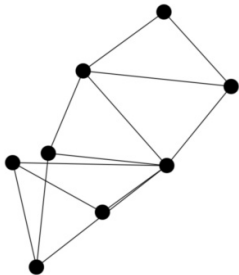
# Realizations of H1 Laman Graphs



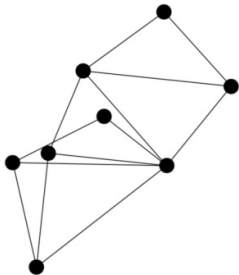
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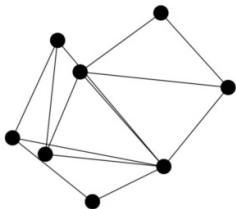
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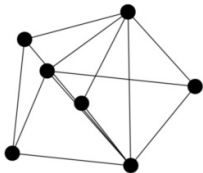


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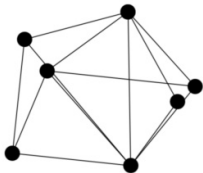




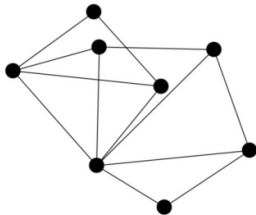
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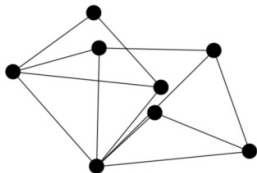
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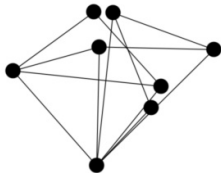
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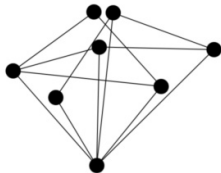
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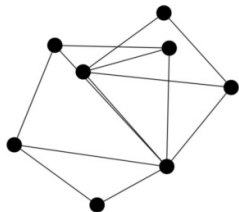
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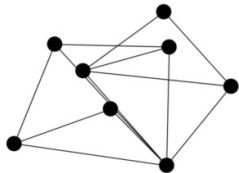
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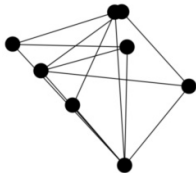
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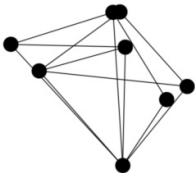
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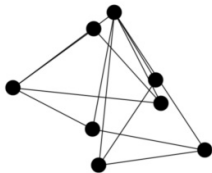
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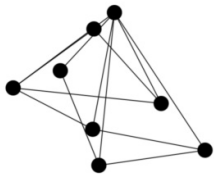
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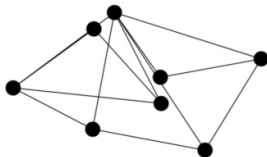
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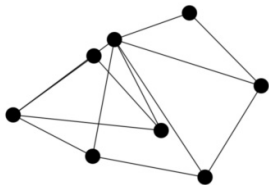
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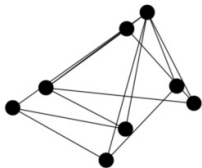
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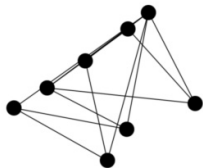
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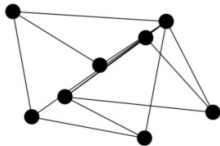
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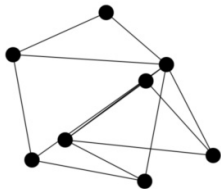
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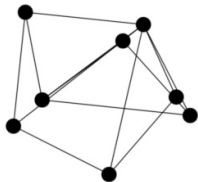
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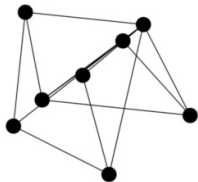
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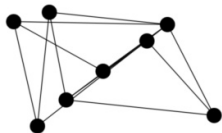
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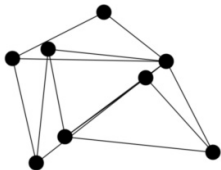
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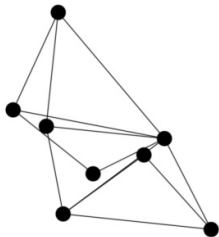
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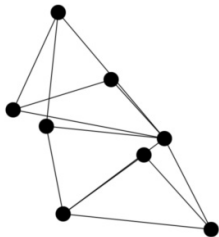
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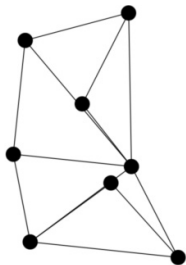
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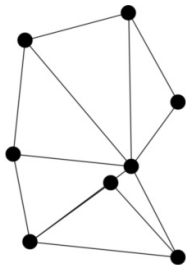
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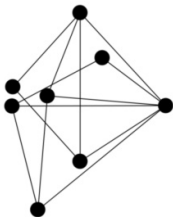
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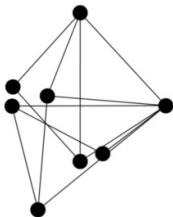
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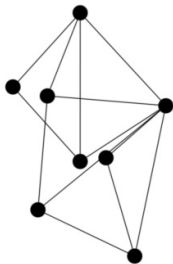
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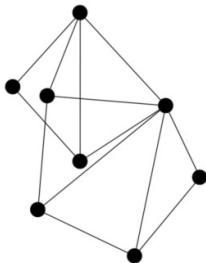
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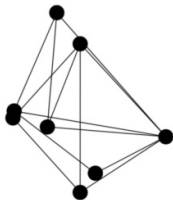
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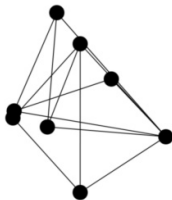
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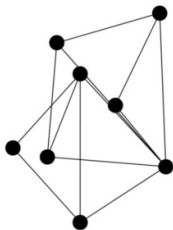
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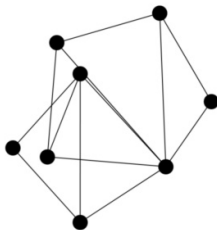
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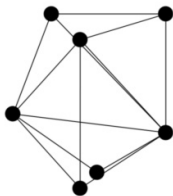
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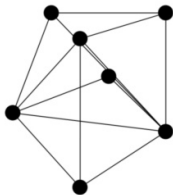
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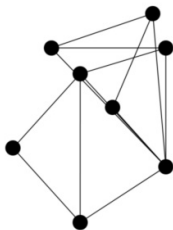
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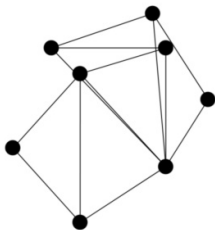
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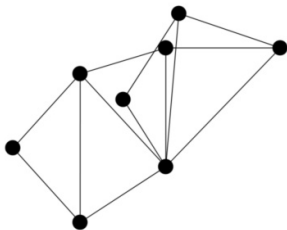
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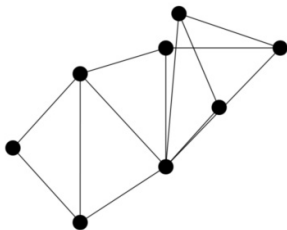
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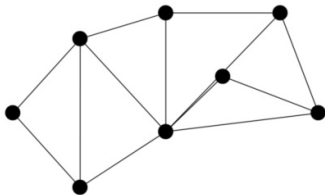
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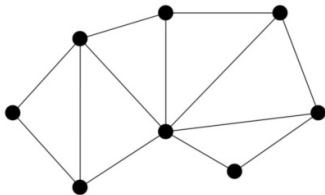
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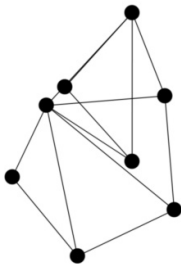
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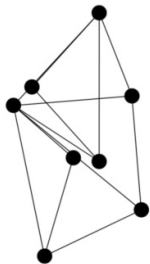
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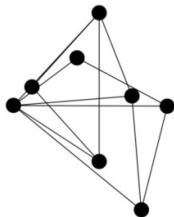


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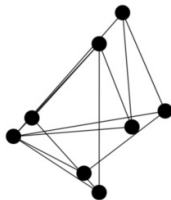




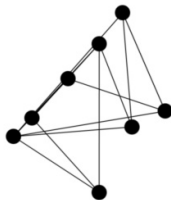
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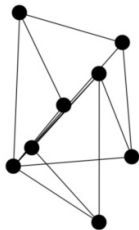
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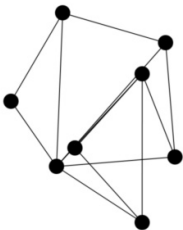
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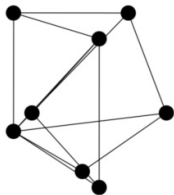
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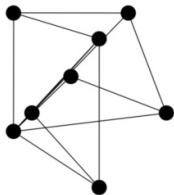
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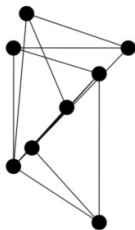
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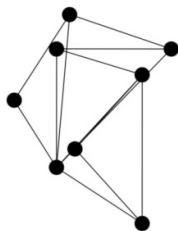
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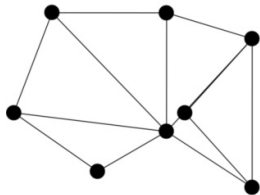
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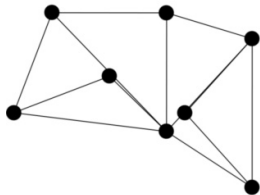
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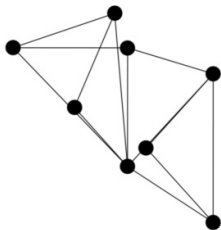
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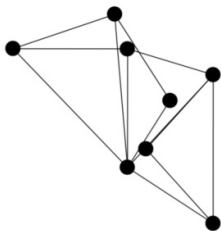
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Number of realizations:

- ▶ Let  $G = (V, E)$  be an H1 Laman graph.
  - ▶ Fix a realizable labeling  $\lambda: E \rightarrow \mathbb{R}_{>0}$ .
  - ▶ Fix the positions of the first two vertices, respecting  $\lambda(1, 2)$ .
  - ▶ Each vertex that is added can be put at two different positions.
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**Definition:** For a Laman graph  $G$  the **Laman number**  $\text{Lam}(G)$  is the number of realizations of  $G$ , for a generic realizable labeling  $\lambda$ .

## More Complex Laman Graphs

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### Previous work:

- ▶ Borcea, Streinu (2004):  $\text{Lam}(G) \leq \binom{2n-4}{n-2}$  where  $n = |V|$ .
- ▶ Steffens, Theobald (2010):  $\text{Lam}(G) < 4^{n-2}$ .
- ▶ Emiris, Despotakis, Psarros, Tsigaridas, Varvitsiotis (2009, 2012, 2013, 2014):

$n$	3	4	5	6	7	8	9	10
lower	2	4	8	24	48	96	288	576
upper	2	4	8	24	64	128	512	2048

## More Complex Laman Graphs

**Question:** What about Laman graphs that are not H1?

Set up a system of equations:

- ▶ Let  $(x_v, y_v)$  be the coordinates of vertex  $v$ .
- ▶ For  $\{u, v\} \in E$ :

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

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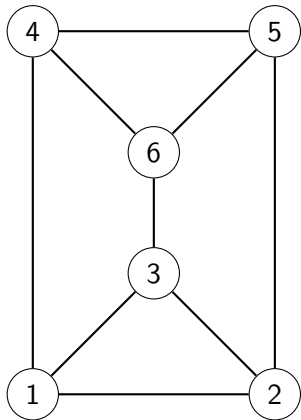
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**Convention:** From now on we work over the complex numbers:

- ▶  $\lambda: E \rightarrow \mathbb{C}$
- ▶  $(x_v, y_v) \in \mathbb{C}^2$

## Example: Three-Prism Graph



$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = \lambda(1, 2)^2$$

$$(x_1 - x_3)^2 + (y_1 - y_3)^2 = \lambda(1, 3)^2$$

$$(x_1 - x_4)^2 + (y_1 - y_4)^2 = \lambda(1, 4)^2$$

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = \lambda(2, 3)^2$$

$$(x_2 - x_5)^2 + (y_2 - y_5)^2 = \lambda(2, 5)^2$$

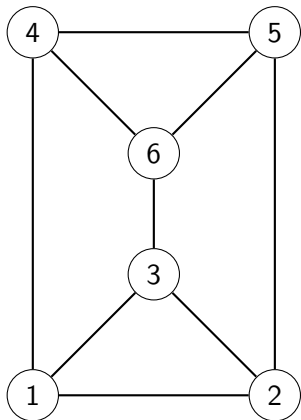
$$(x_3 - x_6)^2 + (y_3 - y_6)^2 = \lambda(3, 6)^2$$

$$(x_4 - x_5)^2 + (y_4 - y_5)^2 = \lambda(4, 5)^2$$

$$(x_4 - x_6)^2 + (y_4 - y_6)^2 = \lambda(4, 6)^2$$

$$(x_5 - x_6)^2 + (y_5 - y_6)^2 = \lambda(5, 6)^2$$

## Example: Three-Prism Graph



$$(x_2)^2 + (y_2)^2 = \lambda(1, 2)^2$$

$$(x_3)^2 + (y_3)^2 = \lambda(1, 3)^2$$

$$(x_4)^2 + (y_4)^2 = \lambda(1, 4)^2$$

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = \lambda(2, 3)^2$$

$$(x_2 - x_5)^2 + (y_2 - y_5)^2 = \lambda(2, 5)^2$$

$$(x_3 - x_6)^2 + (y_3 - y_6)^2 = \lambda(3, 6)^2$$

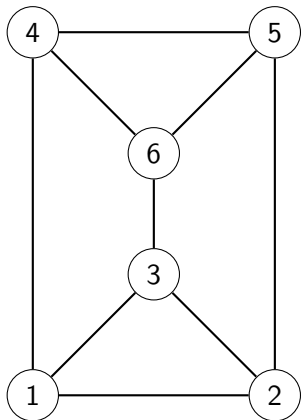
$$(x_4 - x_5)^2 + (y_4 - y_5)^2 = \lambda(4, 5)^2$$

$$(x_4 - x_6)^2 + (y_4 - y_6)^2 = \lambda(4, 6)^2$$

$$(x_5 - x_6)^2 + (y_5 - y_6)^2 = \lambda(5, 6)^2$$

- ▶ Take care of translations:  $(x_1, y_1) = (0, 0)$

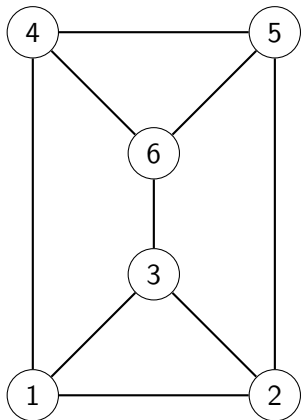
## Example: Three-Prism Graph



$$\begin{aligned} y_2 &= \lambda(1, 2) \\ (x_3)^2 + (y_3)^2 &= \lambda(1, 3)^2 \\ (x_4)^2 + (y_4)^2 &= \lambda(1, 4)^2 \\ (x_3)^2 + (y_2 - y_3)^2 &= \lambda(2, 3)^2 \\ (x_5)^2 + (y_2 - y_5)^2 &= \lambda(2, 5)^2 \\ (x_3 - x_6)^2 + (y_3 - y_6)^2 &= \lambda(3, 6)^2 \\ (x_4 - x_5)^2 + (y_4 - y_5)^2 &= \lambda(4, 5)^2 \\ (x_4 - x_6)^2 + (y_4 - y_6)^2 &= \lambda(4, 6)^2 \\ (x_5 - x_6)^2 + (y_5 - y_6)^2 &= \lambda(5, 6)^2 \end{aligned}$$

- ▶ Take care of translations:  $(x_1, y_1) = (0, 0)$
- ▶ Take care of rotations:  $x_2 = 0$  and  $y_2 > 0$

## Example: Three-Prism Graph



$$\begin{aligned} y_2 &= \lambda(1, 2) \\ (x_3)^2 + (y_3)^2 &= \lambda(1, 3)^2 \\ (x_4)^2 + (y_4)^2 &= \lambda(1, 4)^2 \\ (x_3)^2 + (y_2 - y_3)^2 &= \lambda(2, 3)^2 \\ (x_5)^2 + (y_2 - y_5)^2 &= \lambda(2, 5)^2 \\ (x_3 - x_6)^2 + (y_3 - y_6)^2 &= \lambda(3, 6)^2 \\ (x_4 - x_5)^2 + (y_4 - y_5)^2 &= \lambda(4, 5)^2 \\ (x_4 - x_6)^2 + (y_4 - y_6)^2 &= \lambda(4, 6)^2 \\ (x_5 - x_6)^2 + (y_5 - y_6)^2 &= \lambda(5, 6)^2 \end{aligned}$$

- ▶ Take care of translations:  $(x_1, y_1) = (0, 0)$
- ▶ Take care of rotations:  $x_2 = 0$  and  $y_2 > 0$

**Question:** How many solutions does this system have?

- ▶ Compute a Gröbner basis!

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- ▶ But: not feasible for symbolic parameters  $\lambda(i, j)$

- ▶ Compute a Gröbner basis!
- ▶ But: not feasible for symbolic parameters  $\lambda(i, j)$
- ▶ Replace each  $\lambda(i, j)$  by a random integer!



# Do the computation modulo $p = 2^{31} - 1$ :

$$\begin{aligned}
 & \{y_3 + 1727\,076644, x_5 x_6 + 1\,073\,741\,823 x_0^2 + 240\,407\,003 y_6 + 2147\,476\,519, x_3 y_5 + 1\,449\,935\,236 x_4 y_5 + 87\,139\,559 x_5 y_5 + 821\,582\,392 y_4 x_6 + \\
 & 534\,432\,936 y_5 x_6 + 2\,127\,003\,394 x_3 y_6 + 393\,122\,455 x_4 y_6 + 739\,525\,427 x_5 y_6 + 1\,428\,199\,694 x_6 y_6 + 1\,318\,362\,776 x_3 + 45\,332\,622 x_4 + 1\,666\,067\,743 x_5 + 1\,402\,190\,174 x_6, \\
 & x_2^2 + y_3^2 + 2147\,483\,269 y_3 + 2147\,482\,119, y_4 + 758\,303\,990 x_3 + 504\,327\,305 x_4 + 513\,732\,789 x_5 + 1\,018\,326\,077 x_6, x_4 x_5 + y_4 y_5 + 2147\,483\,458 y_5 + 619\,474\,2715, \\
 & y_4^2 + 544\,418\,756 y_4 + 47\,332\,294 y_5^2 + 1\,603\,064\,889 y_4 y_6 + 1\,508\,400\,303 y_5 y_6 + 591\,751\,051 y_6^2 + 1\,072\,510\,925, x_4 y_4 + 1\,252\,488\,948 x_4 y_5 + 1\,309\,580\,129 x_3 y_5 + 201\,607\,1435 y_4 x_6 + \\
 & 1\,654\,952\,335 y_4 y_6 + 1\,839\,605\,994 x_3 y_6 + 577\,627\,465 y_4 y_6 + 876\,148\,120 x_5 y_6 + 335\,588\,542 x_6 y_6 + 2\,136\,682\,920 x_3 + 1\,038\,483\,051 x_4 + 157\,778\,551 x_5 + 540\,431\,639 x_6 + \\
 & x_3 y_4 + 204\,011\,627 x_4 y_5 + 839\,002\,279 x_3 y_5 + 368\,180\,718 y_4 x_6 + 1\,641\,249\,205 y_5 x_6 + 430\,135\,887 x_3 y_6 + 486\,556\,477 x_4 y_6 + 1\,706\,891\,994 x_5 y_6 + 83\,415\,671 x_6 y_6 + 123\,469\,149 x_3 + \\
 & 554\,422\,930 x_4 + 1\,257\,780\,688 x_5 + 1\,936\,702\,634 x_6, x_4^2 + 1\,603\,064\,891 y_4 y_5 + 2\,100\,151\,353 y_5^2 + 544\,418\,756 y_4 y_6 + 639\,083\,344 y_5 y_6 + 1\,555\,732\,596 y_6^2 + 1\,074\,934\,697, \\
 & x_5^2 + 1527\,353\,090, y_6^2 + 724\,462\,234 x_3 x_4 + 191\,839\,650 x_3 y_6 + 1\,293\,615\,843 y_4 y_5 + 2\,115\,905\,836 y_5^2 + 158\,590\,087 x_6^2 + 808\,924\,606 y_4 y_6 + 945\,043\,470 y_5 y_6 + 57\,464\,572 y_6^2 + \\
 & 1\,051\,760\,435 y_4 + 458\,639\,039 y_5 + 890\,226\,335 y_6 + 306\,458\,357, x_6 y_6^2 + 1\,202\,942\,319 x_4 y_5 + 891\,621\,123 x_3 y_5 + 694\,981\,073 y_4 x_6 + 1\,268\,149\,653 x_5 x_6 + \\
 & 566\,843\,284 x_3 y_6 + 1579\,449\,712 x_4 y_6 + 2\,096\,627\,325 x_5 y_6 + 217\,935\,702 x_6 y_6 + 1\,838\,771\,945 x_3 + 1574\,100\,689 x_4 + 890\,711\,649 x_5 + 527\,754\,025 x_6, \\
 & y_5 y_6^2 + 1\,397\,298\,562 x_3 x_4 + 1\,093\,626\,759 x_3 x_5 + 1\,874\,498\,615 y_4 y_5 + 410\,806\,791 y_5^2 + 34\,715\,881 x_6^2 + 1\,602\,680\,419 y_4 y_6 + 1\,365\,806\,073 y_5 y_6 + 1\,574\,388\,257 y_6^2 + \\
 & 1\,986\,672\,592 y_4 + 1\,454\,700\,418 y_5 + 207\,782\,012 y_6 + 817\,238\,271, x_3 y_6^2 + 906\,551\,028 x_4 y_5 + 2\,088\,326\,233 x_3 y_5 + 983\,660\,499 y_4 x_6 + 2\,020\,744\,231 y_3 x_6 + 438\,982\,960 x_3 y_6 + \\
 & 105\,460\,105 x_4 y_6 + 1791\,795\,415 x_5 y_6 + 752\,681\,903 x_6 y_6 + 1\,243\,232\,341 x_3 + 236\,567\,207 x_4 + 2\,039\,336\,095 x_5 + 204\,724\,127 x_6, y_4 y_6^2 + 1\,798\,033\,564 x_3 x_4 + 1\,368\,970\,181 x_3 x_5 + \\
 & 2111\,288\,438 y_4 y_5 + 211\,652\,589 y_5^2 + 631\,579\,871 x_6^2 + 2\,098\,374\,939 y_4 y_6 + 14\,559\,548 y_5 y_6 + 265\,925\,976 y_6^2 + 768\,097\,244 y_4 + 197\,849\,421 y_5 + 1\,272\,087\,803 y_6 + 1\,950\,925\,264, \\
 & x_4 y_6^2 + 2\,000\,180\,329 x_4 y_5 + 138\,882\,411 x_3 y_5 + 1\,964\,621\,882 y_4 x_6 + 1\,562\,649\,152 y_5 x_6 + 274\,800\,980 x_3 y_6 + 381\,168\,929 x_4 y_6 + 1\,561\,080\,504 x_5 y_6 + 646\,135\,501 x_6 y_6 + \\
 & 1\,252\,024\,999 x_3 + 1\,828\,948\,462 x_4 + 1\,907\,059\,409 x_5 + 1\,062\,878\,925 x_6, x_3 y_6^2 + 1\,940\,064\,434 x_4 y_5 + 1\,699\,323\,466 x_3 y_5 + 2\,767\,389 x_6 + 309\,430\,653 y_3 x_6 + \\
 & 1746\,152\,111 x_3 y_6 + 1\,486\,922\,955 x_4 y_6 + 1\,042\,873\,400 x_5 y_6 + 1\,877\,302\,158 x_6 y_6 + 898\,865\,598 x_3 + 2\,023\,749\,908 x_4 + 1\,369\,459\,334 x_5 + 1\,937\,240\,696 x_6 + \\
 & x_6^2 y_6 + 1\,859\,350\,309 x_3 x_4 + 828\,165\,967 x_3 x_5 + 1\,319\,416\,912 y_4 y_5 + 1\,281\,531\,769 y_5^2 + 41\,645\,396 x_6^2 + 555\,896\,977 y_4 y_6 + 838\,162\,654 y_5 y_6 + 1\,094\,898\,319 y_6^2 + \\
 & 1\,025\,635\,396 y_4 + 758\,820\,774 y_5 + 1\,932\,663\,106 y_6 + 902\,372\,666, y_5 x_6 y_6 + 1\,776\,737\,250 x_4 y_5 + 1\,335\,234\,339 x_3 y_5 + 197\,659\,465 y_4 x_6 + 388\,691\,694 y_3 x_6 + \\
 & 1\,214\,819\,713 y_4 + 1\,236\,939\,013 x_3 y_6 + 1\,895\,585\,096 x_4 y_6 + 1\,457\,663\,787 x_5 y_6 + 1\,908\,824\,636 x_6 + 1\,937\,866\,443 x_3 + 906\,541\,898 x_4 + 1\,779\,256\,072 x_6, \\
 & y_4 x_6 y_6 + 392\,800\,087 x_4 y_5 + 431\,423\,5 x_5 y_6 + 1\,752\,015\,765 y_4 y_6 + 697\,637\,736 y_3 x_6 + 1\,174\,862\,040 x_3 y_6 + 1\,726\,470\,482 x_4 y_6 + 524\,280\,549 x_5 y_6 + 1\,783\,594\,194 x_6 y_6 + \\
 & 777\,027\,038 x_3 + 1\,196\,924\,612 x_4 + 669\,351\,278 x_5 + 1\,365\,654\,514 x_6, y_2^2 y_6 + 769\,157\,270 x_3 x_4 + 301\,291\,773 x_3 x_5 + 147\,541\,859 y_4 y_5 + 696\,342\,885 y_6^2 + 953\,052\,903 x_6^2 + 63\,094\,058 y_4 y_6 + \\
 & 1\,607\,775\,536 y_5 y_6 + 2\,003\,959\,420 y_6^2 + 1\,657\,122\,998 y_4 + 1\,041\,341\,194 y_5 + 643\,382\,090 y_6 + 298\,205\,400, x_5 y_6 y_6 + 1\,361\,368\,571 x_4 y_5 + 143\,005\,480 x_3 y_5 + 749\,246\,637 y_4 x_6 + \\
 & 556\,781\,711 y_3 x_6 + 268\,588\,588 x_3 y_6 + 179\,323\,388 x_4 y_6 + 260\,672\,145 x_5 y_6 + 542\,764\,427 x_6 y_6 + 2\,031\,844\,241 x_3 + 112\,806\,238 x_4 + 2\,024\,968\,156 x_5 + 1\,634\,398\,896 x_6, \\
 & y_4 y_5 y_6 + 1\,495\,723\,262 x_3 x_4 + 11\,505\,525\,105 x_3 x_5 + 647\,627\,904 y_4 y_5 + 834\,052\,394 y_5^2 + 680\,400\,990 x_6^2 + 703\,082\,161 y_4 y_6 + 1\,261\,907\,640 y_3 y_6 + 1\,146\,890\,666 y_6^2 + \\
 & 339\,024\,153 y_4 + 1\,829\,077\,048 y_5 + 11\,210\,614\,065 y_6 + 420\,664\,718, x_4 y_5 y_6 + 391\,789\,202 x_4 y_5 + 1\,778\,622\,432 x_3 y_5 + 32\,574\,434 x_6 y_6 + 638\,864\,222 y_3 x_6 + \\
 & 2\,008\,976\,092 x_3 y_6 + 1\,158\,838\,637 x_4 y_6 + 298\,082\,231 x_5 y_6 + 579\,017\,100 x_6 y_6 + 541\,015\,847 x_3 + 1\,347\,513\,279 x_4 + 1\,774\,560\,872 x_5 + 1\,614\,705\,109 x_6, \\
 & x_3 y_5 y_6 + 1\,581\,681\,716 x_3 x_4 + 486\,946\,881 x_3 x_5 + 421\,622\,009 y_4 y_5 + 1\,075\,313\,850 y_5^2 + 156\,480\,523 x_6^2 + 198\,951\,616 y_4 y_6 + 1\,466\,002\,977 y_3 y_6 + 932\,669\,036 y_6^2 + 248\,319\,512 y_4 + \\
 & 862\,020\,011 y_5 + 649\,537\,600 y_6 + 815\,933\,435, x_3 x_4 y_6 + 1\,343\,545\,648 x_3 x_5 + 1\,023\,324\,514 x_3 x_6 + 40\,371\,239 y_4 y_5 + 1\,905\,289\,341 y_5^2 + 1\,639\,954\,889 x_6^2 + 786\,545\,101 y_4 y_6 + \\
 & 121\,919\,433 y_3 y_6 + 321\,512\,152 y_4 y_6 + 1\,631\,889\,154 x_4 + 850\,776\,521 y_5 + 530\,499\,711 y_6 + 20\,867\,437\,747, x_6^2 + 1\,359\,379\,754 x_4 y_5 + 4\,699\,239 x_3 y_5 + 1\,446\,967\,796 y_4 x_6 + \\
 & 260\,472\,488 y_3 x_6 + 701\,675\,423 y_3 y_6 + 1\,899\,155\,319 x_4 y_6 + 112\,548\,169 x_5 y_6 + 1\,629\,096\,917 x_6 y_6 + 658\,580\,665 x_3 + 820\,850\,436 x_4 + 656\,336\,977 x_5 + 1\,707\,190\,979 x_6, \\
 & y_3 x_6^2 + 1\,013\,046\,601 x_3 x_4 + 969\,596\,453 x_3 x_5 + 1\,553\,889\,292 y_4 y_5 + 1\,185\,309\,841 y_5^2 + 1\,987\,921\,573 x_6^2 + 1\,033\,458\,441 y_4 y_6 + 1\,320\,068\,753 y_3 y_6 + 1\,102\,491\,211 y_6^2 + \\
 & 110\,941\,459 y_4 + 1\,375\,118\,664 y_5 + 672\,833\,739 y_6 + 626\,376\,047, y_4 x_6^2 + 2\,009\,134\,087 x_3 x_4 + 1\,611\,713\,763 x_3 x_5 + 168\,461\,479 y_4 y_5 + 1\,706\,153\,267 y_5^2 + \\
 & 178\,915\,690 x_6^2 + 1\,182\,579\,576 y_4 y_6 + 557\,864\,255 y_3 y_6 + 503\,714\,053 y_6^2 + 176\,291\,393 y_4 + 1\,354\,065\,871 y_5 + 954\,347\,026 y_6 + 734\,410\,570, \\
 & y_2^2 x_6 + 619\,010\,252 x_4 y_3 + 916\,211\,455 x_3 y_3 + 1\,431\,371\,638 y_4 x_6 + 969\,212\,309 y_5 x_6 + 1\,949\,990\,023 x_3 y_6 + 414\,782\,496 x_4 y_6 + 1\,907\,745\,509 x_5 y_6 + 970\,368\,126 x_6 y_6 + \\
 & 1740\,320\,236 x_3 + 1\,975\,330\,810 x_4 + 2143\,293\,978 x_5 + 252\,311\,982 x_6, y_4 y_5 x_6 + 558\,167\,467 x_4 y_5 + 433\,016\,430 x_3 y_5 + 2\,075\,138\,717 y_4 x_6 + 1\,434\,835\,475 y_3 x_6 + \\
 & 531\,264\,210 x_4 y_6 + 427\,467\,244 x_5 y_6 + 1374\,860\,777 x_3 x_4 + 149\,117\,380 x_3 y_6 + 1\,826\,680\,361 x_4 y_6 + 969\,629\,736 x_5 + 766\,694\,650 x_6 + 1\,666\,548\,268 x_6, \\
 & y_3^2 + 649\,714\,439 x_3 x_4 + 1\,076\,476\,457 x_3 x_5 + 1\,435\,812\,662 y_4 y_5 + 2\,053\,151\,093 y_6^2 + 280\,374\,149 x_6^2 + 1\,469\,939\,973 y_4 y_6 + 1\,400\,337\,770 y_3 y_6 + 1\,634\,063\,342 y_6^2 + \\
 & 354\,162\,717 y_4 + 737\,661\,535 y_5 + 816\,931\,778 y_6 + 1\,428\,529\,698, x_3 y_2^2 + 1136\,639\,110 x_4 y_5 + 121\,108\,532 x_3 y_5 + 212\,027\,098 y_4 x_6 + 701\,800\,649 y_3 x_6 + \\
 & 1281\,723\,728 x_3 y_6 + 2\,092\,528\,324 x_4 y_6 + 1\,816\,317\,333 x_5 y_6 + 1524\,717\,023 x_6 y_6 + 737\,384\,683 x_3 + 261\,085\,830 x_4 + 712\,596\,842 x_5 + 1219\,275\,979 x_6, \\
 & y_4 y_2^2 + 1\,382\,099\,903 x_3 x_4 + 1\,674\,451\,197 x_3 x_5 + 1\,964\,164\,303 y_4 y_5 + 610\,824\,582 y_5^2 + 1726\,175\,807 x_6^2 + 1\,045\,412\,838 y_4 y_6 + 1\,328\,732\,288 y_3 y_6 + 1\,416\,893\,499 y_6^2 + \\
 & 509\,989\,107 y_4 + 356\,562\,705 y_5 + 701\,591\,991 y_6 + 907\,910\,056, x_4 y_2^2 + 1125\,381\,690 x_4 y_5 + 343\,309\,511 x_3 y_5 + 412\,315\,532 y_4 x_6 + 392\,837\,310 y_3 x_6 + \\
 & 185\,974\,430 x_3 y_6 + 1289\,634\,195 x_4 y_6 + 511\,405\,427 x_5 y_6 + 2104\,680\,646 x_6 y_6 + 1304\,660\,656 x_3 + 1431\,387\,822 x_4 + 2142\,663\,821 x_5 + 395\,031\,648 x_6\}
 \end{aligned}$$

# Do the computation modulo $p = 2^{31} - 1$ :

$\{1727076644, 1073741823 x_0^2 + 1073741823 y_0^2 + 2147483458, 1073741823 x_0^2 + y_4 y_6 + 1073741823 y_0^2 + 2147472199,$   
 $1073741823 x_0^2 + 1073741823 y_0^2 + 420407003 y_6 + 2147476519, 1449935236 x_4 y_5 + 871395959 x_5 y_3 + 821582392 y_4 x_6 +$   
 $534432936 y_5 x_6 + 2127003394 x_3 y_6 + 393122455 x_4 y_6 + 739525427 x_5 y_6 + 1428199694 x_6 y_6 + 1318362776 x_3 + 45332622 x_4 + 1666067743 x_5 + 1402190174 x_6,$   
 $+ y_0^2 + 2147483269 y_3 + 2147482119, 1431835485 x_4 y_5 + 1585512332 x_5 y_5 + 209945504 y_4 x_6 + 1274481640 y_5 x_6 + 1926461619 y_4 y_6 + 1819204411 x_4 y_6 +$   
 $2064309228 x_5 y_6 + 1860755017 x_6 y_6 + 758303990 x_3 + 504327305 x_4 + 513732789 x_5 + 1018326077 x_6, y_4 y_5 + 2147483458 y_3 + 2147472715,$   
 $544418756 y_4 y_5 + 47332294 y_5^2 + 160360488 y_4 y_6 + 1508400303 y_5 y_6 + 591751051 y_6^2 + 1072510925, 1252484848 x_4 y_5 + 1309580129 x_5 y_3 + 2016071435 y_4 x_6 +$   
 $165495235 x_5 y_6 + 1839606594 x_3 y_6 + 577627465 y_4 y_6 + 876148120 x_5 y_6 + 335588542 x_6 y_6 + 2136682920 x_3 + 103848051 x_4 + 157778551 x_5 + 540431639 x_6 +$   
 $204101627 x_4 y_3 + 839020279 x_5 y_3 + 368180718 y_4 x_6 + 1641249205 y_5 x_6 + 430135887 x_6 y_6 + 486556477 x_4 y_6 + 1706899194 x_5 y_6 + 83415671 x_6 y_6 + 123469149 x_3 +$   
 $554422930 x_4 + 1257780688 x_5 + 1936702634 x_6, 1603604891 y_4 y_5 + 2100151353 y_5^2 + 544418756 y_4 y_6 + 639083344 y_5 y_6 + 1555732950 y_6^2 + 1074934697,$   
 $+ 1527353090, 72446234 x_3 x_4 + 191839560 x_3 y_4 + 1293615843 y_4 y_3 + 2119505836 y_4^2 + 158590087 x_0^2 + 808924606 y_4 y_6 + 945043470 y_5 y_6 + 57464572 y_6^2 +$   
 $1051760435 y_4 + 458639039 y_5 + 890226336 y_6 + 306458357, 2102942319 x_4 y_5 + 891621123 x_5 y_3 + 694981073 y_4 x_6 + 1268149853 x_5 x_6 +$   
 $566843284 x_3 y_4 + 1579449712 x_4 y_6 + 2096627235 x_5 y_6 + 217935702 x_6 y_6 + 1838771945 x_3 + 1574100689 x_4 + 890711649 x_5 + 52754025 x_6,$   
 $+ 1397298562 x_3 x_4 + 1093626759 x_3 x_5 + 1874498615 y_4 y_3 + 410806791 y_5^2 + 34715881 x_0^2 + 1602680419 y_4 y_6 + 1365806073 y_5 y_6 + 1574388257 y_6^2 +$   
 $1986672592 y_4 + 1454700418 y_5 + 207782012 y_6 + 817238271, 906551028 x_4 y_5 + 2088326233 x_5 y_3 + 983660499 y_4 x_6 + 2020744231 y_5 x_6 + 438982960 x_3 y_6 +$   
 $105460105 x_4 y_6 + 1791795415 x_3 y_6 + 752681903 x_6 y_6 + 1243232341 x_3 + 236567207 x_4 + 2039336095 x_5 + 204724127 x_6, 1798033564 x_3 x_4 + 1368970181 x_3 x_5 +$   
 $2111288438 y_4 y_3 + 2116525889 y_5^2 + 631579871 x_0^2 + 2098734939 y_4 y_6 + 14559548 y_5 y_6 + 265925976 y_6^2 + 768097244 y_4 + 197849421 y_5 + 1272087003 y_6 + 1950925264,$   
 $+ 2000180329 x_4 y_5 + 138882411 x_4 y_6 + 196461821882 y_4 x_6 + 1562649152 y_5 x_6 + 274800980 x_3 y_6 + 381168929 x_4 y_6 + 1561080504 x_5 y_6 + 646135501 x_6 y_6 +$   
 $1252024999 x_3 + 1828948462 x_4 + 1907054909 x_5 + 1062878925 x_6, 1940064434 x_4 y_5 + 1699323466 x_5 y_3 + 27673891 x_6 y_6 + 309430653 y_3 x_6 +$   
 $1746152111 x_3 y_6 + 1486922955 x_4 y_6 + 1042873400 x_5 y_6 + 1877302158 x_6 y_6 + 8988657598 x_3 + 2023749908 x_4 + 1369459334 x_5 + 1937240606 x_6 +$   
 $+ 1859350309 x_3 x_4 + 828165967 x_3 x_5 + 1319416912 y_4 y_5 + 1281531769 y_5^2 + 416445396 x_0^2 + 555896977 y_4 y_6 + 838162654 y_5 y_6 + 1904689319 y_6^2 +$   
 $1025635396 y_4 + 758820774 y_5 + 1932663106 y_6 + 912372666, 1776737250 x_4 y_5 + 1335234339 x_5 y_3 + 197659465 y_4 x_6 + 388691694 y_5 x_6 + 1218189713 x_6 y_6 +$   
 $1236399013 x_4 y_6 + 1895585096 x_5 y_6 + 1457663787 x_6 y_6 + 1908824636 x_3 + 1937866443 x_4 + 906541898 x_5 + 1779256072 x_6,$   
 $+ 392800087 x_4 y_5 + 4314235 x_5 y_6 + 1752051765 y_4 y_6 + 69763736 y_5 x_6 + 1174862040 x_3 y_6 + 1726470482 x_4 y_6 + 524280549 x_5 y_6 + 1783594194 x_6 y_6 +$   
 $77027038 x_3 + 1196924612 x_4 + 669351278 x_5 + 136564514 x_6, 769157270 x_3 x_4 + 30129177 x_3 x_5 + 147541859 y_4 y_5 + 696342885 y_6^2 + 953052903 x_0^2 + 63094058 y_4 y_6 +$   
 $160776536 y_5 y_6 + 2003959420 y_6^2 + 1657122998 y_4 + 1041341194 y_5 + 643382090 y_6 + 29820540, 1361368571 x_4 y_5 + 443005480 x_3 y_6 + 7492646637 y_4 x_6 +$   
 $556781711 y_5 x_6 + 268588588 x_3 y_6 + 179323388 x_4 y_6 + 260672145 x_5 y_6 + 542764427 x_6 y_6 + 2031844241 x_3 + 112806238 x_4 + 202496158 x_5 + 1634398896 x_6,$   
 $+ 1495723262 x_3 x_4 + 11505552105 x_3 x_5 + 647627904 y_4 y_5 + 834052394 y_5^2 + 680400990 x_0^2 + 703082161 y_4 y_6 + 1261907640 y_5 y_6 + 1146890666 y_6^2 x_6 +$   
 $339024153 y_4 + 182907048 y_5 + 1120614465 y_6 + 420646718, 391789202 x_4 y_5 + 1778622432 x_5 y_3 + 325744304 x_6 y_6 + 638864222 y_3 x_6 +$   
 $2008976092 x_3 y_6 + 1158838637 x_4 y_6 + 298082231 x_5 y_6 + 579017100 x_6 y_6 + 541015847 x_3 + 1347513279 x_4 + 1774560872 x_5 + 1614705109 x_6,$   
 $+ 1581681716 x_3 x_4 + 486946881 x_3 x_5 + 421622009 y_4 y_5 + 107531850 y_5^2 + 1564800523 x_0^2 + 198951616 y_4 y_6 + 1466602977 y_5 y_6 + 932669036 y_6^2 + 248319152 y_4 +$   
 $862020011 y_5 + 649537600 y_6 + 81593435, 1343545648 x_3 x_4 + 1023324514 x_3 x_5 + 40371239 y_4 y_5 + 1905289341 y_5^2 + 1639954889 x_0^2 + 786545101 y_4 y_6 +$   
 $1219192433 y_5 y_6 + 321512152 y_6^2 + 1631889158 y_4 + 850776521 y_5 + 530499711 y_6 + 2038743747, 1359379754 x_4 y_5 + 4699239 x_5 y_3 + 1446967796 y_4 x_6 +$   
 $260472488 y_5 x_6 + 701675423 y_6 x_6 + 1897859319 x_4 y_6 + 1125484169 x_5 y_6 + 1629096917 x_6 y_6 + 65850665 x_3 + 820850436 x_4 + 656336977 x_5 + 1707190979 x_6 +$   
 $+ 1013046601 x_3 x_4 + 969596453 x_3 x_5 + 1553889292 y_4 y_5 + 1185309841 y_5^2 + 1987921573 x_0^2 + 1033458441 y_4 y_6 + 1320068753 y_5 y_6 + 1102491211 y_6^2 +$   
 $1104911459 y_4 + 1375118664 y_5 + 672833739 y_6 + 626367304, 2009134087 x_3 x_4 + 1611713763 x_3 x_5 + 168461479 y_4 y_5 + 1706153267 y_5^2 +$   
 $1789051690 x_0^2 + 1182597576 y_4 y_6 + 557864255 y_5 y_6 + 507414053 y_6^2 + 176291393 y_4 y_5 + 1354068781 y_5 + 954347026 y_6 + 734410570,$   
 $+ 619010252 x_4 y_3 + 916121455 x_5 y_3 + 1431371638 y_4 x_6 + 969212309 y_5 x_6 + 1949990023 x_3 y_6 + 414782496 x_4 y_6 + 1907745509 x_5 y_6 + 970368126 x_6 y_6 +$   
 $1740320236 x_3 + 1975308010 x_4 + 2143293978 x_5 + 252311982 x_6, 558167467 x_4 y_5 + 433016430 x_5 y_3 + 2075138717 y_4 x_6 + 14348335475 y_5 x_6 +$   
 $5312642106 y_6 + 427467244 x_4 y_6 + 1374860777 x_5 y_6 + 149117380 x_6 y_6 + 1826680361 x_3 + 969629736 x_4 + 766694650 x_5 + 1666548268 x_6,$   
 $+ 649714439 x_3 x_4 + 1076476457 x_3 x_5 + 1435812662 y_4 y_5 + 2053151093 y_5^2 + 2803741449 x_0^2 + 1469399373 y_4 y_6 + 140033770 y_5 y_6 + 1634063342 y_6^2 +$   
 $354162717 y_4 + 737861553 y_5 + 816931778 y_6 + 1428529698, 1136639110 x_4 y_5 + 12110832 x_5 y_3 + 212702098 x_6 y_6 + 701800649 y_5 x_6 +$   
 $1281723728 x_3 y_6 + 2092528324 x_4 y_6 + 1816317333 x_5 y_6 + 1524717023 x_6 y_6 + 737384683 x_3 + 261085830 x_4 + 712596842 x_5 + 1219275979 x_6,$   
 $+ 1382099903 x_3 x_4 + 1674451197 x_3 x_5 + 1964164303 y_4 y_5 + 610824582 y_5^2 + 1726175807 x_0^2 + 1045412838 y_4 y_6 + 1328732288 y_5 y_6 + 1416893499 y_6^2 +$   
 $509989107 y_4 + 356562705 y_5 + 701591991 y_6 + 90791906, 1125381690 x_4 y_5 + 343309511 x_5 y_3 + 412315532 y_4 x_6 + 392837310 y_5 x_6 +$   
 $1859774430 x_3 y_6 + 1289634195 x_4 y_6 + 511405427 x_5 y_6 + 2104680646 x_6 y_6 + 1304660656 x_3 + 1431387822 x_4 + 2142663821 x_5 + 395031648 x_6\}$

## Determine the Number of Solutions

Leading monomials:

$y_3$	$x_5x_6$	$x_4x_6$	$x_3x_6$	$x_3y_5$	$x_5^2$	$y_4x_5$	$x_4x_5$
$y_4^2$	$x_4y_4$	$x_3y_4$	$x_4^2$	$x_3^2$	$y_6^3$	$x_6y_6^2$	$y_5y_6^2$
$x_5y_6^2$	$y_4y_6^2$	$x_4y_6^2$	$x_3y_6^2$	$x_6^2y_6$	$y_5x_6y_6$	$y_4x_6y_6$	$y_5^2y_6$
$x_5y_5y_6$	$y_4y_5y_6$	$x_4y_5y_6$	$x_3x_5y_6$	$x_3x_4y_6$	$x_6^3$	$y_5x_6^2$	$y_4x_6^2$
$y_5^2x_6$	$y_4y_5x_6$	$y_5^3$	$x_5y_5^2$	$y_4y_5^2$	$x_4y_5^2$		

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$$\begin{array}{cccccccc} y_3 & x_5x_6 & x_4x_6 & x_3x_6 & x_3y_5 & x_5^2 & y_4x_5 & x_4x_5 \\ y_4^2 & x_4y_4 & x_3y_4 & x_4^2 & x_3^2 & y_6^3 & x_6y_6^2 & y_5y_6^2 \\ x_5y_6^2 & y_4y_6^2 & x_4y_6^2 & x_3y_6^2 & x_6^2y_6 & y_5x_6y_6 & y_4x_6y_6 & y_5^2y_6 \\ x_5y_5y_6 & y_4y_5y_6 & x_4y_5y_6 & x_3x_5y_6 & x_3x_4y_6 & x_6^3 & y_5x_6^2 & y_4x_6^2 \\ y_5^2x_6 & y_4y_5x_6 & y_5^3 & x_5y_5^2 & y_4y_5^2 & x_4y_5^2 & & \end{array}$$

Monomials under the staircase:

$$\begin{array}{cccccccc} 1 & y_6 & x_6 & y_5 & x_5 & y_4 & x_4 & x_3 \\ y_6^2 & x_6y_6 & y_5y_6 & x_5y_6 & y_4y_6 & x_4y_6 & x_3y_6 & x_6^2 \\ y_5x_6 & y_4x_6 & y_5^2 & x_5y_5 & y_4y_5 & x_4y_5 & x_3x_5 & x_3x_4 \end{array}$$

→ 24 complex solutions.

## Gröbner Basis Approach

Compute a Gröbner basis of the ideal generated by

$$\begin{aligned} & \{(x_u - x_v)^2 + (y_u - y_v)^2 - \lambda(u, v)^2 \mid \{u, v\} \in E\} \\ & \cup \{x_1, y_1, x_2, y_2 - \lambda(1, 2)\}. \end{aligned}$$

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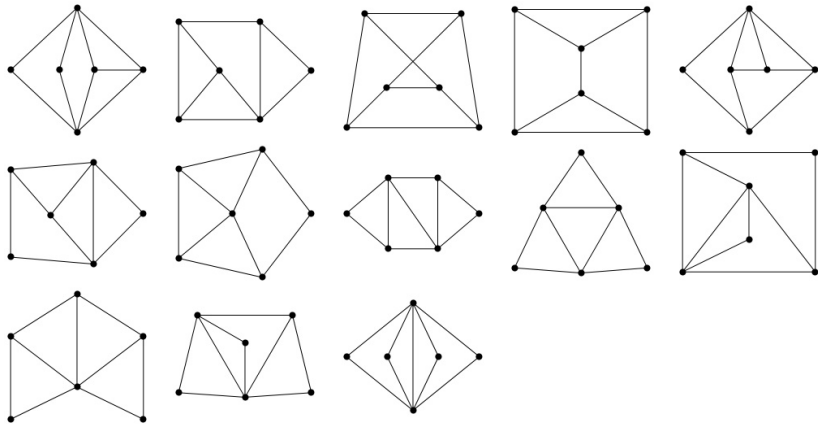
Some tricks for speed-up:

- ▶ Preprocessing: undo Henneberg steps of type I.
- ▶ Assign random values to  $\lambda(u, v)$ .
- ▶ Discard vertices that are “fixed” by a triangle.
- ▶ Compute modulo a prime number.

→ This way, we were able to compute the Laman numbers of all (approximately 118,000) Laman graphs with  $\leq 10$  vertices (using Maple's fgb).

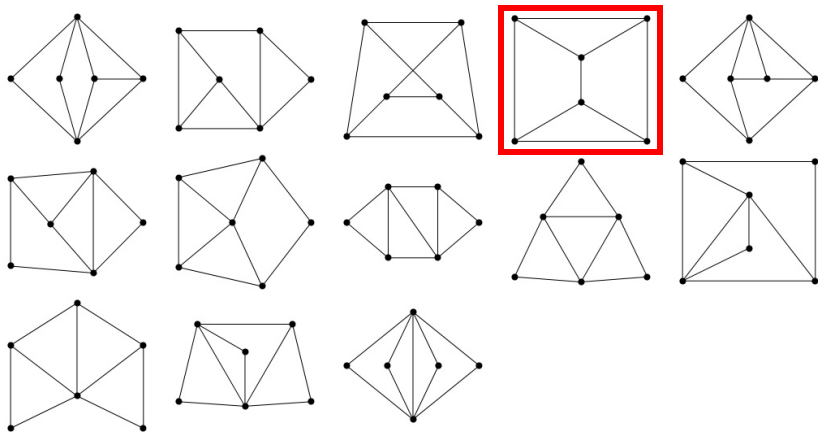
# Laman Numbers

All but one Laman graphs with 6 vertices have Laman number 16.



## Laman Numbers

All but one Laman graphs with 6 vertices have Laman number 16.



The only exception is the three-prism graph with  $\text{Lam}(\blacksquare) = 24$ .

## Laman Number as Degree

**Recall:** For each edge  $\{u, v\} \in E$  we get an equation

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

**Idea:**  $\text{Lam}(G)$  is given by the **degree** of the map

$$f_G: \mathbb{C}^V \times \mathbb{C}^V \rightarrow \mathbb{C}^E,$$

$$(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \left( (x_u - x_v)^2 + (y_u - y_v)^2 \right)_{\{u,v\} \in E}$$

i.e., by the number how often a generic  $(\lambda(u, v))_{\{u,v\} \in E} \in \mathbb{C}^E$  is hit by the map  $f_G$  (modulo translations and rotations).

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i.e., by the number how often a generic  $(\lambda(u, v))_{\{u,v\} \in E} \in \mathbb{C}^E$  is hit by the map  $f_G$  (modulo translations and rotations).

In order to apply methods from algebraic geometry, we want:

- ▶ to work in projective space;
- ▶  $f_G$  then should be a homogeneous map.

## Laman Number as Degree

“Homogenize” the map  $f_G$  by a change of coordinates:

$$\begin{aligned}(x_u - x_v)^2 + (y_u - y_v)^2 &= \\ ((x_u - x_v) + i(y_u - y_v)) \cdot ((x_u - x_v) - i(y_u - y_v)) &= \\ \underbrace{(x_u + iy_u)}_{\downarrow x_u} - \underbrace{(x_v + iy_v)}_{\downarrow x_v} \cdot \underbrace{(x_u - iy_u)}_{\downarrow y_u} - \underbrace{(x_v - iy_v)}_{\downarrow y_v} &= \end{aligned}$$

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Hence our map becomes

$$\begin{aligned}f_G: \mathbb{C}^V \times \mathbb{C}^V &\rightarrow \mathbb{C}^E, \\ (x_1, \dots, x_n, y_1, \dots, y_n) &\mapsto ((x_u - x_v) \cdot (y_u - y_v))_{\{u,v\} \in E}\end{aligned}$$

## Laman Number as Degree

In order to handle translations and rotations, we

- ▶ move one vertex to the origin (for each connected component),
- ▶ fix the position of another vertex (using projective space  $\mathbb{P}$ ),
- ▶ and fix the length of one edge (again, by projectivization).

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**Proposition:** The Laman number  $\text{Lam}(G)$  of  $G = (V, E)$  is given by the degree of the map

$$f_G: \mathbb{P}(\mathbb{C}^V / L_G) \times \mathbb{P}(\mathbb{C}^V / L_G) \rightarrow \mathbb{P}^{|E|-1}, \\ [(x_v)_{v \in V}], [(y_v)_{v \in V}] \mapsto ((x_u - x_v) \cdot (y_u - y_v))_{\{u,v\} \in E}$$

# Bigraphs

**Definition:** A **bigraph**  $B = (G, H)$  is a pair of graphs  $G = (V, \mathcal{E})$  and  $H = (W, \mathcal{E})$ , allowing several components, multiple edges and self-loops. The set  $\mathcal{E}$  is called the set of **biedges**.

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We define the corresponding map  $f_B$  for a bigraph:

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**Proposition:** For  $B = (G, G)$  we have  $\text{Lam}(B) = \text{Lam}(G)$ .

## Puiseux Series

- ▶  $\mathbb{K} = \mathbb{C}\{\{s\}\}$ : field of Puiseux series with coefficients in  $\mathbb{C}$
- ▶ This field comes with a valuation  $\nu: \mathbb{K} \setminus \{0\} \rightarrow \mathbb{Q}$ :

$$\nu\left(\sum_{i=k}^{+\infty} c_i s^{i/n}\right) = \frac{k}{n} \quad \text{if } c_k \neq 0,$$

i.e., the order of a Puiseux series.

- ▶  $\nu(a \cdot b) = \nu(a) + \nu(b)$  and  $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$

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For the map  $f_{B,\mathbb{K}}: \mathbb{P}_{\mathbb{K}}^{(\dots)} \times \mathbb{P}_{\mathbb{K}}^{(\dots)} \rightarrow \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$ , obtained as the extension of scalars from  $f_B$ , we have  $\deg(f_{B,\mathbb{K}}) = \deg(f_B)$ .

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Study the preimage of a “perturbed” point in  $\mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$ :

$$f_{B,\mathbb{K}}^{-1}\left(\left(\lambda_e s^{\text{wt}(e)}\right)_{e \in \mathcal{E}}\right) \quad \text{for some } \text{wt} \in \mathbb{Q}^{\mathcal{E}} \text{ and } \lambda \in \mathbb{C}^{\mathcal{E}},$$

instead of studying the preimage  $f_B^{-1}(p)$  for some  $p \in \mathbb{P}_{\mathbb{C}}^{|\mathcal{E}|-1}$ .

## New Coordinates, New Equations

Introduce new coordinates

- ▶  $x_{uv}$  for all  $u, v \in V$  that are connected by an edge in  $G$
  - ▶  $y_{tw}$  for all  $t, w \in W$  that are connected by an edge in  $H$
- They correspond to the factors  $(x_u - x_v)$  resp.  $(y_t - y_w)$ .

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Select a distinguished biedge  $\bar{e} \in \mathcal{E}$ . Then these coordinates satisfy the system of equations:

$$\begin{aligned}x_{\bar{u}\bar{v}} &= y_{\bar{t}\bar{w}} = 1 \\x_{uv} y_{tw} &= \lambda_e s^{\text{wt}(e)} && \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\} \\ \sum_{\mathcal{C}} x_{uv} &= 0 && \text{for all cycles } \mathcal{C} \text{ in } G \\ \sum_{\mathcal{D}} y_{tw} &= 0 && \text{for all cycles } \mathcal{D} \text{ in } H\end{aligned}$$

In particular,  $x_{uv} = -x_{vu}$ .

# Tropicalization

**Goal:** For a fixed point  $p = (\lambda_e s^{\text{wt}(e)})_{e \in \mathcal{E}} \in \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$  we want to determine its preimages  $f_{B, \mathbb{K}}^{-1}(p)$ .

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## Idea:

- ▶ Apply **tropicalization**: look only at the valuations!
- ▶ An algebraic relation between Puiseux series implies a piecewise linear relation between their orders.
- ▶ For  $q \in f_{B, \mathbb{K}}^{-1}(p)$  let  $d_V(u, v) = \nu(q_{x_{uv}})$ ,  $d_W(t, w) = \nu(q_{y_{tw}})$ .
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- ▶ This way we obtain a discrete object, a pair of functions  $(d_V, d_W)$ , that we call **bidistance**.

**Gain:** We can then partition the set  $f_{B, \mathbb{K}}^{-1}(p)$  according to the bidistances that are determined by its elements.

# Bidistances

The functions  $d_V$  and  $d_W$  satisfy

- ▶  $d_V(u, v) = d_V(v, u)$  for all  $(u, v)$ , and similarly for  $d_W$
- ▶  $d_V(u, v) + d_W(t, w) = \text{wt}(e)$  for all  $e \in \mathcal{E} \setminus \{\bar{e}\}$
- ▶  $d_V(\bar{u}, \bar{v}) = d_W(\bar{t}, \bar{w}) = 0$
- ▶ for every cycle  $\mathcal{C}$  in  $G$ , the minimum of the values of  $d_V$  on the pairs of vertices  $(u, v)$  appearing in  $\mathcal{C}$  is attained at least twice, and similarly for  $d_W$ .

**Definition:** Every pair of functions  $(d_V, d_W)$  satisfying the above conditions is called a **bidistance** compatible with  $\text{wt} \in \mathbb{Q}^{|\mathcal{E}|-1}$ .

## Recursion for the Laman number

**Idea:** We partition the set  $f_{B,\mathbb{K}}^{-1}(p)$  according to the bidistances.

**Lemma:** The number of preimages sharing the same bidistance  $d$  can be obtained as the Laman number of a “simpler” Graph  $B_d$ .

Hence we obtain the following recursion:

**Theorem:**

$$\text{Lam}(B) = \sum_d \text{Lam}(B_d).$$

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$$\text{Lam}(B) = \sum_d \text{Lam}(B_d).$$

Unfortunately, it is not very useful for practical purposes:

1. Enumeration of bidistances  $d$ : **difficult**
2. Computation of  $\text{Lam}(B_d)$ : **difficult**

Two specializations in order to get more explicit formulas...

## First Strategy

By choosing a general weight vector  $\text{wt} \in \mathbb{Q}^{|\mathcal{E}|-1}$ , one can show that  $\text{Lam}(B_d) = 1$  for every bidistance  $d$  compatible with  $\text{wt}$ .

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Hence  $\text{Lam}(B)$  equals the number of such bidistances.

The computation of  $\text{Lam}(B)$  is therefore reduced to a piecewise linear problem:

1. Enumeration of bidistances  $d$ : **difficult**
2. Computation of  $\text{Lam}(B_d)$ : **trivial**

## Second Strategy

**Idea:** We choose the special weight vector  $(1, \dots, 1) \in \mathbb{Q}^{|\mathcal{E}|-1}$ .

We can show that in this case the values of  $d_V$  and  $d_W$  are

- ▶ integers
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- ▶ integers
- ▶ moreover: only the values 0 and 1 can occur.

Hence, each bidistance can be characterized by a single vector in  $\{0, 1\}^{|\mathcal{E}|-1}$  (since  $d_V + d_W = 1$  for all  $e \in \mathcal{E} \setminus \{\bar{e}\}$ ).

1. Enumeration of bidistances  $d$ : **easy**
2. Computation of  $\text{Lam}(B_d)$ : **feasible**

# Operations on Graphs

For constructing the graph  $B_d$ , we need to introduce two operations on graphs:

- ▶ complement
- ▶ quotient

## Graph Complement

Let  $G = (V, E)$  be a graph and let  $E' \subseteq E$ .

**Definition:** The **graph complement**  $G \setminus E'$  is defined as

$$G \setminus E' := (V, E \setminus E').$$

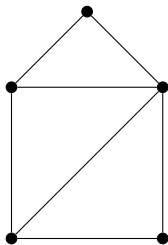
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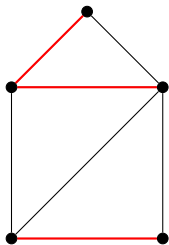
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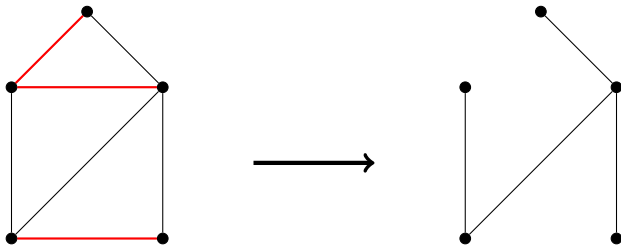
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## Graph Quotient

Let  $G = (V, E)$  be a graph and let  $E' \subseteq E$ .

**Definition:** The **graph quotient**  $G / E'$  is constructed as follows:

- ▶ Connected components of  $(V, E')$  become vertices of  $G / E'$ .
- ▶ Each edge in  $E \setminus E'$  induces an edge of  $G / E'$ .

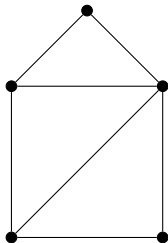
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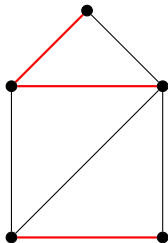
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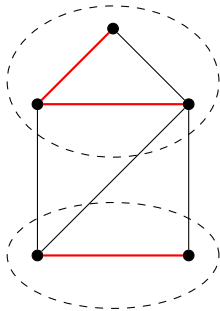
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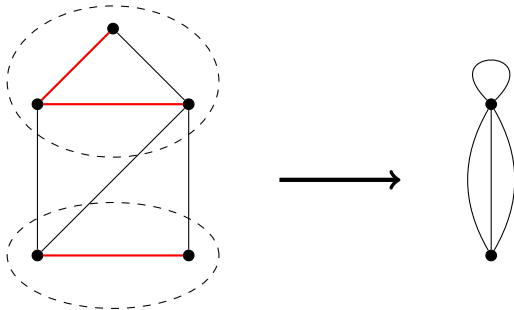
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**Example:**



## Operations on Bigraphs

We define the following two operations on a bigraph  $B = (G, H)$ :

For a subset  $\mathcal{M} \subseteq \mathcal{E}$  of the bigraph edges  $\mathcal{E}$  let

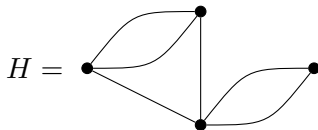
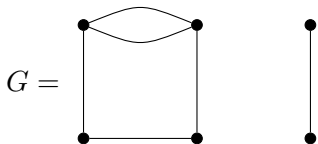
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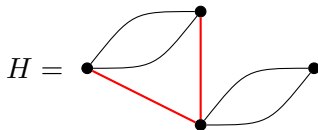
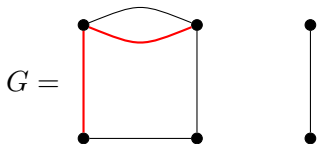
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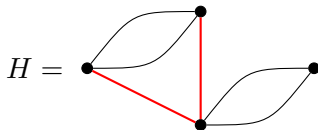
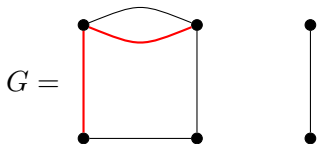


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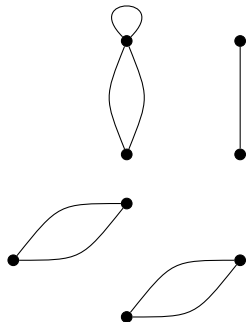
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$B = (G, H), \mathcal{M} \subseteq \mathcal{E}$



The bigraph  ${}^{\mathcal{M}}B$

# The Combinatorial Algorithm

**Theorem.** Let  $B = (G, H)$  be a bigraph with  $G = (V, \mathcal{E})$  and  $H = (W, \mathcal{E})$ . Choose  $\bar{e} \in \mathcal{E}$ . Then

$$\text{Lam}(B) = \text{Lam}(\{\bar{e}\}B) + \text{Lam}(B^{\{\bar{e}\}}) + \sum_{\substack{\mathcal{M} \cup \mathcal{N} = \mathcal{E} \\ \mathcal{M} \cap \mathcal{N} = \{\bar{e}\}}} \text{Lam}(\mathcal{M}B) \cdot \text{Lam}(B^{\mathcal{N}}).$$

Initial conditions:

- ▶  $\text{Lam}(G) = \text{Lam}(G, G)$
- ▶  $\text{Lam}(B) = 0$  if  $G$  or  $H$  contains a loop.
- ▶  $\text{Lam}(B) = 0$  if  $|V| - |\text{Comp}(G)| + |W| - |\text{Comp}(H)| \neq |\mathcal{E}| + 1$ .
- ▶  $\text{Lam}(B) = 1$  if  $|\mathcal{E}| = 1$  and if there are no loops.

## Laman Graphs with Most Realizations

**Question:** Among all Laman graphs with  $n$  vertices, which one has the largest number of realizations?

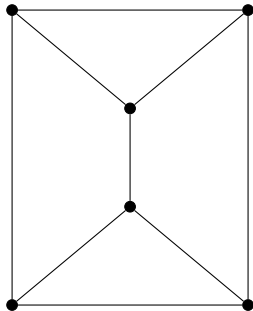
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## Laman Graphs with Most Realizations

**Question:** Among all Laman graphs with  $n$  vertices, which one has the largest number of realizations?

$n$		6
#		24

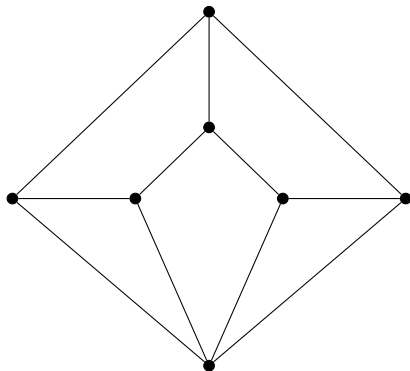
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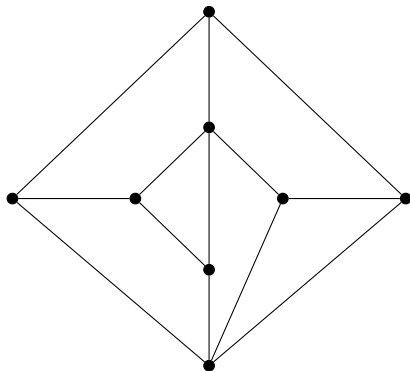
$n$	6	7
#	24	56



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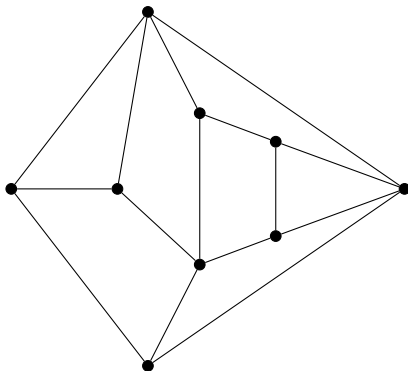
$n$	6	7	8
#	24	56	136



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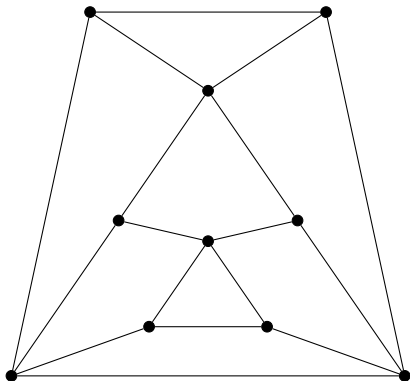
$n$	6	7	8	9
#	24	56	136	344



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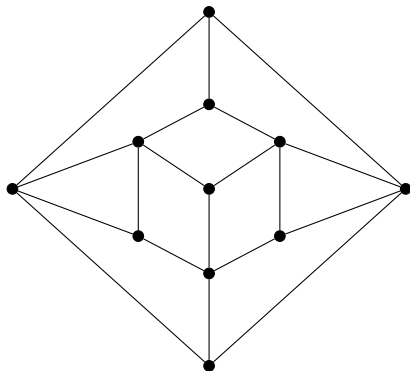
$n$	6	7	8	9	10
#	24	56	136	344	880



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$n$	6	7	8	9	10	11
#	24	56	136	344	880	2288



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$n$	6	7	8	9	10	11
#	24	56	136	344	880	2288
$\frac{1}{8}$	3	7	17	43	110	286

A114589 Number of hill-free Dyck paths of semilength  $n+3$  and having no peaks at even levels (a hill in <sup>1</sup> a Dyck path is a peak at level 1).

1, 1, 3, 7, 17, 43, 110, 286, 753, 2003, 5376, 14540, 39589, 108427, 298512, 825664, 2293271, 6393539, 17885835, 50191175, 141247519, 398537101, 1127203038, 3195229662, 9076078057, 25830193513, 73643406563, 210312889095 ([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

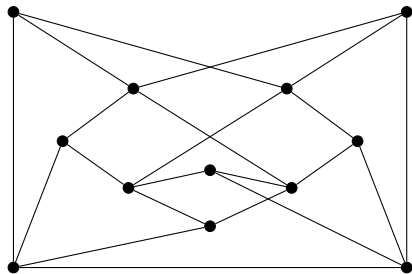
OFFSET 0,3

COMMENTS Column 0 of [A114588](#). The number of hill-free Dyck paths having no peaks at odd level are given by the Riordan numbers ([A005043](#)).  
 contribution from [Paul Barry](#), Jul 05 2009: (start)  
 The sequence 1,0,0,1,1,3,7,... has g.f.  $((1+x)(1+2x)-\sqrt{(1+x)(1-3x)})/(2x(2+2x+x^2))$ . It is the inverse binomial transform of [A035929](#)( $n+1$ ). (End)

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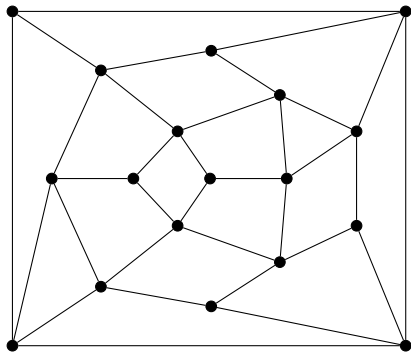
$n$	6	7	8	9	10	11	12
#	24	56	136	344	880	2288	6180
$\frac{1}{8}$	3	7	17	43	110	286	772.5



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$n$	6	7	8	9	10	11	12		18
#	24	56	136	344	880	2288	6180	...	$\geq 1953816$
$\frac{1}{8}$	3	7	17	43	110	286	772.5		$\geq 244227$



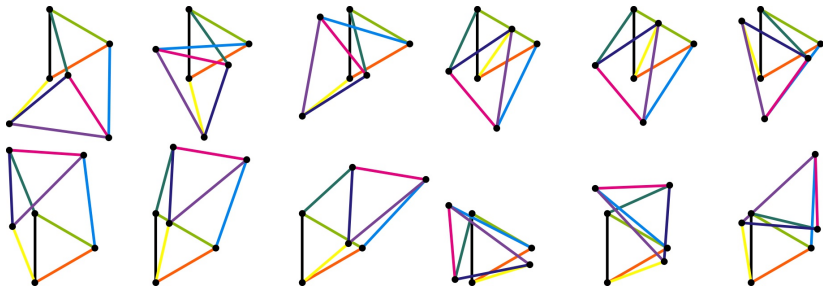
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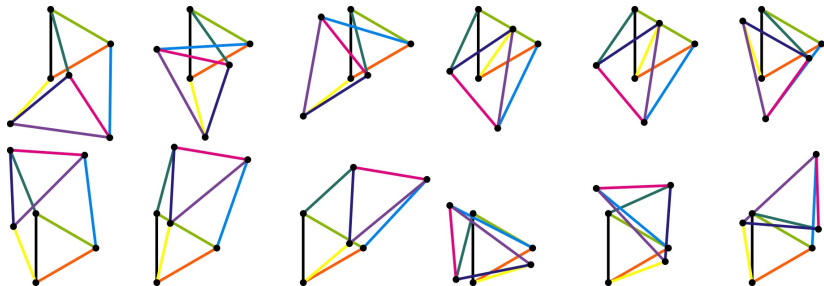
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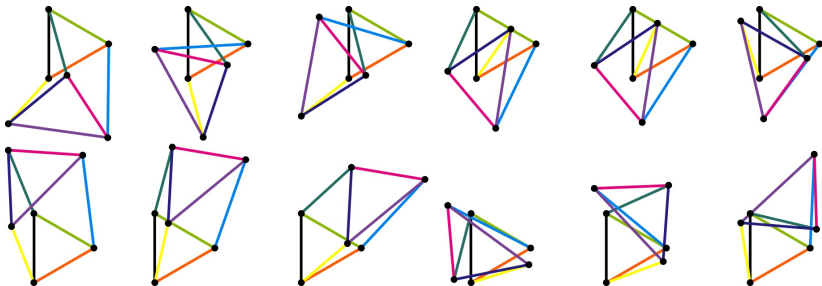


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**Answer:** Show movie.