Binomial Determinants for Tiling Problems Yield to the Holonomic Ansatz

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Motivation

“Conjecture 37” (Lascoux/Krattenthaler 2005)

Let μ be an indeterminate and $m, r \in \mathbb{Z}$. If $m \geqslant r \geqslant 1$, then

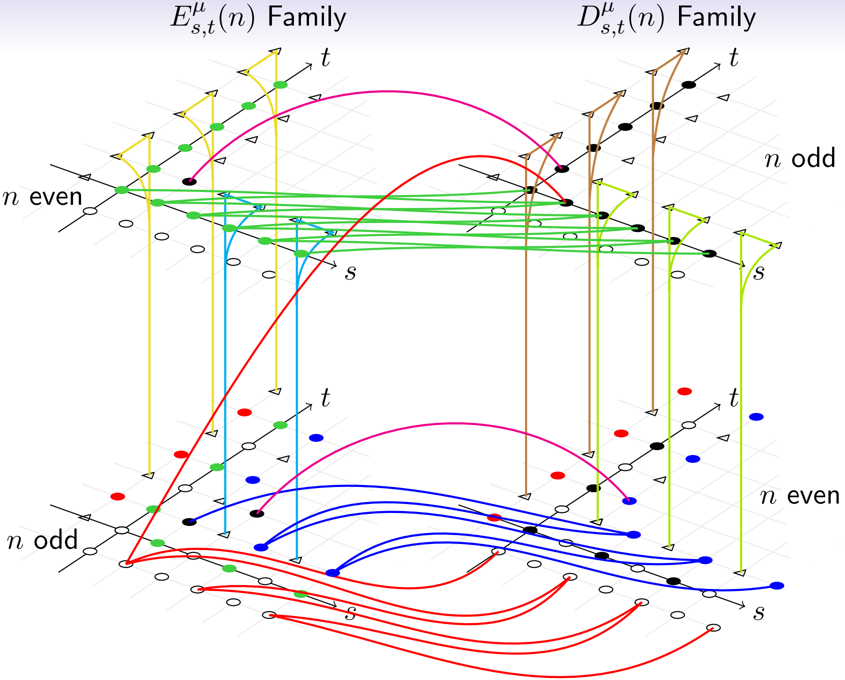
$$\det_{\substack{1 \leqslant i \leqslant 2m-1 \\ 1 \leqslant j \leqslant 2m-1}} \left(\binom{\mu + i + j + 2r - 4}{j + 2r - 2} - \delta_{i,j+2r-2} \right) =$$

$$(-1)^{m-r} \cdot 2^{4m+(m-r)(m-r-1)-3r} \cdot \left(\prod_{i=0}^{2r-3} i! \right) \cdot \left(\prod_{i=0}^{m-1} \frac{i!\,(i+1)!}{(2i)!\,(2i+2)!} \right)$$

$$\times (\mu - 1) \cdot \left(\tfrac{\mu}{2} + r - \tfrac{1}{2} \right)_{m-r} \cdot \left(\prod_{i=1}^{2r-2} (\mu + i - 1)_{2m+2r-2i-1} \right)$$

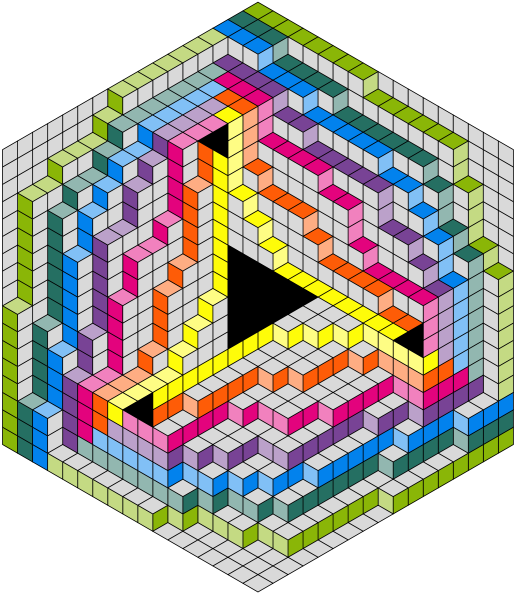
$$\times \left(\prod_{i=0}^{r-2} \frac{\big((2m-2i-3)!\big)^2}{\big((m-i-2)!\big)^2 \,(2m+2i-1)!\,(2m+2i+1)!} \right)$$

$$\times \left(\prod_{i=0}^{\lfloor \frac{m-r-1}{2} \rfloor} \left(\tfrac{\mu}{2} + 3i + 3r - \tfrac{1}{2} \right)_{m-r-2i-1}^2 \left(-\tfrac{\mu}{2} - 3m + 3i + 3 \right)_{m-r-2i}^2 \right)$$



$E^{\mu}_{s,t}(n)$ Family
$D^{\mu}_{s,t}(n)$ Family
t
s
n even
n odd
t
s
n odd
n even

Cylically Symmetric Rhombus Tilings of a Holey Hexagon



Families of Binomial Determinants

Definition: For $n \in \mathbb{N}$, for $s, t \in \mathbb{Z}$, and for μ an indeterminate, define the following $(n \times n)$-determinants:

$$D_{s,t}^{\mu}(n) := \det_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} \left(\binom{\mu + i + j + s + t - 4}{j + t - 1} + \delta_{i+s,j+t} \right),$$

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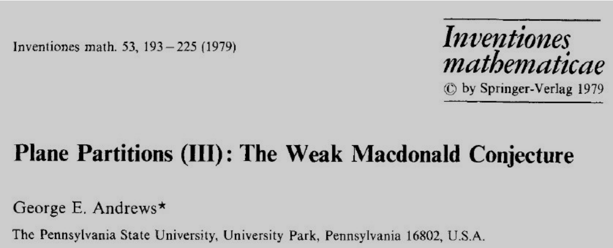
History: $D^{\mu}_{0,0}(n)$ was introduced in the work of Andrews in 1979 – 1980 in the context of descending plane partitions:


Inventiones math. 53, 193 – 225 (1979)

Inventiones mathematicae
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Plane Partitions (III): The Weak Macdonald Conjecture

George E. Andrews*
The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

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Example: $D^{\mu}_{4,6}(5)$ is the determinant of the matrix

$$\begin{pmatrix} \binom{\mu+8}{6} & \binom{\mu+9}{7} & \binom{\mu+10}{8} & \binom{\mu+11}{9} & \binom{\mu+12}{10} \\ \binom{\mu+9}{6} & \binom{\mu+10}{7} & \binom{\mu+11}{8} & \binom{\mu+12}{9} & \binom{\mu+13}{10} \\ \binom{\mu+10}{6}+1 & \binom{\mu+11}{7} & \binom{\mu+12}{8} & \binom{\mu+13}{9} & \binom{\mu+14}{10} \\ \binom{\mu+11}{6} & \binom{\mu+12}{7}+1 & \binom{\mu+13}{8} & \binom{\mu+14}{9} & \binom{\mu+15}{10} \\ \binom{\mu+12}{6} & \binom{\mu+13}{7} & \binom{\mu+14}{8}+1 & \binom{\mu+15}{9} & \binom{\mu+16}{10} \end{pmatrix}$$

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Example: $D^{\mu+2}_{3,5}(5)$ is the determinant of the matrix

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Example: $D^{\mu-1}_{5,6}(5)$ is the determinant of the matrix

$$\begin{pmatrix} \binom{\mu+8}{6} & \binom{\mu+9}{7} & \binom{\mu+10}{8} & \binom{\mu+11}{9} & \binom{\mu+12}{10} \\ \binom{\mu+9}{6} + 1 & \binom{\mu+10}{7} & \binom{\mu+11}{8} & \binom{\mu+12}{9} & \binom{\mu+13}{10} \\ \binom{\mu+10}{6} & \binom{\mu+11}{7} + 1 & \binom{\mu+12}{8} & \binom{\mu+13}{9} & \binom{\mu+14}{10} \\ \binom{\mu+11}{6} & \binom{\mu+12}{7} & \binom{\mu+13}{8} + 1 & \binom{\mu+14}{9} & \binom{\mu+15}{10} \\ \binom{\mu+12}{6} & \binom{\mu+13}{7} & \binom{\mu+14}{8} & \binom{\mu+15}{9} + 1 & \binom{\mu+16}{10} \end{pmatrix}$$

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Ugly Example

$$D^{\mu}_{3,3}(7) = \frac{1}{22122558259200000}(\mu^{25} + 335\mu^{24} + 53170\mu^{23} + 5219210\mu^{22} + 353884975\mu^{21} + 17654136185\mu^{20} + 675334978420\mu^{19} + 20393582102960\mu^{18} + 496547143637215\mu^{17} + 9902234513723585\mu^{16} + 163628567918015170\mu^{15} + 2259409940615500610\mu^{14} + 26220413043850095745\mu^{13} + 256610136017431510535\mu^{12} + 2120496573913057782520\mu^{11} + 14782628582961949481060\mu^{10} + 86673574436964799906960\mu^{9} + 425074251314867787511760\mu^{8} + 1729277578550904467089920\mu^{7} + 5765988504741630995828160\mu^{6} + 15490845170481326463535104\mu^{5} + 32714130921152175099417600\mu^{4} + 52316431952932423423180800\mu^{3} + 59610947649553163501568000\mu^{2} + 43184734857314137866240000\mu + 14982065085331066060800000)$$

Nice Example

$$E^{\mu}_{1,5}(7) = \frac{1}{148354371231500206080000000}(-\mu^{36} - 200\mu^{35} - 19159\mu^{34} - 1171436\mu^{33} - 51394672\mu^{32} - 1724423456\mu^{31} - 46048129836\mu^{30} - 1005506521104\mu^{29} - 18305564269902\mu^{28} - 281867442349584\mu^{27} - 3711004634220450\mu^{26} - 42124413821616840\mu^{25} - 414889994727011100\mu^{24} - 3562629353787488640\mu^{23} - 26764739256385498620\mu^{22} - 176333020693153028880\mu^{21} - 1020132335713727670105\mu^{20} - 5184100772592640581480\mu^{19} - 23125258686352150100735\mu^{18} - 90390456977427664781740\mu^{17} - 308644189797756712933964\mu^{16} - 916403980791449441431840\mu^{15} - 2350093624208246581241696\mu^{14} - 5154412290653177844256384\mu^{13} - 9525414800317726242119808\mu^{12} - 14472568507785350993547264\mu^{11} - 17255372452899442525004544\mu^{10} - 14360009990096346869615616\mu^{9} - 4453778806728199172840448\mu^{8} + 8910764739632797324222464\mu^{7} + 18620314976835976877015040\mu^{6} + 19676254731549280468992000\mu^{5} + 13589211129691858698240000\mu^{4} + 6195378277541943705600000\mu^{3} + 1707950341804208947200000\mu^{2} + 216751516409659392000000\mu) =$$

Nice Example

$$
\begin{aligned}
E^{\mu}_{1,5}(7) = {} & \tfrac{1}{148354371231500206080000000}(-\mu^{36} - 200\mu^{35} - 19159\mu^{34} - \\
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& 23125258686352150100735\mu^{18} - 90390456977427664781740\mu^{17} - \\
& 308644189797756712933964\mu^{16} - 916403980791449441431840\mu^{15} - \\
& 2350093624208246581241696\mu^{14} - 5154412290653177844256384\mu^{13} - \\
& 9525414800317726242119808\mu^{12} - 14472568507785350993547264\mu^{11} - \\
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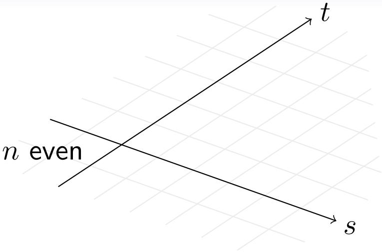
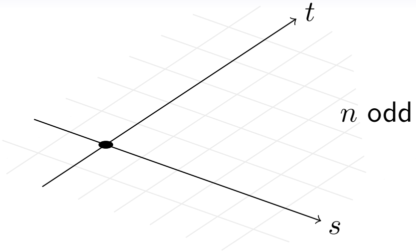
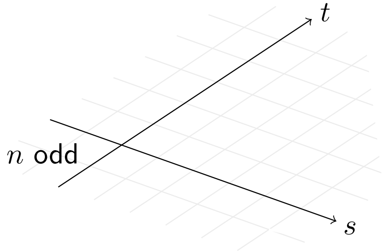
& 1707950341804208947200000\mu^{2} + 216751516409659392000000\mu) = \\
& \tfrac{1}{148354371231500206080000000}(1-\mu)\mu(\mu+1)^2(\mu+2)^3(\mu+3)^4(\mu+4)^4 \\
& (\mu+5)^5(\mu+6)^4(\mu+7)^4(\mu+8)^3(\mu+9)^2(\mu+10)(\mu+18)^2
\end{aligned}
$$

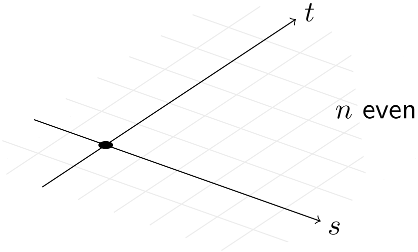
$E^{\mu}_{s,t}(n)$ Family

$D^{\mu}_{s,t}(n)$ Family









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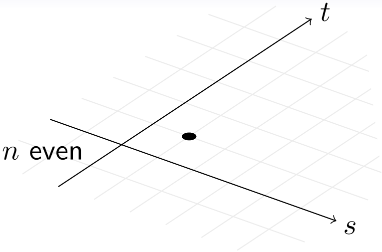

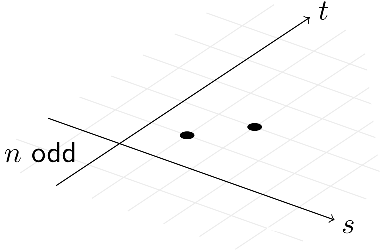

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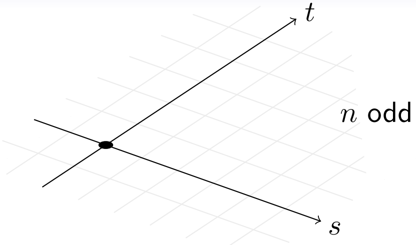

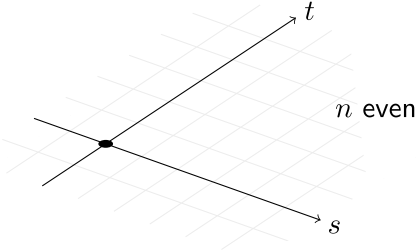

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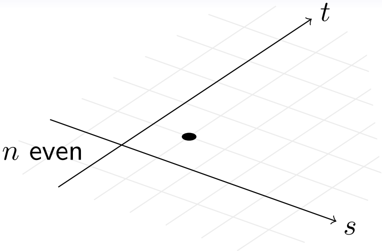

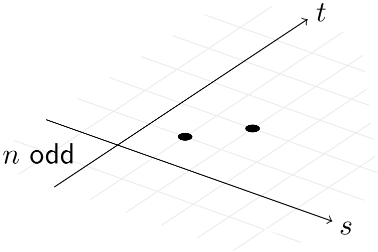

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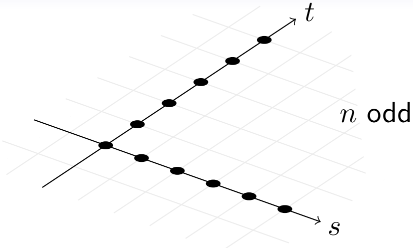

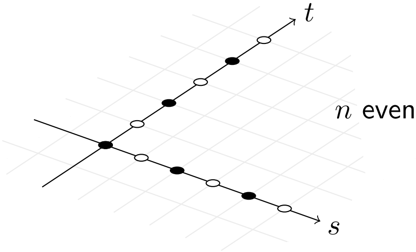

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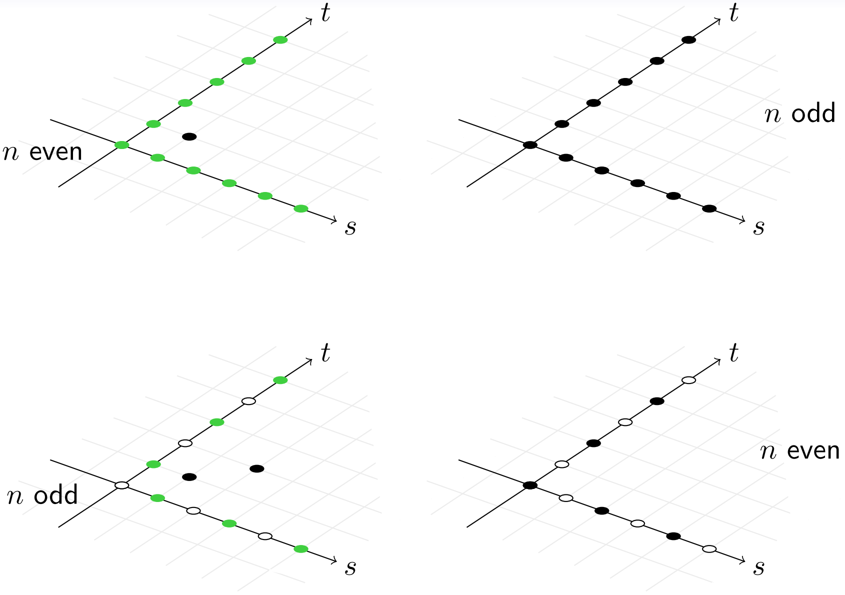

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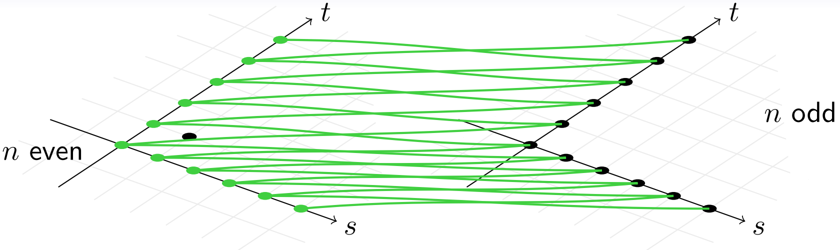

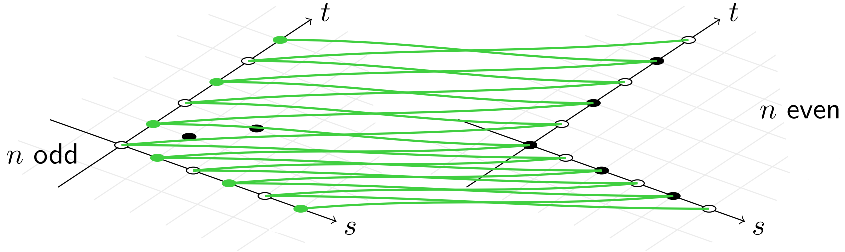

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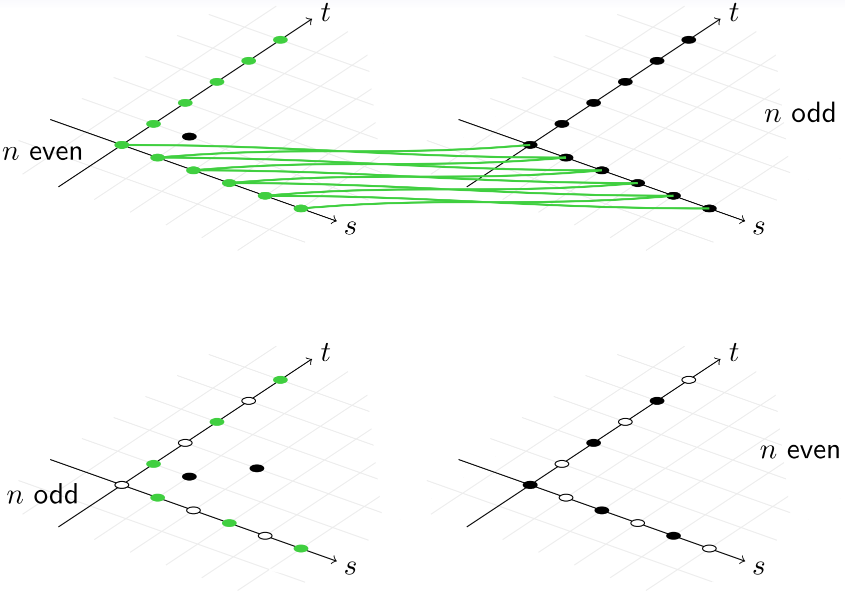

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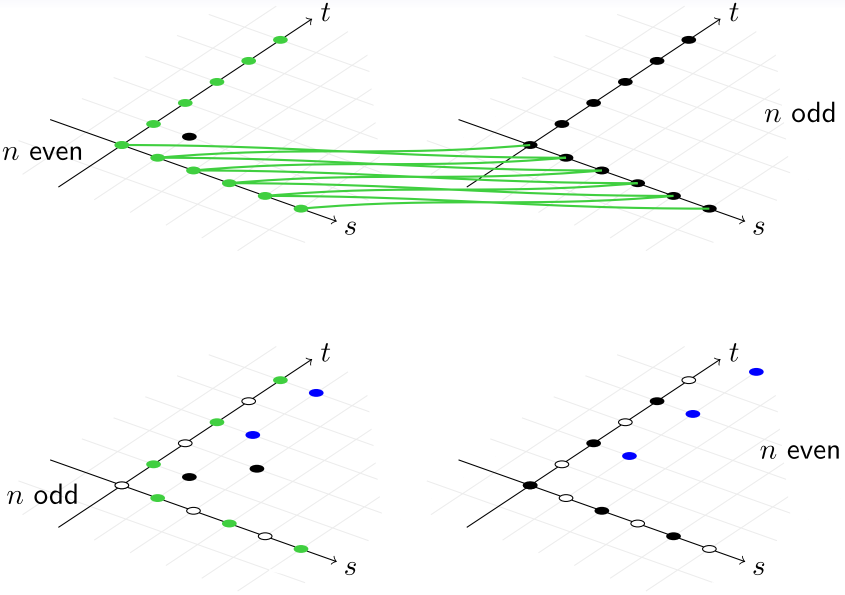

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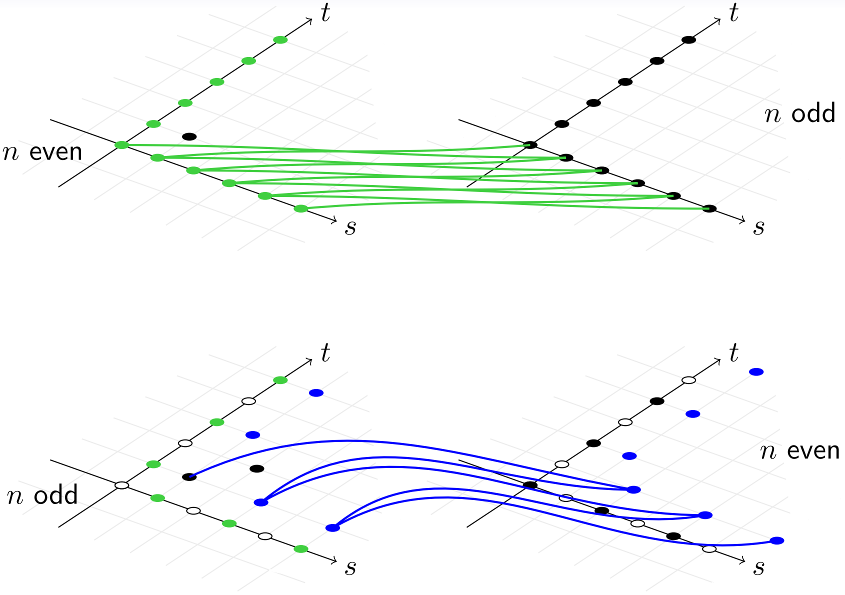

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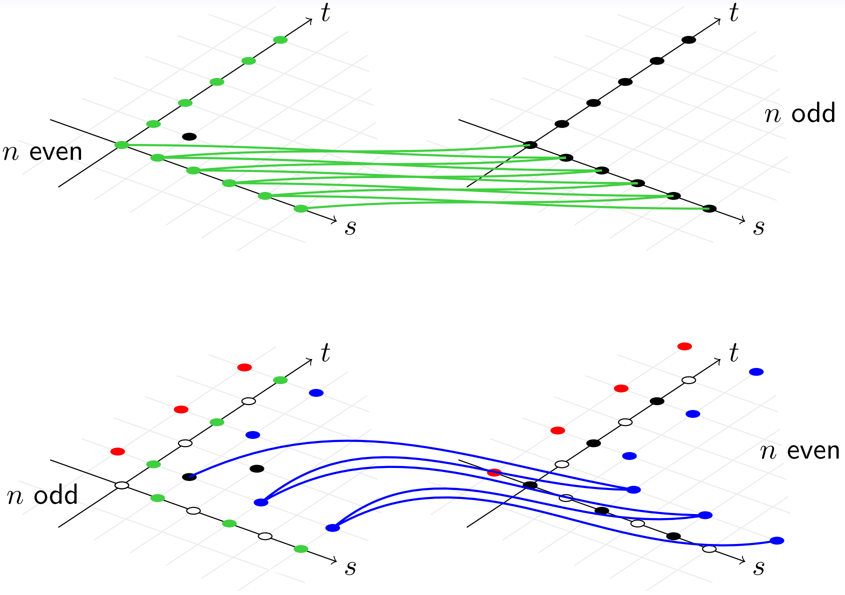

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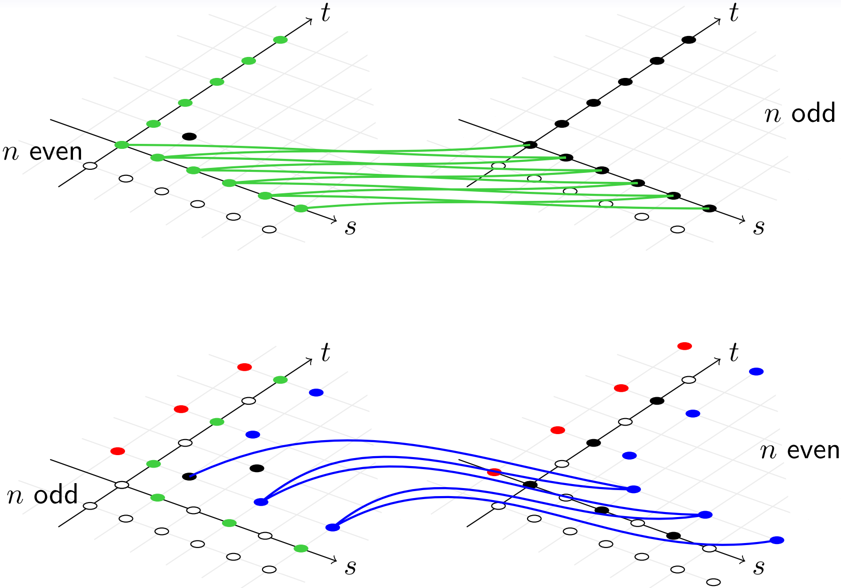

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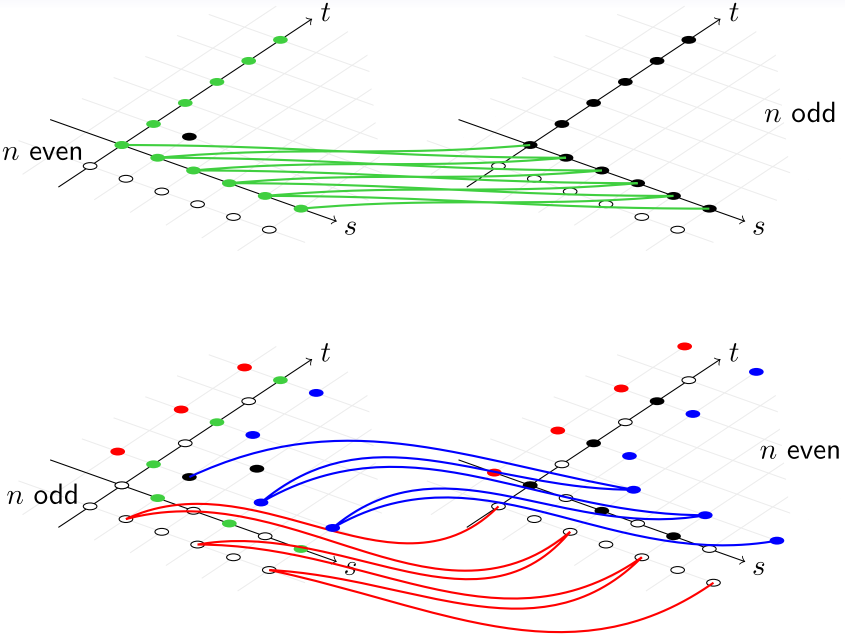

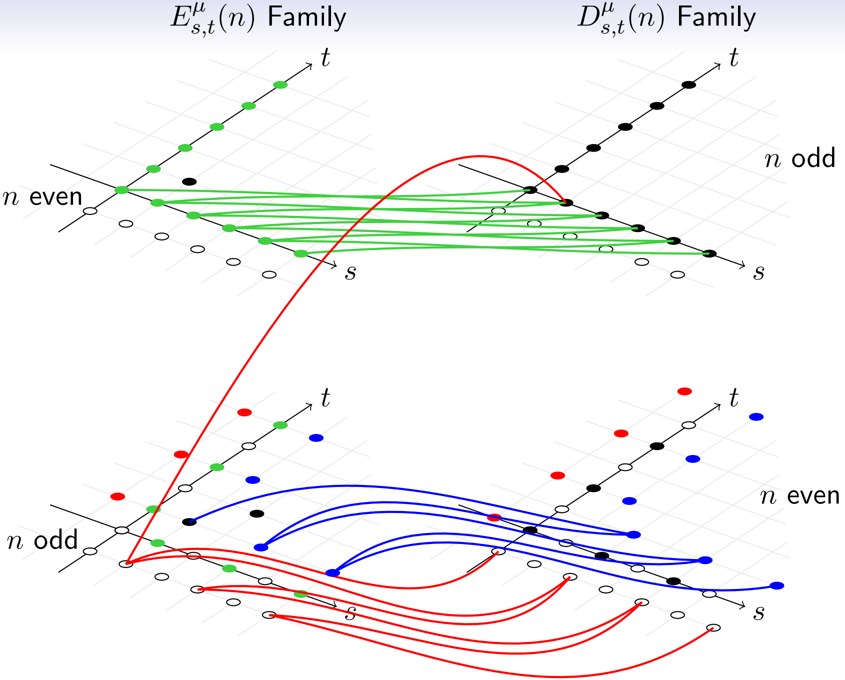
$E^{\mu}_{s,t}(n)$ Family
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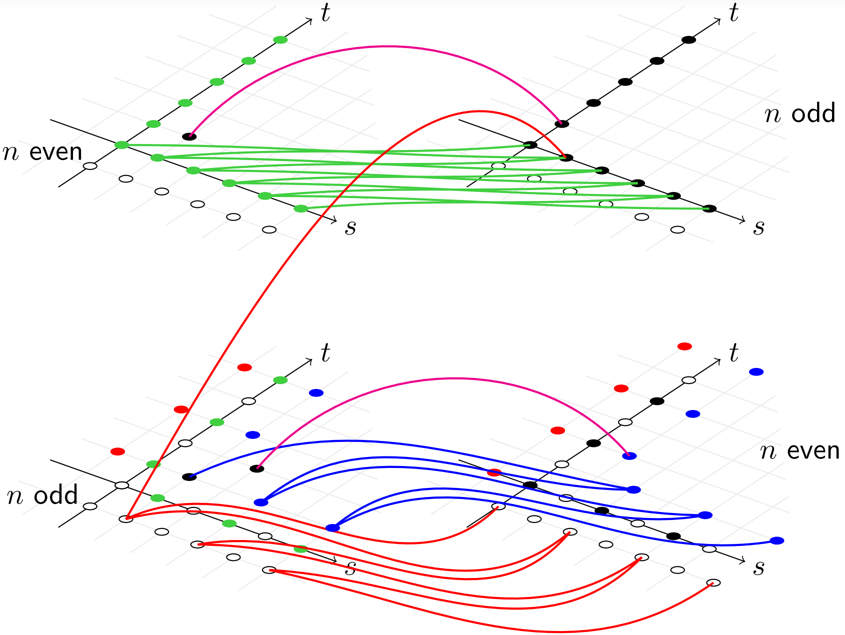
$E^{\mu}_{s,t}(n)$ Family $D^{\mu}_{s,t}(n)$ Family



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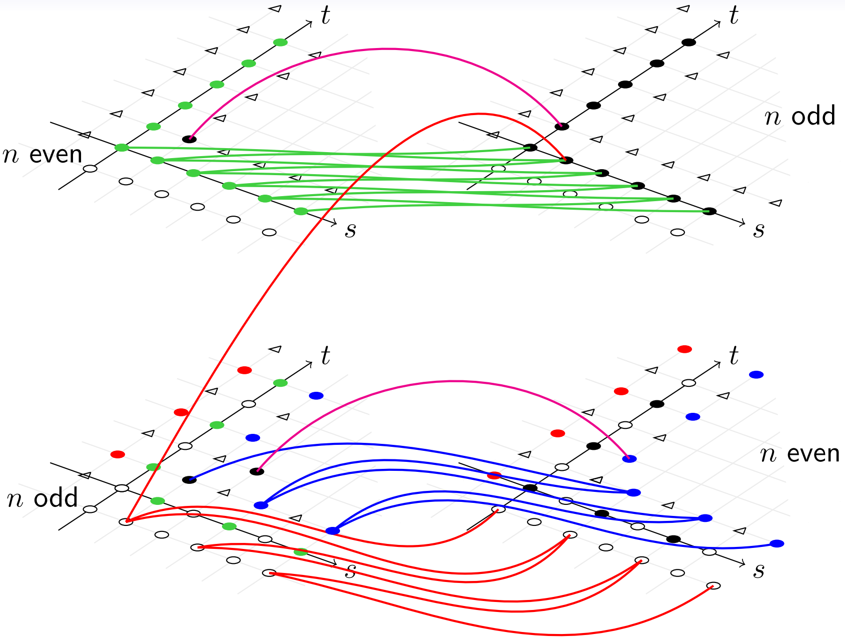

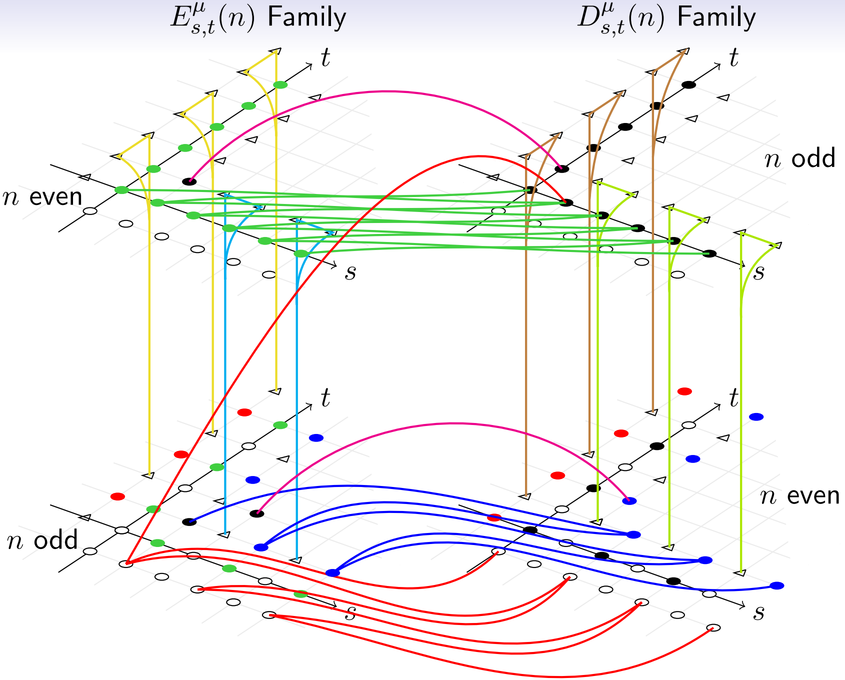
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t
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t
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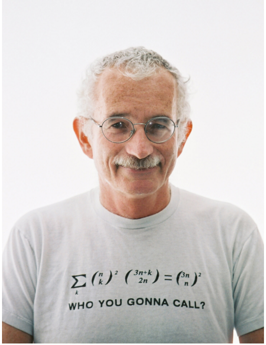
HOLONOMIC ANSATZ!!!

Problem: Prove a determinantal identity of the form $\det_{1 \leqslant i,j \leqslant n} (a_{i,j}) = b_n$



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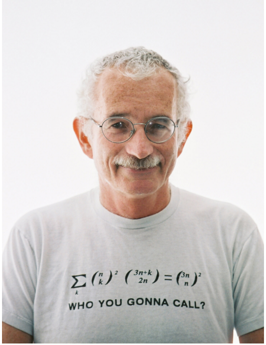
- $a_{i,j}$ is a holonomic sequence



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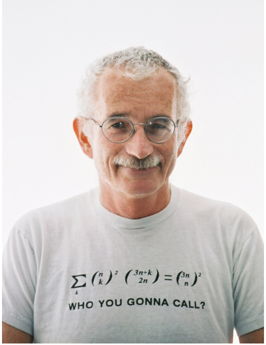
- b_n is a closed form ($b_n \neq 0$ for all n)



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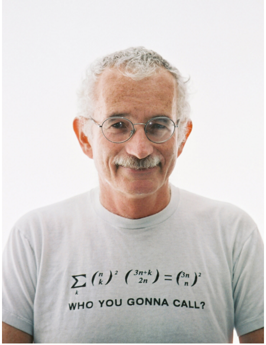
$$\mathcal{A}_n = \left(\begin{array}{ccc:c} & \mathcal{A}_{n-1} & & \\ \hdashline a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \end{array}\right)$$



Laplace expansion:

$$\det(\mathcal{A}_n) = a_{n,1}\mathrm{Cof}_{n,1} + \ldots + a_{n,n-1}\mathrm{Cof}_{n,n-1} + a_{n,n}\det(\mathcal{A}_{n-1})$$

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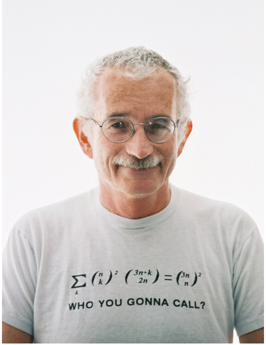

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$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1}\frac{\mathrm{Cof}_{n,1}}{\det(\mathcal{A}_{n-1})} + \ldots + a_{n,n-1}\frac{\mathrm{Cof}_{n,n-1}}{\det(\mathcal{A}_{n-1})} + a_{n,n}$$

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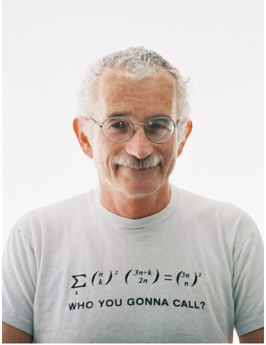

Laplace expansion:

$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1}c_{n,1} + \ldots + a_{n,n-1}c_{n,n-1} + a_{n,n}c_{n,n}$$

Problem: Prove a determinantal identity of the form $\det_{1 \leqslant i,j \leqslant n} (a_{i,j}) = b_n$, where

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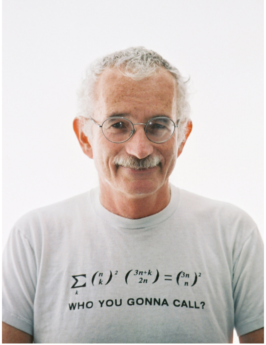

Laplace expansion:

$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = \sum_{j=1}^{n} a_{n,j} c_{n,j}$$

HOLONOMIC ANSATZ!!!

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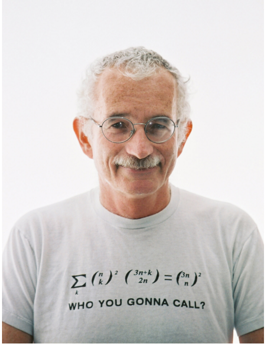

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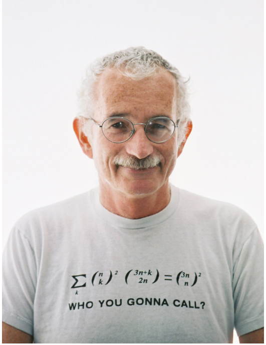

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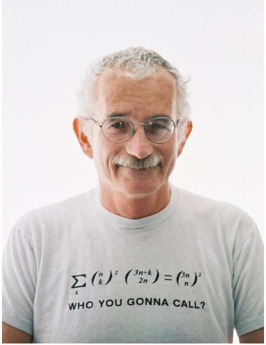

Laplace expansion:

$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j} \quad (1 \leqslant i < n), \quad c_{n,n} = 1$$

E-Mail from Doron during LP conference

From Doron Zeilberger
Subject **challenge**
To Me <christoph.koutschan@ricam.oeaw.ac.at>
Cc philippe@illinois.edu

Dear Christoph,

Philippe Di Francesco just gave a great talk at the Lattice path conference mentioning, inter alia, a certain conjectured determinant.

It is

Conj. 8.1 (combined with Th. 8.2) in
https://arxiv.org/pdf/2102.02920.pdf

I am curious if you can prove it by the Koutschan-Zeilberger-Aek holonomic ansatz method.

If you can do it before Friday, June 25, 2021, 17:00 Paris time, I will mention it in my talk in that conference.

Best wishes

Doron

E-Mail from Doron during LP conference

Conjecture 8.1. *The total numbers* $Z_n^{20V} = Z_n^{DT}$ *of configurations of the 20V model on the quadrangle* $\mathcal{Q}_n$ *and of domino tilings of the Aztec triangle* $\mathcal{T}_n$ *read:*

$$Z_n^{20V} = Z_n^{DT} = 2^{n(n-1)/2} \prod_{i=0}^{n-1} \frac{(4i+2)!}{(n+2i+1)!} \tag{8.1}$$

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Theorem 8.2. *The number Z_n^{20V} of configurations of the 20V model on the quadrangle $\mathcal{Q}_n$ reads:*

$$Z_n^{20V} = \det_{0 \le i \le j \le n-1} \left(2^i \binom{i+2j+1}{2j+1} - \binom{i-1}{2j+1} \right) \tag{8.2}$$

with the convention $\binom{m}{p} = 0$ for all $-1 \le m < p$.

Doron Zeilberger

Outline and Links for Doron Zeilberger's Talk , June 25, 2021, CIRM

1. Thanks Cyril et. al.

2. Warning: not a proper math talk (quote Kimmo)

3. The triumphs of "Guess and Check":

   - Comment on MBM's talk: This simple-minded approach that ultimately lead to the FIRST proof of Gessel's conjecture, gives a (very ELEMENTARY!) one-line proof of the Kreweras walk formula (for the quarter plane) (mentioned in this masterpiece), a similar proof should exist for the three-quarter-plane.
   - Comment on Mourad I.'s talk: Christian Krathenthaler noticed that the expression of Brennan that he conjectured to be a polynomial in q, is nothing but MacMahon's box formula that immediately proves that not only are they polynomials, but they have positive coefficients. (e.g. see here)
   - Comment on Philippe Di-F's great talk. Using this nice experimental-yet-rigorous approach (that ultimately lead to the proof of the qTSPP conjecture) Christoph Koutschan proved Philippe Di-F's determinant conjecture! (that another CK could not do) (see certificate)

4. Congratulate MBM, implicit honors

5. Memory Lane: FPSAC 1991 (show proceedings), MBM's defense (the one who laughed, show thesis)

6. my report

7. shocking shortcut

Switching Lemma

Recall:

$$D^{\mu}_{s,t}(n) := \det_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \left(\binom{\mu+i+j+s+t-4}{j+t-1} + \delta_{i+s,j+t} \right),$$

$$E^{\mu}_{s,t}(n) := \det_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \left(\binom{\mu+i+j+s+t-4}{j+t-1} - \delta_{i+s,j+t} \right).$$

Switching Lemma

Recall:

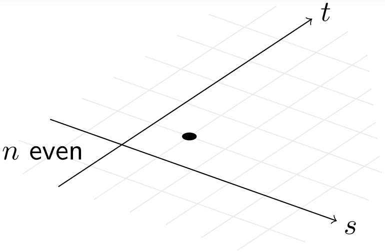
$$D_{s,t}^{\mu}(n) := \det_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \left(\binom{\mu+i+j+s+t-4}{j+t-1} + \delta_{i+s,j+t} \right),$$

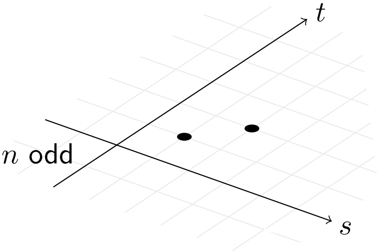
$$E_{s,t}^{\mu}(n) := \det_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \left(\binom{\mu+i+j+s+t-4}{j+t-1} - \delta_{i+s,j+t} \right).$$

Lemma: Let $A_{s,t}^{\mu}(n)$ be either $D_{s,t}^{\mu}(n)$ or $E_{s,t}^{\mu}(n)$. For real numbers $s, t \notin \{-1, -2, \ldots\}$ with $t - s \in \mathbb{N}$ and $n \in \mathbb{Z}^{+}$,

$$A_{s,t}^{\mu}(n) = \prod_{i=0}^{t-s-1} \frac{(\mu+s+i-1)_n}{(i+s+1)_n} \cdot A_{t,s}^{\mu}(n).$$

$E^{\mu}_{s,t}(n)$ Family





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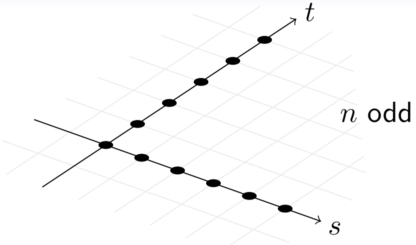

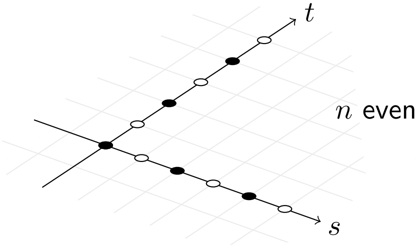

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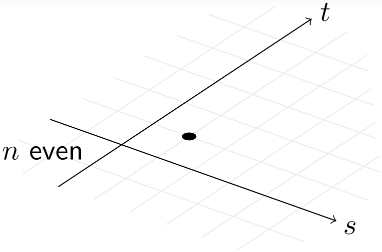

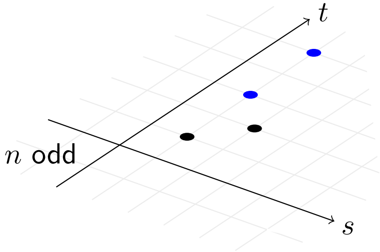

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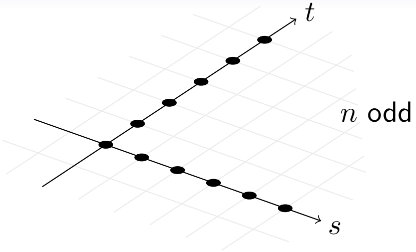

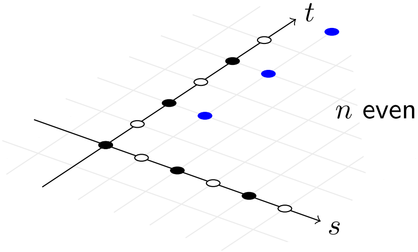

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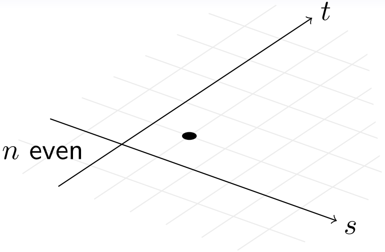

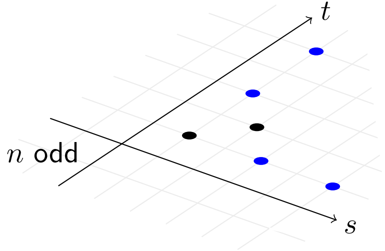

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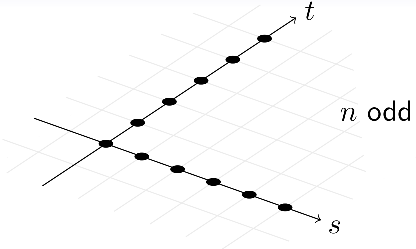
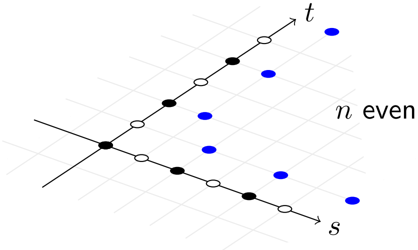
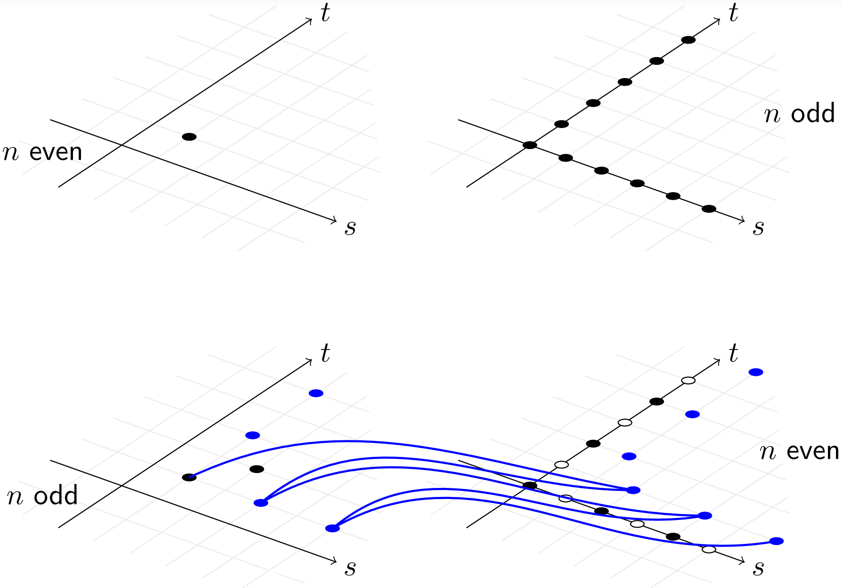




$E^{\mu}_{s,t}(n)$ Family | $D^{\mu}_{s,t}(n)$ Family



Proof of the Lascoux/Krattenthaler Conjecture

$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} =$$

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Proof of the Lascoux/Krattenthaler Conjecture

Lemma: For $m, r \in \mathbb{Z}$ and $m \geqslant r \geqslant 1$,

$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} = \frac{(m+r-1)(\mu-1)(\mu+2m+1)(\mu+2r)}{2m(2r-1)(\mu+2)(\mu+2m+2r-1)},$$

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Theorem: For $m, r \in \mathbb{Z}$ and $m \geqslant r \geqslant 1$,

$$E^{\mu}_{2r-1,1}(2m-1) = \frac{(-1)^{m-r}\,(\mu-1)\,\left(\mu+2r-1\right)_{2m-2}}{(2r-2)!\,\left(m+r-1\right)_{m-r+1}\left(\frac{\mu}{2}+r\right)_{m-r}}$$

$$\times \prod_{i=1}^{m-r} \frac{\left(\mu+2i+6r-5\right)^{2}_{i-1}\left(\frac{\mu}{2}+2i+3r-2\right)^{2}_{i}}{\left(i\right)^{2}_{i}\left(\frac{\mu}{2}+i+3r-2\right)^{2}_{i-1}}.$$

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Corollary: Apply the switching lemma to obtain $E^{\mu}_{1,2r-1}(2m-1)$.

Proof of the Lascoux/Krattenthaler Conjecture

Lemma: For $n, s \in \mathbb{Z}$ and $n \geqslant s \geqslant 1$,

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)},$$

where (A, B, s, n) is $(D, E, 2r, 2m)$ or $(E, D, 2r+1, 2m+1)$.

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Corollary: Apply the switching lemma to obtain $E^{\mu}_{1,2r-1}(2m-1)$.

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2}+1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+2}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3}+1 & \binom{\mu+5}{4} \\ \binom{\mu+3}{1} & \binom{\mu+4}{2} & \binom{\mu+5}{3} & \binom{\mu+6}{4}+1 \\ \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} & \binom{\mu+7}{4} \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+2}{1} - \binom{\mu+1}{1} & \binom{\mu+3}{2} - \binom{\mu+2}{2} - 1 & \binom{\mu+4}{3} - \binom{\mu+3}{3} + 1 & \binom{\mu+5}{4} - \binom{\mu+4}{4} \\ \binom{\mu+3}{1} - \binom{\mu+2}{1} & \binom{\mu+4}{2} - \binom{\mu+3}{2} & \binom{\mu+5}{3} - \binom{\mu+4}{3} - 1 & \binom{\mu+6}{4} - \binom{\mu+5}{4} + 1 \\ \binom{\mu+4}{1} - \binom{\mu+3}{1} & \binom{\mu+5}{2} - \binom{\mu+4}{2} & \binom{\mu+6}{3} - \binom{\mu+5}{3} & \binom{\mu+7}{4} - \binom{\mu+6}{4} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2}+1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1}-1 & \binom{\mu+3}{2}+1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} & \binom{\mu+4}{2}-1 & \binom{\mu+5}{3}+1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3}-1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + \binom{\mu+1}{1} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1} + \binom{\mu+1}{0} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} + \binom{\mu+2}{0} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} + \binom{\mu+3}{0} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2}+\binom{\mu+2}{1} & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1}+\binom{\mu+2}{0}-1 & \binom{\mu+3}{2}+1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1}+\binom{\mu+3}{0} & \binom{\mu+4}{2}-1 & \binom{\mu+5}{3}+1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1}+\binom{\mu+4}{0} & \binom{\mu+5}{2} & \binom{\mu+6}{3}-1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+3}{3}+\binom{\mu+3}{2} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1}-1 & \binom{\mu+3}{2}+\binom{\mu+3}{1} & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+4}{2}+\binom{\mu+4}{1}-1 & \binom{\mu+5}{3}+1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+5}{2}+\binom{\mu+5}{1} & \binom{\mu+6}{3}-1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+4}{4} + \binom{\mu+4}{3} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+4}{3} + \binom{\mu+4}{2} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+5}{3} + \binom{\mu+5}{2} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+6}{3} + \binom{\mu+6}{2} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+5}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+5}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+6}{3} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+7}{3} - 1 \end{pmatrix}$$

Matrix Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+5}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+5}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+6}{3} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+7}{3} - 1 \end{pmatrix}$$

$$\mathcal{L} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \mathcal{R} = \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 1 & & & \\ 1 & & \mathcal{E}^{\mu+3}_{1,1}(3) & \\ 1 & & & \end{array}\right)$$

Proof via the Holonomic Ansatz

To show:

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)}$$

Proof via the Holonomic Ansatz

To show:

$$\frac{A_{s,1}^{\mu}(n)}{B_{s-1,1}^{\mu+3}(n-1)} = \frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)}$$

Laplace expansion:

$$A_{s,1}^{\mu}(n) = \det \left(\begin{array}{c|cccc} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \cdots & & \tilde{a}_{1,n} \\ \hline 1 & & & & \\ 1 & & \mathcal{B}_{s-1,1}^{\mu+3}(n-1) & & \\ 1 & & & & \end{array} \right)$$

$$= \tilde{a}_{1,1} \cdot \mathrm{Cof}_{1,1}(n-1) + \ldots + \tilde{a}_{1,n} \cdot \mathrm{Cof}_{1,n}(n-1).$$

Proof via the Holonomic Ansatz

To show:

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$$= \tilde{a}_{1,1} \cdot \mathrm{Cof}_{1,1}(n-1) + \ldots + \tilde{a}_{1,n} \cdot \mathrm{Cof}_{1,n}(n-1).$$

With $c_{n,j} := \mathrm{Cof}_{1,j}(n-1)/\mathrm{Cof}_{1,1}(n-1)$, we obtain

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \sum_{j=1}^{n} \tilde{a}_{1,j} \cdot c_{n,j}$$

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To show:

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \underbrace{\frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)}}_{=:R^{\mu}_{s,1}(n)}$$

Laplace expansion:

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$$= \tilde{a}_{1,1} \cdot \mathrm{Cof}_{1,1}(n-1) + \ldots + \tilde{a}_{1,n} \cdot \mathrm{Cof}_{1,n}(n-1).$$

With $c_{n,j} := \mathrm{Cof}_{1,j}(n-1)/\mathrm{Cof}_{1,1}(n-1)$, we obtain

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \sum_{j=1}^{n} \tilde{a}_{1,j} \cdot c_{n,j} \stackrel{!}{=} R^{\mu}_{s,1}(n).$$

Proof via the Holonomic Ansatz

Guess: $c_{n,j}$ satisfies a holonomic system of recurrence equations.

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$$\begin{aligned}
p^{[1]}_{0,2} \cdot c_{n,j+2} + p^{[1]}_{1,0} \cdot c_{n+1,j} + p^{[1]}_{0,1} \cdot c_{n,j+1} + p^{[1]}_{0,0} \cdot c_{n,j} &= 0 \\
p^{[2]}_{1,1} \cdot c_{n+1,j+1} + p^{[2]}_{1,0} \cdot c_{n+1,j} + p^{[2]}_{0,1} \cdot c_{n,j+1} + p^{[2]}_{0,0} \cdot c_{n,j} &= 0 \\
p^{[3]}_{2,0} \cdot c_{n+2,j} + p^{[3]}_{1,0} \cdot c_{n+1,j} + p^{[3]}_{0,1} \cdot c_{n,j+1} + p^{[3]}_{0,0} \cdot c_{n,j} &= 0
\end{aligned}$$

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$$
\begin{aligned}
p^{[1]}_{0,2} \cdot c_{n,j+2} + p^{[1]}_{1,0} \cdot c_{n+1,j} + p^{[1]}_{0,1} \cdot c_{n,j+1} + p^{[1]}_{0,0} \cdot c_{n,j} = 0 \\
p^{[2]}_{1,1} \cdot c_{n+1,j+1} + p^{[2]}_{1,0} \cdot c_{n+1,j} + p^{[2]}_{0,1} \cdot c_{n,j+1} + p^{[2]}_{0,0} \cdot c_{n,j} = 0 \\
p^{[3]}_{2,0} \cdot c_{n+2,j} + p^{[3]}_{1,0} \cdot c_{n+1,j} + p^{[3]}_{0,1} \cdot c_{n,j+1} + p^{[3]}_{0,0} \cdot c_{n,j} = 0
\end{aligned}
$$

$p^{[3]}_{2,0} = -(j-2n-4)(j-2n-3)(\mu+6n+5)(\mu+6n+7)(\mu+6n+9)(n+r-1)(n+r)(j+\mu+2n+3)(j+\mu+2n+4)(2j^4+3j^3\mu-6j^3n+j^3+j^2\mu^2-12j^2\mu n-3j^2\mu+12j^2n^2-30j^2n-8j^2-4j\mu^2n-2j\mu^2+24j\mu n^2-8j\mu n-6j\mu+72jn^2+12jn-4j+8\mu^2n^2+4\mu^2n+40\mu n^2+20\mu n+48n^2+24n)(\mu+2n+2r)(\mu+2n+2r+1)(\mu+2n+2r+2)(\mu+2n+2r+3)(\mu+4n+2r+1)$

Proof via the Holonomic Ansatz

Prove: in the case where (A, B, s, n) is $(D, E, 2r, 2m)$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

Proof via the Holonomic Ansatz

Prove: in the case where (A, B, s, n) is $(D, E, 2r, 2m)$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

- Abandon the original definition $c_{n,j} := \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.

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- Abandon the original definition $c_{n,j} := \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.
- Use the conjectured holonomic description for $c_{n,j}$ instead.

Proof via the Holonomic Ansatz

Prove: in the case where (A, B, s, n) is $(D, E, 2r, 2m)$

$$c_{2m,1} = 1,$$

$$\sum_{j=1}^{2m} \binom{\mu + i + j + 2r - 3}{j - 1} \cdot c_{2m,j} - c_{2m,i+2r-2} = 0, \qquad (2 \leqslant i \leqslant 2m),$$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

- Abandon the original definition $c_{n,j} := \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.
- Use the conjectured holonomic description for $c_{n,j}$ instead.
- The first two identities prove $c_{n,j} = \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.

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$$c_{2m,1} = 1,$$

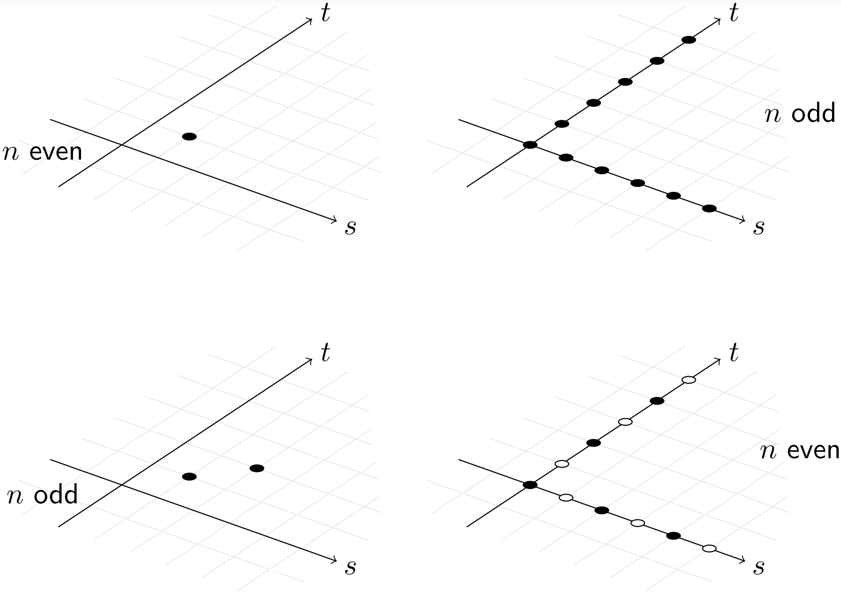
$$\sum_{j=1}^{2m} \binom{\mu + i + j + 2r - 3}{j - 1} \cdot c_{2m,j} - c_{2m,i+2r-2} = 0, \qquad (2 \leqslant i \leqslant 2m),$$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

- Abandon the original definition $c_{n,j} := \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.
- Use the conjectured holonomic description for $c_{n,j}$ instead.
- The first two identities prove $c_{n,j} = \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.
- The third identity proves the claimed quotient of determinants.

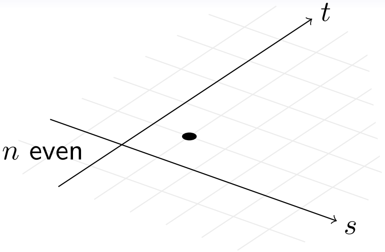
$E^{\mu}_{s,t}(n)$ Family

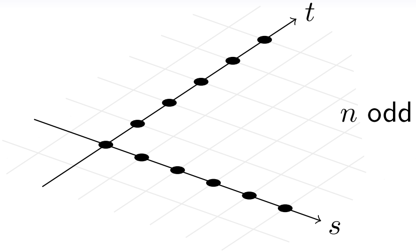
$D^{\mu}_{s,t}(n)$ Family

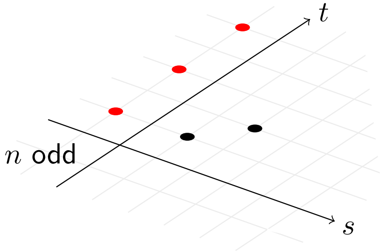


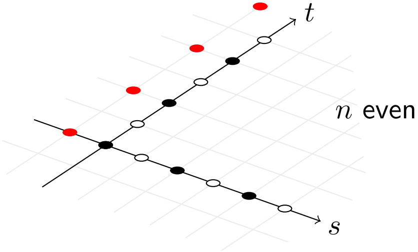
$E^{\mu}_{s,t}(n)$ Family

$D^{\mu}_{s,t}(n)$ Family



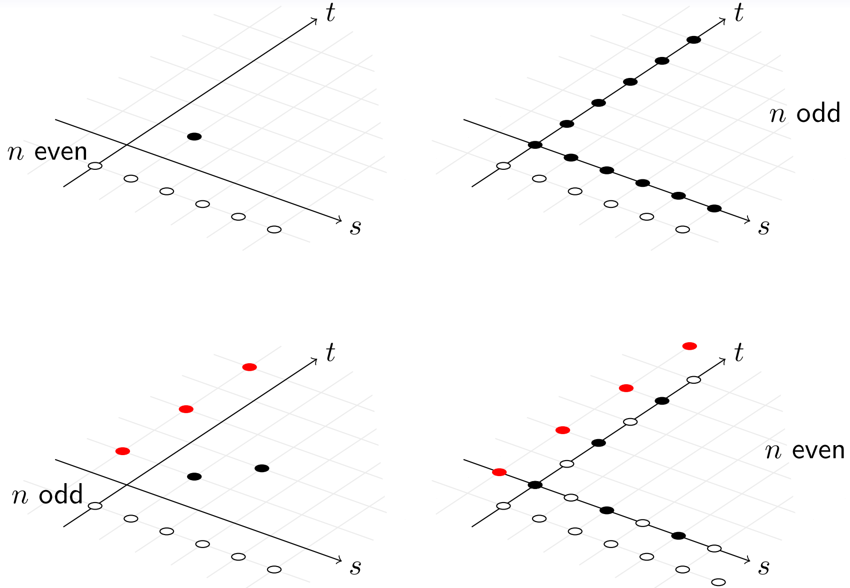




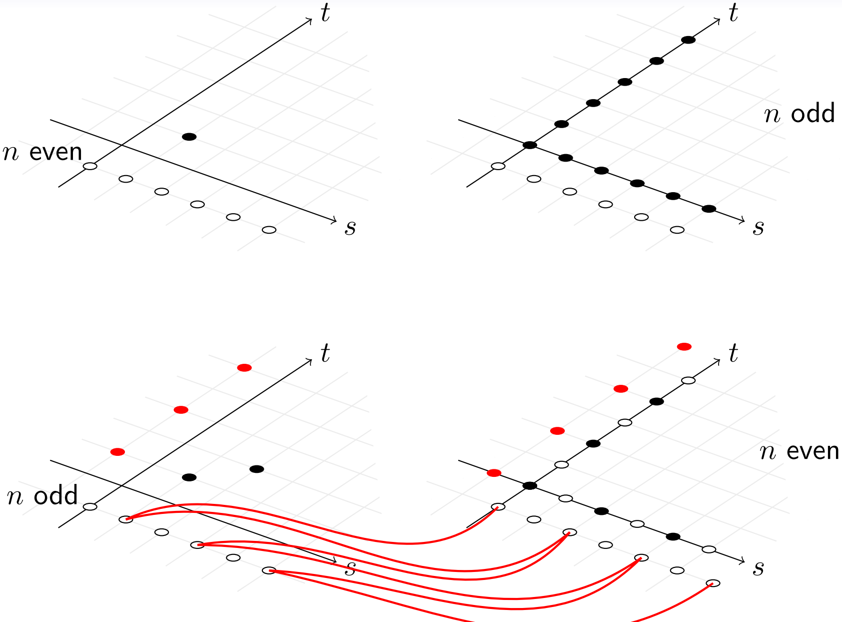


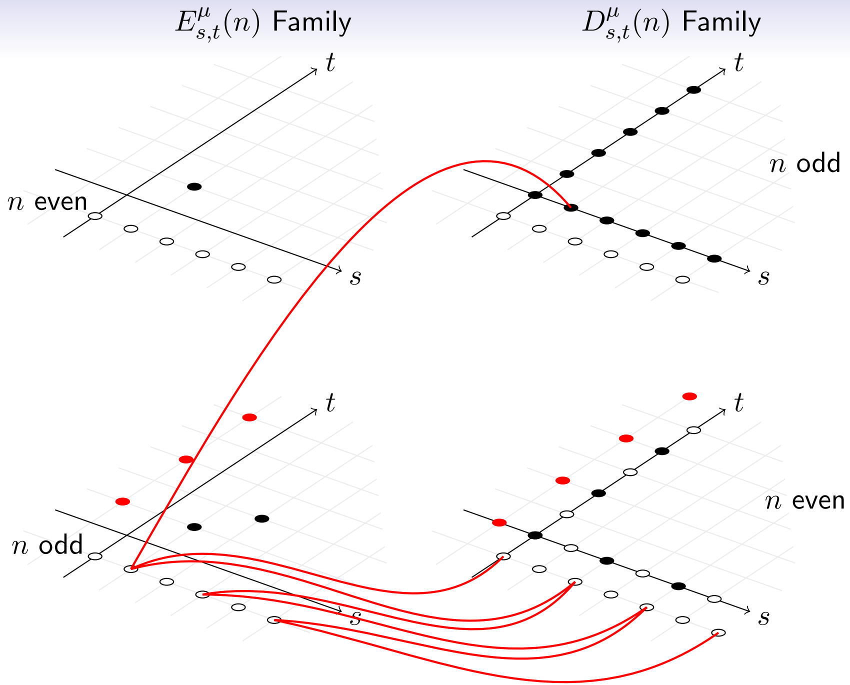
$E^{\mu}_{s,t}(n)$ Family

$D^{\mu}_{s,t}(n)$ Family



$E^{\mu}_{s,t}(n)$ Family $\quad$ $D^{\mu}_{s,t}(n)$ Family





$E^{\mu}_{s,t}(n)$ Family
$D^{\mu}_{s,t}(n)$ Family
t
s
n even
n odd
t
s
n odd
t
s
n even
t
s

Proof of a Conjecture for $s = -1$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geqslant r \geqslant 1$,

$$E^{\mu}_{-1,2r-1}(2m+1) =$$

$$\frac{(-1)^{m-r}\,(3-\mu)\,(m+r+1)_{m-r}}{2^{2m-2r+1}\left(\frac{\mu}{2}+r-\frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}}\right)$$

$$\times \left(\prod_{i=1}^{m-r} \frac{\left(\mu+2i+6r-3\right)_i^2 \left(\frac{\mu}{2}+2i+3r-1\right)_{i-1}^2}{\left(i\right)_i^2 \left(\frac{\mu}{2}+i+3r-1\right)_{i-1}^2}\right).$$

Proof of a Conjecture for $s = -1$

$$\frac{D^{\mu}_{-1,2r}(2m)}{E^{\mu+3}_{-1,2r-1}(2m-1)}$$

$$\frac{E^{\mu}_{-1,2r+1}(2m+1)}{D^{\mu+3}_{-1,2r}(2m)}$$

Theorem: For $m, r \in \mathbb{Z}$ and $m \geqslant r \geqslant 1$,

$$E^{\mu}_{-1,2r-1}(2m+1) = \frac{(-1)^{m-r}\,(3-\mu)\,(m+r+1)_{m-r}}{2^{2m-2r+1}\left(\frac{\mu}{2}+r-\frac{3}{2}\right)_{m-r+1}} \cdot \left(\prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}}\right) \times \left(\prod_{i=1}^{m-r} \frac{\left(\mu+2i+6r-3\right)_i^2 \left(\frac{\mu}{2}+2i+3r-1\right)_{i-1}^2}{(i)_i^2\left(\frac{\mu}{2}+i+3r-1\right)_{i-1}^2}\right).$$

Proof of a Conjecture for $s = -1$

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Proof of a Conjecture for $s = -1$

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Proof of a Conjecture for $s = -1$

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Proof of a Conjecture for $s = -1$

Lemma: For $m, r \in \mathbb{Z}$ and $m > r \geqslant 1$,

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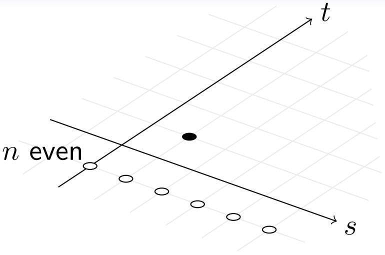
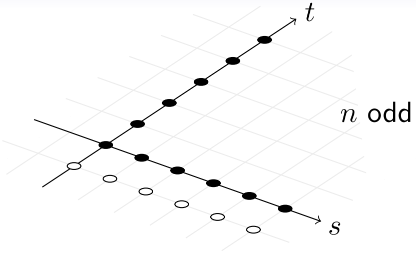
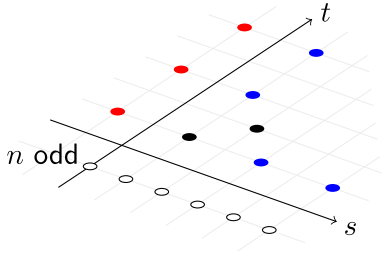
- Additional sums coming from the Kronecker deltas in combination with the row and column operations, some of which having non-natural boundaries.

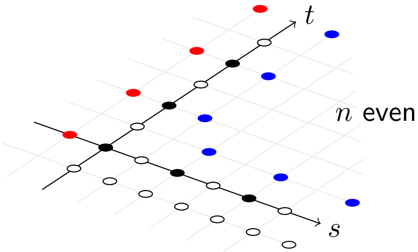
$E^{\mu}_{s,t}(n)$ Family

$D^{\mu}_{s,t}(n)$ Family









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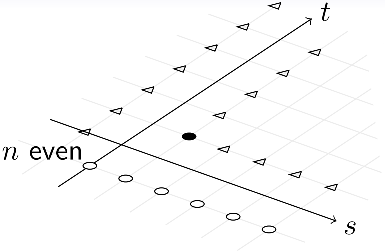

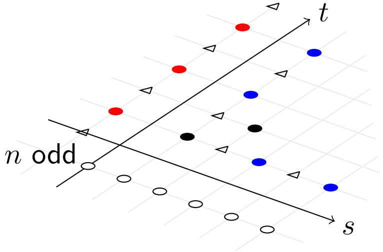

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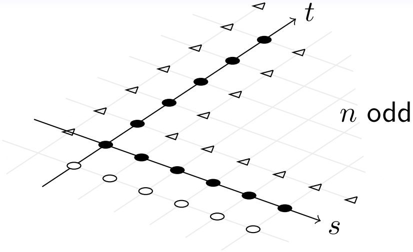

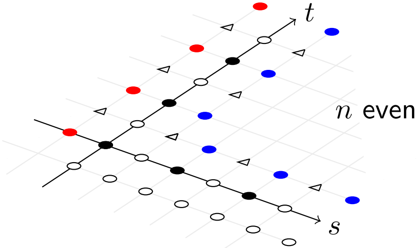

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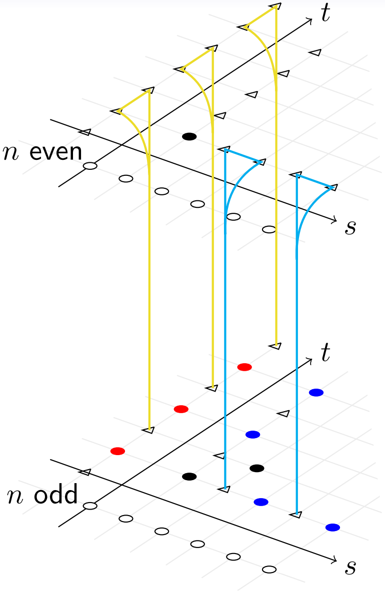

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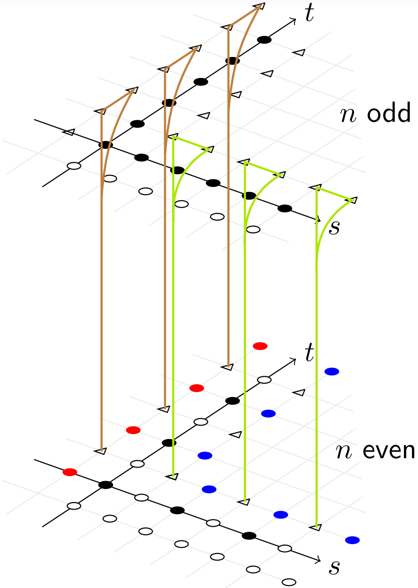

Triangle Relations

Corollary: Let μ be an indeterminate, and let $m, r \in \mathbb{Z}$.
If $m > r \geqslant 1$, then

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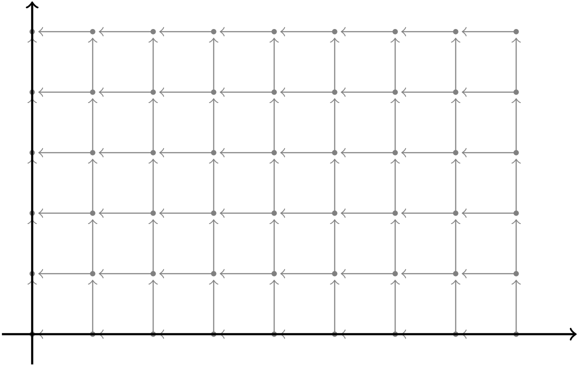
Lindström-Gessel-Viennot Lemma

It implies that the determinant without the Kronecker delta

$$\det_{1 \leqslant i,j \leqslant n} \binom{\mu + i + j + s + t - 4}{j + t - 1}$$

counts n-tuples of non-intersecting paths in the lattice $\mathbb{N}^2$:



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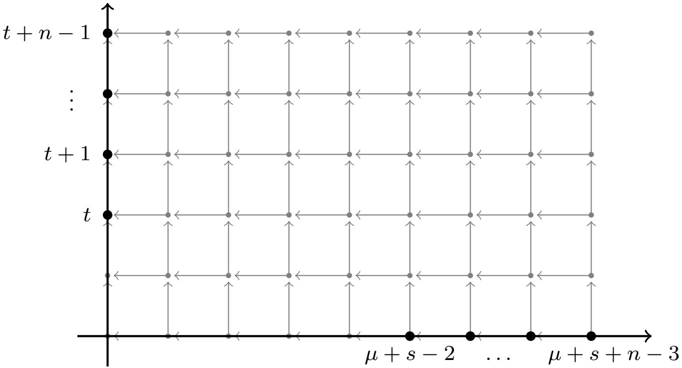

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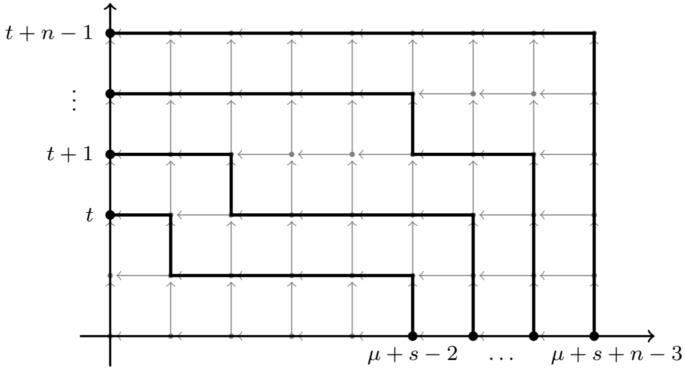
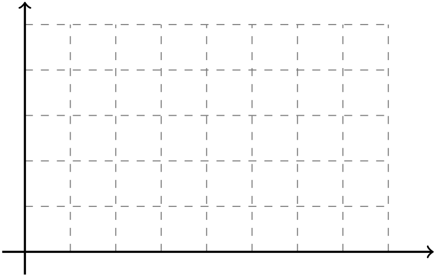
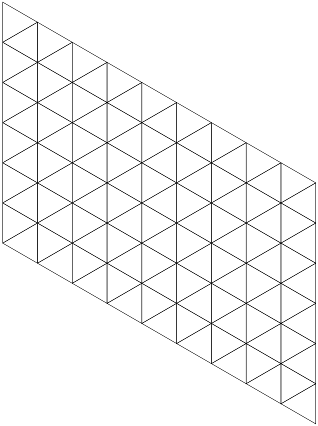


Lattice Paths ⟷ Rhombus Tilings





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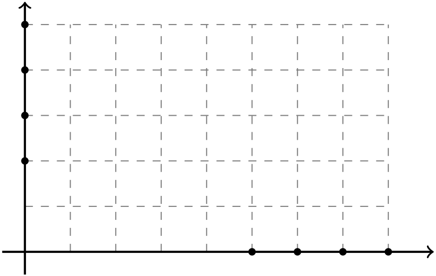

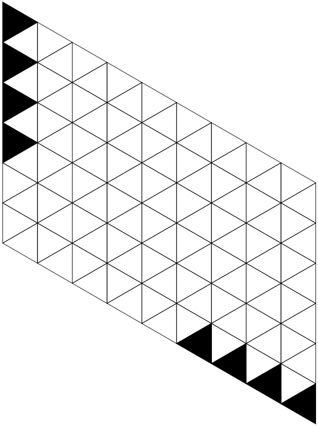

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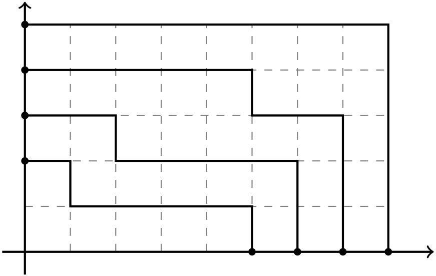
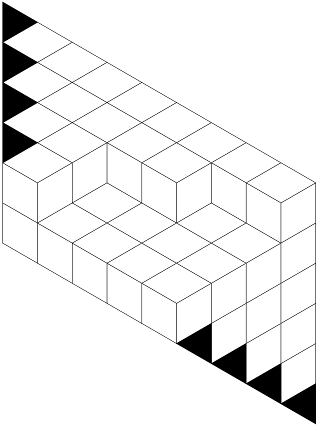
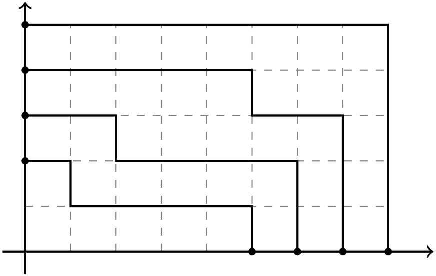
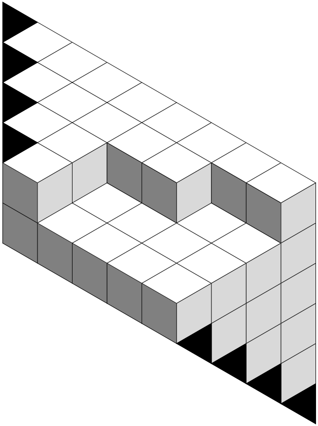




Lattice Paths $\longleftrightarrow$ Rhombus Tilings





Expansion of Kronecker Deltas

Laplace expansion:
$$\begin{vmatrix} \cdots & b_{1,6} & b_{1,7}+1 & b_{1,8} & \cdots \\ \cdots & b_{2,6} & b_{2,7} & b_{2,8}+1 & \cdots \\ & \vdots & \vdots & \vdots & \end{vmatrix} =$$

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By applying this procedure recursively, one obtains

$$D^{\mu}_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-s+t\}} (-1)^{(s-t)\cdot|I|} \det\left(B^{I}_{I+s-t}\right) \qquad (s \geqslant t),$$

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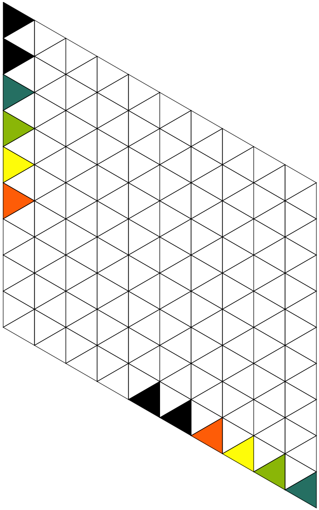
Hexagonal Fusion

$s = 1$

$t = 3$

$n = 6$

$\mu = 5$



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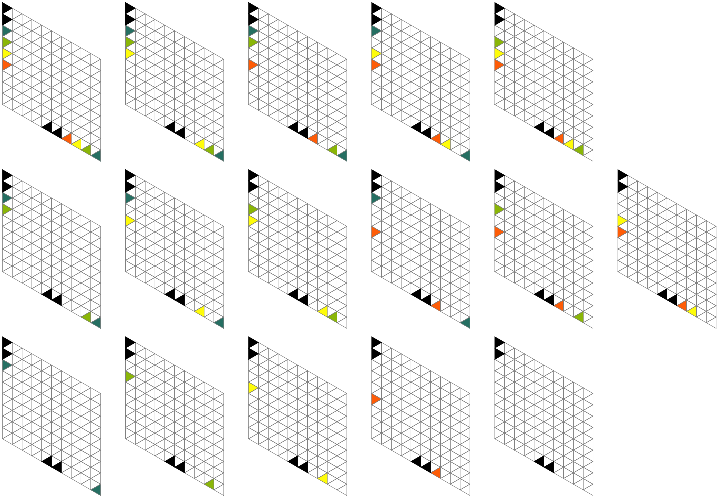

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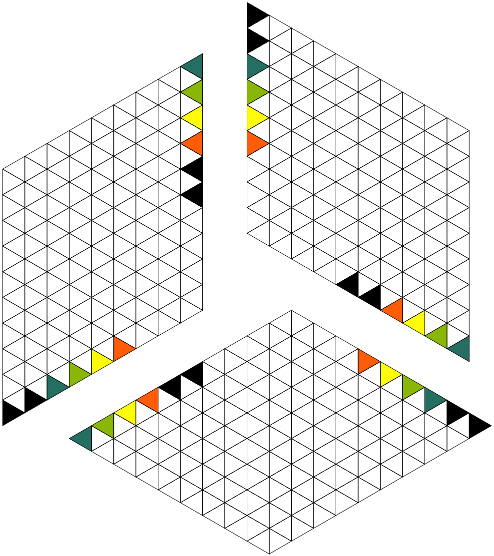

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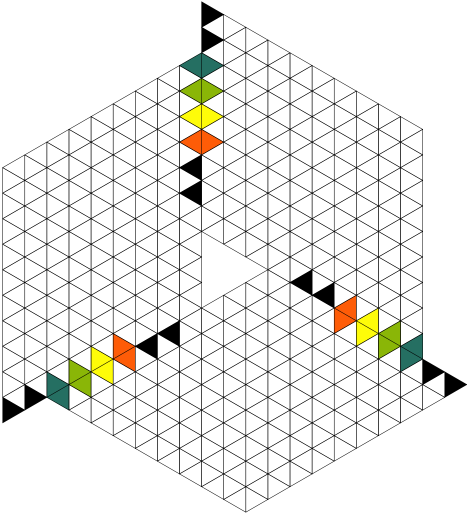

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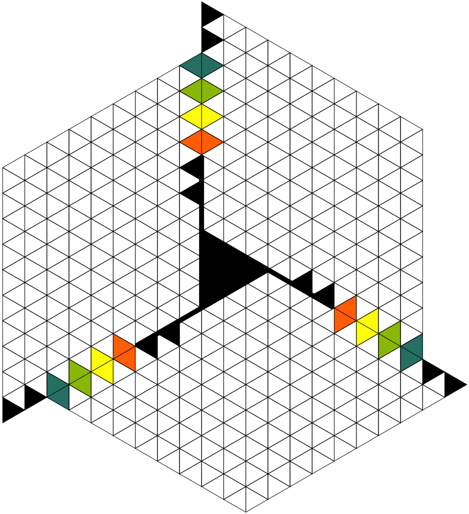

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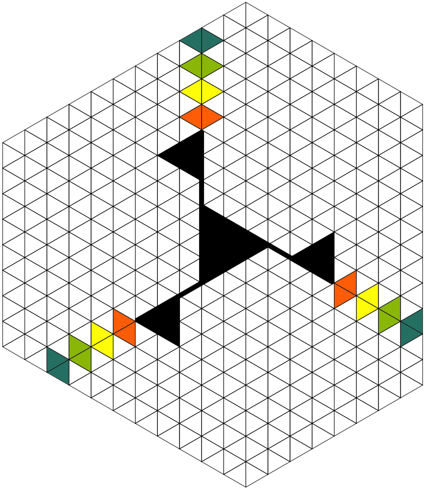

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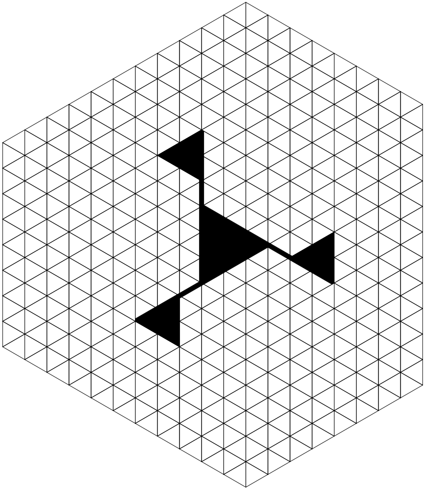

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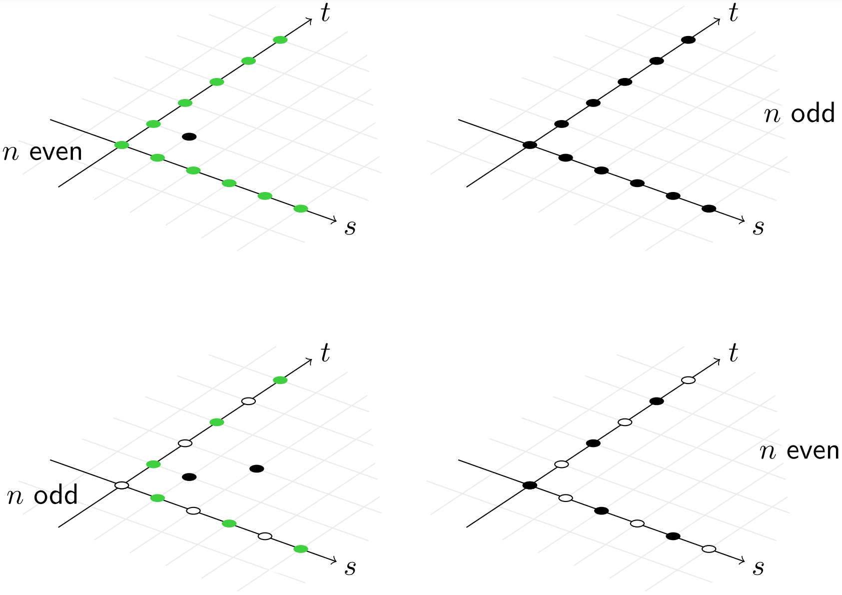

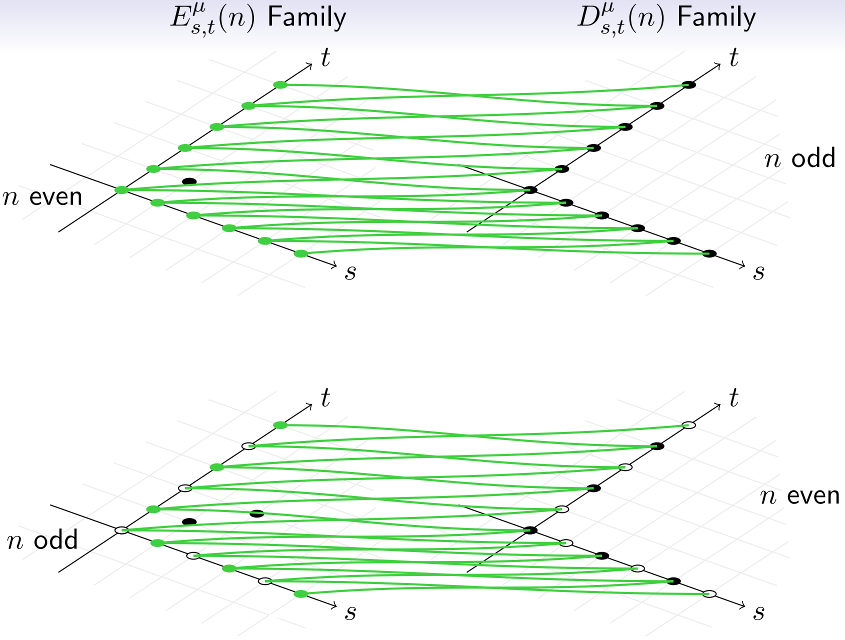
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t
t
n odd
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s
s
t
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n odd
s
s

A Combinatorial Proof

Lemma: For $n, s \in \mathbb{Z}$ such that $n \geqslant s \geqslant 1$ and $n > 1$,

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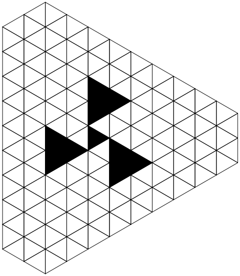
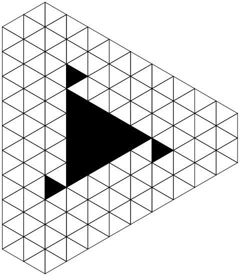
Proof (by example):

$D^{3}_{2,0}(4)$



$E^{6}_{1,0}(3)$



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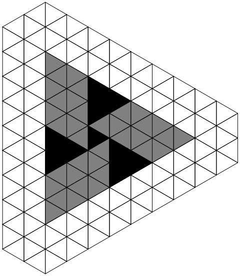
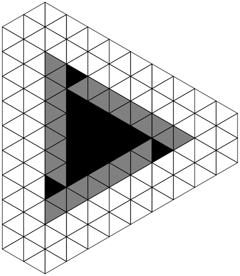
Proof (by example):

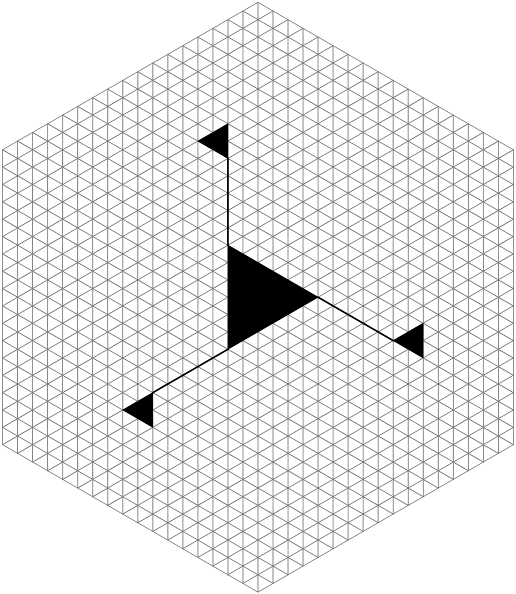
$D^{3}_{2,0}(4)$



$E^{6}_{1,0}(3)$



Happy Birthday!



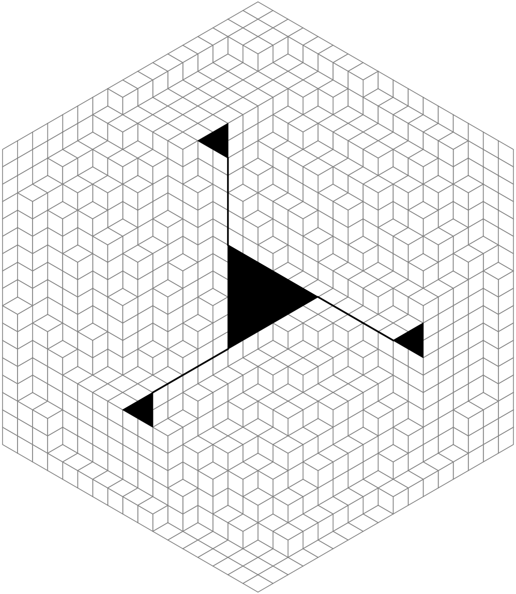
$s = 5$

$t = 7$

$n = 8$

$\mu = 8$

Happy Birthday!



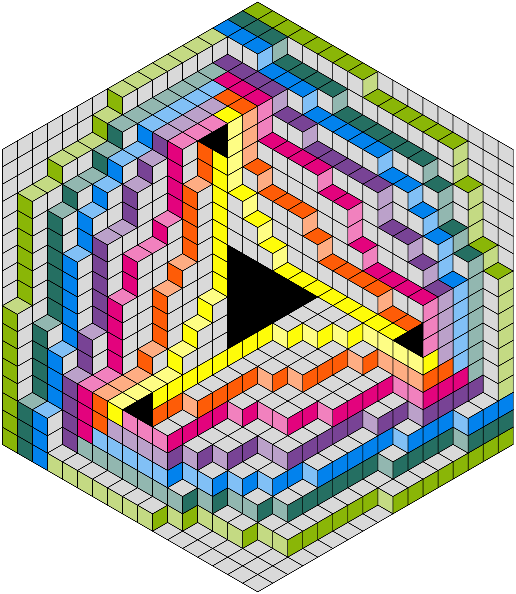
$s = 5$

$t = 7$

$n = 8$

$\mu = 8$

Happy Birthday!



$s = 5$

$t = 7$

$n = 8$

$\mu = 8$