Two applications of automated holonomic function manipulation:
Simulation of electromagnetic waves
and
Counting lattice paths

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January 27, 2009
Introduction

Past work:

- Zeilberger: “Holonomic systems approach to special function identities” (1990)
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Future work:
- Finish the implementation...
The universe of holonomic functions
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Holonomic

$\delta_{m,n}$

$\partial$-finite

$\frac{1}{k^2 + n^2}$
The universe of holonomic functions
The universe of holonomic functions

- Holonomic
- Pochhammer
- $\partial$-finite
- HarmonicNumber
- Factorial2
- Fibonacci
- Binomial
- CatalanNumber
- Factorial
- $\delta_{m,n}$
- $\frac{1}{k^2 + n^2}$
The universe of holonomic functions

- Pochhammer
- HarmonicNumber
- GegenbauerC
- LucasL
- Binomial
- ChebyshevT
- CatalanNumber
- Factorial
- JacobiP
- LegendreP
- δ_{m,n}
- \frac{1}{k^2 + n^2}
- Factorial2
- Fibonacci
- HermiteH
- ChebyshevU
- LaguerreL

Holonomic

∂-finite
The universe of holonomic functions

Holonomic

Pochhammer

Log
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GegenbauerC
Cos
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CatalanNumber
Factorial
JacobiP
LegendreP

SqIt
Exp
Factorial2
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Sin

\delta_{m,n}

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\text{LaguerreL}
The universe of holonomic functions
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Holonomic

Pochhammer

\( \delta_{m,n} \)

HarmonicNumber

ArcCos

Log

GegenbauerC

Cos

ArcCosh

CatalanNumber

Factorial

Binomial

ArcCot

LaguerreL

Sinh

\( k^2 + n^2 \)

ArcCsc

ArcCoth

ArcTanh

ArcTan

ArcSec

Sqrt

Exp

Factorial2

Fibonacci

ArcCsch

Cosh

HermiteH

ChebyshevT

JacobiP

LegendreP

ArcSinh

Sin
The universe of holonomic functions
The universe of holonomic functions
The universe of holonomic functions

\[ \delta_{m,n} = \frac{1}{k^2 + n^2} \]
Holonomic functions

Let $A_n$ denote the $n$-th Weyl algebra.

**Definition:**
A function $f(x_1, \ldots, x_n)$ is said to be holonomic if $A_n / \text{Ann}_{A_n} f$ is a holonomic $A_n$-module, i.e., the Bernstein dimension is minimal (according to Bernstein’s inequality).
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**Definition:**
A sequence $f(k_1, \ldots, k_n) \in \mathbb{C}^{\mathbb{N}^n}$ is holonomic if its multivariate generating function

$$F(x_1, \ldots, x_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f(k_1, \ldots, k_n)x_1^{k_1} \cdots x_n^{k_n}$$

is a holonomic function.
Properties of holonomic functions

Closure properties:

- sum
- product
- definite integration
- ...

Elimination property:

Given a left ideal \( I \) in \( A^n \) such that \( A^n/I \) is holonomic; then for any choice of \( n+1 \) among the \( 2^n \) generators of \( A^n \) there exists a nonzero operator in \( I \) that depends only on these. In other words, we can eliminate \( n-1 \) variables.
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In order to deal with differentiation and shift operators at the same time we use the generalizing concept of Ore algebras:

- $\mathcal{O} = R[D_x; 1, D_x]$ is an Ore extension by the differential operator $D_x$. 

Example:

$$K[x][D_x; 1, D_x] = A_1$$
Ore algebras (in short)

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- $\mathcal{O} = \mathbb{R}[D_x; 1, D_x]$ is an Ore extension by the differential operator $D_x$.
- $\mathcal{O} = \mathbb{R}[S_n; S_n, 0]$ is an Ore extension by the shift operator $S_n$.

Example: $\mathbb{K}\left[x\right][D_x; 1, D_x] = \mathbb{A}_1$. 

Meaning: Polynomials in $\partial$ with coefficients from $\mathbb{R}$.

Note: Noncommutativity between $\partial$ and the coefficients (determined by $\sigma$ and $\delta$)! E.g., $D_x D_x = x D_x + 1$ and $S_n S_n = n S_n + S_n$. 
Ore algebras (in short)

In order to deal with differentiation and shift operators at the same time we use the generalizing concept of Ore algebras:

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- $\mathcal{O} = R[S_n; S_n, 0]$ is an Ore extension by the shift operator $S_n$.
- $\mathcal{O} = R[\partial; \sigma, \delta]$ is the general notation for an Ore extension. Meaning: Polynomials in $\partial$ with coefficients from $R$.

**Note:** Noncommutativity between $\partial$ and the coefficients (determined by $\sigma$ and $\delta$)! E.g.,

$$D_x x = xD_x + 1 \quad \text{and} \quad S_n n = nS_n + S_n$$

Example: $\mathbb{K}[x][D_x; 1, D_x] = A_1$
Definite integration via elimination

**Given:** Integration bounds \( a, b \in \mathbb{R} \cup \{-\infty, \infty\} \) and \( \text{Ann}_\mathcal{O} f \), the annihilator of a holonomic function \( f(x, y) \) in the Ore algebra \( \mathcal{O} = \mathbb{K}[x, y][D_x; 1, D_x][D_y; 1, D_y] = A_2 \)

**Find:** A differential equation for \( F(y) = \int_a^b f(x, y) \, dx \)
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**Find:** A differential equation for \( F(y) = \int_a^b f(x, y) \, dx \)

Since \( \mathcal{O}/\text{Ann}_\mathcal{O} f \) is holonomic, there exists \( P \in \text{Ann}_\mathcal{O} f \) that does not contain \( x \) (by elimination property). Write

\[ P(y, D_x, D_y) = Q(y, D_y) + D_x \cdot R(y, D_x, D_y) \]
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\[
P(y, D_x, D_y) = Q(y, D_y) + D_x \cdot R(y, D_x, D_y)
\]

Apply \(\int_a^b \ldots \, dx\) to \(P \cdot f = 0\):

\[
\int_a^b Q(y, D_y) \cdot f \, dx + \int_a^b D_x R(y, D_x, D_y) \cdot f \, dx = 0
\]

\[
Q(y, D_y) \cdot F(y) + \left[ R(y, D_x, D_y) \cdot f \right]_{x=a}^{x=b} = 0
\]
\(\partial\)-finite functions

Definition: A function \(f(x_1, \ldots, x_n)\) is called \(\partial\)-finite w.r.t. \(\mathcal{O} = \mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]\) if \(\mathcal{O}/\text{Ann}_\mathcal{O} f\) is a finite-dimensional \(\mathbb{K}(x_1, \ldots, x_n)\)-vector space.

Example: All derivatives (w.r.t. \(x\) and \(y\)) of \(\sin(x + y x - y)\) are of the form
\[r_1(x, y) \sin(x + y x - y) + r_2(x, y) \cos(x + y x - y),\]
where \(r_1, r_2 \in \mathbb{Q}(x, y)\).

For instance,
\[D_3 x D_2 y \cdot \sin(x + y x - y) = 32 \left(3x^4 + 12yx^3 - 30y^2x^2 - 4y^3x + 9y^4\right) (x - y)^9 \sin(x + y x - y) - 16 \left(6x^5 - 33yx^4 + 80y^3x^2 - 54y^4x + 3y^5\right) (x - y)^{10} \cos(x + y x - y)\].
**∂-finite functions**

**Definition:** A function $f(x_1, \ldots, x_n)$ is called ∂-finite w.r.t. $\mathcal{O} = K(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ if $\mathcal{O}/\text{Ann}_\mathcal{O} f$ is a finite-dimensional $K(x_1, \ldots, x_n)$-vector space.

In other words, $f$ is ∂-finite if all its derivatives span a finite-dimensional $K(x_1, \ldots, x_n)$-vector space.
$\partial$-finite functions

**Definition:** A function $f(x_1, \ldots, x_n)$ is called $\partial$-finite w.r.t. $\mathcal{O} = \mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ if $\mathcal{O}/\text{Ann}_{\mathcal{O}} f$ is a finite-dimensional $\mathbb{K}(x_1, \ldots, x_n)$-vector space.

In other words, $f$ is $\partial$-finite if all its derivatives span a finite-dimensional $\mathbb{K}(x_1, \ldots, x_n)$-vector space.

**Example:** All derivatives (w.r.t. $x$ and $y$) of $\sin \left( \frac{x+y}{x-y} \right)$ are of the form

$$r_1(x, y) \sin \left( \frac{x+y}{x-y} \right) + r_2(x, y) \cos \left( \frac{x+y}{x-y} \right), \quad r_1, r_2 \in \mathbb{Q}(x, y)$$

e.g.,

$$D_x^3 D_y^2 \bullet \sin \left( \frac{x+y}{x-y} \right) \quad = \quad \frac{32(3x^4+12yx^3-30y^2x^2-4y^3x+9y^4)}{(x-y)^9} \sin \left( \frac{x+y}{x-y} \right)$$

$$- \quad 16(6x^5-33yx^4+80y^3x^2-54y^4x+3y^5) \cos \left( \frac{x+y}{x-y} \right)$$
Closure properties of $\partial$-finite functions

Closure properties:

- sum
- product
- application of an Ore operator
- algebraic substitution (of a continuous variable)
- subsequence / $\mathbb{Q}$-linear substitution (of a discrete variable)
- definite summation and integration

In contrast to holonomic closure properties, the closure properties for $\partial$-finite functions can be computed quite easily (using linear algebra and an FGLM-like algorithm).
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Creative telescoping: Chyzak’s extension of Zeilberger’s fast algorithm

**Given:** \( \text{Ann}_\mathcal{O} f \), the annihilator of a \( \partial \)-finite function \( f(x, y) \) in the Ore algebra \( \mathcal{O} = \mathbb{K}(x, y)[D_x; 1; D_x][D_y; 1, D_y] \).

**Find:** Operators \( Q(y, D_y) \) and \( R(x, y, D_x, D_y) \) such that \( Q + D_x \cdot R \in \text{Ann}_\mathcal{O} f \).

1. compute a Gröbner basis \( G \) of \( \text{Ann}_\mathcal{O} f \) in order to know the set \( U = \{u_1, \ldots, u_k\} \) of monomials that can not be reduced by \( \text{Ann}_\mathcal{O} f \), i.e., the elements under the stairs of \( G \)

2. make an ansatz for \( Q(y, D_y) = \sum_{i=0}^{d} \eta_i(y) D_y^i \) and \( R(x, y, D_x, D_y) = \sum_{j=1}^{k} \phi_j(x, y) u_j \)

3. reduce \( Q + D_x \cdot R \) with \( G \) and set all coefficients to zero

4. solve the corresponding coupled system of differential equations (for rational solutions)

5. if there is no solution, increase \( d \)
Holonomic vs. $\partial$-finite

holonomic description: $\mathbb{K}[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$

$\partial$-finite description: $\mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$

In a pure differential setting, both notions coincide.

In the shift case there are only some subtle exceptions:

• $\partial^2 + n^2$ is $\partial$-finite but not holonomic.
• $\delta_{i,j}$ is holonomic but not $\partial$-finite.

In practice, we consider only functions that are both holonomic and $\partial$-finite:

• $\partial$-finite: better algorithmic treatment
• holonomic: guarantees termination of algorithms
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Example: One of Olver’s problems (1)

From a letter by Frank Olver to my advisor Peter Paule:

“The writing of DLMF Chapter BS by Leonard Maximon and myself is now largely complete; ... However, a problem has arisen in connection with about a dozen formulas from Chapter 10 of Abramowitz and Stegun for which we have not yet tracked down proofs, and the author of this chapter, Henry Antosiewiecz, died about a year ago. Since it is the editorial policy for the DLMF not to state formulas without indications of proofs, I am hoping that you will be willing to step into the breach and supply verifications by computer algebra methods ... I will fax you the formulas later today ...”
Example: One of Olver’s problems (2)

Prove the following identity:

\[
\frac{1}{z} \sinh \sqrt{z^2 - 2izt} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \sqrt{\frac{\pi}{2z}} I_{\frac{1}{2}-n}(z)
\]
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\]

For the left hand side, we can immediately compute annihilating operators (using closure properties + database):

\[
F[t_,z_] := \text{Sinh[Sqrt[z^2 - 2*I*z*t]]}/z
\]
\[
lhs = \text{Annihilator}[F[t,z], \{\text{Der}[t], \text{Der}[z]\}]
\]
\[
\{(t + iz)D_t + zD_z + 1,
(-z^4 + 3itz^3 + 2t^2z^2)D_z^2 + (-2z^3 + 6itz^2 + 5t^2z)D_z
+(z^4 - 3itz^3 - 3t^2z^2 + it^3z + t^2)\}
\]
Example: One of Olver’s problems (3)

On the right hand side \( \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \sqrt{\frac{\pi}{2z}} I_{\frac{1}{2}-n}(z) \), we perform creative telescoping:

\[
f[n_,t_,z_]:= (-I*t)^n/n!*Sqrt[Pi/2/z]*BesselI[-n+1/2,z];
{opQ, opR} =
    CreativeTelescoping[f[n,t,z], S[n]-1, {Der[t],Der[z]}]
\]

We obtain two operators \( Q_i + (S_n - 1) \cdot R_i \in \text{Ann } f \) where

\[
\begin{align*}
Q_1 &= -t(t + iz)D_t + tzD_z + t, \\
Q_2 &= (t + iz)(2t + iz)z^2D_z^2 - z(-5t^2 - 6itz + 2z^2)D_z \\
    & \quad + i(-iz^4 - 3tz^3 + 3it^2z^2 + t^3z - it^2), \\
R_1 &= -inz, \\
R_2 &= i(n^2 + n)(t + iz)zS_n + 2t^2n^2 - z^2n^2 + 3itzn^2 - t^2n \\
    & \quad + z^2n - 3itzn
\end{align*}
\]
Example: One of Olver’s problems (4)

Next verify that \([R_1 \cdot f]_{n=0} = 0\) and that \(R_1 \cdot f\) tends to 0 when \(n\) goes to infinity (the same for \(R_2\)):

\[
\text{ApplyOreOperator}[\text{opR}, f[n,t,z]] /. n->0
\]

\[
\{0, 0\}
\]

\(\longrightarrow\) The delta part vanishes.
Hence \(Q_1\) and \(Q_2\) are annihilating operators for the sum. In fact, we find that they agree with the annihilating operators that we computed for the left hand side.
Example: One of Olver’s problems (5)

In order to establish equality, we have to compare initial values. Look at the vector space under the stairs of the Gröbner basis:

\[ u = \text{UnderTheStaircase}[\text{lhs}] \]

\[ \{1, D_z\} \]

This means we have to compute two initial values:

\[
\text{ApplyOreOperator}[u, F[t,z]] /. \{t->0,z->1\} //\text{FullSimplify}
\]

\[ \{\sinh(1), \frac{1}{e}\} \]

\[
\text{ApplyOreOperator}[u, f[n,t,z]] /. \{t->0,z->1\}
\]

\[
\begin{array}{l}
0^n \sqrt{\frac{\pi}{2}} I_{\frac{1}{2}-n}(1) - 0^n \sqrt{\frac{\pi}{2}} \left( I_{-n-\frac{1}{2}}(1) + I_{\frac{3}{2}-n}(1) \right) - 0^n \sqrt{\frac{\pi}{2}} I_{\frac{1}{2}-n}(1) \\
\frac{n!}{2n!} - \frac{n!}{2n!}
\end{array}
\]

\%

\[
\text{/. (0^n)->1 /. n->0 // FullSimplify}
\]

\[ \{\sinh(1), \frac{1}{e}\} \]
First Application: Simulation of electromagnetic waves
(joint work with J. Schöberl, RWTH Aachen)

We use the package HolonomicFunctions for finding certain relations between expressions involving orthogonal polynomials (Legendre, Jacobi). These relations are badly needed in numerical simulations with finite element methods.
Finite Element Method (FEM)

Numerical method for finding approximate solutions to partial differential equations on non-trivial domains:
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Numerical method for finding approximate solutions to partial differential equations on non-trivial domains:

Divide the domain into small finite elements, i.e., triangles in the 2D case or tetrahedrons in the 3D case. Approximate the solution by certain basis functions that are defined on each finite element (locally supported piecewise polynomial basis functions).
Simulation of electromagnetic waves

**Task:** Simulate electromagnetic waves using the Maxwell equations:

\[
\frac{dH}{dt} = \text{curl} \ E, \quad \frac{dE}{dt} = -\text{curl} \ H
\]

where \(H\) and \(E\) are the magnetic and the electric field respectively.

**Method:**

In this application we define the basis functions as follows:

\[
\varphi_{i,j}(x, y) := (1 - x)^i P_{2i+1,0}^{(2i+1,0)}(2x - 1)P_i \left( \frac{2y}{1-x} - 1 \right)
\]

In order to solve the above equations, one needs to represent the partial derivatives of \(\varphi_{i,j}(x, y)\) in the basis (i.e., as linear combinations of shifts of the \(\varphi_{i,j}(x, y)\) itself).
First Try

\[
\phi[i_,j_,x_,y_] := 
\\text{LegendreP}[i,2*y/(1-x)-1]*(1-x)^i*\text{JacobiP}[j,2*i+1,0,2*x-1]
\]

\[\text{ann} = \text{Annihilator}[\phi[i,j,x,y], \{\text{Der}[x], S[i], S[j]\}]\]

\[\langle \text{quite big output} \rangle\]

In order to see better the structure of the output, we look only at the support of each operator:

\[
\text{Support @ ann}
\]

\[
\{\{S_j^2, S_j, 1\}, \{S_i S_j, D_x, S_i, S_j, 1\}, \{S_i^2, D_x, S_i, S_j, 1\}, \\
\{D_x S_j, D_x, S_i, S_j, 1\}, \{D_x S_i, D_x, S_i, S_j, 1\}, \{D_x^2, D_x, S_i, S_j, 1\}\}
\]

\[\rightarrow\] We see that the second and the third operator match exactly our needs!
**BUT:** The numerists need a relation that is free of $x$ and $y$! In change, they allow also shifts in the derivative, i.e., we are now looking for a relation of the following form:

$$\sum_{(k,l) \in A} a_{k,l}(i, j) \frac{d}{dx} \varphi_{i+k,j+l}(x, y) = \sum_{(m,n) \in B} b_{m,n}(i, j) \varphi_{i+m,j+n}(x, y),$$

where $A, B \subseteq \mathbb{N}^2$ are finite index sets.
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$$

where $A, B \subset \mathbb{N}^2$ are finite index sets.

- Use Gröbner basis computation in order to eliminate $x$ and $y$.
- After some time we get an operator of the desired form, that is even not too big (about 2 pages).
- Because of extension/contraction problem we can not be sure that we obtain the smallest operator.
Application in Finite Element Methods (4)

\[ \sum_{(k,l) \in A} a_{k,l}(i,j) \frac{d}{dx} \varphi_{i+k,j+l}(x,y) = \sum_{(m,n) \in B} b_{m,n}(i,j) \varphi_{i+m,j+n}(x,y), \]

**New idea:** Similar approach as in creative telescoping.

1. we work in \( \mathcal{O} = \mathbb{K}(i, j, x, y)[D_x; 1, D_x][S_i; S_i, 0][S_j; S_j, 0] \)
2. choose index sets \( A \) and \( B \)
3. reduce the corresponding ansatz with the Gröbner basis of \( \text{Ann}_\mathcal{O} \varphi \)
4. do coefficient comparison w.r.t. \( x \) and \( y \)
5. solve the resulting linear system for \( a_{k,l} \) and \( b_{m,n} \) in \( \mathbb{K}(i, j) \)
With this method, we find in short time the following relation:

\[
(2i + j + 3)(2i + 2j + 7) \frac{d}{dx} \varphi_{i,j+1}(x, y) \\
+ 2(2i + 1)(i + j + 3) \frac{d}{dx} \varphi_{i,j+2}(x, y) \\
-(j + 3)(2i + 2j + 5) \frac{d}{dx} \varphi_{i,j+3}(x, y) \\
+(j + 1)(2i + 2j + 7) \frac{d}{dx} \varphi_{i+1,j}(x, y) \\
-2(2i + 3)(i + j + 3) \frac{d}{dx} \varphi_{i+1,j+1}(x, y) \\
-(2i + j + 5)(2i + 2j + 5) \frac{d}{dx} \varphi_{i+1,j+2}(x, y) = \\
2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i,j+2}(x, y) \\
+ 2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i+1,j+1}(x, y)
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2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i,j+2}(x, y) \\
+2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i+1,j+1}(x, y)
\]

Joachim Schöberl’s answer: “jetzt bin ich echt beeindruckt... Genau so eine Relation brauche ich!”
3D case

**Note:** Using the formulas from the 2D case gave already a speed-up of 20 percent in the numerical simulations!

But we would like to have an analog of the 2D formulae in 3D.
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Problem: Now the basis functions $\varphi(i, j, k, x, y, z)$ contain 6 variables and computations become too big and too slow.
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**Problem:** Now the basis functions $\varphi(i, j, k, x, y, z)$ contain 6 variables and computations become too big and too slow.

**Improvements:**

- In a first step determine only the support of the operator, without the coefficients
- Reduce each monomial separately
- Use modular techniques (tricky because of noncommutative algebra!)
A first result for 3D

On of the supports looks as follows:

\[
\{ S_j S_k^4, S_j^2 S_k^3, S_j S_k^2, S_j S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j^2 S_k^4, S_j S_k^3, S_j S_k^2, S_j S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j^2 S_k^3, S_j^2 S_k^2, S_j S_k^3, S_j S_k^2, S_j S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j^2 S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j S_k^4, S_j S_k^3, S_j S_k^2, S_j S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j S_k^3, S_j S_k^2, S_j S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j S_k^2, S_j S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j S_k, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \\
S_j, D_x S_j S_k^3, D_x S_j^2 S_k, D_x S_j S_k^3, D_x S_j^4, S_j S_k^5, \}
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S_j^2 S_k^4, S_j^3 S_k^3, S_j^4 S_k^2, S_i S_k^5, S_i S_j S_k^4, S_i S_j^2 S_k^3, S_i S_j^3 S_k^2, D_x S_j S_k^4, D_x S_j^2 S_k^3, \\
D_x S_j^3 S_k, D_x S_j^4 S_k, D_x S_i S_k^4, D_x S_i S_j S_k^3, D_x S_i S_j^2 S_k^2, D_x S_i S_j^3 S_k, S_j S_k^6, \\
S_j^2 S_k^5, S_j^3 S_k^4, S_j^4 S_k^3, S_i S_k^6, S_i S_j S_k^5, S_i S_j^2 S_k^4, S_i S_j^3 S_k^3, D_x S_j S_k^5, D_x S_j^2 S_k^4, \\
D_x S_j^3 S_k, D_x S_j^4 S_k, D_x S_i S_k^5, D_x S_i S_j S_k^4, D_x S_i S_j^2 S_k^3, D_x S_i S_j^3 S_k^2, S_j S_k^7, \\
S_j^2 S_k^6, S_j^3 S_k^5, S_j^4 S_k^4, S_i S_k^7, S_i S_j S_k^6, S_i S_j^2 S_k^5, S_i S_j^3 S_k^4, D_x S_j S_k^6, D_x S_j^2 S_k^5, \\
D_x S_j^3 S_k, D_x S_j^4 S_k, D_x S_i S_k^6, D_x S_i S_j S_k^5, D_x S_i S_j^2 S_k^4, D_x S_i S_j^3 S_k^3, S_j S_k^8, \\
S_j^2 S_k^7, S_j^3 S_k^6, S_j^4 S_k^5, D_x S_j S_k^7, D_x S_j^2 S_k^6, D_x S_j^3 S_k^5, D_x S_j^4 S_k^4, D_x S_i S_k^7, \\
D_x S_i S_j S_k^6, D_x S_i S_j^2 S_k^5, D_x S_i S_j S_k^4, D_x S_j S_k^8, D_x S_j^2 S_k^7, D_x S_j^3 S_k^6, \\
D_x S_j^4 S_k^5, D_x S_i S_k^8, D_x S_i S_j S_k^7, D_x S_i S_j^2 S_k^6, D_x S_i S_j^3 S_k^5, D_x S_j S_k^9, \\
D_x S_j^2 S_k^8, D_x S_j^3 S_k^7, D_x S_j^4 S_k^6 \}\n
Joachim Schöberl was impressed but not too happy about these results...
New ideas for the 3D case (1)

**Next idea:** Write $\varphi = u \cdot v \cdot w$ and use the product rule

$$\frac{d\varphi}{dx} = \frac{du}{dx} vw + u \frac{dv}{dx} w + uv \frac{dw}{dx}$$

This means, we have to find a relation between e.g. $uvw$ and $\frac{du}{dx} vw$. 
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Trivial solution: $\text{op}_1 \in \text{Ann } f$ and $\text{op}_2 \in \text{Ann } g$. But since $f$ and $g$ are closely related we expect that there exists something “better”. 
New ideas for the 3D case (2)

The natural way to express a relation like

$$\text{op}_1 \bullet f = \text{op}_2 \bullet g$$

is by introducing the module $M = \mathbb{O} \times \mathbb{O}$ which we let act on $\mathcal{F} \times \mathcal{F}$ by

$$P \bullet f = (P_1, P_2) \bullet (f, g) := P_1 \bullet f + P_2 \bullet g,$$

where $P \in M$, $f \in \mathcal{F} \times \mathcal{F}$

But how to compute a Gröbner basis for the ideal of relations between $f$ and $g$, i.e. the annihilator $\text{Ann}_M(f, g)$?
Assume that $f = uvw$ and $g = \frac{du}{dx}vw$.
We start with $u$ and $u' = \frac{du}{dx}$:
$$\text{Ann}_M(u, u') =$$

$$\big\langle \{(p, 0) | p \in \text{Ann}_\oplus u\} \cup \{(0, p) | p \in \text{Ann}_\oplus u'\} \cup \{(D_x, -1)\} \big\rangle$$

After computing a Gröbner basis of the above, we can perform the closure property “multiplication by $vw$” in a very similar fashion as usual.
Finally we can use the ansatz technique as before in order to find an \( \{x, y, z\} \)-free operator:

\[
-2(1 + 2i)(2 + j)(3 + 2i + j)(7 + 2i + 2j)(5 + i + j + k) \\
(7 + i + j + k)(8 + i + j + k)(8 + 2i + 2j + k)(9 + 2i + 2j + k) \\
(11 + 2i + 2j + 2k)(15 + 2i + 2j + 2k)f(i, j + 1, k + 3) + \\
\vdots \\
\langle 31 \text{ similar terms} \rangle \\
\vdots \\
-2(4 + 2i + j)(5 + 2i + j)(5 + 2i + 2j)(5 + i + j + k) \\
(6 + i + j + k)(8 + i + j + k)(10 + 2i + 2j + k) \\
(11 + 2i + 2j + k)(11 + 2i + 2j + 2k)(15 + 2i + 2j + 2k) \\
g(i + 1, j + 2, k + 3) = 0
\]

where \( f = uvw \) and \( g = \frac{dv}{dx} vw \).
Second Application: Gessel walks

(joint work with M. Kauers and D. Zeilberger)

We use CKs package HolonomicFunctions.m in order to prove an open conjecture about certain lattice paths...
Gessel walks

- walks in the integer lattice $\mathbb{N}^2$
Gessel walks

- walks in the integer lattice $\mathbb{N}^2$
- start at $(0, 0)$
Gessel walks

- walks in the integer lattice $\mathbb{N}^2$
- start at $(0, 0)$
- do not leave $\mathbb{N}^2$
Gessel walks

- walks in the integer lattice \( \mathbb{N}^2 \)
- start at \((0, 0)\)
- do not leave \(\mathbb{N}^2\)
- only certain steps are allowed:

\[
G := \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{\leftarrow, \rightarrow, \swarrow, \nearrow\} \]
Gessel walks — Example
Gessel walks — Example
Gessel walks — Example
Gessel walks — Example
Gessel walks — Example

Diagram showing a path on a grid, illustrating a Gessel walk.
Gessel walks — Example
Gessel walks — Example
Gessel walks — Example
Gessel walks — Example
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Let $f(n; i, j)$ denote the number of Gessel walks

- with exactly $n$ steps
- starting at the origin $(0, 0)$
- ending at the point $(i, j)$
Ira Gessel in 2001 conjectured that

\[ f(n; 0, 0) = \begin{cases} 
16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\
0 & \text{if } n \text{ is odd}
\end{cases} \]
Ira Gessel’s conjecture

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The function \( f(n; 0, 0) \) counts the number of closed Gessel walks.
Get ready for the proof!

**Need:** relations (linear recurrences with polynomial coefficients) for \( f(n; i, j) \)
Get ready for the proof!

**Need:** relations (linear recurrences with polynomial coefficients) for \( f(n; i, j) \)

The step set \( \{←, →, ↑, ↓\} \) gives readily rise to the recurrence

\[
f(n + 1; i, j) = \]

\[
\begin{align*}
&f(n; i + 1, j) + f(n; i - 1, j) + f(n; i, j + 1) + f(n; i, j - 1)
\end{align*}
\]
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The step set \{←, →, ↑, ↓\} gives readily rise to the recurrence

\[
 f(n + 1; i, j) = f(n; i + 1, j)
\]
Get ready for the proof!

**Need:** relations (linear recurrences with polynomial coefficients) for $f(n; i, j)$

The step set $\{\leftarrow, \rightarrow, \uparrow, \downarrow\}$ gives readily rise to the recurrence

$$f(n + 1; i, j) =$$

$$f(n; i + 1, j)$$

$$+ f(n; i - 1, j)$$
Get ready for the proof!

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  f(n + 1; i, j) = \\
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  + f(n; i - 1, j) \\
  + f(n; i + 1, j + 1)
\]
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The step set \{←, →, ↑, ↓\} gives readily rise to the recurrence

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f(n + 1; i, j) =
\]

\[
f(n; i + 1, j) 
+ f(n; i - 1, j) 
+ f(n; i + 1, j + 1) 
+ f(n; i - 1, j - 1)
\]
More recurrences

**Question:** How to find more such recurrences?

**Answer:**
- With guessing!
  - ansatz with unspecified coefficients
  - plug in small values for $n$, $i$, $j$
  - solve the corresponding linear system

**Remark:** We have to prove that the guessed recurrences are indeed correct!
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The recurrence

\[ f(n + 1; i, j) = f(n; i + 1, j) + f(n; i - 1, j) \]
\[ + f(n; i + 1, j + 1) + f(n; i - 1, j - 1) \]

translates to the annihilating Ore operator

\[ S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1 \]

in the Ore algebra \( \mathcal{O} = \mathbb{Q}(i, j, n)[S_i; S_i, 0][S_j; S_j, 0][S_n; S_n, 0] \).
Our guessing resulted in a set $A$ of 68 operators:

$$A = \{ S_n S_i S_j - S_i^2 S_j - S_j S_i^2 - 1, (i+1)(i-2j-3n-20)(i-2j-n-12)S_i S_n^3 S_j^4 - 2(i-2j-7)(2i-4j-3n-26)(i-2j-n-12)S_n^2 S_j^4 - 32(i-2j-7)(i-2j-3n-13)(n+1)S_j^4 + 16(i+1)(i^2-4ji-4ni-22i+4j^2-3n^2+44j+8jn+14n+89)S_i S_n S_j^4 - (i-n-4)(i-2j-n-12)(i-j-n-7)S_n^4 S_j^3 + (i+1)(11i^2-12ji-4ni-36i+12j^2+21n^2+104j+8jn+204n+596)S_i S_n S_j^3 - 4(6i^3-24ji^2+2ni^2-70i^2+32j^2i-9n^2i+256ji+16jni+19ni+478i-16j^3+8n^3-176j^2+6jn^2+93n^2-544j+58jn+451n-126)S_n^2 S_j^3 - 64(n+1)(2i^2-8ji-3ni-30i+8j^2-4n^2+60j+6jn+3n+96)S_j^3 + 16(i+1)(3i^2-12ji-4ni-42i+12j^2-21n^2+84j+8jn-66n+51)S_i S_n S_j^3 - (i-n-4)(5i^2-4ji-7ni-29i+4j^2+2n^2+20j+5n+16)S_n^4 S_j^2 + (i+1)(11i^2-12ji+4ni+8i+12j^2+21n^2+16j-8jn+164n+376)S_i S_n^3 S_j^2 - 4(4i^3-16ji^2+33ni^2+38i^2+16j^2i-36n^2i+56ji-20jni-154ni+8i+16n^3+24j^2+90n^2+120j+20j^2n+100jn+379n+494)S_n^2 S_j^2 - 64(n+1)(3i^2-12ji-30i+12j^2-8n^2+60j-30n+51)S_j^2 + 16(i+1)(3i^2-12ji+4ni-18i+12j^2-21n^2+36j-8jn-106n-69)S_i S_n S_j^2 + (i-n-4)(j-n-2)(i-2j+n+2)S_n^4 S_j + (i+1)(i-2j+n+2)(i-2j+3n+10)S_i S_n^3 S_j + 4(2i^3-8ji^2-18ni^2-50i^2+16j^2i+3n^2i+64ji+16jni+3ni-14i-16j^3-8n^3-64j^2+6jn^2-63n^2+16j+58jn-161n-194)S_n^2 S_j - 64(n+1)(2i^2-8ji+3ni-10i+8j^2-4n^2+20j-6jn-27n-4)S_j + 16(i+1)(i^2-4ji+4ni+2i+4j^2-3n^2-4j-8jn-26n-31)S_i S_n S_j + 2(i-2j-3)(i-2j+n+2)(2i-4j+3n+6)S_n^2 - 32(i-2j-3)(n+1)(i-2j+3n+3), \ldots \}
Note: The operators in $A$ generate a left ideal, namely $\mathfrak{o}\langle A \rangle$, all of whose elements annihilate $f(n; i, j)$.
Zeilberger’s quasi-holonomic ansatz

**Note:** The operators in $A$ generate a left ideal, namely $\mathcal{O}\langle A \rangle$, all of whose elements annihilate $f(n; i, j)$.

**Idea:** Find an operator $R \in \mathcal{O}\langle A \rangle$ of the form

$$R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j)$$
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\]

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- $R(n, i, j, S_n, S_i, S_j)$ annihilates $f(n; i, j)$
- set $i = j = 0$
Note: The operators in $A$ generate a left ideal, namely $\mathfrak{O}<A>$, all of whose elements annihilate $f(n; i, j)$.

Idea: Find an operator $R \in \mathfrak{O}<A>$ of the form

$$R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j)$$

- $R(n, i, j, S_n, S_i, S_j)$ annihilates $f(n; i, j)$
- set $i = j = 0$
- $P(n, S_n)$ annihilates $f(n; 0, 0)$
Zeilberger’s quasi-holonomic ansatz

**Note:** The operators in $A$ generate a left ideal, namely $\mathfrak{O}\langle A \rangle$, all of whose elements annihilate $f(n; i, j)$.

**Idea:** Find an operator $R \in \mathfrak{O}\langle A \rangle$ of the form

$$R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j)$$

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**Problem:** $R(n, i, j, S_n, S_i, S_j)$ is too big to be computed.
Takayama, Chyzak, and Salvy enter the game

\[ R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j) \]

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We apply the following trick:

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2. eliminate \( S_i \) and \( S_j \)

**Remark:** The result will be \( P(n, S_n) \) as above, but \( Q_1 \) and \( Q_2 \) are not computed at all.

\( \rightarrow \) Computation becomes feasible!
How to eliminate?

**Problem:** After setting $i = 0$, no multiplication by $S_i$ is allowed!
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$P + iQ$
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**Example:**

\[
P + iQ \quad \cdot \quad S_i \quad \rightarrow \quad S_iP + (i + 1)S_iQ
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**Example:**

\[
P + iQ \\
\cdot S_i \\
S_i P + (i + 1)S_i Q \quad \text{or} \quad S_i P \\
i = 0 \\
\]
How to eliminate?

**Problem:** After setting $i = 0$, no multiplication by $S_i$ is allowed!

**Example:**

$$P + iQ$$

$$\cdot S_i$$

$i = 0$

$$S_iP + (i + 1)S_iQ$$

$i = 0$

$$S_iP + S_iQ \neq S_iP$$
A variant of a variant of Takayama’s algorithm

Let $A = \{A_1, \ldots, A_m\}$ be a set of annihilating operators.
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3. do the same for $j$
4. $A := A|_{i \to 0, j \to 0}$
5. translate the elements of $A$ to vectors w.r.t. the basis
   \[\{S_i^\alpha S_j^\beta \mid 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\}\]
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   $\{S_i^\alpha S_j^\beta \mid 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\}$, e.g.,
   $S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$ translates to
   $(-1, -1, 0, 0, S_n, 0, 0, -1, -1)$
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A variant of a variant of Takayama’s algorithm

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6. compute a Gröbner basis of $A$ in this module
7. if no $(P(n, S_n), 0, \ldots, 0)$ is found, increase $d_i$ and $d_j$
The operator $P(n, S_n)$ annihilating $f(n; 0, 0)$ has
• order 32
• polynomial coefficients of degree 172
• and integer coefficients up to 385 digits.

The computation was done with CK’s implementation of noncommutative Gröbner bases and Takayama’s algorithm; it took 7 hours.
Make the proof rigorous!

Verify that $P(n, S_n)$ also annihilates $g(n; 0, 0)$ for

$$g(n; 0, 0) := \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Compare initial values, i.e., $f(n; 0, 0) = g(n; 0, 0)$ for $0 \leq n \leq 31$. 
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Make sure that the leading coefficient of $P(n, S_n)$ (and all contents that have been cancelled out during the computation) do not have positive integer roots (= poles).
Don’t forget: Prove correctness of guessed recurrences!

How to prove that $R \cdot f = R(n, i, j, S_n, S_i, S_j) \cdot f(n; i, j) = 0$?

By division with remainder computation we get $TR = UT + V$ where

$$T = S_n S_i S_j - S_2^i S_j - S_j - S_2^i S_2^j - 1.$$ 

Since $(UT) \cdot f = 0$ for sure, we reduced the problem: We have to show that $V \cdot f = 0$ (which is of smaller degree in $n, i, j$).

Once we know that $(TR) \cdot f = 0$, it can be algorithmically decided whether $R \cdot f = 0$. 
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- $R \cdot f$ satisfies the recurrence $T$
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- Initial values are $(R \cdot f)(0; i, j)$
- $f(n; i, j) = 0$ for $i > n$ or $j > n$
- only finitely many values to check!
Marko Petkovšek and Herb Wilf conjectured that

\[ f(2n; 0, 1) = 16^n \frac{\binom{1}{2}^n}{\binom{3}{n}} \left( \frac{(111n^2 + 183n - 50) \binom{5}{6}^n}{270 \binom{8}{3}^n} + \frac{5 \binom{7}{6}^n}{27 \binom{7}{3}^n} \right) \]
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This conjecture is proven in the same way!
More conjectures, more proofs (2)

Marko Petkovšek and Herb Wilf conjectured that $g(n) := f(2n + 1; 1, 0)$ satisfies the second order recurrence

$$(n + 3)(3n + 7)(3n + 8) g(n + 1) - 8(2n + 3)(18n^2 + 54n + 35) g(n) + 256n(3n + 1)(3n + 2) g(n - 1) = 0$$
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This conjecture is proven in the same way!
Marko Petkovšek and Herb Wilf conjectured that $h(n) := f(2n; 2, 0)$ is not holonomic.
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\[ h(n) := f(2n; 2, 0) \]
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This conjecture is disproven in the same way!
In fact, $h(n) = f(2n; 2, 0)$ satisfies the recurrence

$4096(n + 1)(2n + 1)(2n + 3)(3n + 4)(3n + 5)(6n + 5)(6n + 7)(6144n^7 + 130560n^6 +$

$1169216n^5 + 5718720n^4 + 16490716n^3 + 28015035n^2 + 25933899n + 10077210)h(n) - 128(2n +$

$3)(31850496n^{13} + 1043103744n^{12} + 15528112128n^{11} + 139066675200n^{10} + 835537836288n^9 +$

$3554184658752n^8 + 11003992594864n^7 + 25083927328960n^6 + 42052581871616n^5 +$

$51138759649954n^4 + 43770815405708n^3 + 24915467579665n^2 + 8429189779675n +$

$1274964941250)h(n + 1) + 48(n + 4)(15925248n^{13} + 561364992n^{12} + 9001764864n^{11} +$

$86874808320n^{10} + 562452019584n^9 + 2576877461856n^8 + 8584177057392n^7 +$

$21020268432120n^6 + 37767656881868n^5 + 49065078284877n^4 + 44671143917844n^3 +$

$26891118085035n^2 + 9545234776900n + 1498120123500)h(n + 2) - 8(n + 4)(n + 5)(3n +$}

$13)(3n + 14)(442368n^{10} + 11612160n^9 + 133731840n^8 + 888142080n^7 + 3758533024n^6 +$

$10562908440n^5 + 19901273510n^4 + 24718969695n^3 + 19263730233n^2 + 8437822050n +$

$1558180800)h(n + 3) + (n + 4)(n + 5)(n + 6)(3n + 13)(3n + 14)(3n + 16)(3n + 17)(6144n^7 +$

$87552n^6 + 514880n^5 + 1616000n^4 + 2911836n^3 + 2992423n^2 + 1606825n + 341550)h(n + 4)$
More recent results

In August 2008, Alin Bostan and Manuel Kauers proved that the trivariate generating function of $f(n; i, j)$ is not only holonomic but even algebraic!
“I offer a prize of one hundred (100) US-dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel’s conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel’s conjecture is indeed beyond the scope of humankind.”