Reduction-Based Creative Telescoping for D-Finite Functions

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JOHANN · RADON · INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Motivation:

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The telescoper P gives rise to a differential equation satisfied by the integral.

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- until they become linearly dependent over C.
- The relation $p_0[f] + \cdots + p_r[\partial_t^r \cdot f] = 0$ yields the telescoper $P = p_0 + \cdots + p_r\partial_t^r$, since by linearity we have

$$[p_0 f + \cdots + p_r \partial_t^r \cdot f] = 0,$$

and by definition of $[\cdot]$ we have that $P \cdot f$ is integrable.

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Our task consists in defining a reduction map that satisfies either Property 1 or Property 2 (or both).

Previous Work on Reduction-Based Telescoping

- Bostan, Chen, Chyzak, Li (2010): first reduction-based algorithm, for integrating bivariate rational functions
- Chen, Singer (2012): bivariate rational functions, was then extended to consider also summation and q-summation
- Chen, Kauers, Singer (2012): hybrid algorithm for integrating bivariate algebraic functions
- Bostan, Lairez, Salvy (2013, 2015): reduction technique for multivariate rational functions
- Bostan, Chen, Chyzak, Li, Xin (2013): Hermite reduction for hyperexponential functions
- Chen, Huang, Kauers, Li (2015, 2016): summation algorithm for hypergeometric terms
- Bostan, Dumont, Salvy (2016): integration of bivariate hypergeometric-hyperexponential terms
- Kauers, Koutschan (2015): integral bases for D-finite functions
- Chen, Kauers, Koutschan (2016): algebraic functions
- Chen, van Hoeij, Kauers, Koutschan (2017): fuchsian D-finite

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1. If $[f] \neq 0$ then f is not integrable (in D), since a rational function with a simple pole has no rational antiderivative (one would have to introduce logarithms).
2. Set $h_i = [\partial_t^i f]$. Since differentiating doesn't introduce new poles, there is a (squarefree) polynomial d that is a common denominator for all h_i. Since the h_i are proper, the dimension of the C-vector they generate is at most $\deg_x d$.

Fuchsian Differential Operators

Consider a linear differential operator with polynomial coefficients:

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Definition: The operator L is called **fuchsian at a point** $a \in \bar{C}$ if it admits n linearly independent solutions in

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$$\bar{C}[[[x^{-1}]]] := \bigcup_{\nu \in C} x^{-\nu} \bar{C}[[x^{-1}]][\log(x)].$$

It is simply called **fuchsian** if it is fuchsian at all $a \in \bar{C} \cup \{\infty\}$.

Integral Series

Definition:

- Terms in these power series expansions, i.e., terms of the form

$$(x-a)^\alpha \log(x-a)^\beta \qquad \text{or} \qquad (\tfrac{1}{x})^\alpha \log(x)^\beta$$

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Integral Bases

For a fuchsian operator L, we consider the left $C(x)[\partial_x]$-module $A = C(x)[\partial_x]/\langle L\rangle$, where $\langle L\rangle$ is the left ideal generated by L.

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- algorithm to compute integral bases (Kauers/Koutschan 2015)
- algorithm for algebraic function fields (van Hoeij 1994)

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- "$y \in A$ is integral at 0" means that all solutions of $M = 0$ have Puiseux series expansions at $x = 0$ with exponents $\geqslant 0$.
- "$y^2 \in A$ is integral" means that for any solution y of $M = 0$ and any $a \in \bar{C}$, the Puiseux series of y^2 at $x = a$ is integral.

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Note: An integral basis $\{\omega_1, \ldots, \omega_n\}$ is always also a $C(x)$-vector space basis of A.

- We have that $f \in A$ has a pole at $a \in \bar{C}$ if and only if at least one of the f_i has a pole at a.
- In particular the poles of the coefficients f_i cannot cancel each other.

Hermite-Trager Reduction

Task: For given $f \in A$ find $g, h \in A$ such that

$$f = \sum_{i=1}^{n} \frac{f_i}{d} \omega_i = g' + h \qquad \text{and} \qquad h = \sum_{i=1}^{n} \frac{h_i}{d^*} \omega_i,$$

where $f_i, h_i \in C[x]$ and where d^* is the squarefree part of d.

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where $f_i, h_i \in C[x]$ and where d^* is the squarefree part of d.

One step of the reduction consists in reducing the multiplicity $\mu > 1$ of some nontrivial squarefree factor $v \in C[x]$ of d:

$$\sum_{i=1}^{n} \frac{f_i}{uv^\mu} \omega_i = \left(\sum_{i=1}^{n} \frac{g_i}{v^{\mu-1}} \omega_i \right)' + \sum_{i=1}^{n} \frac{h_i}{uv^{\mu-1}} \omega_i \qquad (d = uv^\mu).$$

Hermite-Trager Reduction

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By a repeated application of such reduction steps one can decompose any $f \in A$ as $f = g' + h$, where the denominators of the coefficients of h are squarefree.

HT Reduction for Fuchsian D-Finite Functions

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In order to determine the unknown polynomials $g_1, \ldots, g_n$, we clear the denominator uv^{μ}:

$$\sum_{i=1}^{n} f_i\omega_i = \sum_{i=1}^{n}\left(uvg_i'\omega_i + uv^{\mu}g_i\left(v^{1-\mu}\omega_i\right)' + vh_i\omega_i\right).$$

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Using $\gcd(u,v)=1$ we can prove that the elements $uv^{\mu}\left(v^{1-\mu}\omega_i\right)'$ form a local integral basis at each root of v.
$\longrightarrow$ This implies that the g_i are uniquely determined modulo v.

Obstacle

Chevalley's theorem: The only algebraic functions which are integral at all finite places and at infinity are the constant ones.

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Fortunately, we can show that there are not too many such cases:

Lemma: Let $\{\omega_1, \ldots, \omega_n\}$ be an integral basis of A that is normal at infinity, and let $\tau_1, \ldots, \tau_n \in \mathbb{Z}$ be such that $\{x^{\tau_1}\omega_1, \ldots, x^{\tau_n}\omega_n\}$ is a local integral basis at infinity. Then the set of all $f \in A$ which are integral everywhere is a C-vector space with basis

$$\{\, x^j \omega_i : i = 1, \ldots, n;\ j = 0, \ldots, \tau_i \,\}.$$

Main Result: Property 1

Theorem:

Suppose that $f \in A$ has at least a double root at infinity, i.e., every series in $\bar{C}[[[x^{-1}]]]$ associated to f only contains monomials $(1/x)^{\alpha}\log(x)^{\beta}$ with $\alpha \geqslant 2$.

Let $W = \{\omega_1, \ldots, \omega_n\}$ be an integral basis for A that is normal at infinity, and let $f = g' + h$ be the result of the Hermite reduction with respect to W.

Let $V \subseteq A$ be the C-vector space of all elements that are integral at all places, including infinity, and let $U = \{v' : v \in V\}$ be the space of all elements of A that are integrable in V.

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Note: In the special case of algebraic functions we get

$$f \text{ is integrable if and only if } h = 0$$

since V is the set of constant functions (according to Chevalley).

Property 2: Confinement

Proposition: Let $W = \{\omega_1, \ldots, \omega_n\}$, $f = \sum_{i=1}^n (f_i/D)\omega_i \in A$, $\tau_1, \ldots, \tau_n \in \mathbb{Z}$, $D, f_1, \ldots, f_n \in C[x]$, $e \in C[x]$, $E = \operatorname{lcm}(e, D^*)$. Then f admits a telescoper of order at most

$$n\left(\deg_x(E) - 1 - \min\bigl(0, \min{}_j(\tau_j)\bigr)\right) + \sum_{i=1}^n \tau_i.$$

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Polynomial Reduction: the idea is to decompose h further:

$$f = g' + h = g' + \sum_{i=1}^{n} \frac{h_i}{de}\omega_i = g' + \sum_{i=1}^{n} \frac{r_i}{d}\omega_i + \sum_{i=1}^{n} \frac{s_i}{e}\omega_i,$$

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Alternative bound: Every $f \in A$ has a telescoper of order at most

$$n\big(\deg_x(d) + \deg_x(e) + \tau + \lambda + 1\big).$$

Example

Let $L = 3(x^3 - x)D_x^2 + 2(3x^2 - 1)D_x$ with solutions

$$y_1(x) = 1 \qquad \text{and} \qquad y_2(x) = x^{1/3}\,{}_2F_1\left(\tfrac{1}{6}, \tfrac{2}{3}; \tfrac{7}{6}; x^2\right).$$

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A straightforward calculation yields

$$W' = \frac{1}{e}MW = \frac{1}{x^3 - x}\begin{pmatrix} 0 & 1 \\ 0 & x^2 - \frac{1}{3} \end{pmatrix} W \quad \text{for } W = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Example (continued)

Consider the following integrand (which has a double root at ∞):

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According to the theorem, f is integrable if this remainder lies in the subspace $U = \{v' : v \in V\}$. Using the matrix M, we find that $\omega_1' = \frac{1}{x^3-x}\omega_2$, which is indeed a scalar multiple of the remainder. Hence, f is integrable:

$$f = \left(-\frac{3(x+1)}{x}\omega_1 - \frac{3(2x+1)}{2(x^3-x)}\omega_2\right)'.$$