Two applications of automated holonomic function manipulation: Simulation of electromagnetic waves and Counting lattice paths

Christoph Koutschan

Research Institute for Symbolic Computation
Johannes Kepler Universität Linz, Austria

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Introduction

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- Zeilberger: "Holonomic systems approach to special function identities" (1990)

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Future work:

- Finish the implementation...

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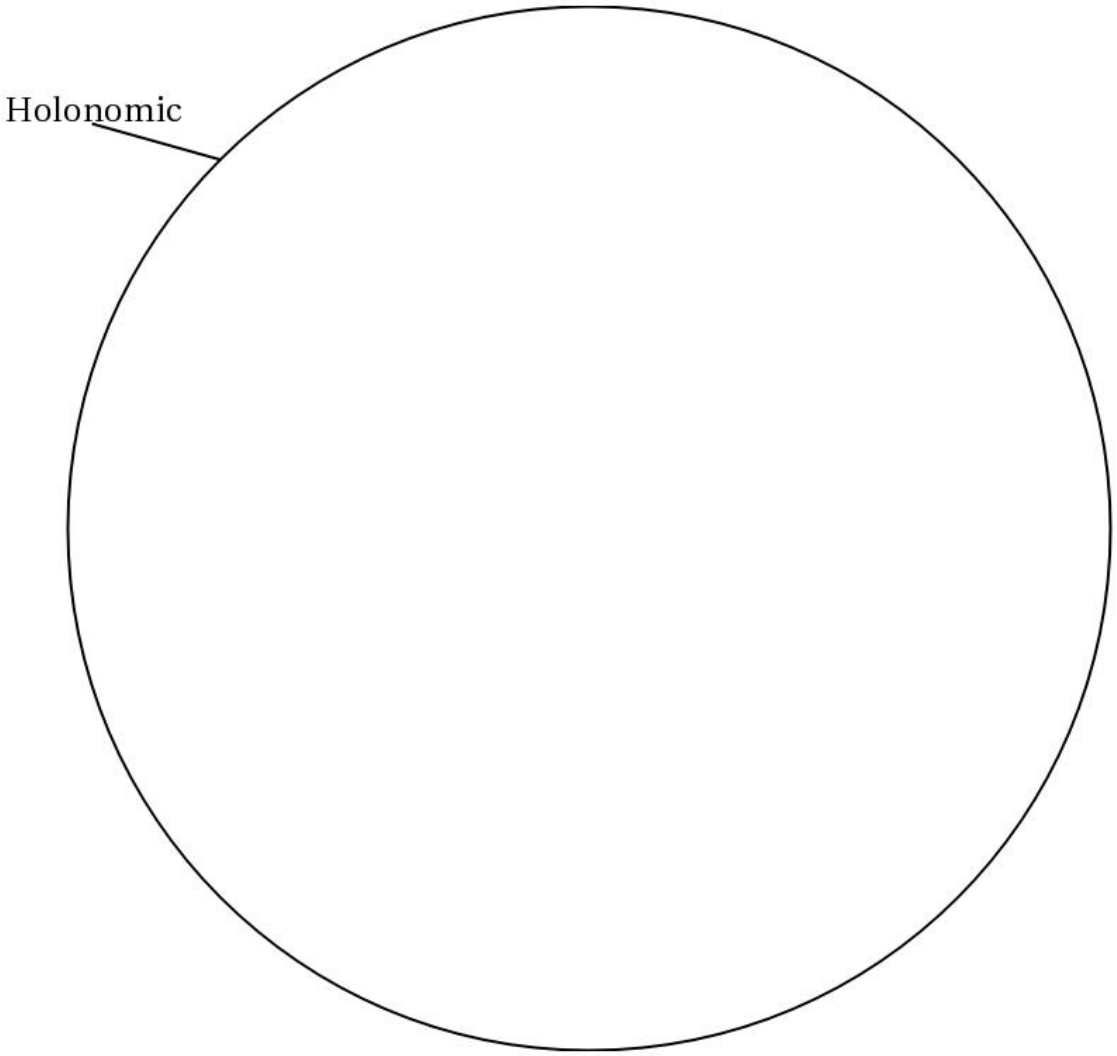

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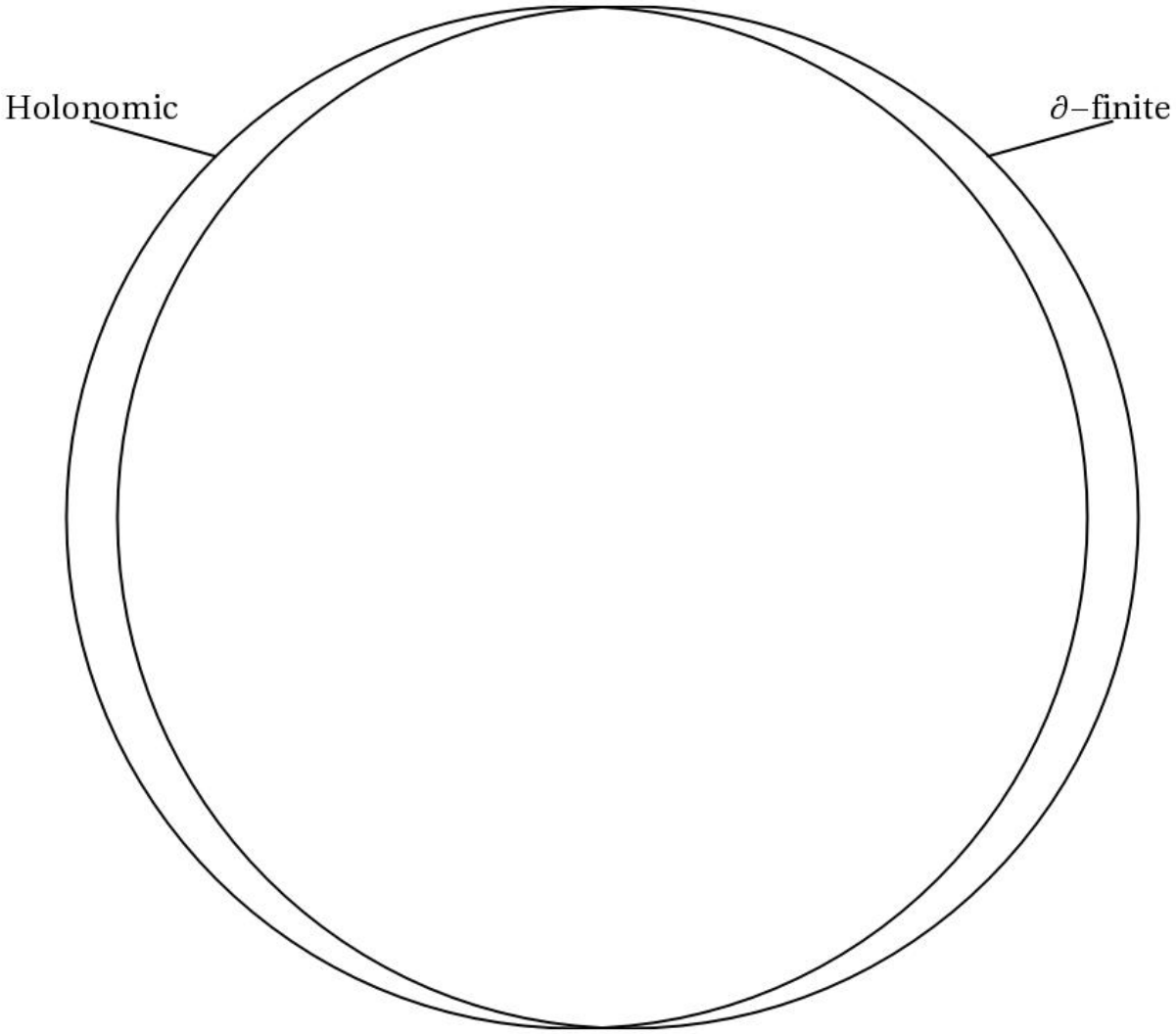

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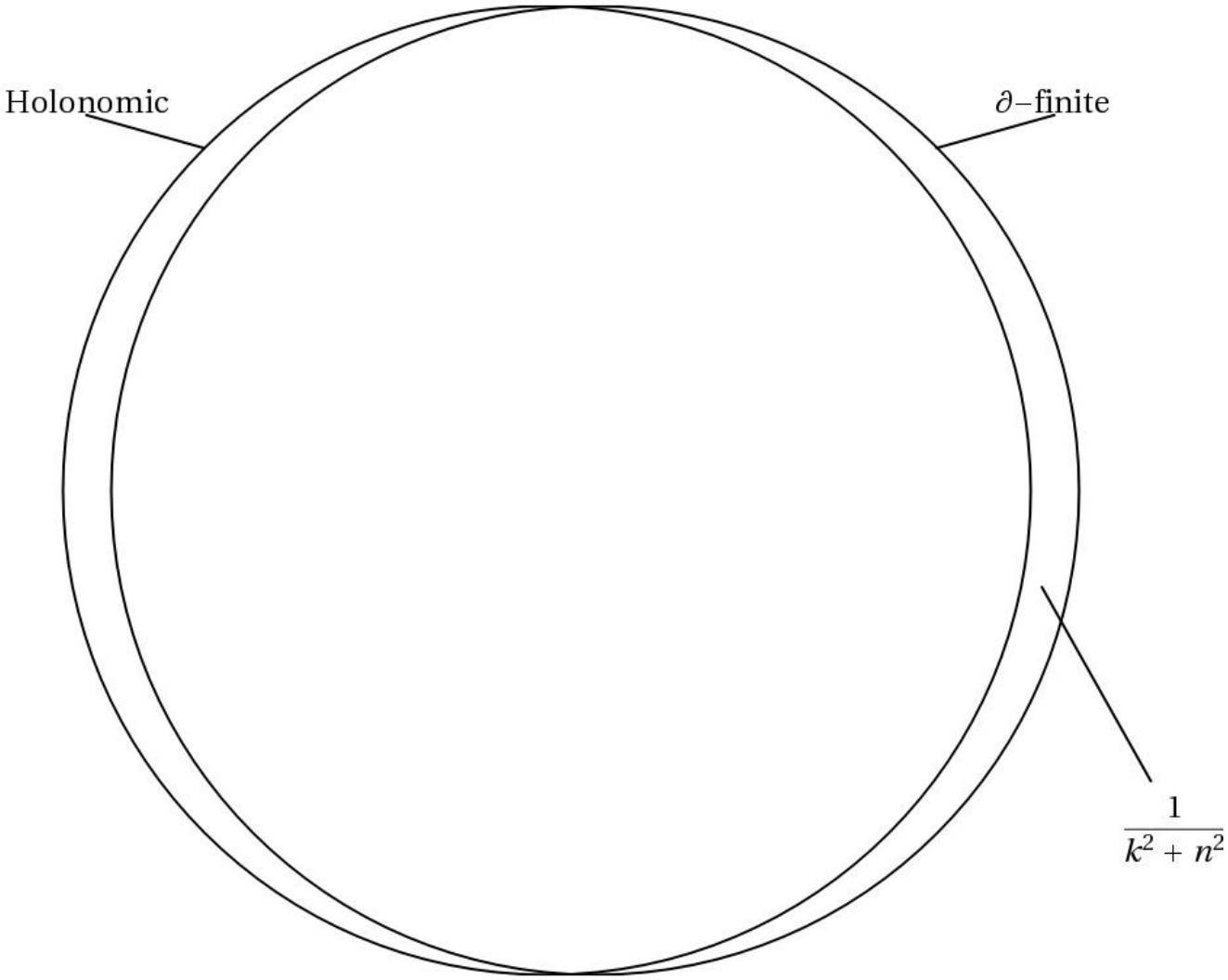

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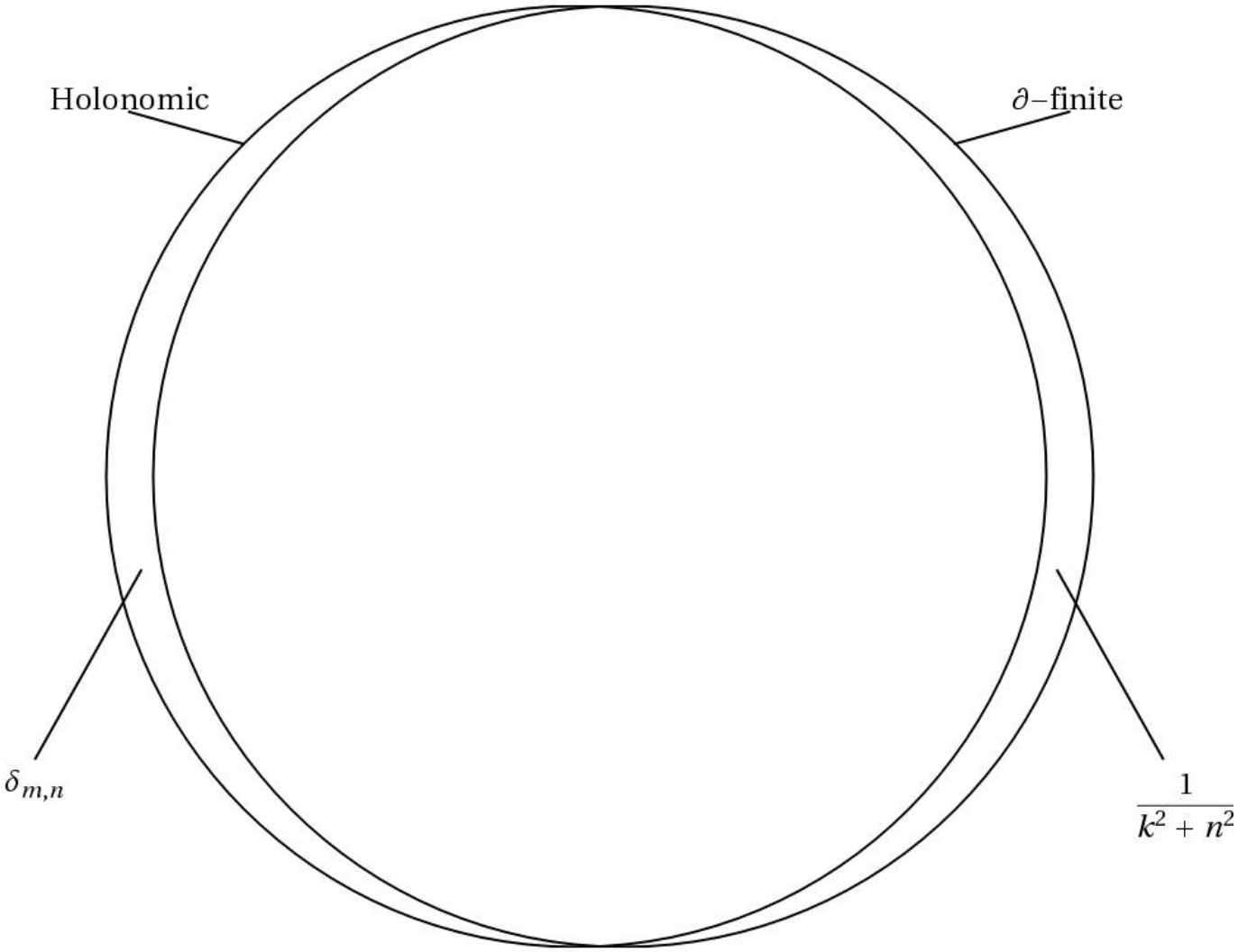

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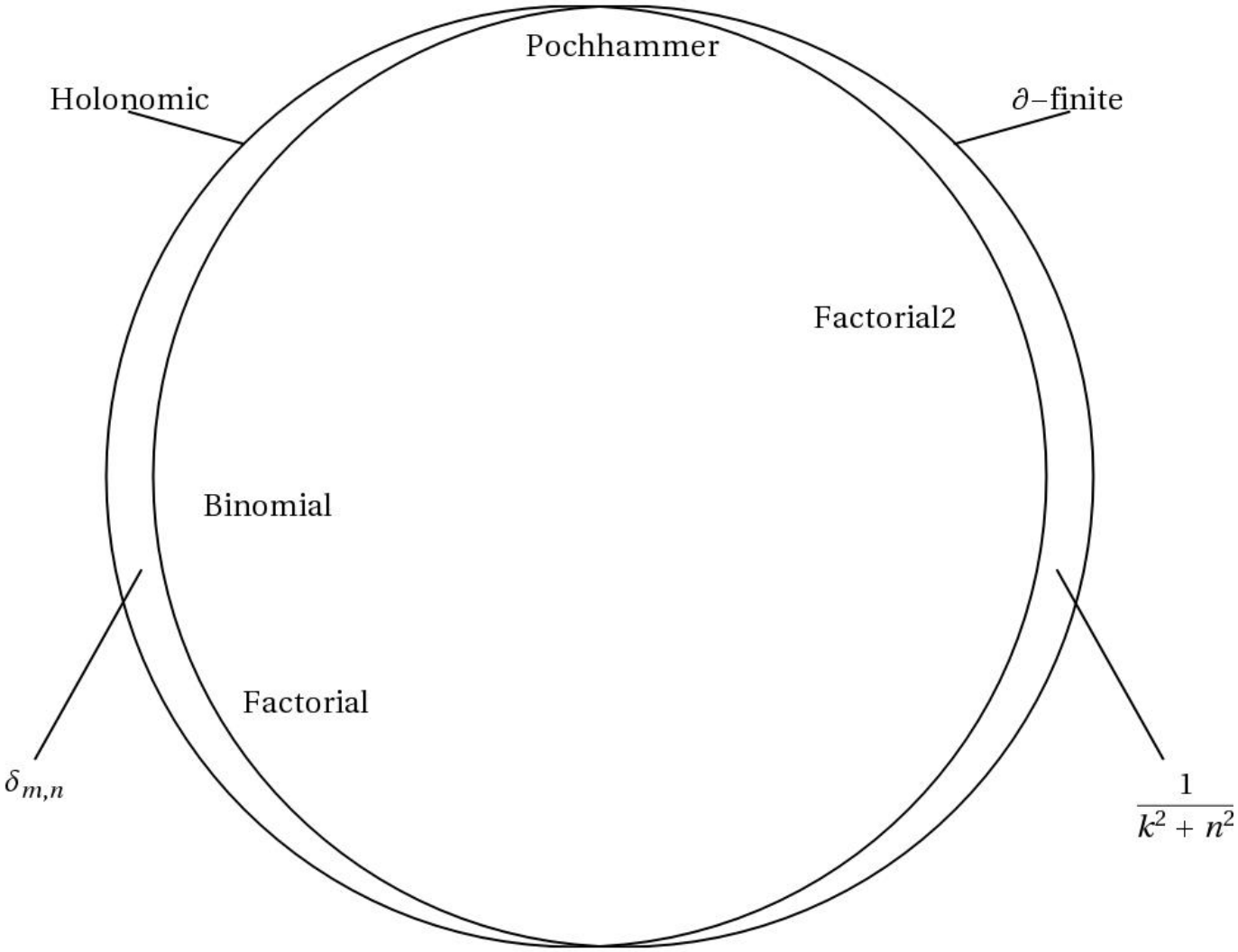

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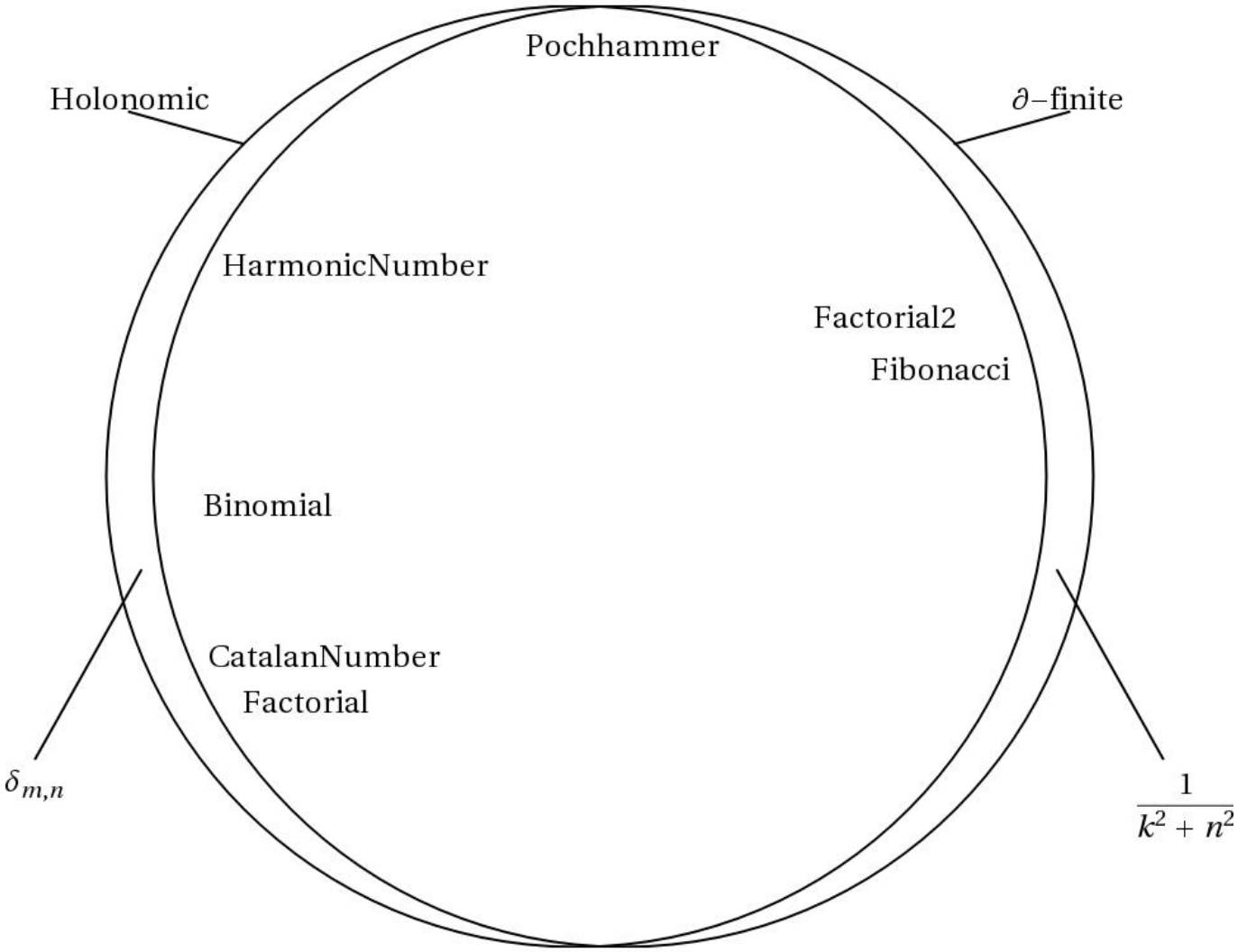

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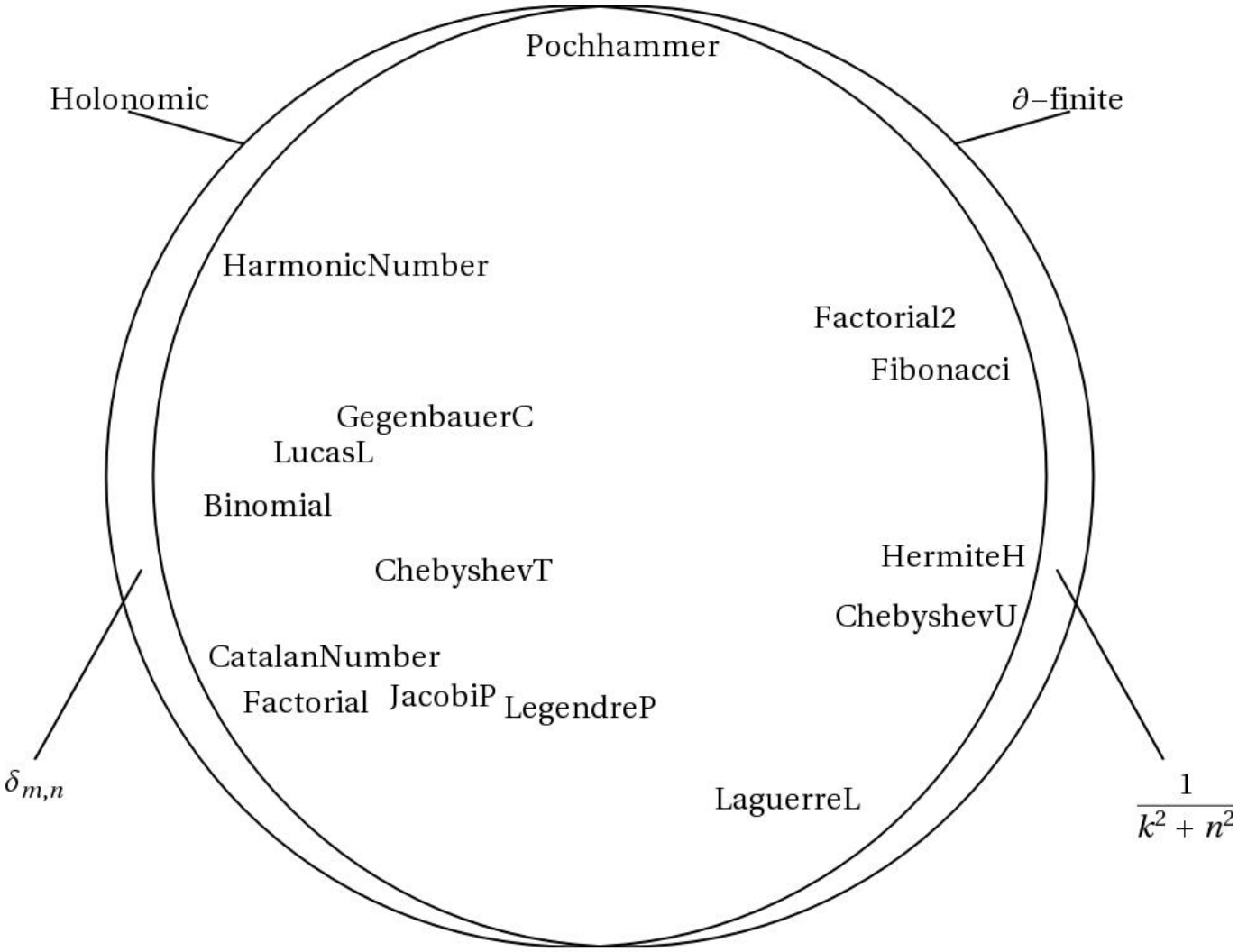

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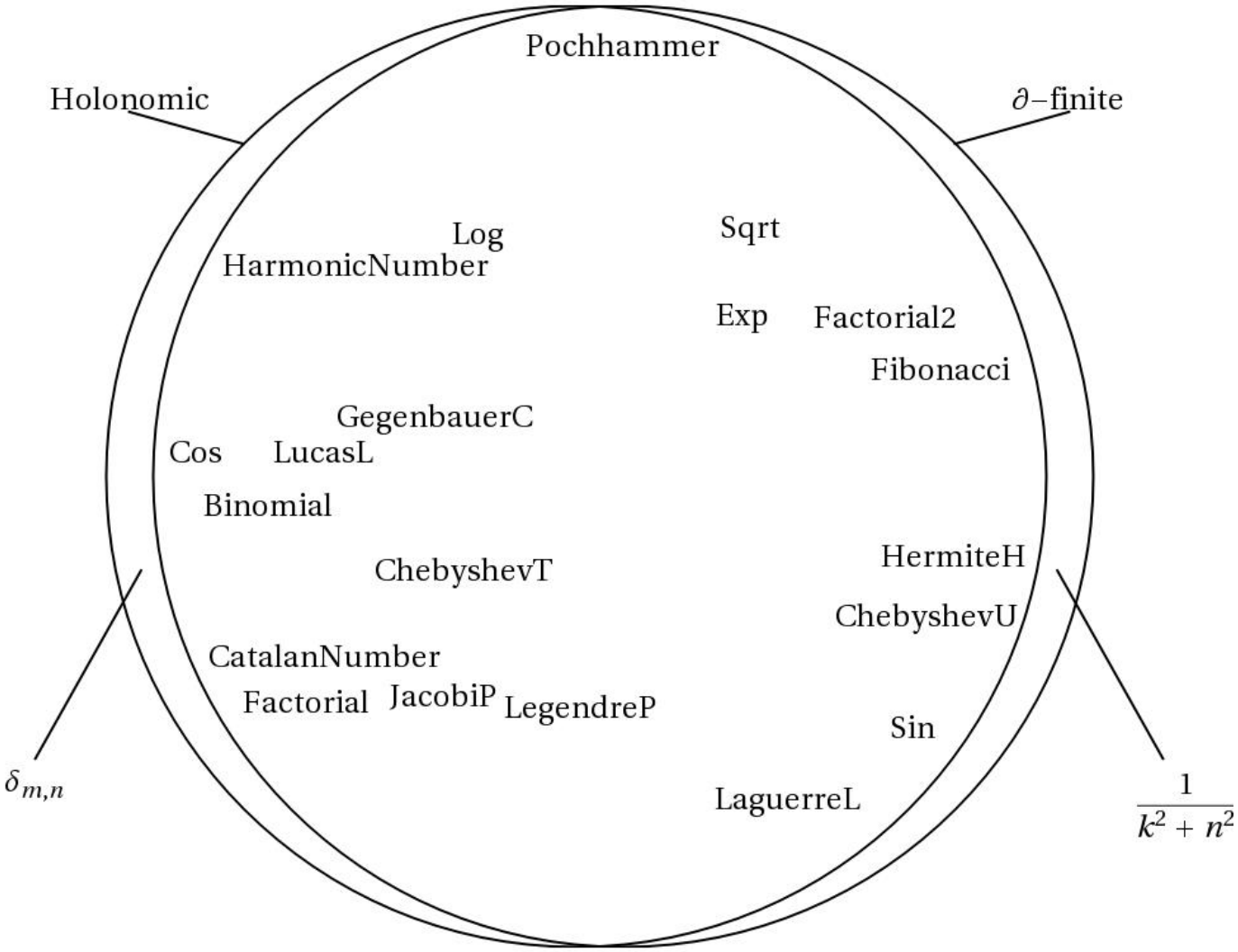

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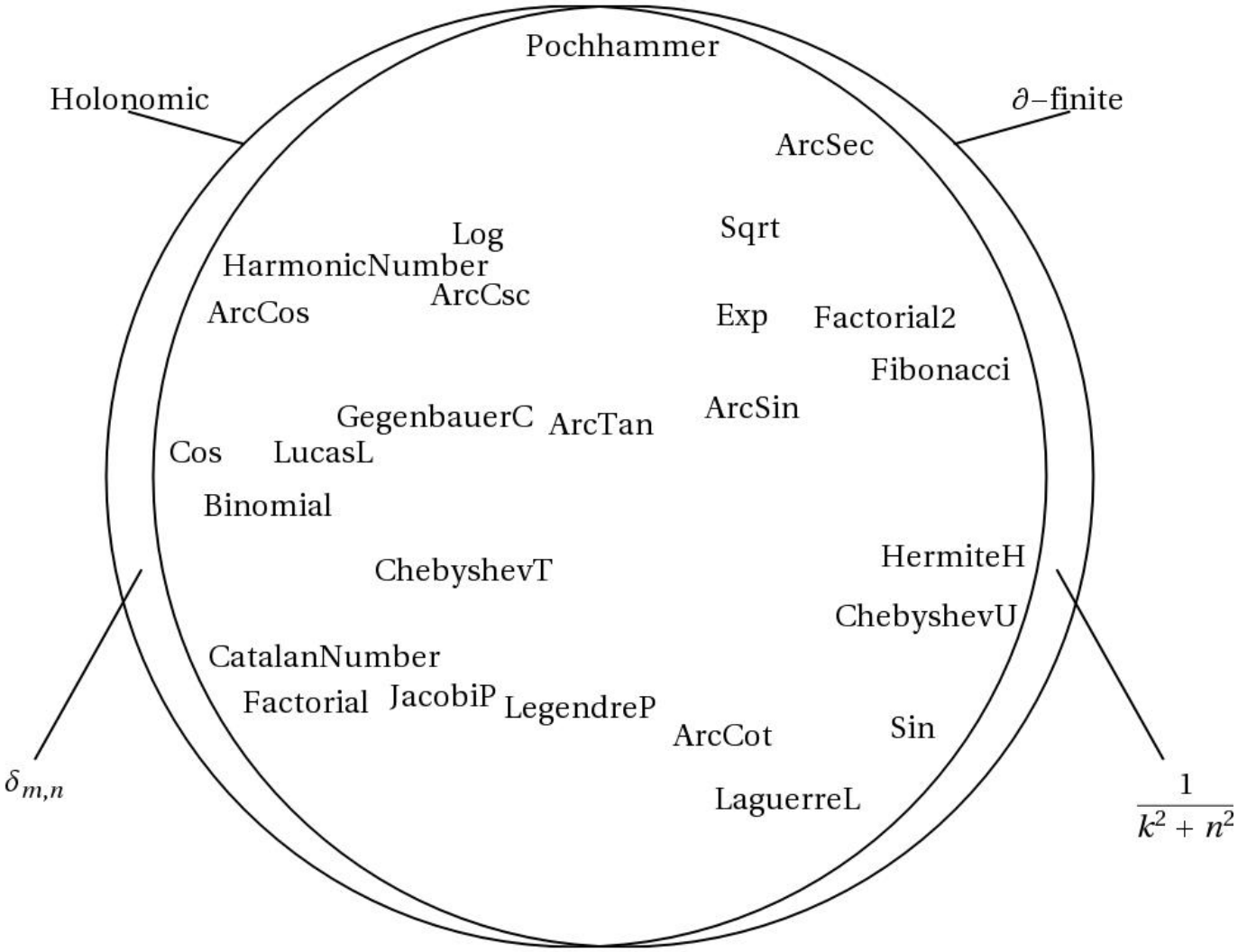

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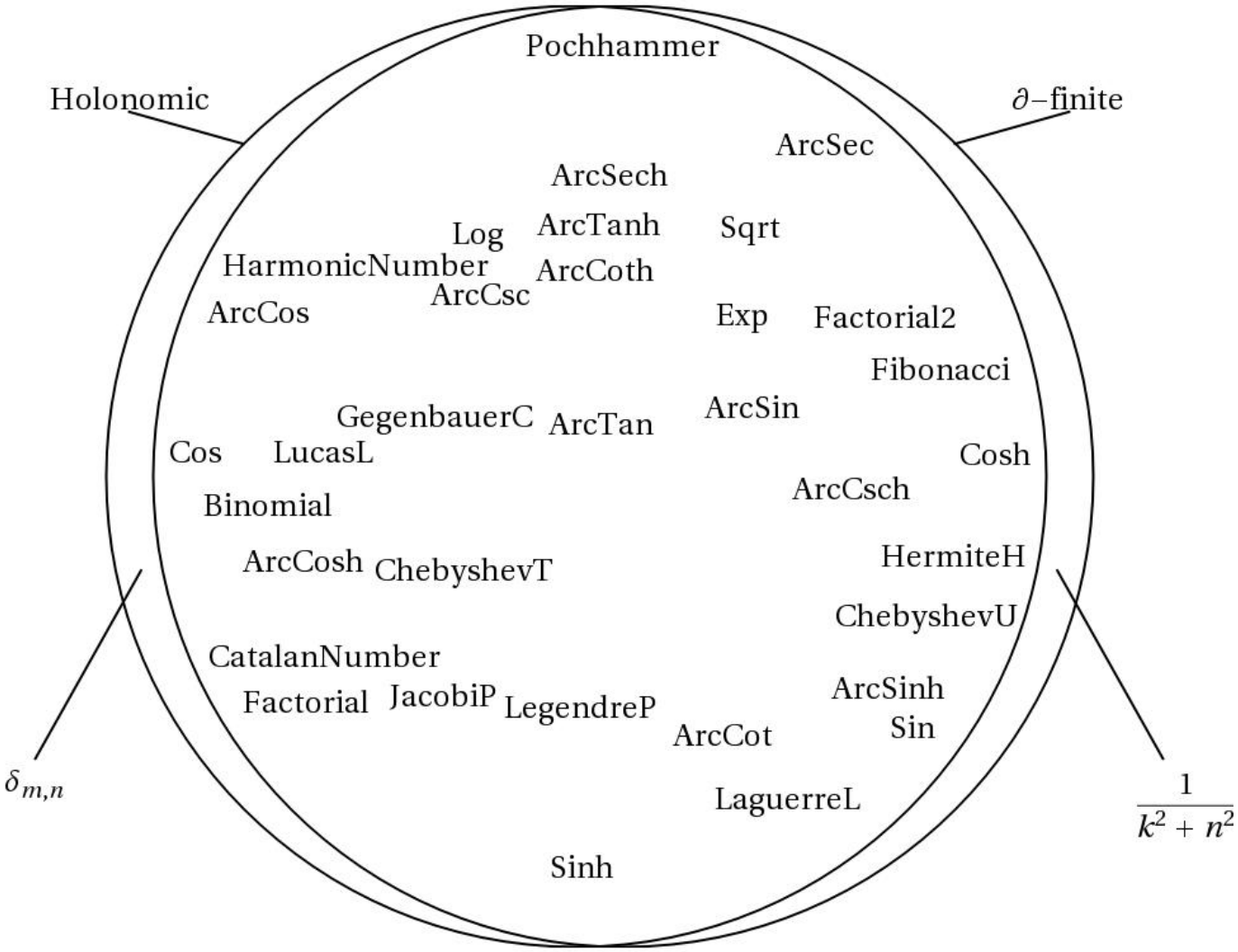

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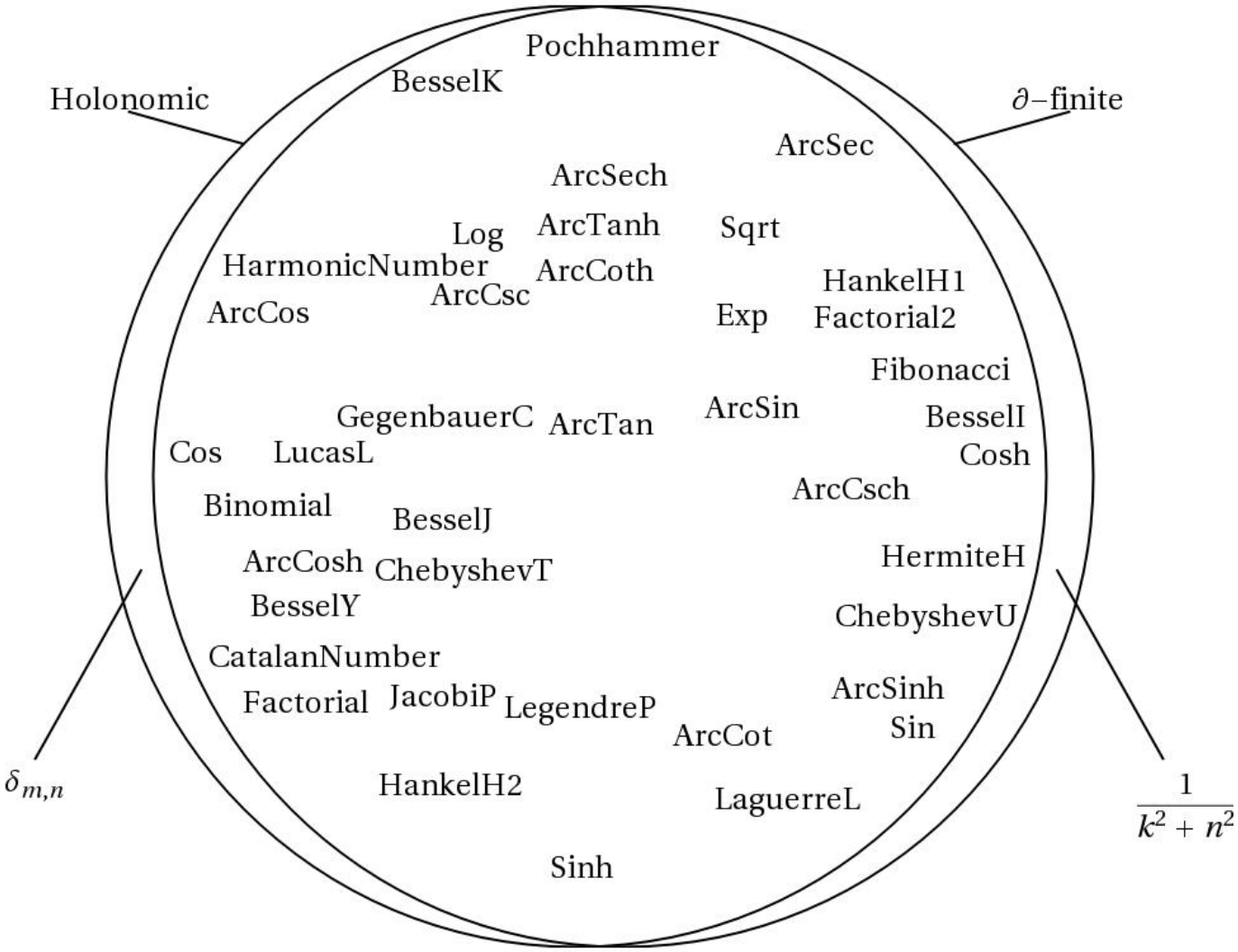

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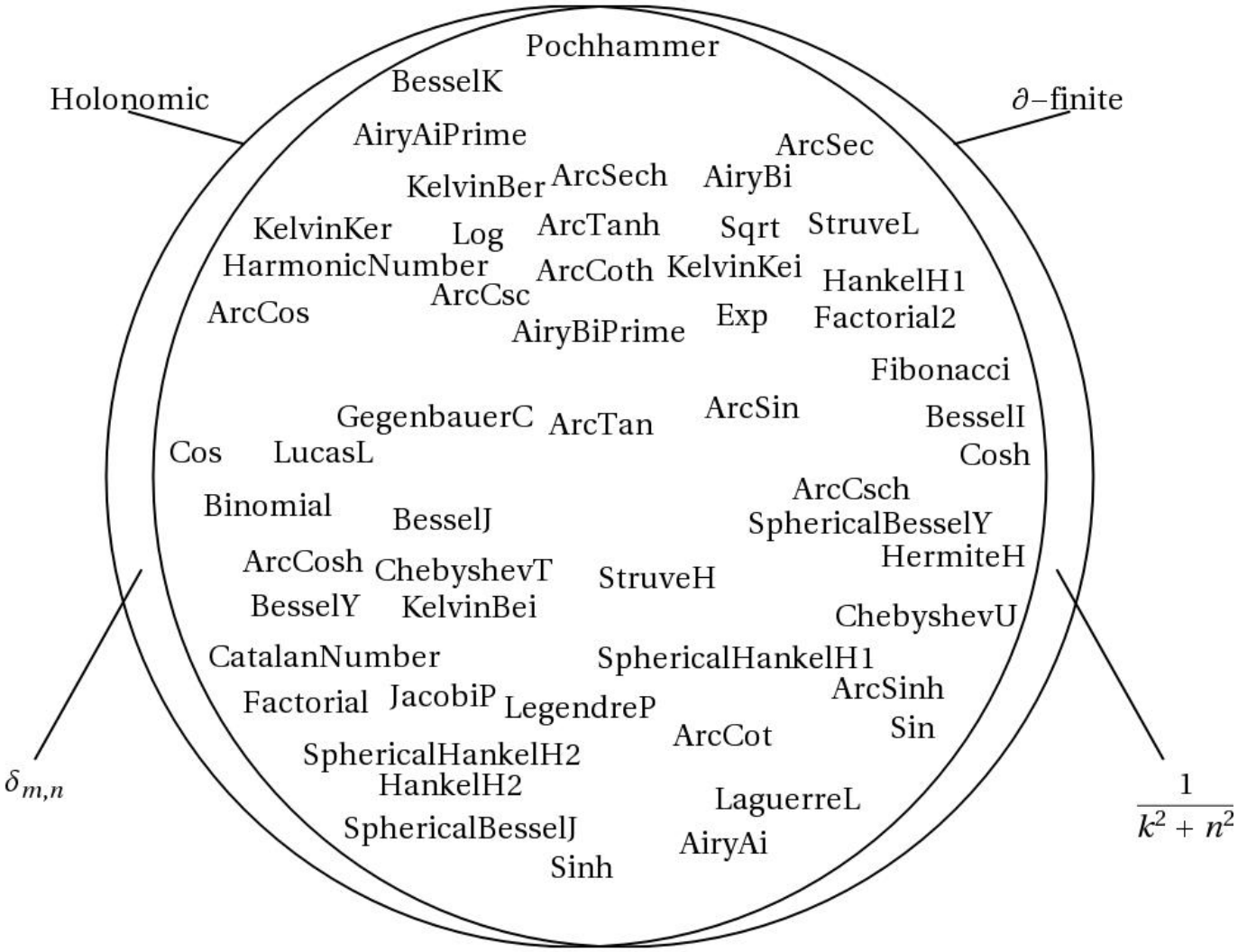

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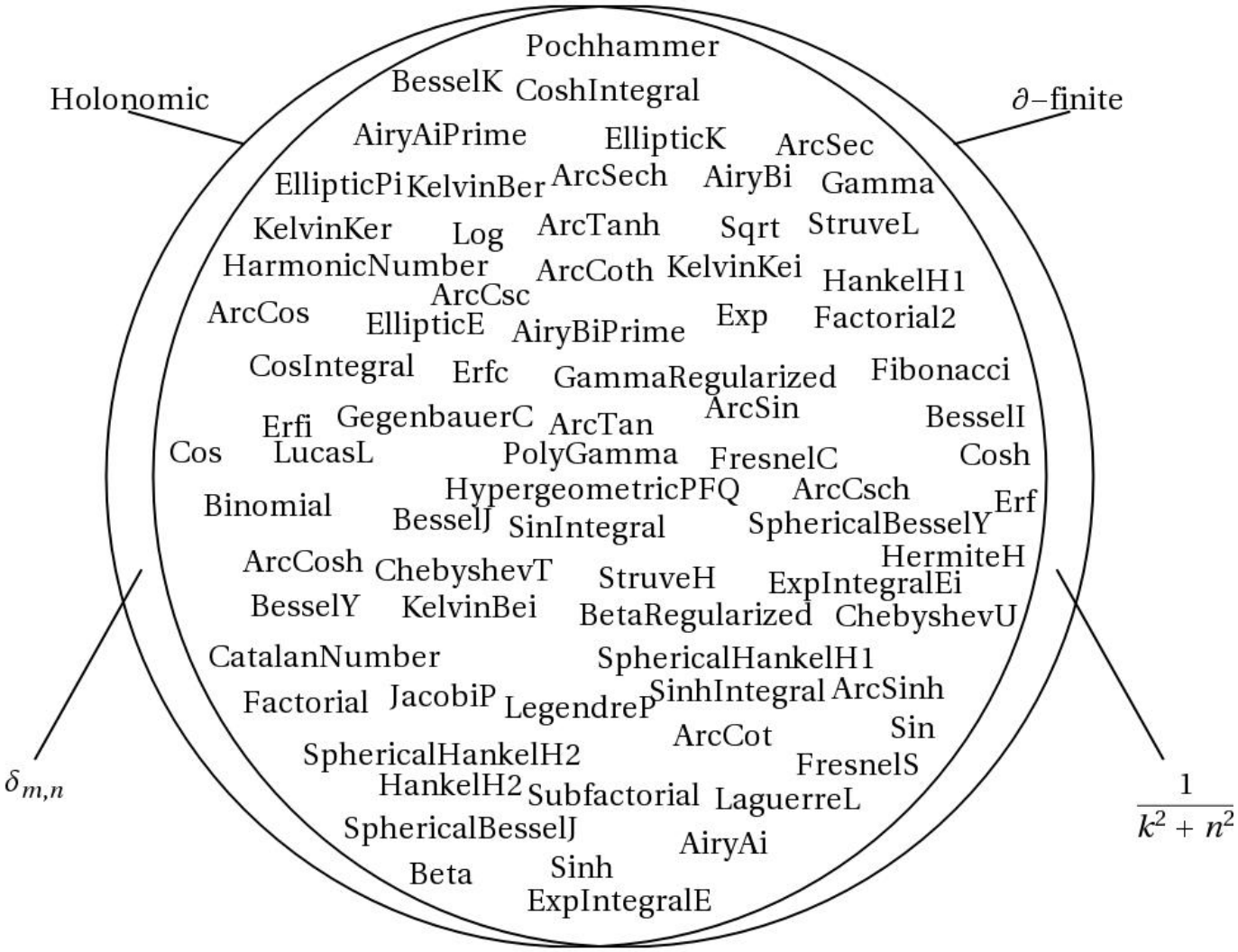

Holonomic functions

Let A_n denote the n-th Weyl algebra.

Definition:
A function $f(x_1, \ldots, x_n)$ is said to be holonomic if $A_n / \operatorname{Ann}_{A_n} f$ is a holonomic A_n-module, i.e., the Bernstein dimension is minimal (according to Bernstein's inequality).

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Definition:
A sequence $f(k_1, \ldots, k_n) \in \mathbb{C}^{\mathbb{N}^n}$ is holonomic if its multivariate generating function

$$F(x_1, \ldots, x_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f(k_1, \ldots, k_n) x_1^{k_1} \cdots x_n^{k_n}$$

is a holonomic function.

Properties of holonomic functions

Closure properties:

- sum
- product
- definite integration
- ...

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Elimination property:
Given a left ideal I in A_n such that A_n/I is holonomic; then for any choice of $n+1$ among the $2n$ generators of A_n there exists a nonzero operator in I that depends only on these. In other words, we can eliminate $n-1$ variables.

Ore algebras (in short)

In order to deal with differentiation and shift operators at the same time we use the generalizing concept of Ore algebras:

- $\mathbb{O} = R[D_x; 1, D_x]$ is an Ore extension by the differential operator D_x.

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- $\mathbb{O} = R[S_n; S_n, 0]$ is an Ore extension by the shift operator S_n.
- $\mathbb{O} = R[\partial; \sigma, \delta]$ is the general notation for an Ore extension.

Meaning: Polynomials in ∂ with coefficients from R.

Note: Noncommutativity between ∂ and the coefficients (determined by σ and δ)! E.g.,

$$D_x x = x D_x + 1 \quad \text{and} \quad S_n n = n S_n + S_n$$

Example: $\mathbb{K}[x][D_x; 1, D_x] = A_1$

Definite integration via elimination

Given: Integration bounds $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic function $f(x, y)$ in the Ore algebra $\mathbb{O} = \mathbb{K}[x, y][D_x; 1, D_x][D_y; 1, D_y] = A_2$

Find: A differential equation for $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$

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Since $\mathbb{O}/\operatorname{Ann}_{\mathbb{O}} f$ is holonomic, there exists $P \in \operatorname{Ann}_{\mathbb{O}} f$ that does not contain x (by elimination property). Write

$$P(y, D_x, D_y) = Q(y, D_y) + D_x \cdot R(y, D_x, D_y)$$

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Apply $\int_a^b \ldots \mathrm{d}x$ to $P \bullet f = 0$:

$$\int_a^b Q(y, D_y) \bullet f\,\mathrm{d}x + \int_a^b D_x R(y, D_x, D_y) \bullet f\,\mathrm{d}x = 0$$

$$Q(y, D_y) \bullet F(y) + \Big[R(y, D_x, D_y) \bullet f \Big]_{x=a}^{x=b} = 0$$

∂-finite functions

Definition: A function $f(x_1, \ldots, x_n)$ is called ∂-finite w.r.t. $\mathbb{O} = \mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ if $\mathbb{O}/\operatorname{Ann}_{\mathbb{O}} f$ is a finite-dimensional $\mathbb{K}(x_1, \ldots, x_n)$-vector space.

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In other words, f is ∂-finite if all its derivatives span a finite-dimensional $\mathbb{K}(x_1, \ldots, x_n)$-vector space.

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Example: All derivatives (w.r.t. x and y) of $\sin\left(\frac{x+y}{x-y}\right)$ are of the form

$$r_1(x,y)\sin\left(\frac{x+y}{x-y}\right) + r_2(x,y)\cos\left(\frac{x+y}{x-y}\right), \quad r_1, r_2 \in \mathbb{Q}(x,y)$$

e.g.,

$$\begin{aligned} D_x^3 D_y^2 \bullet \sin\left(\frac{x+y}{x-y}\right) &= \frac{32\left(3x^4+12yx^3-30y^2x^2-4y^3x+9y^4\right)}{(x-y)^9}\sin\left(\frac{x+y}{x-y}\right) \\ &\quad - \frac{16\left(6x^5-33yx^4+80y^3x^2-54y^4x+3y^5\right)}{(x-y)^{10}}\cos\left(\frac{x+y}{x-y}\right) \end{aligned}$$

Closure properties of ∂-finite functions

Closure properties:

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- application of an Ore operator
- algebraic substitution (of a continuous variable)
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In contrast to holonomic closure properties, the closure properties for ∂-finite functions can be computed quite easily (using linear algebra and an FGLM-like algorithm).

Creative telescoping: Chyzak's extension of Zeilberger's fast algorithm

Given: $\mathrm{Ann}_{\mathbb{O}} f$, the annihilator of a ∂-finite function $f(x, y)$ in the Ore algebra $\mathbb{O} = \mathbb{K}(x, y)[D_x; 1; D_x][D_y; 1, D_y]$.
Find: Operators $Q(y, D_y)$ and $R(x, y, D_x, D_y)$ such that $Q + D_x \cdot R \in \mathrm{Ann}_{\mathbb{O}} f$.

1. compute a Gröbner basis G of $\mathrm{Ann}_{\mathbb{O}} f$ in order to know the set $U = \{u_1, \ldots, u_k\}$ of monomials that can not be reduced by $\mathrm{Ann}_{\mathbb{O}} f$, i.e., the elements under the stairs of G
2. make an ansatz for $Q(y, D_y) = \sum_{i=0}^{d} \eta_i(y) D_y^i$ and $R(x, y, D_x, D_y) = \sum_{j=1}^{k} \phi_j(x, y) u_j$
3. reduce $Q + D_x \cdot R$ with G and set all coefficients to zero
4. solve the corresponding coupled system of differential equations (for rational solutions)
5. if there is no solution, increase d

Holonomic vs. ∂-finite

holonomic description:	$\mathbb{K}[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$
∂-finite description:	$\mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$

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In the shift case there are only some subtle exceptions:

- $\frac{1}{k^2+n^2}$ is ∂-finite but not holonomic.
- $\delta_{i,j}$ is holonomic but not ∂-finite.

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In practice, we consider only functions that are both holonomic and ∂-finite:

- ∂-finite: better algorithmic treatment
- holonomic: guarantees termination of algorithms

Example: One of Olver's problems (1)

From a letter by Frank Olver to my advisor Peter Paule:

"The writing of DLMF Chapter BS by Leonard Maximon and myself is now largely complete; ... However, a problem has arisen in connection with about a dozen formulas from Chapter 10 of Abramowitz and Stegun for which we have not yet tracked down proofs, and the author of this chapter, Henry Antosiewiecz, died about a year ago. Since it is the editorial policy for the DLMF not to state formulas without indications of proofs, I am hoping that you will be willing to step into the breach and supply verifications by computer algebra methods ... I will fax you the formulas later today ..."

Example: One of Olver's problems (2)

Prove the following identity:

$$\frac{1}{z}\sinh\sqrt{z^2-2izt}=\sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\sqrt{\frac{\pi}{2z}}I_{\frac{1}{2}-n}(z)$$

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Prove the following identity:

$$\frac{1}{z}\sinh\sqrt{z^2-2izt} = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\sqrt{\frac{\pi}{2z}}I_{\frac{1}{2}-n}(z)$$

For the left hand side, we can immediately compute annihilating operators (using closure properties + database):

```
F[t_,z_] := Sinh[Sqrt[z^2 - 2*I*z*t]]/z
lhs = Annihilator[F[t,z], {Der[t], Der[z]}]
```

$$\begin{aligned}&\{(-t-iz)D_t + zD_z + 1,\\ &(-z^4+3itz^3+2t^2z^2)D_z^2 + (-2z^3+6itz^2+5t^2z)D_z\\ &+(z^4-3itz^3-3t^2z^2+it^3z+t^2)\}\end{aligned}$$

Example: One of Olver's problems (3)

On the right hand side $\sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \sqrt{\frac{\pi}{2z}} I_{\frac{1}{2}-n}(z)$, we perform creative telescoping:

```
f[n_,t_,z_]:= (-I*t)^n/n!*Sqrt[Pi/2/z]*BesselI[-n+1/2,z];
{opQ, opR} =
  CreativeTelescoping[f[n,t,z], S[n]-1, {Der[t],Der[z]}]
```

We obtain two operators $Q_i + (S_n - 1) \cdot R_i \in \operatorname{Ann} f$ where

$$
\begin{aligned}
Q_1 &= -t(t+iz)D_t + tzD_z + t, \\
Q_2 &= (t+iz)(2t+iz)z^2 D_z^2 - z(-5t^2 - 6izt + 2z^2)D_z \\
&\quad + i(-iz^4 - 3tz^3 + 3it^2z^2 + t^3z - it^2), \\
R_1 &= -inz, \\
R_2 &= i(n^2+n)(t+iz)zS_n + 2t^2n^2 - z^2n^2 + 3itzn^2 - t^2n \\
&\quad + z^2n - 3itzn
\end{aligned}
$$

Example: One of Olver's problems (4)

Next verify that $[R_1 \bullet f]_{n=0} = 0$ and that $R_1 \bullet f$ tends to 0 when n goes to infinity (the same for R_2):

```
ApplyOreOperator[opR, f[n,t,z]] /. n->0
```

$$\{0, 0\}$$

$\longrightarrow$ The delta part vanishes.
Hence Q_1 and Q_2 are annihilating operators for the sum. In fact, we find that they agree with the annihilating operators that we computed for the left hand side.

Example: One of Olver's problems (5)

In order to establish equality, we have to compare initial values.
Look at the vector space under the stairs of the Gröbner basis:

```
u = UnderTheStaircase[lhs]
```

$$\{1, D_z\}$$

This means we have to compute two initial values:

```
ApplyOreOperator[u, F[t,z]] /. {t->0,z->1} //FullSimplify
```

$$\{\sinh(1), \tfrac{1}{e}\}$$

```
ApplyOreOperator[u, f[n,t,z]] /. {t->0,z->1}
```

$$\left\{\frac{0^n\sqrt{\frac{\pi}{2}}I_{\frac{1}{2}-n}(1)}{n!}, \frac{0^n\sqrt{\frac{\pi}{2}}\left(I_{-n-\frac{1}{2}}(1)+I_{\frac{3}{2}-n}(1)\right)}{2n!} - \frac{0^n\sqrt{\frac{\pi}{2}}I_{\frac{1}{2}-n}(1)}{2n!}\right\}$$

```
% /. (0^n)->1 /. n->0 // FullSimplify
```

$$\{\sinh(1), \tfrac{1}{e}\}$$

First Application: Simulation of electromagnetic waves

(joint work with J. Schöberl, RWTH Aachen)

We use the package `HolonomicFunctions` for finding certain relations between expressions involving orthogonal polynomials (Legendre, Jacobi). These relations are badly needed in numerical simulations with finite element methods.

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Numerical method for finding approximate solutions to partial differential equations on non-trivial domains:

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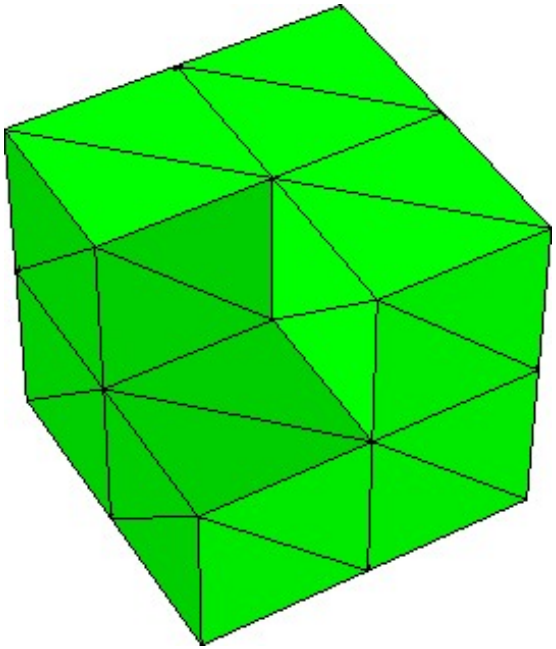

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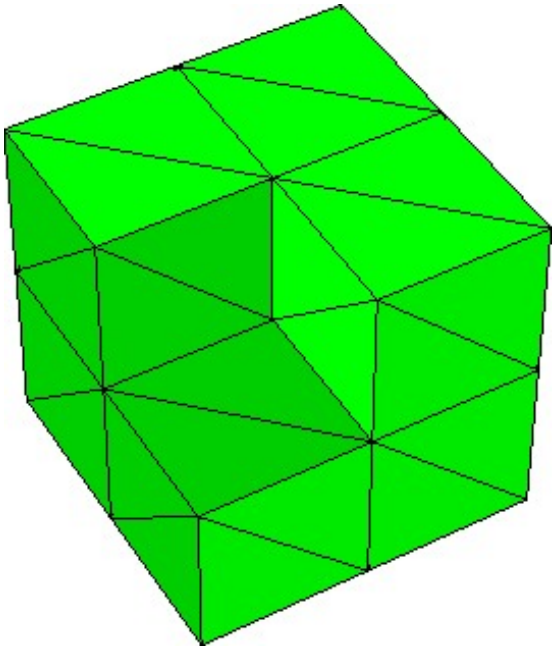

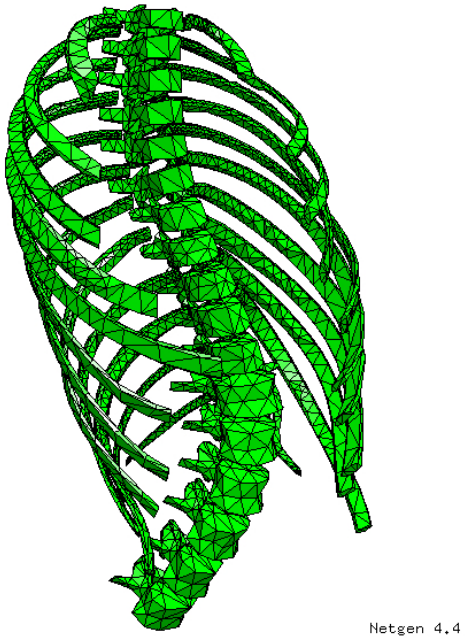

Divide the domain into small finite elements, i.e., triangles in the 2D case or tetrahedrons in the 3D case. Approximate the solution by certain basis functions that are defined on each finite element (locally supported piecewise polynomial basis functions).

Simulation of electromagnetic waves

Task: Simulate electromagnetic waves using the Maxwell equations:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively.

Method:

In this application we define the basis functions as follows:

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\tfrac{2y}{1-x} - 1\right)$$

In order to solve the above equations, one needs to represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself).

First Try

```
phi[i_,j_,x_,y_] :=
 LegendreP[i,2*y/(1-x)-1]*(1-x)^i*JacobiP[j,2*i+1,0,2*x-1]
ann = Annihilator[phi[i,j,x,y], {Der[x], S[i], S[j]}]
```

$\langle$*quite big output*$\rangle$

In order to see better the structure of the output, we look only at the support of each operator:

```
Support /@ ann
```

$$\{\{S_j^2, S_j, 1\}, \{S_i S_j, D_x, S_i, S_j, 1\}, \{S_i^2, D_x, S_i, S_j, 1\},$$
$$\{D_x S_j, D_x, S_i, S_j, 1\}, \{D_x S_i, D_x, S_i, S_j, 1\}, \{D_x^2, D_x, S_i, S_j, 1\}\}$$

$\longrightarrow$ We see that the second and the third operator match exactly our needs!

Application in Finite Element Methods (3)

BUT: The numerists need a relation that is free of x and y! In change, they allow also shifts in the derivative, i.e., we are now looking for a relation of the following form:

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

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where $A, B \subset \mathbb{N}^2$ are finite index sets.

- Use Gröbner basis computation in order to eliminate x and y.
- After some time we get an operator of the desired form, that is even not too big (about 2 pages).
- Because of extension/contraction problem we can not be sure that we obtain the smallest operator.

Application in Finite Element Methods (4)

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

New idea: Similar approach as in creative telescoping.

1. we work in $\mathbb{O} = \mathbb{K}(i,j,x,y)[D_x;1,D_x][S_i;S_i,0][S_j;S_j,0]$
2. choose index sets A and B
3. reduce the corresponding ansatz with the Gröbner basis of $\mathrm{Ann}_{\mathbb{O}}\,\varphi$
4. do coefficient comparison w.r.t. x and y
5. solve the resulting linear system for $a_{k,l}$ and $b_{m,n}$ in $\mathbb{K}(i,j)$

Application in Finite Element Methods (5)

With this method, we find in short time the following relation:

$$
\begin{aligned}
&(2i+j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)\\
&+2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)\\
&-(j+3)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)\\
&+(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)\\
&-2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)\\
&-(2i+j+5)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)=\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)\\
&+2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)
\end{aligned}
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&+(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)\\
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&+2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)
\end{aligned}
$$

Joachim Schöberl's answer: "jetzt bin ich echt beeindruckt... Genau so eine Relation brauche ich!"

3D case

Note: Using the formulas from the 2D case gave already a speed-up of 20 percent in the numerical simulations!

But we would like to have an analog of the 2D formulae in 3D.

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Problem: Now the basis functions $\varphi(i, j, k, x, y, z)$ contain 6 variables and computations become too big and too slow.

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Improvements:

- In a first step determine only the support of the operator, without the coefficients
- Reduce each monomial separately
- Use modular techniques (tricky because of noncommutative algebra!)

A first result for 3D

On of the supports looks as follows:

$$\{S_j S_k^4, S_j^2 S_k^3, S_j^3 S_k^2, S_j^4 S_k, D_x S_j S_k^3, D_x S_j^2 S_k^2, D_x S_j^3 S_k, D_x S_j^4, S_j S_k^5,$$
$$S_j^2 S_k^4, S_j^3 S_k^3, S_j^4 S_k^2, S_i S_k^5, S_i S_j S_k^4, S_i S_j^2 S_k^3, S_i S_j^3 S_k^2, D_x S_j S_k^4, D_x S_j^2 S_k^3,$$
$$D_x S_j^3 S_k^2, D_x S_j^4 S_k, D_x S_i S_k^4, D_x S_i S_j S_k^3, D_x S_i S_j^2 S_k^2, D_x S_i S_j^3 S_k, S_j S_k^6,$$
$$S_j^2 S_k^5, S_j^3 S_k^4, S_j^4 S_k^3, S_i S_k^6, S_i S_j S_k^5, S_i S_j^2 S_k^4, S_i S_j^3 S_k^3, D_x S_j S_k^5, D_x S_j^2 S_k^4,$$
$$D_x S_j^3 S_k^3, D_x S_j^4 S_k^2, D_x S_i S_k^5, D_x S_i S_j S_k^4, D_x S_i S_j^2 S_k^3, D_x S_i S_j^3 S_k^2, S_j S_k^7,$$
$$S_j^2 S_k^6, S_j^3 S_k^5, S_j^4 S_k^4, S_i S_k^7, S_i S_j S_k^6, S_i S_j^2 S_k^5, S_i S_j^3 S_k^4, D_x S_j S_k^6, D_x S_j^2 S_k^5,$$
$$D_x S_j^3 S_k^4, D_x S_j^4 S_k^3, D_x S_i S_k^6, D_x S_i S_j S_k^5, D_x S_i S_j^2 S_k^4, D_x S_i S_j^3 S_k^3, S_j S_k^8,$$
$$S_j^2 S_k^7, S_j^3 S_k^6, S_j^4 S_k^5, D_x S_j S_k^7, D_x S_j^2 S_k^6, D_x S_j^3 S_k^5, D_x S_j^4 S_k^4, D_x S_i S_k^7,$$
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$$D_x S_j^4 S_k^5, D_x S_i S_k^8, D_x S_i S_j S_k^7, D_x S_i S_j^2 S_k^6, D_x S_i S_j^3 S_k^5, D_x S_j S_k^9,$$
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Joachim Schöberl was impressed but not too happy about these results...

New ideas for the 3D case (1)

Next idea: Write $\varphi = u \cdot v \cdot w$ and use the product rule

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} vw + u \frac{\mathrm{d}v}{\mathrm{d}x} w + uv \frac{\mathrm{d}w}{\mathrm{d}x}$$

This means, we have to find a relation between e.g. uvw and $\frac{\mathrm{d}u}{\mathrm{d}x} vw$.

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Trivial solution: $\mathrm{op}_1 \in \mathrm{Ann}\, f$ and $\mathrm{op}_2 \in \mathrm{Ann}\, g$. But since f and g are closely related we expect that there exists something "better".

New ideas for the 3D case (2)

The natural way to express a relation like

$$\mathrm{op}_1 \bullet f = \mathrm{op}_2 \bullet g$$

is by introducing the module $M = \mathbb{O} \times \mathbb{O}$ which we let act on $\mathcal{F} \times \mathcal{F}$ by

$$P \bullet f = (P_1, P_2) \bullet (f, g) := P_1 \bullet f + P_2 \bullet g, \quad \text{where } P \in M, f \in \mathcal{F} \times \mathcal{F}$$

But how to compute a Gröbner basis for the ideal of relations between f and g, i.e. the annihilator $\mathrm{Ann}_M(f, g)$?

Closure properties

Assume that $f = uvw$ and $g = \frac{\mathrm{d}u}{\mathrm{d}x}vw$.
We start with u and $u' = \frac{\mathrm{d}u}{\mathrm{d}x}$:
$\mathrm{Ann}_M(u, u') =$

$$_{\mathbb{O}}\left\langle \left\{(p,0) | p \in \mathrm{Ann}_{\mathbb{O}}\, u\right\} \cup \left\{(0,p) | p \in \mathrm{Ann}_{\mathbb{O}}\, u'\right\} \cup \left\{(D_x, -1)\right\} \right\rangle$$

After computing a Gröbner basis of the above, we can perform the closure property "multiplication by vw" in a very similar fashion as usual.

Result

Finally we can use the ansatz technique as before in order to find an $\{x, y, z\}$-free operator:

$$
\begin{gathered}
-2(1+2i)(2+j)(3+2i+j)(7+2i+2j)(5+i+j+k) \\
(7+i+j+k)(8+i+j+k)(8+2i+2j+k)(9+2i+2j+k) \\
(11+2i+2j+2k)(15+2i+2j+2k)f(i,j+1,k+3)+ \\
\vdots \\
\langle \text{ 31 similar terms } \rangle \\
\vdots \\
-2(4+2i+j)(5+2i+j)(5+2i+2j)(5+i+j+k) \\
(6+i+j+k)(8+i+j+k)(10+2i+2j+k) \\
(11+2i+2j+k)(11+2i+2j+2k)(15+2i+2j+2k) \\
g(i+1,j+2,k+3)=0
\end{gathered}
$$

where $f = uvw$ and $g = \frac{\mathrm{d}u}{\mathrm{d}x}vw$.

Second Application: Gessel walks

(joint work with M. Kauers and D. Zeilberger)

We use CKs package `HolonomicFunctions.m` in order to prove an open conjecture about certain lattice paths...

Gessel walks

- walks in the integer lattice $\mathbb{N}^2$

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- walks in the integer lattice $\mathbb{N}^2$
- start at $(0, 0)$

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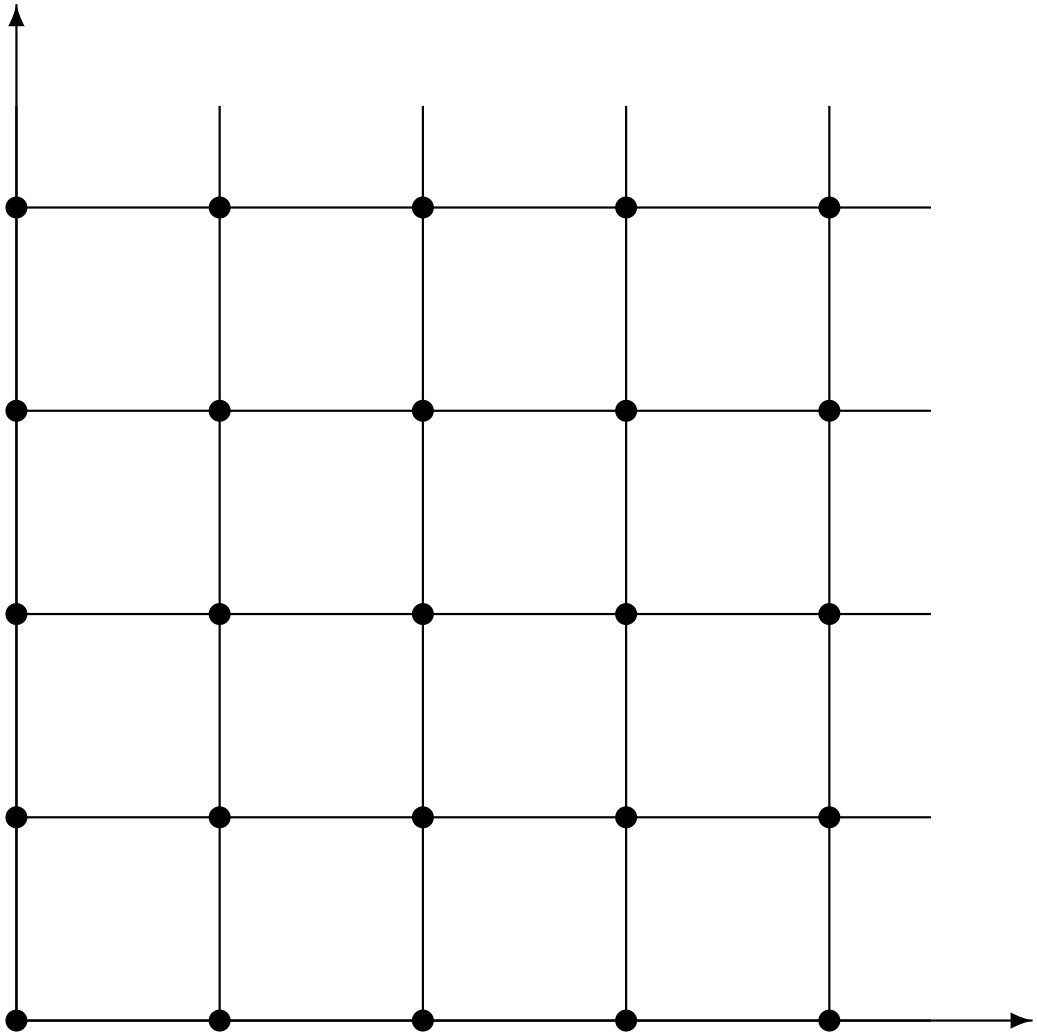
Gessel walks

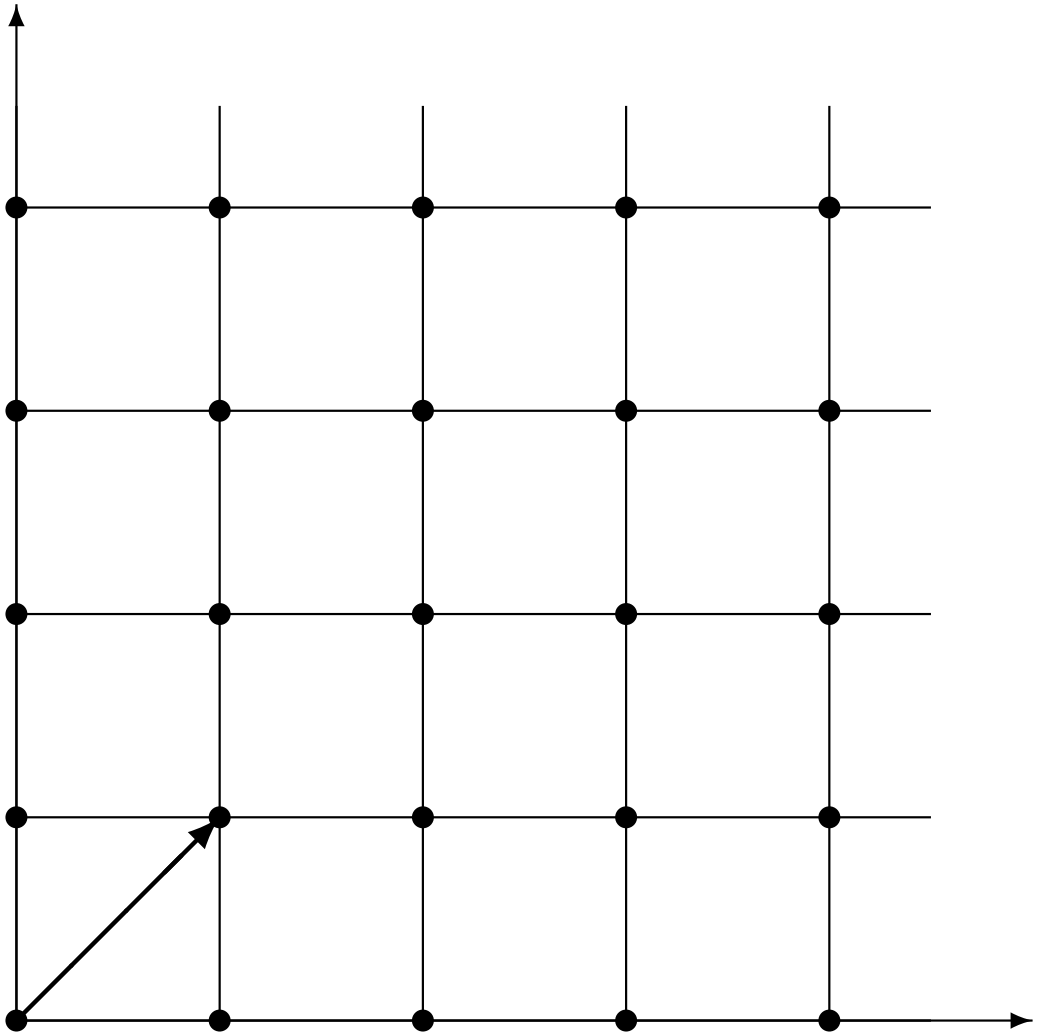
- walks in the integer lattice $\mathbb{N}^2$
- start at $(0,0)$
- do not leave $\mathbb{N}^2$
- only certain steps are allowed:

$$\begin{aligned} G &:= \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \\ &= \{\leftarrow, \rightarrow, \swarrow, \nearrow\} \end{aligned}$$

Gessel walks — Example



Gessel walks — Example



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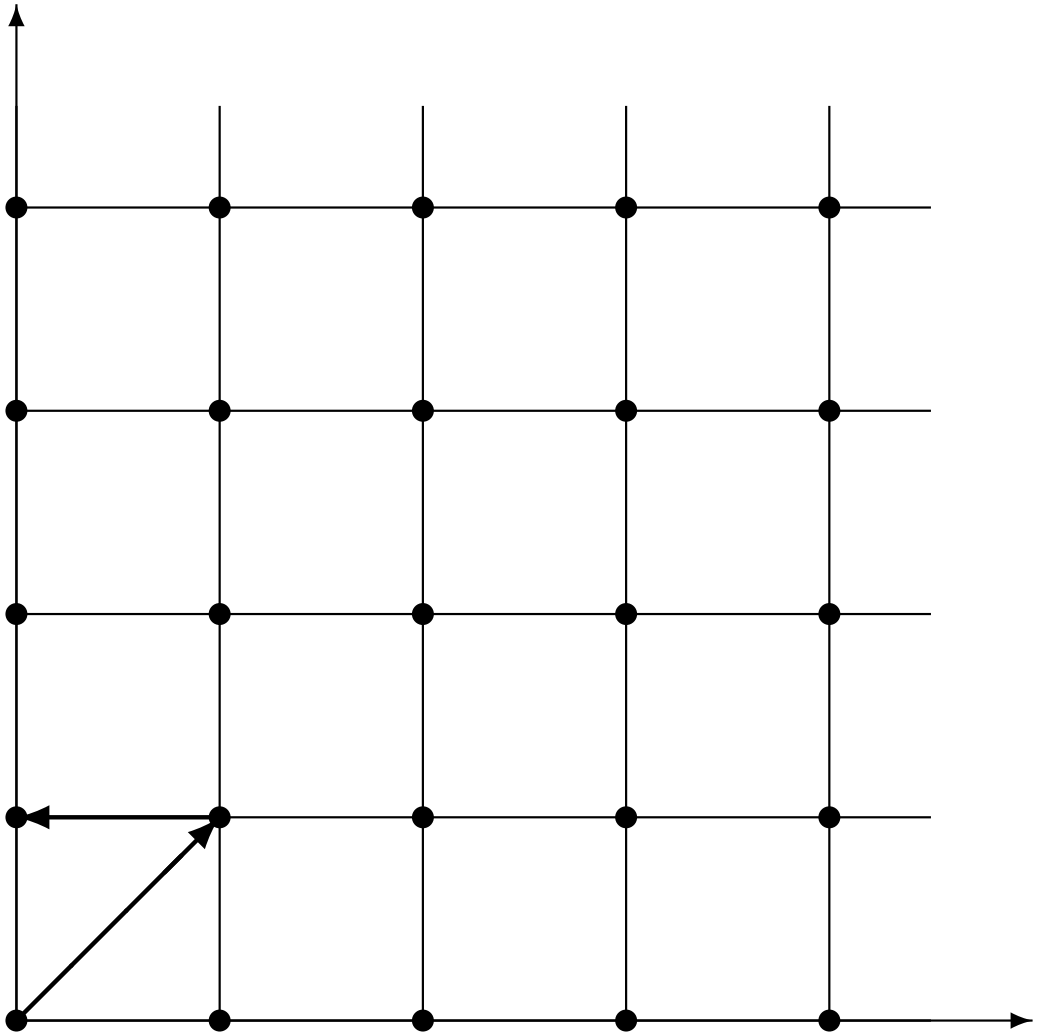

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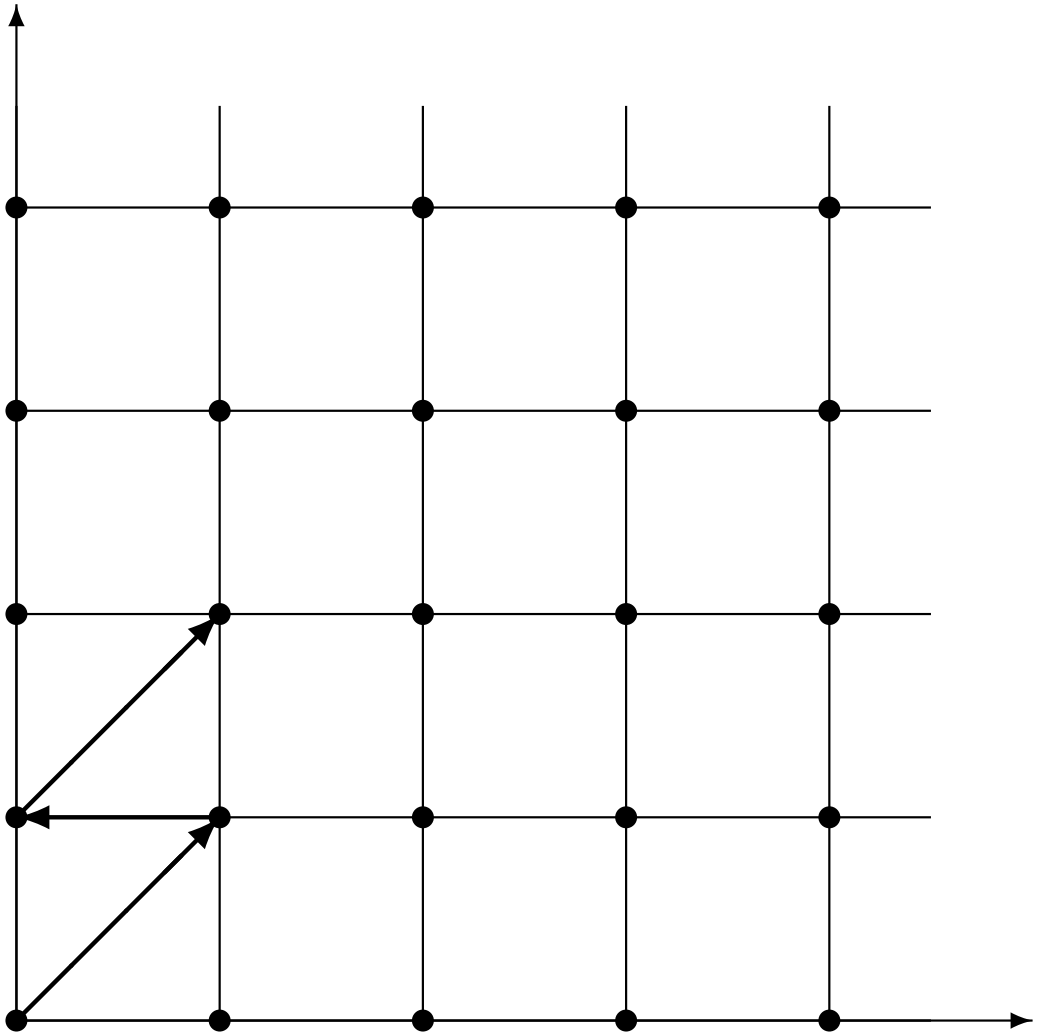

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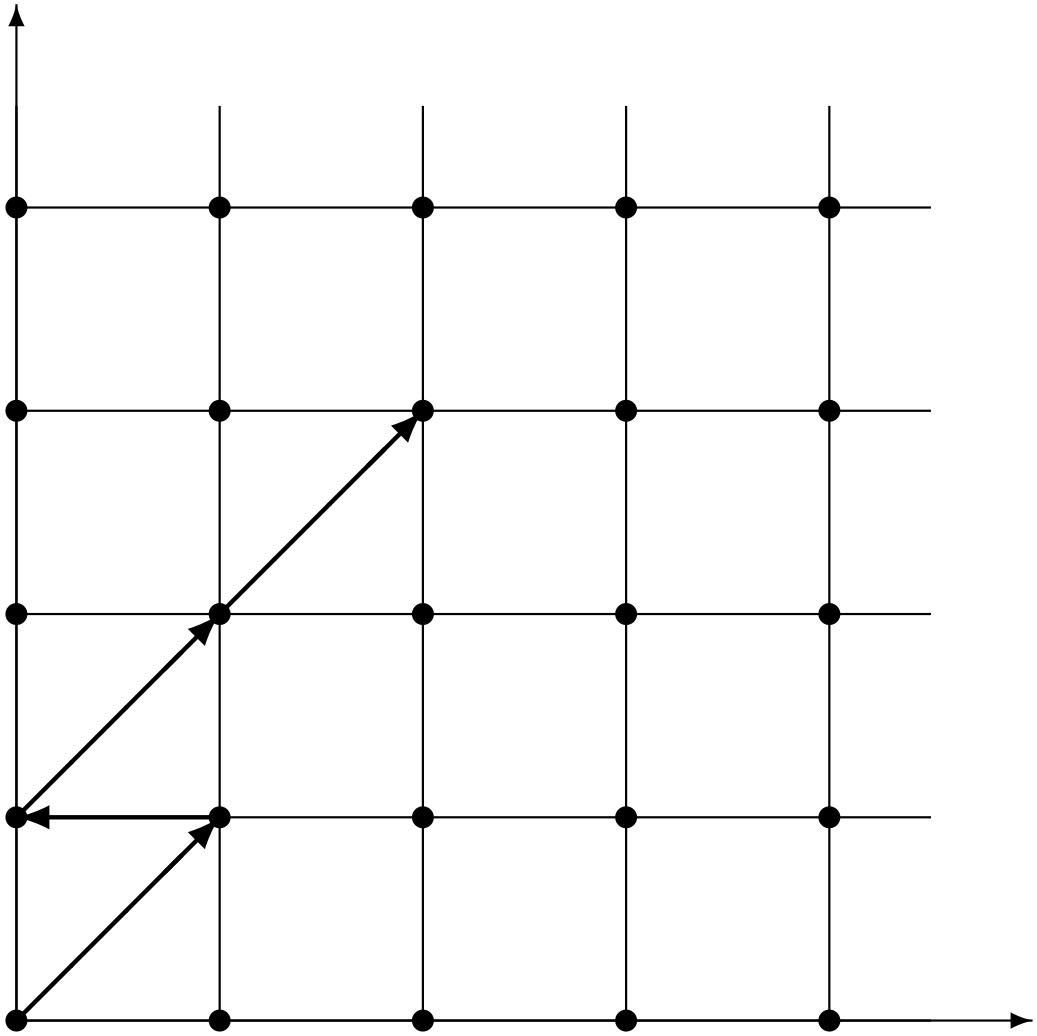

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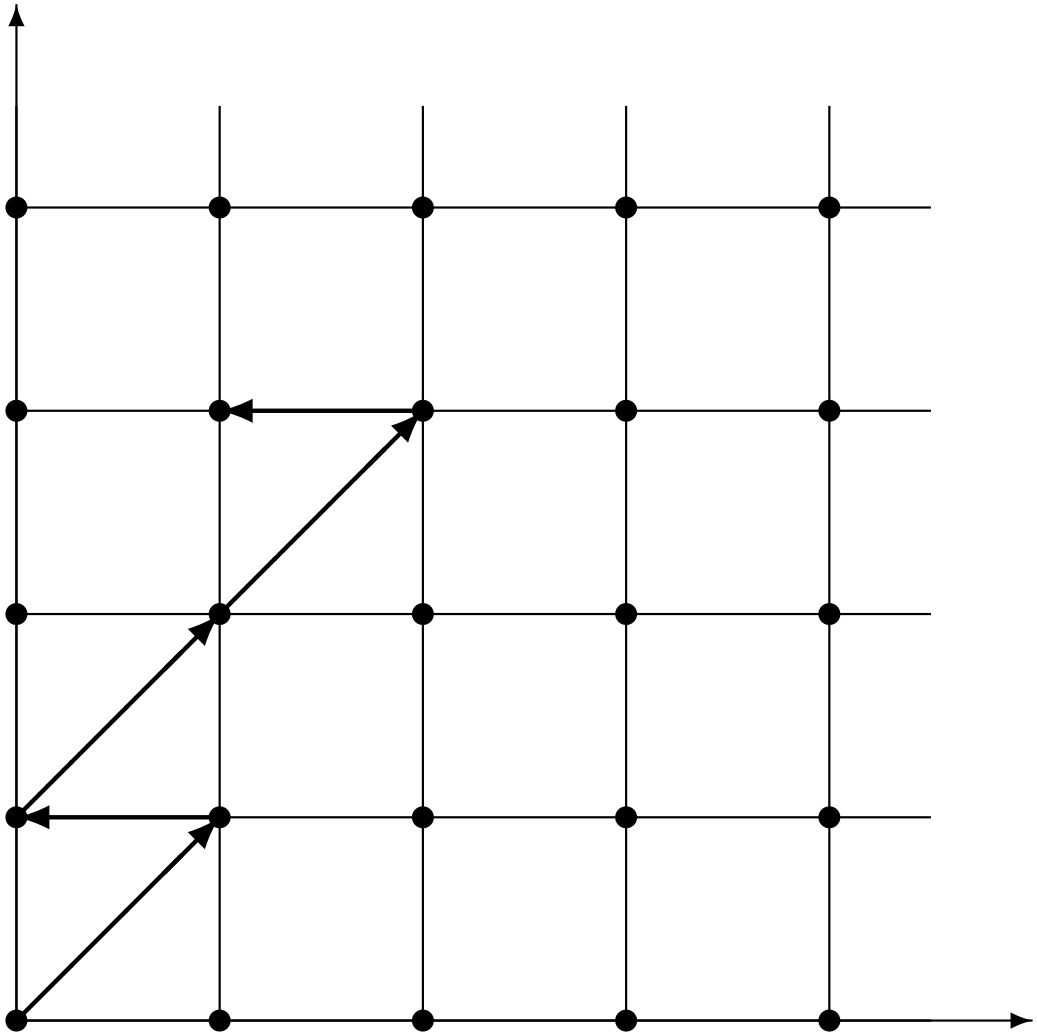

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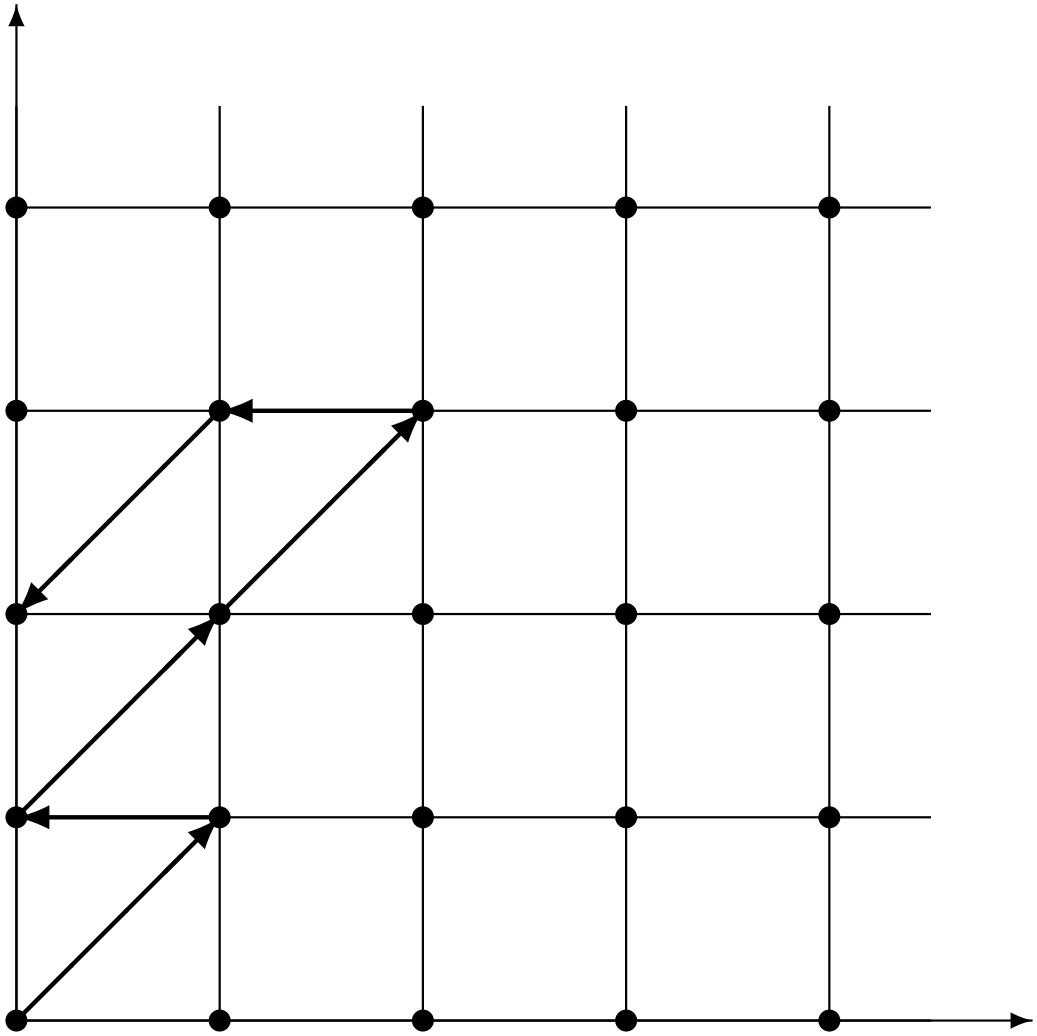

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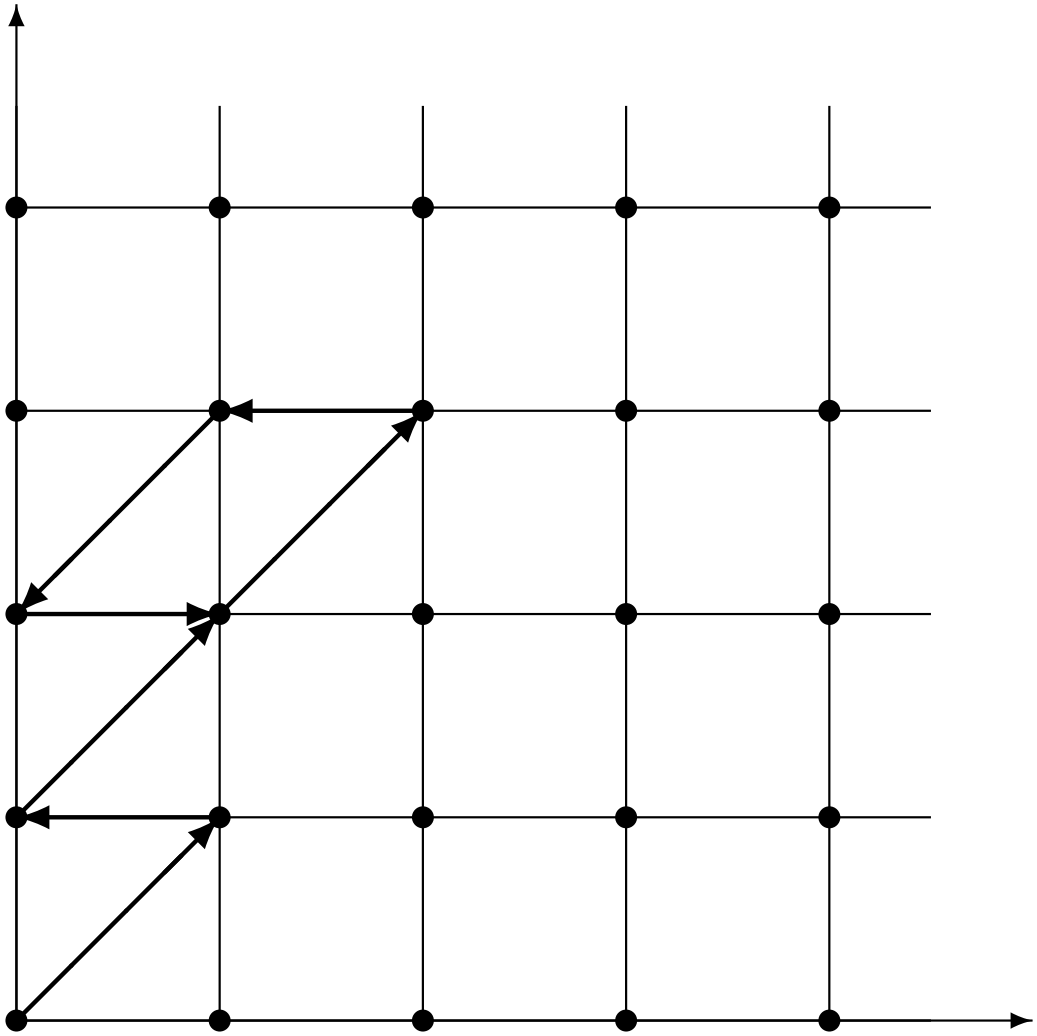

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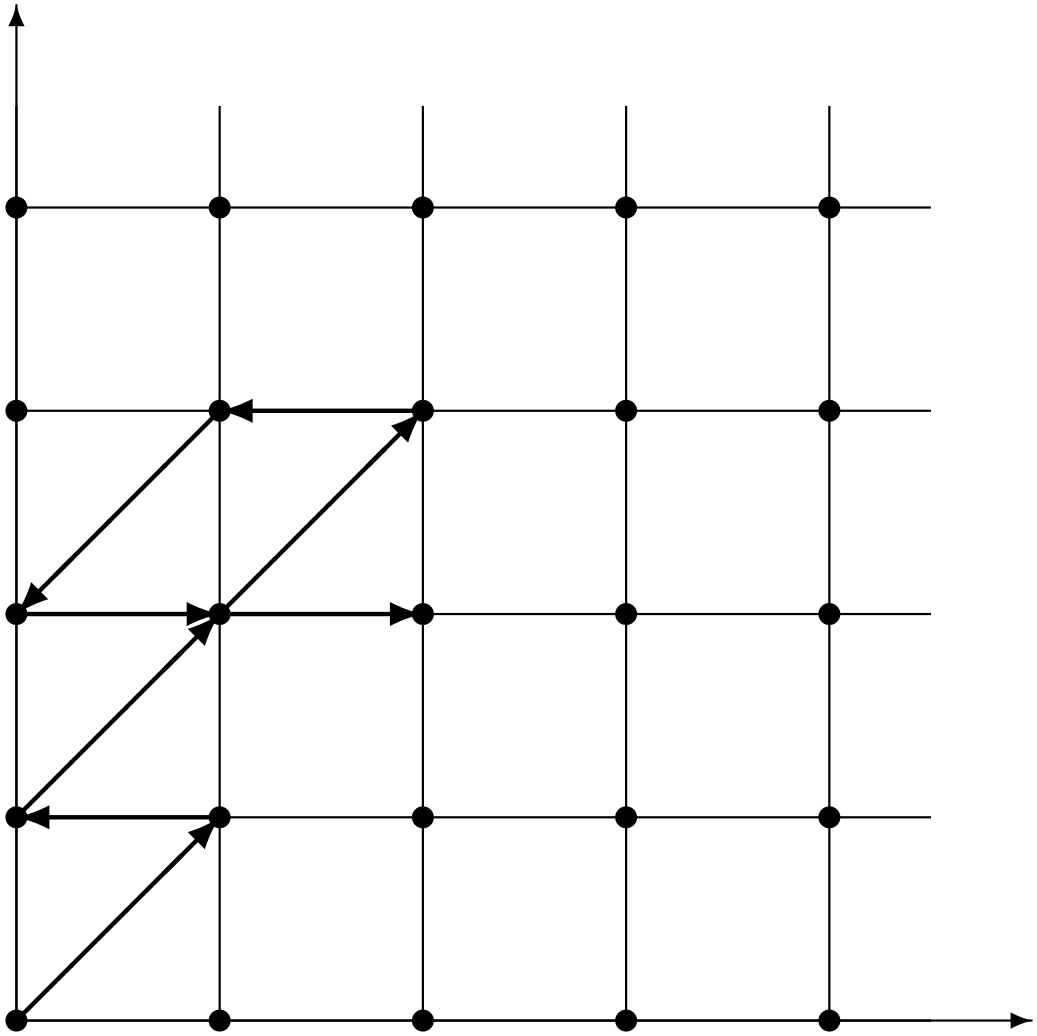

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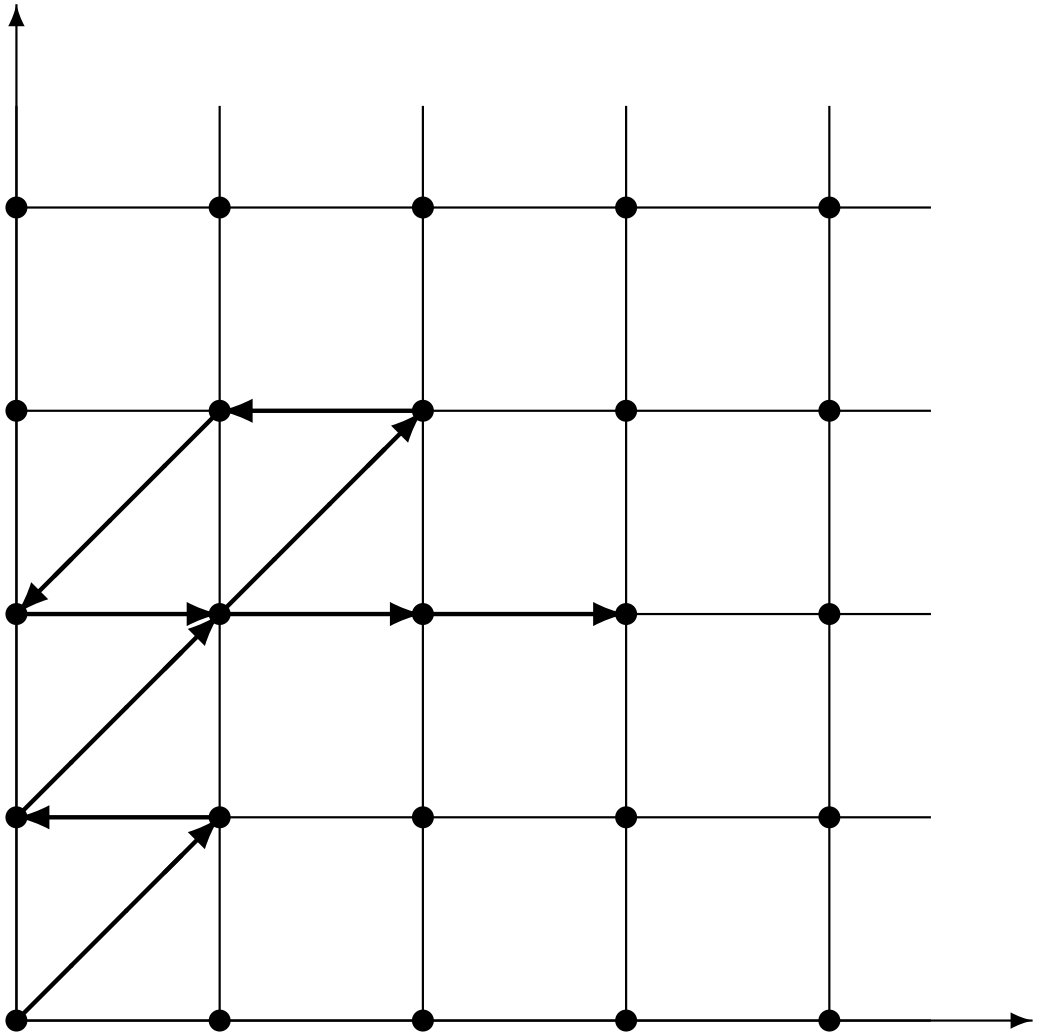

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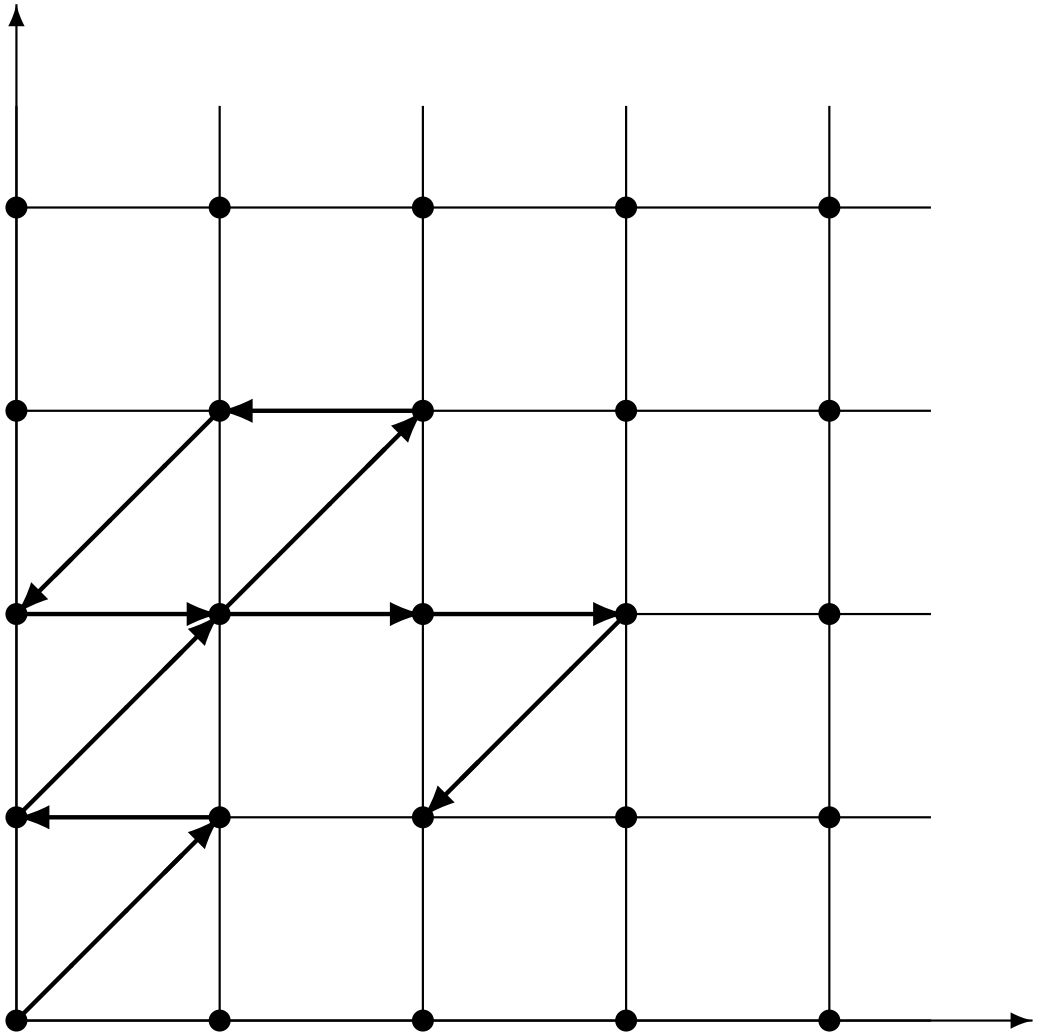

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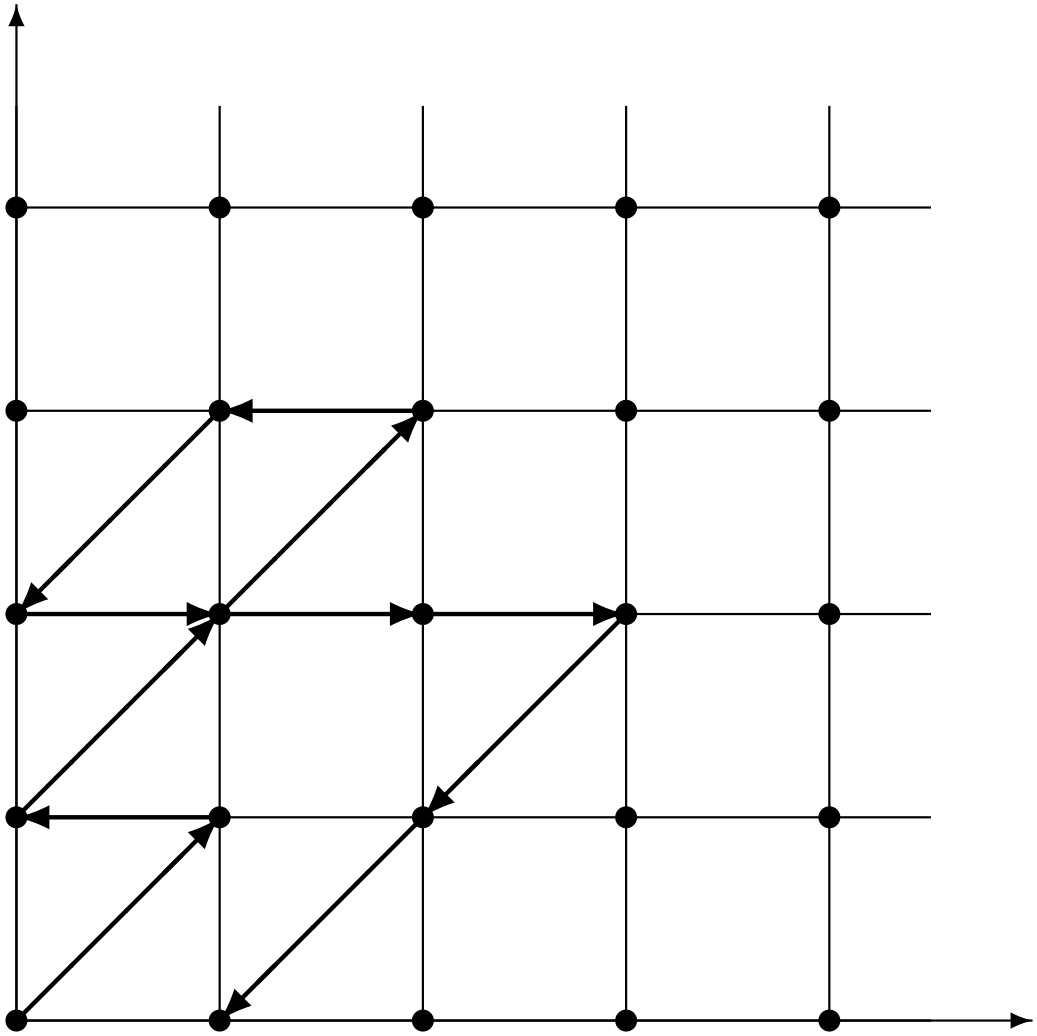

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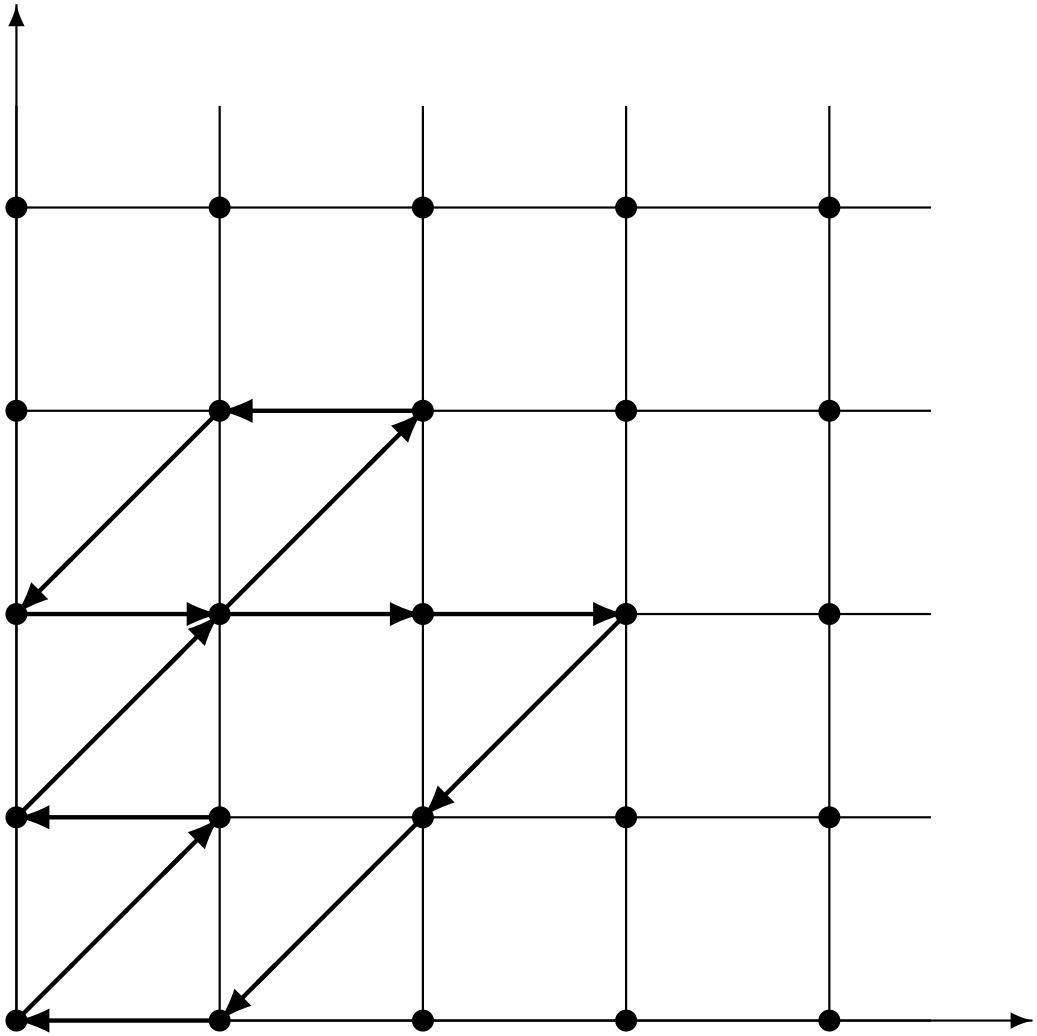

Definition

Let $f(n; i, j)$ denote the number of Gessel walks

- with exactly n steps
- starting at the origin $(0, 0)$
- ending at the point (i, j)

Ira Gessel's conjecture

Ira Gessel in 2001 conjectured that

$$f(n;0,0) = \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

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The function $f(n;0,0)$ counts the number of closed Gessel walks.

Get ready for the proof!

Need: relations (linear recurrences with polynomial coefficients) for $f(n; i, j)$

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The step set $\{\leftarrow, \rightarrow, \nearrow, \swarrow\}$ gives readily rise to the recurrence

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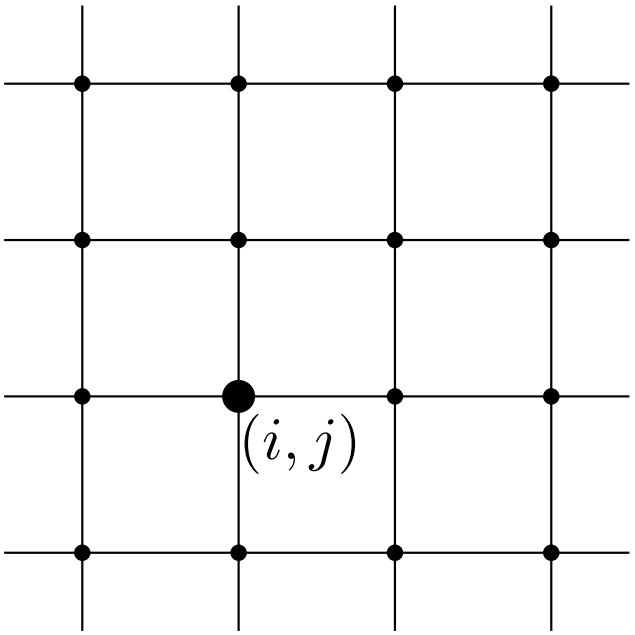

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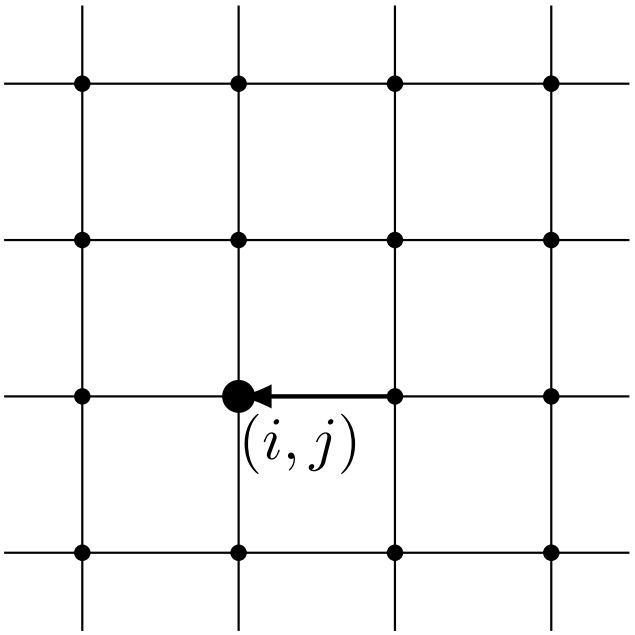

Get ready for the proof!

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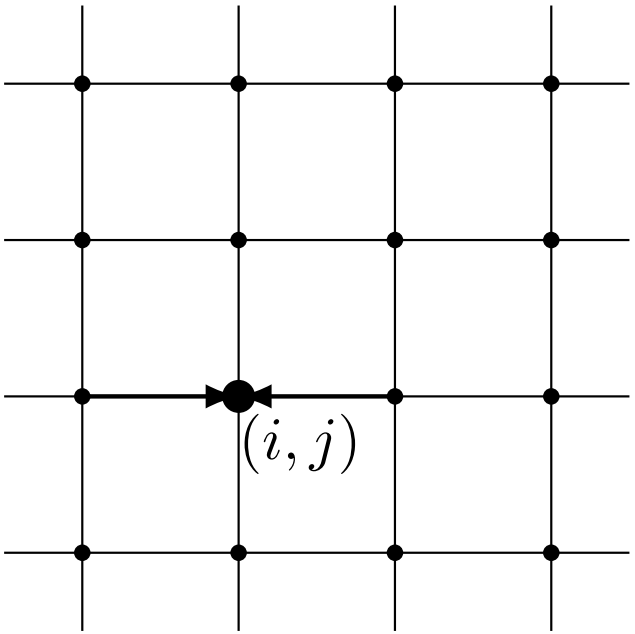
The step set $\{\leftarrow, \rightarrow, \nearrow, \swarrow\}$ gives readily rise to the recurrence

$$f(n+1; i, j) = f(n; i+1, j) + f(n; i-1, j)$$



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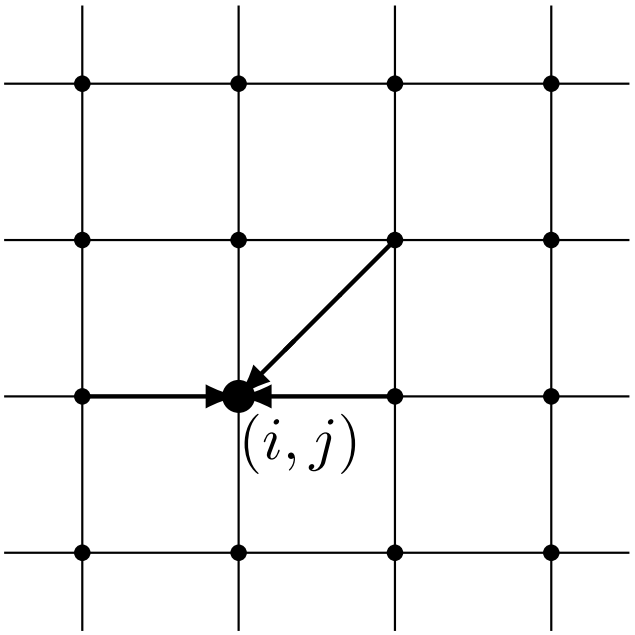

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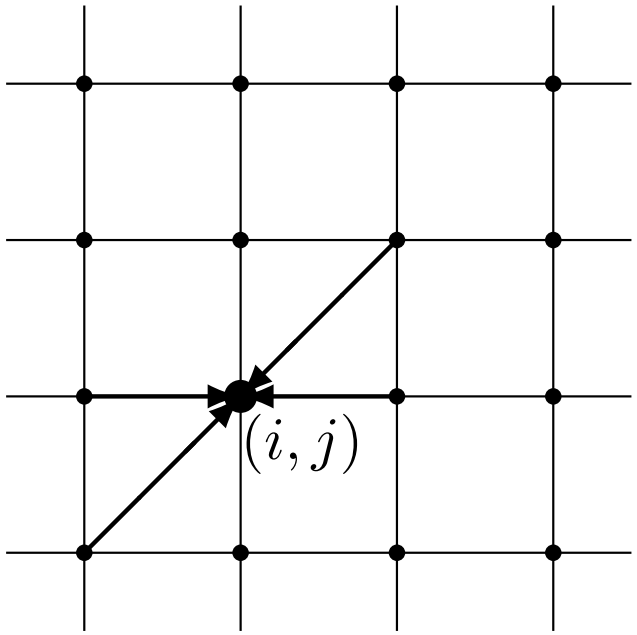

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Remark: We have to prove that the guessed recurrences are indeed correct!

Translation

The recurrence

$$\begin{aligned} f(n+1;i,j) &= f(n;i+1,j)+f(n;i-1,j) \\ &\quad +f(n;i+1,j+1)+f(n;i-1,j-1) \end{aligned}$$

translates to the annihilating Ore operator

$$S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$$

in the Ore algebra $\mathbb{O} = \mathbb{Q}(i,j,n)[S_i;S_i,0][S_j;S_j,0][S_n;S_n,0]$.

Our guessing resulted in a set A of 68 operators:

$$A = \{S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1, (i+1)(i-2j-3n-20)(i-2j-n-12)S_i S_n^3 S_j^4 - 2(i-2j-7)(2i-4j-3n-26)(i-2j-n-12)S_n^2 S_j^4 - 32(i-2j-7)(i-2j-3n-13)(n+1)S_j^4 + 16(i+1)(i^2-4ji-4ni-22i+4j^2-3n^2+44j+8jn+14n+89)S_i S_n S_j^4 - (i-n-4)(i-2j-n-12)(i-j-n-7)S_n^4 S_j^3 + (i+1)(11i^2-12ji-4ni-36i+12j^2+21n^2+104j+8jn+204n+596)S_i S_n^3 S_j^3 - 4(6i^3-24ji^2+2ni^2-70i^2+32j^2 i-9n^2 i+256ji+16jni+19ni+478i-16j^3+8n^3-176j^2+6jn^2+93n^2-544j+58jn+451n-126)S_n^2 S_j^3 - 64(n+1)(2i^2-8ji-3ni-30i+8j^2-4n^2+60j+6jn+3n+96)S_j^3 + 16(i+1)(3i^2-12ji-4ni-42i+12j^2-21n^2+84j+8jn-66n+51)S_i S_n S_j^3 - (i-n-4)(5i^2-4ji-7ni-29i+4j^2+2n^2+20j+5n+16)S_n^4 S_j^2 + (i+1)(11i^2-12ji+4ni+8i+12j^2+21n^2+16j-8jn+164n+376)S_i S_n^3 S_j^2 - 4(4i^3-16ji^2+33ni^2+38i^2+16j^2 i-36n^2 i+56ji-20jni-154ni+8i+16n^3+24j^2+90n^2+120j+20j^2 n+100jn+379n+494)S_n^2 S_j^2 - 64(n+1)(3i^2-12ji-30i+12j^2-8n^2+60j-30n+51)S_j^2 + 16(i+1)(3i^2-12ji+4ni-18i+12j^2-21n^2+36j-8jn-106n-69)S_i S_n S_j^2 + (i-n-4)(j-n-2)(i-2j+n+2)S_n^4 S_j + (i+1)(i-2j+n+2)(i-2j+3n+10)S_i S_n^3 S_j + 4(2i^3-8ji^2-18ni^2-50i^2+16j^2 i+3n^2 i+64ji+16jni+3ni-14i-16j^3-8n^3-64j^2+6jn^2-63n^2+16j+58jn-161n-194)S_n^2 S_j - 64(n+1)(2i^2-8ji+3ni-10i+8j^2-4n^2+20j-6jn-27n-4)S_j + 16(i+1)(i^2-4ji+4ni+2i+4j^2-3n^2-4j-8jn-26n-31)S_i S_n S_j + 2(i-2j-3)(i-2j+n+2)(2i-4j+3n+6)S_n^2 - 32(i-2j-3)(n+1)(i-2j+3n+3), \ldots\}$$

Zeilberger's quasi-holonomic ansatz

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Problem: $R(n,i,j,S_n,S_i,S_j)$ is too big to be computed.

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Remark: The result will be $P(n, S_n)$ as above, but Q_1 and Q_2 are not computed at all.

$\longrightarrow$ Computation becomes feasible!

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S_i P + S_i Q & \neq & S_i P
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6. compute a Gröbner basis of A in this module
7. if no $(P(n, S_n), 0, \ldots, 0)$ is found, increase d_i and d_j

Result

The operator $P(n, S_n)$ annihilating $f(n; 0, 0)$ has

- order 32
- polynomial coefficients of degree 172
- and integer coefficients up to 385 digits.

The computation was done with CK's implementation of noncommutative Gröbner bases and Takayama's algorithm; it took 7 hours.

Make the proof rigorous!

Verify that $P(n, S_n)$ also annihilates $g(n; 0, 0)$ for

$$g(n;0,0) := \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Compare initial values, i.e., $f(n; 0, 0) = g(n; 0, 0)$ for $0 \leq n \leq 31$.

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Make sure that the leading coefficient of $P(n, S_n)$ (and all contents that have been cancelled out during the computation) do not have positive integer roots ($=$ poles).

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How to prove that $R \bullet f = R(n, i, j, S_n, S_i, S_j) \bullet f(n; i, j) = 0$?

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By division with remainder computation we get

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where $T = S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$.

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Once we know that $(TR) \bullet f = 0$, it can be algorithmically decided whether $R \bullet f = 0$.

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- only finitely many values to check!

More conjectures, more proofs (1)

Marko Petkovšek and Herb Wilf conjectured that

$$f(2n;0,1) = 16^n \frac{\left(\frac{1}{2}\right)_n}{(3)_n} \left(\frac{\left(111n^2 + 183n - 50\right)\left(\frac{5}{6}\right)_n}{270\left(\frac{8}{3}\right)_n} + \frac{5\left(\frac{7}{6}\right)_n}{27\left(\frac{7}{3}\right)_n} \right)$$

More conjectures, more proofs (1)

Marko Petkovšek and Herb Wilf conjectured that

$$f(2n;0,1) = 16^n \frac{\left(\frac{1}{2}\right)_n}{(3)_n} \left(\frac{\left(111n^2 + 183n - 50\right)\left(\frac{5}{6}\right)_n}{270\left(\frac{8}{3}\right)_n} + \frac{5\left(\frac{7}{6}\right)_n}{27\left(\frac{7}{3}\right)_n} \right)$$

This conjecture is proven in the same way!

More conjectures, more proofs (2)

Marko Petkovšek and Herb Wilf conjectured that
$g(n) := f(2n+1;1,0)$ satisfies the second order recurrence

$$\begin{aligned} &(n+3)(3n+7)(3n+8)\ g(n+1) \\ -8&(2n+3)(18n^2+54n+35)\ g(n) \\ &+256n(3n+1)(3n+2)\ g(n-1) = 0 \end{aligned}$$

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This conjecture is **dis**proven in the same way!

More conjectures, more proofs (4)

In fact, $h(n) = f(2n; 2, 0)$ satisfies the recurrence

$$4096(n+1)(2n+1)(2n+3)(3n+4)(3n+5)(6n+5)(6n+7)(6144n^7 + 130560n^6 + 1169216n^5 + 5718720n^4 + 16490716n^3 + 28015035n^2 + 25933899n + 10077210)h(n) - 128(2n+3)(31850496n^{13} + 1043103744n^{12} + 15528112128n^{11} + 139066675200n^{10} + 835537836288n^9 + 3554184658752n^8 + 11003992594864n^7 + 25083927328960n^6 + 42052581871616n^5 + 51138759649954n^4 + 43770815405708n^3 + 24915467579665n^2 + 8429189779675n + 1274964941250)h(n+1) + 48(n+4)(15925248n^{13} + 561364992n^{12} + 9001764864n^{11} + 86874808320n^{10} + 562452019584n^9 + 2576877461856n^8 + 8584177057392n^7 + 21020268432120n^6 + 37767656881868n^5 + 49065078284877n^4 + 44671143917844n^3 + 26891118085035n^2 + 9545234776900n + 1498120123500)h(n+2) - 8(n+4)(n+5)(3n+13)(3n+14)(442368n^{10} + 11612160n^9 + 133731840n^8 + 888142080n^7 + 3758533024n^6 + 10562908440n^5 + 19901273510n^4 + 24718969695n^3 + 19263730233n^2 + 8437822050n + 1558180800)h(n+3) + (n+4)(n+5)(n+6)(3n+13)(3n+14)(3n+16)(3n+17)(6144n^7 + 87552n^6 + 514880n^5 + 1616000n^4 + 2911836n^3 + 2992423n^2 + 1606825n + 341550)h(n+4)$$

More recent results

In August 2008, Alin Bostan and Manuel Kauers proved that the trivariate generating function of $f(n; i, j)$ is not only holonomic but even algebraic!

Doron Zeilberger's bet

"I offer a prize of one hundred (100) US-dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel's conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel's conjecture is indeed beyond the scope of humankind."