

Symbolic evaluation of determinants and rhombus tilings of holey hexagons

Christoph Koutschan
(joint work with Thotsaporn Thanatipanonda)

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

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Beginning of the Story

Inventiones math. 53, 193–225 (1979)

*Inventiones
mathematicae*

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Plane Partitions (III): The Weak Macdonald Conjecture

George E. Andrews*

The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Dedicated to the memory of Alfred Young and F.J.W. Whipple

Determinant that counts descending plane partitions:

$$D_{0,0}(n) := \det_{1 \leq i, j \leq n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j - 1} \right),$$

where $\delta_{i,j}$ denotes the Kronecker delta function.

Andrews's Result

Theorem. We have

$$D_{0,0}(n) = 2 \prod_{i=1}^{n-1} R_{0,0}(i),$$

in other words $R_{0,0}(n) = D_{0,0}(n+1)/D_{0,0}(n)$, where

$$R_{0,0}(2n) = \frac{(\mu + 2n)_n \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{(n)_n \left(\frac{\mu}{2} + n + \frac{1}{2}\right)_{n-1}},$$

$$R_{0,0}(2n-1) = \frac{(\mu + 2n - 2)_{n-1} \left(\frac{\mu}{2} + 2n - \frac{1}{2}\right)_n}{(n)_n \left(\frac{\mu}{2} + n - \frac{1}{2}\right)_{n-1}},$$

and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n := a \cdot (a+1) \cdots (a+n-1).$$

Another question is the possibility of other general determinants of this nature. At first glance

$$E_m(\mu) = \det \left(\delta_{ij} + \binom{\mu+i+j}{i+1} \right)_{0 \leq i, j \leq m-1}$$

looks interesting. Indeed it turns out that

$$E_1(\mu) = \mu + 1,$$

$$E_2(\mu) = (\mu+2)(\mu+1),$$

$$E_3(\mu) = \frac{(\mu+14)(\mu+3)(\mu+2)(\mu+1)}{12},$$

$$E_4(\mu) = \frac{(\mu+14)(\mu+9)(\mu+4)(\mu+3)(\mu+2)(\mu+1)}{72},$$

$$E_5(\mu) = \frac{(\mu+9)(\mu+5)(\mu+4)(\mu+3)(\mu+2)(\mu+1)(\mu^3+45\mu^2+722\mu+3432)}{8640}.$$

Empirically it seems reasonable to guess that

$$\frac{E_{2m}(\mu)}{E_{2m-1}(\mu)} = f_{2m, 2m}(\mu-2),$$

George Andrews (1980):
Macdonald's conjecture and
descending plane partitions

Andrews's Conjecture (1980)

Let $D_{1,1}(n)$ denote Andrews's interesting-looking determinant:

$$D_{1,1}(n) := \det_{1 \leq i, j \leq n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right).$$

Conjecture. The following holds:

$$\begin{aligned} \frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} &= (-1)^{(n-1)(n-2)/2} 2^n \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\binom{n}{n} \left(-\frac{\mu}{2} - 2n + \frac{3}{2}\right)_{\lfloor (n-1)/2 \rfloor}} \\ &= 2^n \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\binom{n}{n} \left(\frac{\mu}{2} + \lfloor \frac{3n}{2} \rfloor + \frac{1}{2}\right)_{\lfloor (n-1)/2 \rfloor}} \\ &= \frac{(\mu + 2n)_n \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{\binom{n}{n} \left(\frac{\mu}{2} + n + \frac{1}{2}\right)_{n-1}}. \end{aligned}$$

Andrews's Conjecture (1980)

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$$D_{1,1}(n) := \det_{1 \leq i, j \leq n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right).$$

Theorem. The following holds:

$$\begin{aligned} \frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} &= (-1)^{(n-1)(n-2)/2} 2^n \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\binom{n}{n} \left(-\frac{\mu}{2} - 2n + \frac{3}{2}\right)_{\lfloor (n-1)/2 \rfloor}} \\ &= 2^n \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\binom{n}{n} \left(\frac{\mu}{2} + \lfloor \frac{3n}{2} \rfloor + \frac{1}{2}\right)_{\lfloor (n-1)/2 \rfloor}} \\ &= \frac{(\mu + 2n)_n \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{\binom{n}{n} \left(\frac{\mu}{2} + n + \frac{1}{2}\right)_{n-1}}. \end{aligned}$$

→ Proven by us in 2013.

$$D_{1,1}(1) = \mu + 1$$

$$D_{1,1}(2) = (\mu + 1)(\mu + 2)$$

$$D_{1,1}(3) = \frac{1}{12}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 14)$$

$$D_{1,1}(4) = \frac{1}{72}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 4)(\mu + 9)(\mu + 14)$$

$$D_{1,1}(5) = \frac{1}{8640}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 4)(\mu + 5)(\mu + 9) \\ \times (\mu^3 + 45\mu^2 + 722\mu + 3432)$$

$$D_{1,1}(6) = \frac{1}{518400}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 4)(\mu + 5)(\mu + 6) \\ \times (\mu + 8)(\mu + 13)(\mu + 15)(\mu^3 + 45\mu^2 + 722\mu + 3432)$$

$$D_{1,1}(7) = \frac{1}{870912000}(\mu + 1) \circ \circ \circ (\mu + 34)(\mu^3 + 47\mu^2 + 954\mu + 5928)$$

$$D_{1,1}(8) = \frac{1}{7315666080000}(\mu + 1) \circ \circ \circ (\mu + 34)(\mu^3 + 47\mu^2 + 954\mu + 5928)$$

$$D_{1,1}(9) = \frac{1}{22122558259200000}(\mu + 1)(\mu + 2) \circ \circ \circ (\mu + 21)^2 \\ \times (\mu^6 + 142\mu^5 + 8505\mu^4 + 277100\mu^3 + 5253404\mu^2 + 52937808\mu$$

$$D_{1,1}(10) = \frac{1}{334493080879104000000}(\mu + 1)(\mu + 2) \circ \circ \circ (\mu + 25)(\mu + 27) \\ \times (\mu^6 + 142\mu^5 + 8505\mu^4 + 277100\mu^3 + 5253404\mu^2 + 52937808\mu$$

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Our Conjecture

We found a beautiful formula for Andrews's determinant $D_{1,1}(n)$.

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$$E(n) = (\mu + 1)_n \left(\prod_{i=1}^{\lfloor \frac{3}{2} \lfloor \frac{1}{2}(n-1) \rfloor - 2 \rfloor} (\mu + 2i + 6)^{2 \lfloor \frac{1}{3}(i+2) \rfloor} \right) \\ \times \left(\prod_{i=1}^{\lfloor \frac{3}{2} \lfloor \frac{n}{2} \rfloor - 2 \rfloor} (\mu + 2i + 2 \lfloor \frac{3}{2} \lfloor \frac{n}{2} + 1 \rfloor \rfloor - 1)^{2 \lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor - \frac{1}{3}(i-1) \rfloor - 1} \right),$$

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$$F_m(n) = \left(\prod_{i=1}^{\lfloor \frac{1}{4}(n-1) \rfloor} (\mu + 2i + n + m)^{1-2i-m} \right) \\ \times \left(\prod_{i=1}^{\lfloor \frac{n}{4} - 1 \rfloor} (\mu - 2i + 2n - 2m + 1)^{1-2i-m} \right),$$

Our Conjecture

... further let ...

$$F(n) = \begin{cases} E(n)F_0(n), & \text{if } n \text{ is even,} \\ E(n)F_1(n) \prod_{i=1}^{\frac{1}{2}(n-5)} (\mu + 2i + 2n - 1), & \text{if } n \text{ is odd,} \end{cases}$$

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$$\begin{aligned} T(k) = & 55296k^6 + 41472(\mu - 1)k^5 + 384(30\mu^2 - 66\mu + 53)k^4 \\ & + 96(\mu - 1)(15\mu^2 - 42\mu + 61)k^3 \\ & + 4(19\mu^4 - 122\mu^3 + 419\mu^2 - 544\mu + 72)k^2 \\ & + (\mu - 1)(\mu^4 - 14\mu^3 + 101\mu^2 - 160\mu - 84)k \\ & + 2(\mu - 3)(\mu - 2)(\mu - 1)(\mu + 1), \end{aligned}$$

Our Conjecture

... and let ...

$$S_1(n) = \sum_{k=1}^{n-1} \left(2^{6k} (\mu + 8k - 1) \left(\frac{1}{2}\right)_{2k-1}^2 \left(\frac{1}{2}(\mu + 5)\right)_{2k-3} \right. \\ \times \left. \left(\frac{1}{2}(\mu + 4k + 2)\right)_{k-2} \left(\frac{1}{2}(\mu + 4k + 2)\right)_{2n-2k-2} T(k) \right) \\ \left/ \left((2k)! \left(\frac{1}{2}(\mu + 6k - 3)\right)_{3k+4} \right), \right.$$
$$S_2(n) = \sum_{k=1}^{n-1} \left(2^{6k} (\mu + 8k + 3) \left(\frac{1}{2}\right)_{2k}^2 \left(\frac{1}{2}(\mu + 5)\right)_{2k-2} \right. \\ \times \left. \left(\frac{1}{2}(\mu + 4k + 4)\right)_{k-2} \left(\frac{1}{2}(\mu + 4k + 4)\right)_{2n-2k-2} T\left(k + \frac{1}{2}\right) \right) \\ \left/ \left((2k + 1)! \left(\frac{1}{2}(\mu + 6k + 1)\right)_{3k+5} \right), \right.$$

Our Conjecture

$$P_1(n) = 2^{3n-1} \frac{\left(\frac{1}{2}(\mu + 6n - 3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu + 5)\right)_{2n-3}} \\ \times \left(\frac{\left(\frac{1}{2}(\mu + 2)\right)_{2n-2}}{(\mu + 3)^2} + \frac{\mu(\mu - 1)}{2^{13}} S_1(n) \right),$$

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$$G(n) = \begin{cases} P_1\left(\frac{1}{2}(n + 1)\right), & \text{if } n \text{ is odd,} \\ P_2\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Then for every positive integer n we have

$$D_{1,1}(n) = C(n) F(n) G\left(\left\lfloor \frac{1}{2}(n + 1) \right\rfloor\right).$$

Generalization

Definition: For $n, s, t \in \mathbb{Z}$, $n \geq 1$, and μ an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$ -determinant:

$$\begin{aligned} D_{s,t}(n) &:= \det_{\substack{s \leq i < s+n \\ t \leq j < t+n}} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right) \\ &= \det_{1 \leq i, j \leq n} \left(\delta_{i+s-1, j+t-1} + \binom{\mu + i + j + s + t - 4}{j + t - 1} \right) \end{aligned}$$

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Known special cases:

- ▶ closed form for $D_{0,0}(n)$ (Andrews 1979)
- ▶ closed form for $D_{1,1}(2n)/D_{1,1}(2n-1)$ (Andrews 1980)
- ▶ monstrous conjecture for $D_{1,1}(n)$ (K-T 2013)

Desnanot-Jacobi-Carroll Identity (DJC)

Theorem. Let $(m_{i,j})_{i,j \in \mathbb{Z}}$ be an infinite sequence and denote by $M_{s,t}(n)$ the determinant of the $(n \times n)$ -matrix whose upper left entry is $m_{s,t}$, more precisely the matrix $(m_{i,j})_{s \leq i < s+n, t \leq j < t+n}$.
Then:

$$M_{s,t}(n)M_{s+1,t+1}(n-2) = \\ M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

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Schematically:

The diagram illustrates the DJC identity using matrix blocks. On the left, a solid black square is multiplied by a smaller solid black square with a gray border. This is equal to the difference of two products. The first product consists of a solid black square with a gray border on its bottom and right sides, multiplied by a solid black square with a gray border on its top and left sides. The second product consists of a solid black square with a gray border on its top and left sides, multiplied by a solid black square with a gray border on its bottom and right sides.

DJC for $D_{1,1}(n)$

$$\blacksquare \times \square = \square \times \square - \square \times \square$$

By (DJC) we obtain a recurrence equation for $D_{1,1}(n)$:

$$D_{0,0}(n+1)D_{1,1}(n-1) = D_{0,0}(n)D_{1,1}(n) - D_{1,0}(n)D_{0,1}(n).$$

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$$\begin{bmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix} \times \begin{bmatrix} \blacksquare \\ \blacksquare \end{bmatrix} = \begin{bmatrix} \blacksquare \\ \blacksquare \end{bmatrix} \times \begin{bmatrix} \blacksquare \\ \blacksquare \end{bmatrix} - \begin{bmatrix} \blacksquare \\ \blacksquare \end{bmatrix} \times \begin{bmatrix} \blacksquare \\ \blacksquare \end{bmatrix}$$

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We rewrite it slightly:

$$D_{1,1}(n) = \underbrace{\frac{D_{0,0}(n+1)}{D_{0,0}(n)}}_{= R_{0,0}(n)} D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

→ Hence we need to know $D_{1,0}(n)$ and $D_{0,1}(n)$.

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→ Hence we need to know $D_{1,0}(n)$ and $D_{0,1}(n)$.

Question: What is the combinatorial interpretation of the general determinant $D_{s,t}(n)$?

Lindström-Gessel-Viennot Lemma

Let G be a directed acyclic graph and consider base vertices $A = \{a_1, \dots, a_n\}$ and destination vertices $B = \{b_1, \dots, b_n\}$.

Lindström-Gessel-Viennot Lemma

Let G be a directed acyclic graph and consider base vertices $A = \{a_1, \dots, a_n\}$ and destination vertices $B = \{b_1, \dots, b_n\}$. For each path P , let $\omega(P)$ be the product of its edge weights. Let

$$e(a, b) = \sum_{P:a \rightarrow b} \omega(P) \quad \text{and}$$
$$M = \begin{pmatrix} e(a_1, b_1) & e(a_1, b_2) & \cdots & e(a_1, b_n) \\ e(a_2, b_1) & e(a_2, b_2) & \cdots & e(a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n, b_1) & e(a_n, b_2) & \cdots & e(a_n, b_n) \end{pmatrix}.$$

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Then the determinant of M is the signed sum over all n -tuples $P = (P_1, \dots, P_n)$ of non-intersecting paths from A to B :

$$\det(M) = \sum_{(P_1, \dots, P_n): A \rightarrow B} \text{sign}(\sigma(P)) \prod_{i=1}^n \omega(P_i).$$

where σ denotes a permutation that is applied to B .

Lindström-Gessel-Viennot Lemma

Application: In our context, the lemma implies the following.

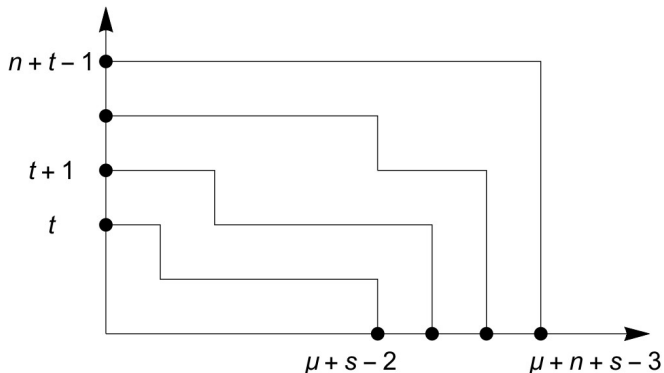
Look at the determinant without the Kronecker-Delta:

$$\det_{1 \leq i, j \leq n} \begin{pmatrix} \mu + i + j + s + t - 4 & \\ & j + t - 1 \end{pmatrix}.$$

It counts n -tuples of non-intersecting paths in the lattice \mathbb{N}^2 :

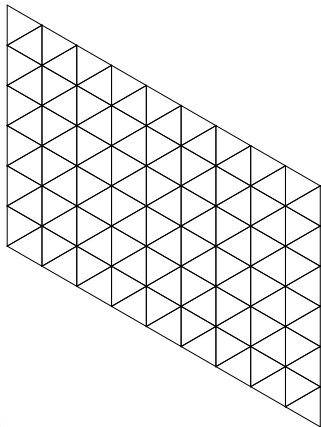
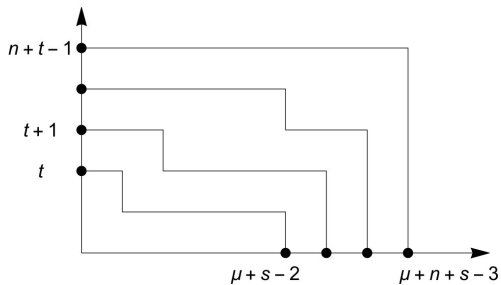
- ▶ The starting points are $(0, t), (0, t + 1), \dots, (0, t + n - 1)$.
- ▶ The end points are $(\mu + s - 2, 0), \dots, (\mu + s + n - 3, 0)$.
- ▶ The allowed steps are $(1, 0)$ and $(0, -1)$.

Non-intersecting Lattice Paths

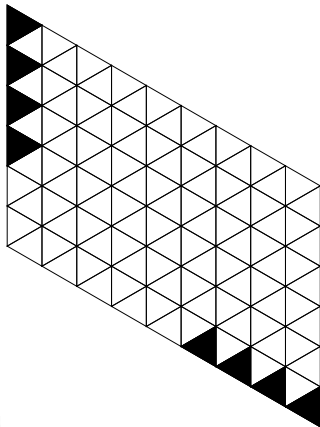
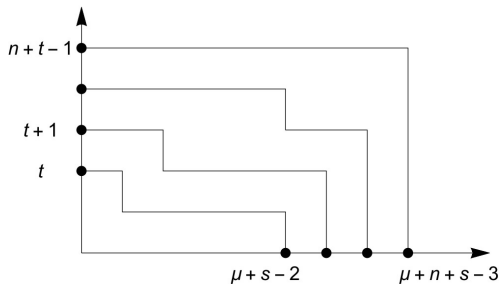


For $1 \leq i, j \leq n$ the number of paths from $(0, t + j - 1)$ to $(\mu + s + i - 3, 0)$ is given by $\binom{\mu+i+j+s+t-4}{j+t-1}$, which is precisely the (i, j) -entry of our matrix.

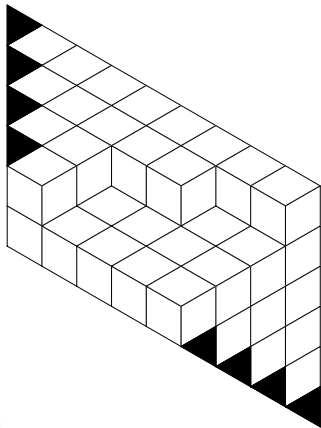
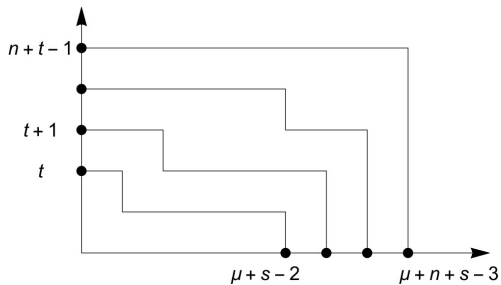
Lattice Paths — Rhombus Tilings



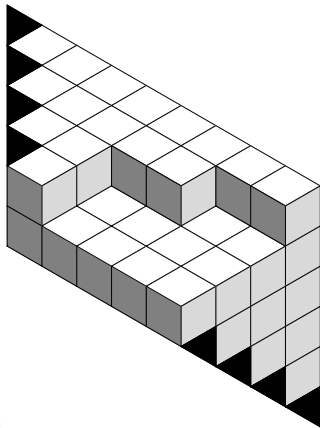
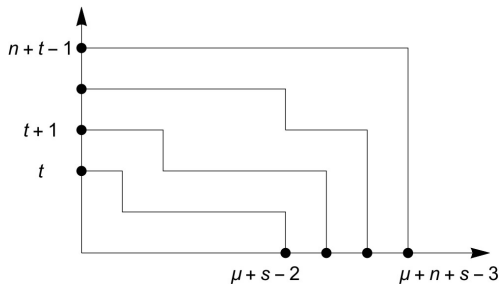
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Determinant with Kronecker-Delta

From the Laplace expansion one immediately sees that

$$\begin{vmatrix} \cdots & b_{1,j} + 1 & b_{1,j+1} & \cdots \\ \cdots & b_{2,j} & b_{2,j+1} + 1 & \cdots \\ & \vdots & \vdots & \end{vmatrix} = \begin{vmatrix} \cdots & b_{1,j} & b_{1,j+1} & \cdots \\ \cdots & b_{2,j} & b_{2,j+1} + 1 & \cdots \\ & \vdots & \vdots & \end{vmatrix} \pm \begin{vmatrix} \cdots & b_{2,j-1} & b_{2,j+1} + 1 & \cdots \\ & \vdots & \vdots & \end{vmatrix}$$

By applying this procedure recursively, one obtains

$$D_{s,t}(n) = \sum_{I \subseteq \{1, \dots, n-s+t\}} (-1)^{(s-t) \cdot |I|} \det(M_{I+s-t}^I) \quad (s \geq t),$$

where M_J^I denotes the matrix that is obtained by deleting all rows with indices in I and all columns with indices in J from the matrix

$$\left(\binom{\mu + i + j + s + t - 4}{j + t - 1} \right)_{1 \leq i, j \leq n}.$$

Kronecker-Deltas on the Main Diagonal

General formula:

$$D_{s,t}(n) = \sum_{I \subseteq \{1, \dots, n-s+t\}} (-1)^{(s-t) \cdot |I|} \det(M_{I+s-t}^I) \quad (s \geq t)$$

Special case: If $s = t$ we obtain

$$D_{s,s}(n) = \sum_{I \subseteq \{1, \dots, n\}} \det(M_I^I),$$

i.e., $D_{s,s}(n)$ is the sum of principal minors of the binomial matrix.

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Hence: $D_{s,s}(n)$ counts all k -tuples of non-intersecting lattice paths, where k runs from 0 to n , and where the start and end points are given by the same k -subset.

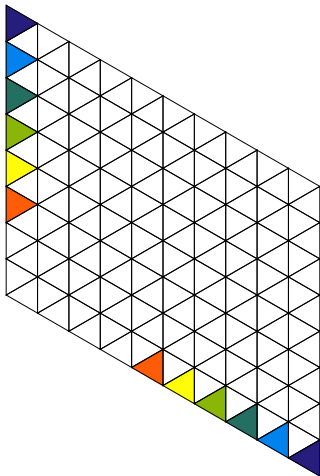
Kronecker-Deltas on the Main Diagonal

$$s = 2$$

$$t = 2$$

$$n = 6$$

$$\mu = 4$$



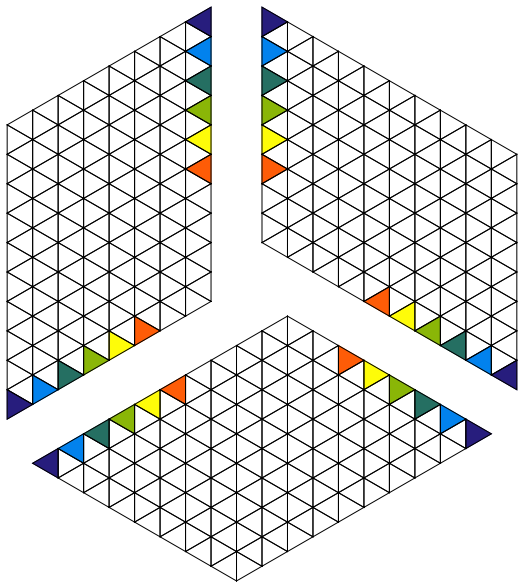
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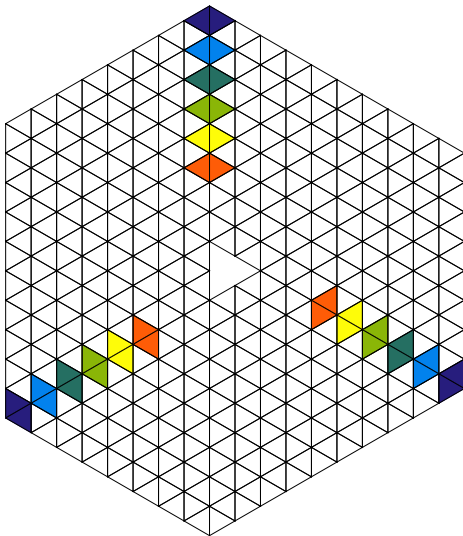
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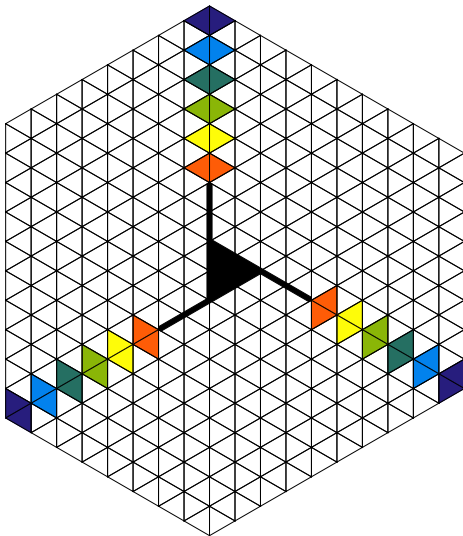
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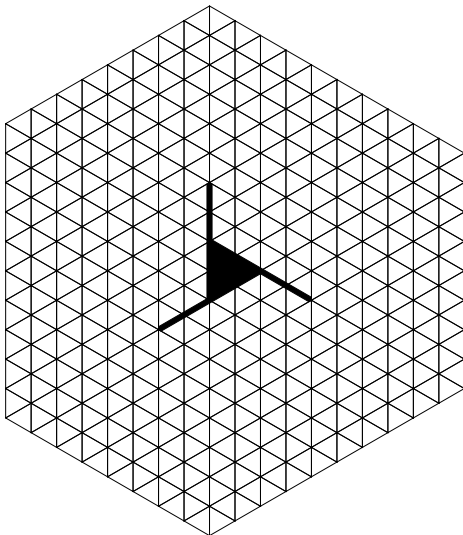
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Rhombus Tilings

Finding: The determinant $D_{s,s}(n)$ counts

- ▶ rhombus tilings
- ▶ of a hexagon with a funny-shaped hole (“holey hexagon”)
- ▶ that are cyclically symmetric.
- ▶ The hole has the shape of a triangle (of size $\mu - 2$) with “boundary lines” (of length s) sticking out of its corners.

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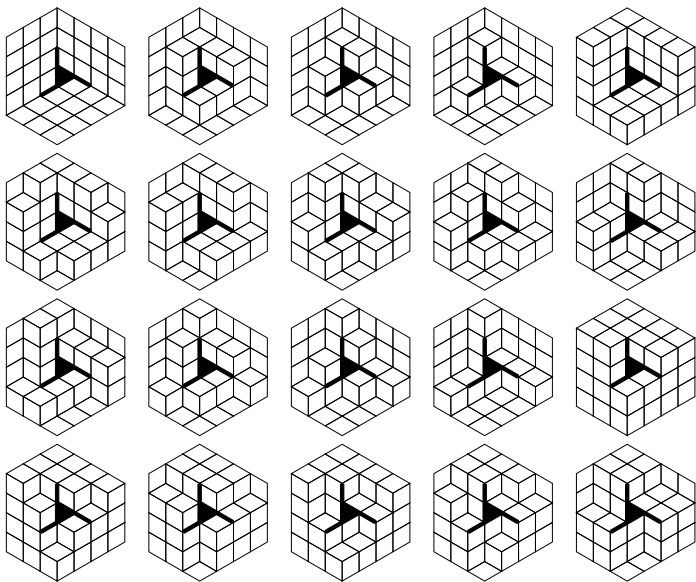
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Example: For $s = t = 1$, $n = 2$, and $\mu = 3$ we obtain

$$D_{1,1}(2)|_{\mu \rightarrow 3} = \begin{vmatrix} 4 & 6 \\ 4 & 11 \end{vmatrix} = 20.$$

Cyclically Symmetric Rhombus Tilings of a Holey Hexagon



Off-Diagonal Kronecker-Deltas

Now let's look at the situation $s \neq t$.

General formula:

$$D_{s,t}(n) = \sum_{I \subseteq \{1, \dots, n+s-t\}} (-1)^{(s-t) \cdot |I|} \det(M_I^{I+t-s}) \quad (t \geq s)$$

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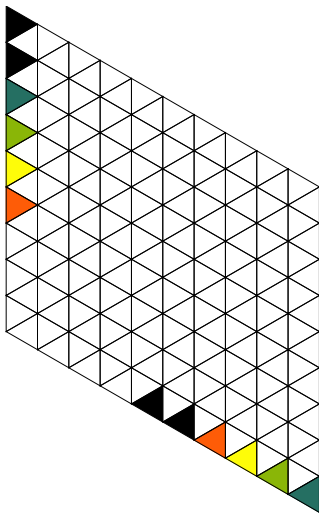
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Remark: If $s - t$ is odd, we perform a weighted count with weights $+1$ and -1 , according to the length of the tuples of paths.

Off-Diagonal Kronecker-Deltas

$$\begin{aligned} s &= 1 \\ t &= 3 \\ n &= 6 \\ \mu &= 5 \end{aligned}$$



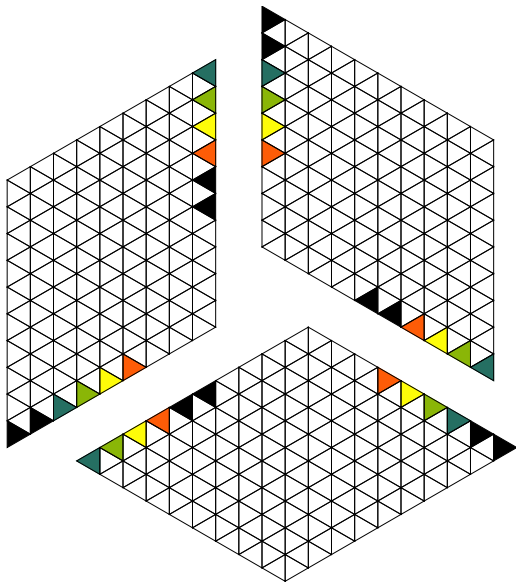
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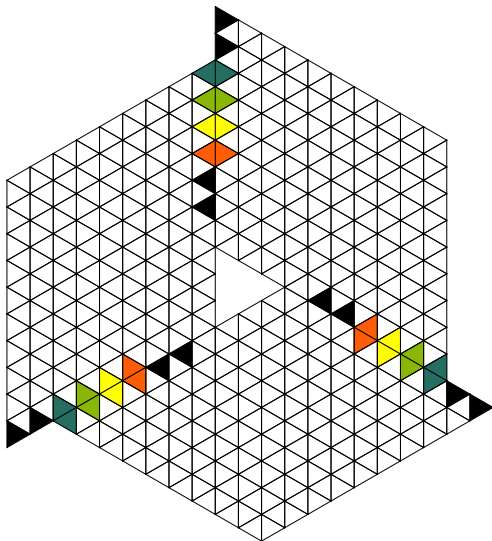
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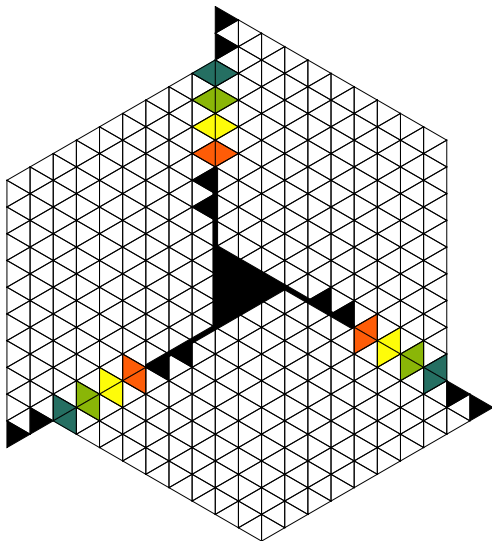
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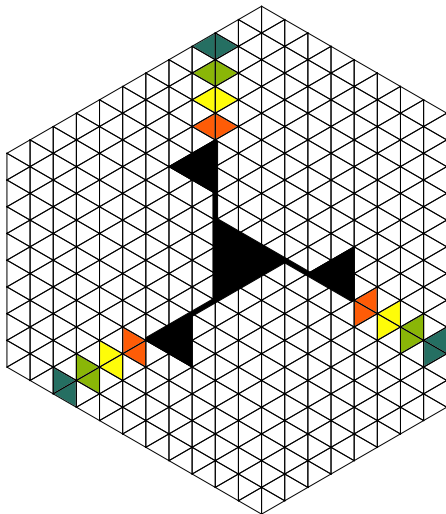
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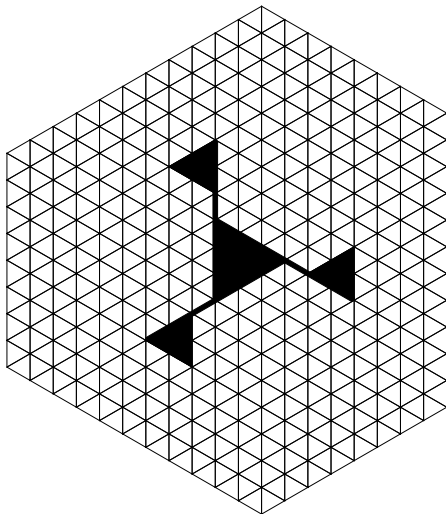
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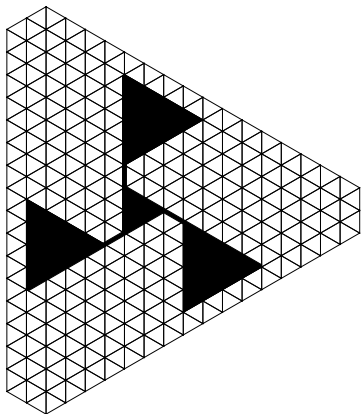
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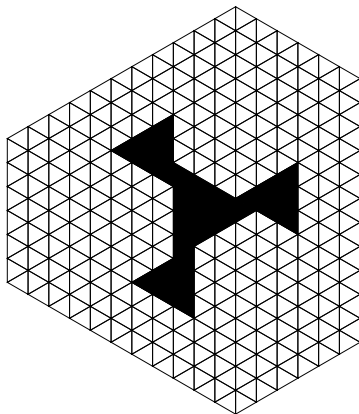


Off-Diagonal Kronecker-Deltas

Example: Shapes for different choices of the parameters



$$s = 5, t = 1, n = 5, \mu = 4$$



$$s = -1, t = 2, n = 6, \mu = 6$$

Back to $D_{1,1}(n)$

Recall: We wanted to evaluate $D_{1,1}(n)$ using (DJC):

$$D_{1,1}(n) = \underbrace{\frac{D_{0,0}(n+1)}{D_{0,0}(n)}}_{= R_{0,0}(n)} D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

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- ▶ Normalize each generator c_n (last component = 1).
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- ▶ Use the holonomic systems approach (Zeilberger) to prove that $D_{0,1}(2n) \cdot c_n = 0$ for all n .

The HOLONOMIC ANSATZ II.
Automatic DISCOVERY(!) and PROOF (!!)
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Expansion Formula

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

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The induction step is completed by proving that this is equal to b_n .

Recipe for the Holonomic Ansatz

Problem: Given $a_{i,j}$ and $b_n \neq 0$. Show that $\det (a_{i,j})_{1 \leq i,j \leq n} = b_n$.

Method: “Pull out of the hat” a function $c_{n,j}$ and prove

$$c_{n,n} = 1 \quad (n \geq 1),$$

$$\sum_{j=1}^n c_{n,j} a_{i,j} = 0 \quad (1 \leq i < n),$$

$$\sum_{j=1}^n c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \quad (n \geq 1).$$

Then $\det (a_{i,j})_{1 \leq i,j \leq n} = b_n$ holds.

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Example: For $D_{0,0}(2n)$ we obtain the following holonomic system of recurrence relations for $c_{n,j}$.

$$\begin{aligned}
& \{ (j + \mu + 2n - 3)(2\mu j^6 + 8nj^6 - 2j^6 + 3\mu^2 j^5 - 48n^2 j^5 - 12\mu j^5 - 24nj^5 + 9j^5 + \\
& \mu^3 j^4 + 48n^3 j^4 - 11\mu^2 j^4 - 84\mu n^2 j^4 + 204n^2 j^4 + 21\mu j^4 - 20\mu^2 n j^4 + 38\mu n j^4 - \\
& 10n j^4 - 11j^4 + 216n^4 j^3 - 2\mu^3 j^3 + 312\mu n^3 j^3 - 408n^3 j^3 + 7\mu^2 j^3 + 28\mu^2 n^2 j^3 + \\
& 122\mu n^2 j^3 - 198n^2 j^3 - 2\mu j^3 - 9\mu^3 n j^3 + 68\mu^2 n j^3 - 113\mu n j^3 + 78n j^3 - 3j^3 - \\
& 864n^5 j^2 - 756\mu n^4 j^2 + 432n^4 j^2 - \mu^3 j^2 - 112\mu^2 n^3 j^2 - 308\mu n^3 j^2 + 600n^3 j^2 + \\
& 11\mu^2 j^2 - 3\mu^3 n^2 j^2 - 66\mu^2 n^2 j^2 + 189\mu n^2 j^2 - 168n^2 j^2 - 23\mu j^2 - 2\mu^4 n j^2 + \\
& 15\mu^3 n j^2 - 28\mu^2 n j^2 + 33\mu n j^2 - 34n j^2 + 13j^2 + 864n^6 j + 432\mu n^5 j + 432n^5 j - \\
& 144\mu^2 n^4 j + 1116\mu n^4 j - 1104n^4 j + 2\mu^3 j - 88\mu^3 n^3 j + 384\mu^2 n^3 j - 392\mu n^3 j - \\
& 36n^3 j - 10\mu^2 j - 14\mu^4 n^2 j + 45\mu^3 n^2 j + 40\mu^2 n^2 j - 317\mu n^2 j + 270n^2 j + 14\mu j - \\
& \mu^5 n j + 3\mu^4 n j + 17\mu^3 n j - 89\mu^2 n j + 112\mu n j - 42n j - 6j + 432\mu n^6 - 864n^6 + \\
& 432\mu^2 n^5 - 1080\mu n^5 + 432n^5 + 144\mu^3 n^4 - 324\mu^2 n^4 - 156\mu n^4 + 456n^4 + 20\mu^4 n^3 - \\
& 18\mu^3 n^3 - 220\mu^2 n^3 + 470\mu n^3 - 204n^3 + \mu^5 n^2 + 3\mu^4 n^2 - 37\mu^3 n^2 + 57\mu^2 n^2 + \\
& 36\mu n^2 - 60n^2 + 2\mu^4 n - 18\mu^3 n + 54\mu^2 n - 62\mu n + 24n) c_{n,j} - (j + \mu - 3)(2j + \mu - \\
& 3)(j - 2n + 1)(\mu + 4n - 1)(j^4 + 2\mu j^3 - 6j^3 + \mu^2 j^2 - 12n^2 j^2 - 9\mu j^2 - 6\mu n j^2 + \\
& 6n j^2 + 13j^2 - 3\mu^2 j - 12\mu n^2 j + 36n^2 j + 13\mu j - 6\mu^2 n j + 24\mu n j - 18n j - 12j + \\
& 2\mu^2 - 2\mu^2 n^2 + 20\mu n^2 - 24n^2 - 6\mu - \mu^3 n + 11\mu^2 n - 22\mu n + 12n + 4) c_{n,j+1} + \\
& 2(2j + \mu - 2)n(2n + 1)(-j + 2n + 1)(-j + 2n + 2)(j + \mu + 2n - 1)(\mu + 4n - 3)(\mu + \\
& 4n - 1) c_{n+1,j} - (j + 1)(2j + \mu)(j - 2n)(j + \mu + 2n - 3) c_{n,j} + (4j^4 + 8\mu j^3 - 8j^3 + \\
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\end{aligned}$$

Back to $D_{1,1}(n)$

Using Zeilberger's method, we obtain product formulas for the missing determinants.

$$D_{1,1}(n) = R_{0,0}(n)D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

Since $D_{0,1}(n) = D_{1,0}(n) = 0$ for even n , the recurrence simplifies:

$$D_{1,1}(n) = R_{0,0}(n)D_{1,1}(n-1) \quad (n \text{ even}).$$

For odd n we obtain $D_{1,1}(n) =$

$$\begin{aligned} &= R_{0,0}(n)D_{1,1}(n-1) + (\mu - 1) \frac{\left(\prod_{j=1}^{\frac{n-1}{2}} R_{1,0}(j)\right)\left(\prod_{j=1}^{\frac{n-1}{2}} R_{0,1}(j)\right)}{2 \prod_{j=1}^{n-1} R_{0,0}(j)} \\ &= R_{0,0}(n)D_{1,1}(n-1) + \frac{(\mu - 1)}{2} \prod_{j=1}^{(n-1)/2} \frac{R_{1,0}(j)R_{0,1}(j)}{R_{0,0}(2j-1)R_{0,0}(2j)}. \end{aligned}$$

Main Result

Theorem. Let μ be an indeterminate and let ρ_k be defined as $\rho_0(a, b) = a$ and $\rho_k(a, b) = b$ for $k > 0$. If n is an odd positive integer then

$$\begin{aligned}
 D_{1,1}(n) = & \sum_{k=0}^{(n+1)/2} \rho_k \left(4(\mu - 2), \frac{1}{(2k-1)!} \right) \frac{(\mu - 1)_{3k-2}}{2 \left(\frac{\mu}{2} + k - \frac{1}{2} \right)_{k-1}} \\
 & \times \left(\prod_{j=1}^{k-1} \frac{(\mu + 2j + 1)_{j-1} \left(\frac{\mu}{2} + 2j + \frac{1}{2} \right)_{j-1}}{(j)_{j-1} \left(\frac{\mu}{2} + j + \frac{1}{2} \right)_{j-1}} \right)^2 \\
 & \times \left(\prod_{j=k}^{(n-1)/2} \frac{(\mu + 2j)_j^2 \left(\frac{\mu}{2} + 2j - \frac{1}{2} \right)_j \left(\frac{\mu}{2} + 2j + \frac{3}{2} \right)_{j+1}}{(j)_j (j+1)_{j+1} \left(\frac{\mu}{2} + j + \frac{1}{2} \right)_j^2} \right)
 \end{aligned}$$

If n is an even positive integer then... [similar formula]

More Results

We can give closed-form evaluations of some infinite one-dimensional families of $D_{s,t}(n)$.

$t \setminus s$	\dots	-3	-2	-1	0	1	2	3	4	5	6	\dots
\vdots				\vdots	\vdots	\vdots						
6				D	A	C						
5				F	B	E						
4				D	A	C						
3				F	B	E						
2				D	A	C						
1				F	B	E	C	E	C	E	C	\dots
0				D	A	B	A	B	A	B	A	\dots
-1				A'	0	0	0	0	0	0	0	\dots
-2			A'	0	0	0	0	0	0	0	0	\dots
-3		A'	0	0	0	0	0	0	0	0	0	\dots
\vdots	\ddots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Example of an Infinite Family (A)

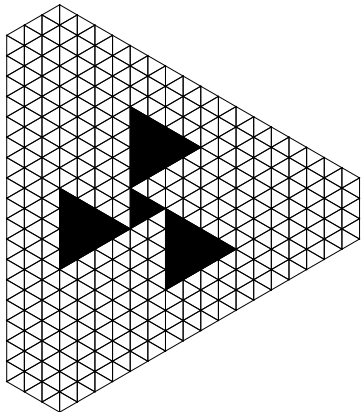
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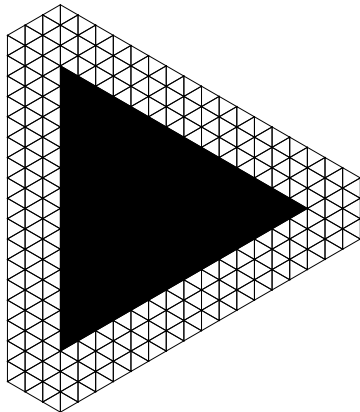
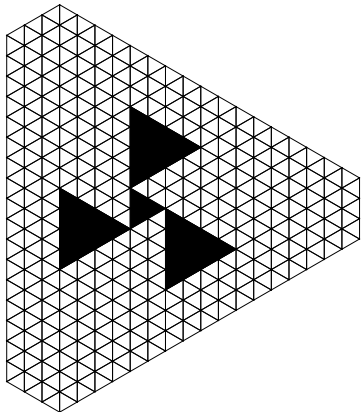
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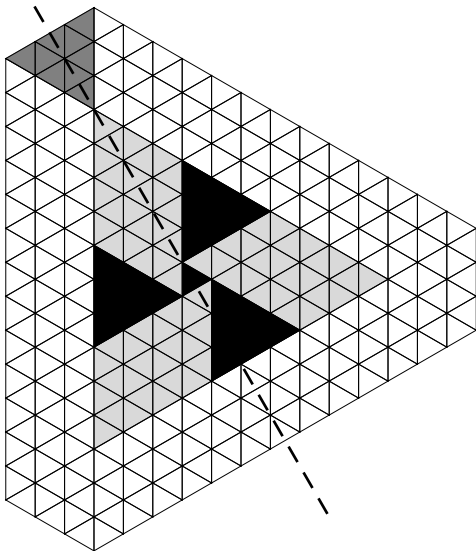


Example of an Infinite Family (B)

Family B: If $n \geq 2r$ is an even number, then $D_{2r-1,0}(n) = 0$.

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Reference

Christoph Koutschan and Thotsaporn Thanatipanonda:
A curious family of binomial determinants that count rhombus
tilings of a holey hexagon

- ▶ Technical report no. 2017-30 in the RICAM Reports Series
- ▶ arxiv:1709.02616
- ▶ <http://www.koutschan.de/data/det2/>