Constructing Linkages for Drawing Plane Curves

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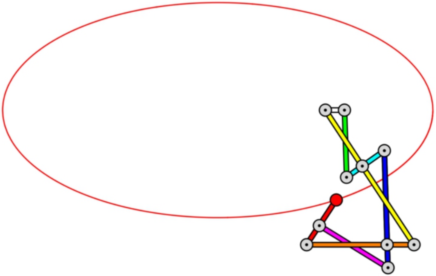
Motivation



Material (paper, pictures, movies) is available at

`http://www.koutschan.de/data/link/`

Linkages

Definition. A **linkage** is a mechanism which consists of

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Restriction: We consider only **planar linkages**, i.e., all links move in parallel planes.

Kempe's Universality Theorem

Goal: For a given planar curve, construct a linkage that draws it.

- Motivation from engineering, dates back to 18th century
- Example: Watt's linkage ("one of the most ingenious simple pieces of mechanics I have invented")

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Theorem. (Kempe 1876)
Let $f \in \mathbb{R}[x, y]$ be a polynomial, and let $B \subseteq \mathbb{R}^2$ be a closed disk. Then there exists a planar linkage which draws the curve

$$B \cap \left\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\right\}.$$

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- Proof of the theorem is constructive.
- "parsing algorithm" with input $f(x,y)$
- Kempe's constructions yield very complicated linkages.
- Can be applied to any algebraic curve.

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3. Answer to the collision problem.
4. A prototype implementation.

Mathematical model for linkages

1. Self-collisions of the links are not taken into account, i.e., the joints are the only constraints for the motion of the links.
2. Thus the actual shape of the links doesn't matter, just the position of the joints.
3. Not a single frame of reference for the configuration of a linkage, but each link has its own frame of reference.

Linkages

Link graph: encodes the “topological information” of a linkage.

- Each link corresponds to a vertex.
- Each joint corresponds to an edge.

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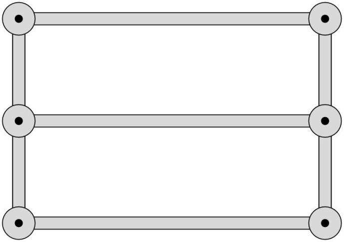

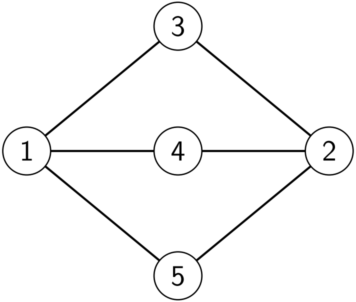

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Definition. A **planar linkage** with revolute joints is a connected undirected graph $G = (V, E)$ without self-loops, together with a map $\rho\colon E \to \mathbb{R}^2$.

- The point $\rho(e)$ is the position of the joint e in the "initial configuration" of the linkage.
- W.l.o.g. assume that V is of the form $\{1, \ldots, n\}$.
- Elements of E are given by unordered pairs $\{i, j\} \subseteq V$.

In the following let $L = (G, \rho)$ with $G = (V, E)$ be a linkage.

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Relative position: Let $\sigma_i, \sigma_j \in \mathrm{SE}_2$ describe the absolute positions of the links i and j. Then $\sigma_{i,j} := \sigma_i \circ \sigma_j^{-1}$ gives the **relative position** of link i w.r.t. link j.

Recapitulation and outlook

Have: Mathematical model for linkages that uses direct isometries in an essential way.

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Outline of solution:

- Embed SE_2 in the real projective space $\mathbb{P}^3_{\mathbb{R}}$.
- Interpret the points in $\mathbb{P}^3_{\mathbb{R}}$ as elements of some ring $\mathbb{K}$.
- The multiplication in $\mathbb{K}$ will correspond to the group operation $\circ$ in SE_2.
- Introduce the polynomial ring $\mathbb{K}[t]$ to describe (rational) **motions**.

Embedding of SE_2 in $\mathbb{P}^3_{\mathbb{R}}$

Definition. The n-dimensional **real projective space** is the set

$$\mathbb{P}^n_{\mathbb{R}} := \big(\mathbb{R}^{n+1} \setminus \{(0,\ldots,0)\}\big)/\sim,$$

where

$$(x_0,\ldots,x_n) \sim (y_0,\ldots,y_n) \;:\Longleftrightarrow \\ \exists\, c \in \mathbb{R}^*\colon (x_0,\ldots,x_n) = c\cdot(y_0,\ldots,y_n).$$

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Write a point in $\mathbb{P}^3_{\mathbb{R}}$ with the coordinates $(x_1 : x_2 : y_1 : y_2)$.

Embedding: We embed SE_2 in $\mathbb{P}^3_{\mathbb{R}}$ as the open subset

$$\mathcal{U} = \mathbb{P}^3_{\mathbb{R}} \setminus \big\{(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_{\mathbb{R}} \mid x_1^2 + x_2^2 = 0\big\}.$$

Action

Let $\sigma \in \mathrm{SE}_2$ be given by the point $(x_1 : x_2 : y_1 : y_2) \in \mathcal{U} \subset \mathbb{P}^3_{\mathbb{R}}$.

The action of σ on a point $(x, y) \in \mathbb{R}^2$ is given by

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Exercise. What kind of isometry is given by $(x_1 : x_2 : 0 : 0)$?

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Exercise. Which points correspond to the identity isometry?

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- If $x_2 = 0$ then we have a pure translation.
- The identity isometry is given by $(x_1 : 0 : 0 : 0)$.

Product

With this action the product in SE_2 becomes a bilinear map:

$$(x_1 : x_2 : y_1 : y_2) \cdot (x'_1 : x'_2 : y'_1 : y'_2) =$$

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Notation 1. Write a point $(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_{\mathbb{R}}$ as a pair of complex numbers $(z, w) = (x_1 + i\,x_2, y_1 + i\,y_2) \in \mathbb{C}^2$.

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$$z\,\eta = \eta\,\overline{z} \quad \text{for all } z \in \mathbb{C} \quad \text{and} \quad \eta^2 = 0.$$

Hence: $$\begin{aligned}(z + \eta\,w) \cdot (z' + \eta\,w') &= z\,z' + z\,\eta\,w' + \eta\,w\,z' + \eta\,w\,\eta\,w' \\ &= z\,z' + \eta\big(\overline{z}\,w' + z'\,w\big).\end{aligned}$$

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Notation 3. Denote by $\mathbb{K}$ the $\mathbb{R}$-algebra $\mathbb{C}[\eta]/\langle i\, \eta + \eta\, i, \eta^2\rangle$, i.e., the ring of dual complex numbers.

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Notation. A rational motion is written as a polynomial $P \in \mathbb{K}[t]$

$$P(t) = Z(t) + \eta\, W(t) \quad Z, W \in \mathbb{C}[t],$$

where $Z = X_1 + i\, X_2$ and $W = Y_1 + i\, Y_2$.

$\longrightarrow$ A polynomial in $\mathbb{K}[t]$ is called a **motion polynomial**.

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$\longrightarrow$ A polynomial in $\mathbb{K}[t]$ is called a **motion polynomial**.

$\longrightarrow$ The multiplication of $P \in \mathbb{K}[t]$ by a real polynomial $R \in \mathbb{R}[t]$ gives a new motion polynomial $RP = PR$, which however describes the **same** rational motion.

Connection to rational curves

Proposition. Let $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ be a rational parametrization,

$$\varphi(t) = \left(\frac{f(t)}{h(t)}, \frac{g(t)}{h(t)}\right), \quad \text{for some } f, g, h \in \mathbb{R}[t],$$

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$\longrightarrow$ If P is monic then $\lim_{t \to +\infty} P(t) = (1 : 0 : 0 : 0) \in \mathbb{P}^3_{\mathbb{R}}$, which corresponds to the identity element in SE_2.

Characterization of simple motions

Lemma. Let $P \in \mathbb{K}[t]$ be a monic motion polynomial of degree 1, i.e., $P(t) = t + i\,x_2 + \eta\,(y_1 + i\,y_2)$ with $x_2, y_1, y_2 \in \mathbb{R}$. Then:

1. if $x_2 = 0$ then P gives a translational motion in direction $\left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$.
2. if $x_2 \neq 0$ then P gives a revolution around the point $\frac{1}{2x_2}\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)$.

$\longrightarrow$ Linear motion polynomials describe exactly those motions that are realized by joints.

Weak and strong realization

Let $L = \big((V, E), \rho\big)$ be a linkage, $\phi\colon \mathbb{R} \to \mathbb{P}^3_{\mathbb{R}}$ a rational motion.

Let $\mathrm{RP}(i, j) \subseteq \mathrm{SE}_2$ denote the set of **relative positions** of link j with respect to the link i.

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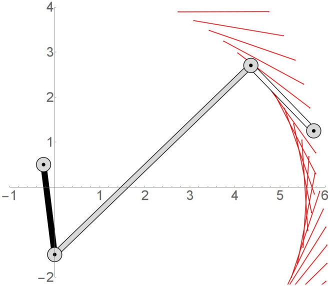

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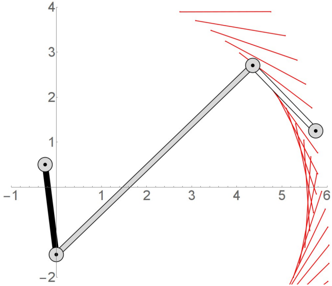

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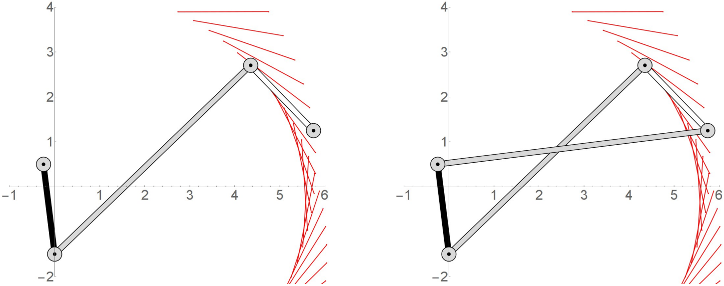

Overview

Task: Construct a linkage that realizes a given rational motion ϕ (special case: that draws a given rational curve).

Solution strategy:

1. The motion ϕ is described by a motion polynomial $P \in \mathbb{K}[t]$.
2. Factor P into linear factors.
3. Each linear factor represents a revolute motion, which can be realized by a single joint.
4. A factorization of P gives rise to an open chain of links, which weakly realizes the motion ϕ.
5. Insert more links in order to restrain the mobility of the linkage so that it strongly realizes the motion ϕ.

Factorization into linear factors

Let $P = Z + \eta W \in \mathbb{K}[t]$ be a monic motion polynomial of degree n.

Goal: Factor P into monic linear motion polynomials, i.e.,

$$P = P_1 \cdot P_2 \cdots P_n, \quad P_i = t - z_i + \eta\, w_i, \quad w_i, z_i \in \mathbb{C}.$$

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By expanding the ansatz we obtain:

$$P = (t - z_1 + \eta\, w_1) \cdot (t - z_2 + \eta\, w_2) \cdots (t - z_n + \eta\, w_n) =$$

$$= \underbrace{\prod_{i=1}^{n} (t - z_i)}_{Z(t)} + \eta \underbrace{\sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} (t - \overline{z_j}) \right) \left(\prod_{j=k+1}^{n} (t - z_j) \right) w_k}_{W(t)} .$$

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⟶ The z_i are precisely the complex roots of $Z(t)$.

⟶ The w_i can be found by ansatz and solving a linear system.

⟶ The order of $z_1, \ldots, z_n$ matters!

No factorization?

Problem: Consider the motion polynomial $t^2 + 1 + \eta$.

- There are two permutations of the z_i: $(i, -i)$ and $(-i, i)$
- For none of them, a solution (w_1, w_2) exists.

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Solution: Multiply P by some real polynomial $R \in \mathbb{R}[t]$!

- Note that this doesn't change the motion itself.
- In our example, we can multiply with $t^2 + 1 = (t + i)(t - i)$ to get the following factorization, corresponding to the permutation $(i, -i, -i, i)$:

$$\left(t + i + \tfrac{1}{2} i\,\eta\right) \cdot \left(t - i\right) \cdot \left(t - i - \tfrac{1}{2} i\,\eta\right) \cdot \left(t + i\right)$$

⟶ Caveat: this trick works only for non-real roots!

Bounded motions

Definition. Let $P = Z + \eta W$ be a motion polynomial. We say that P is **bounded** if it is monic and if Z does not have real roots.

Theorem. Let $P \in \mathbb{K}[t]$ be a bounded motion polynomial. Then there exists a real polynomial $R \in \mathbb{R}[t]$ such that RP can be factored into linear polynomials.

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- This factorization gives rise to a weak linkage realizing the elliptic translation.

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Goal: “Rigidify” the weak linkage in order to get a strong realization (mobility one).

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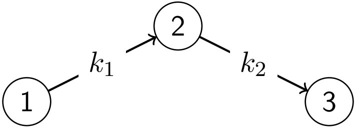

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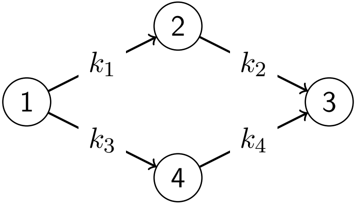

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- Let $k_1 = z_1 + \eta\, w_1$ and $k_2 = z_2 + \eta\, w_2 \quad (z_i, w_i \in \mathbb{C})$.
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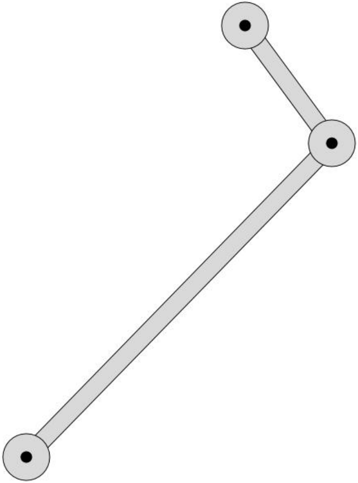

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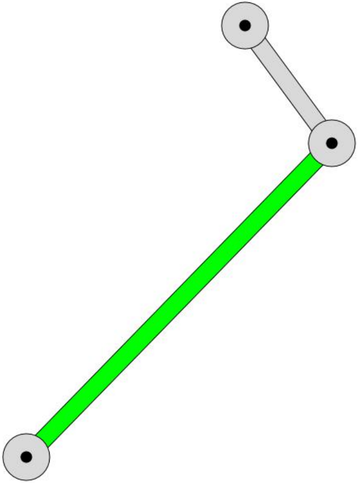

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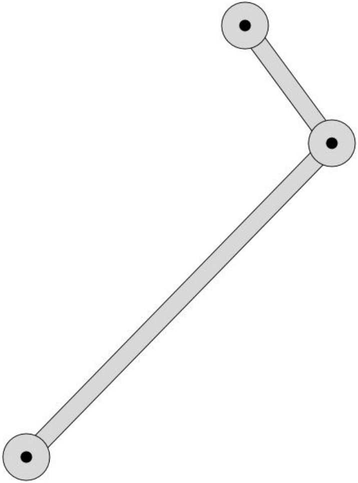

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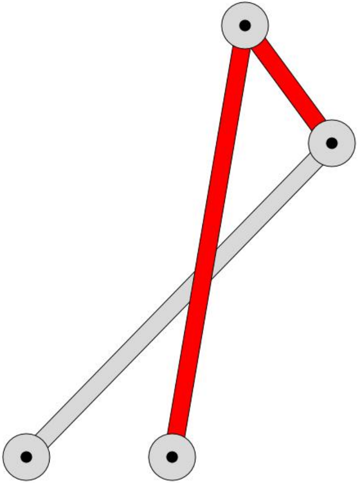

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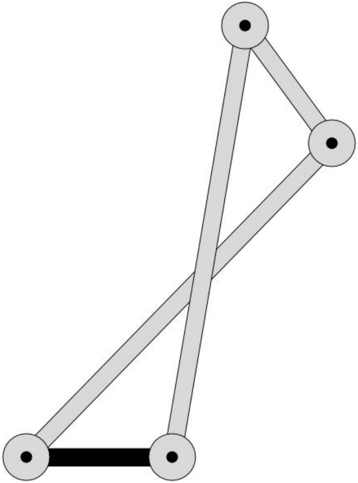
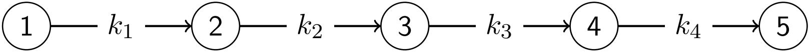
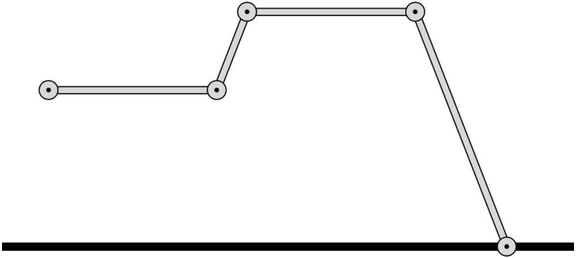


Iterated flips





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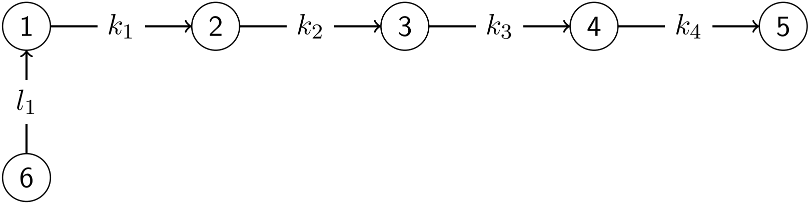

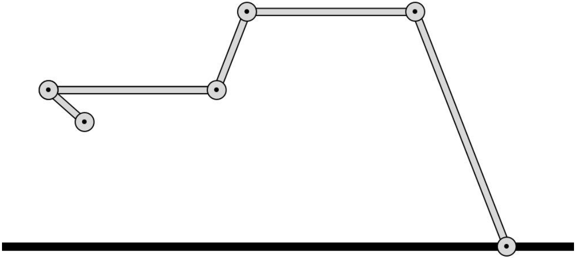

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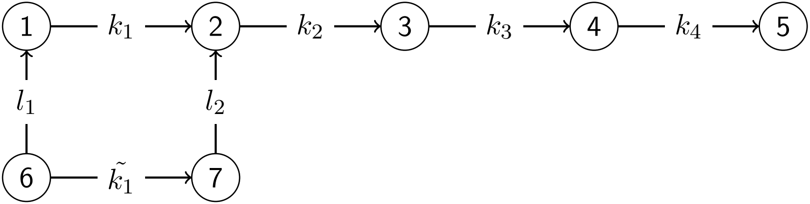

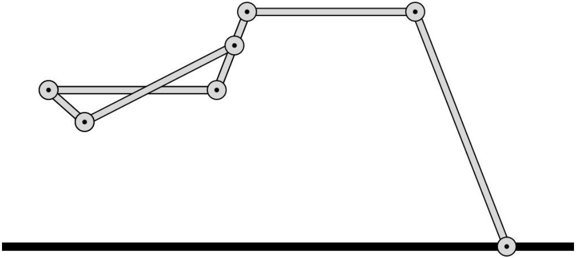

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$$\begin{tikzcd}
① \arrow[r, "k_1"] & ② \arrow[r, "k_2"] & ③ \arrow[r, "k_3"] & ④ \arrow[r, "k_4"] & ⑤ \\
⑥ \arrow[u, "l_1"] \arrow[r, "\tilde{k}_1"] & ⑦ \arrow[u, "l_2"] \arrow[r, "\tilde{k}_2"] & ⑧ \arrow[u, "l_3"] & &
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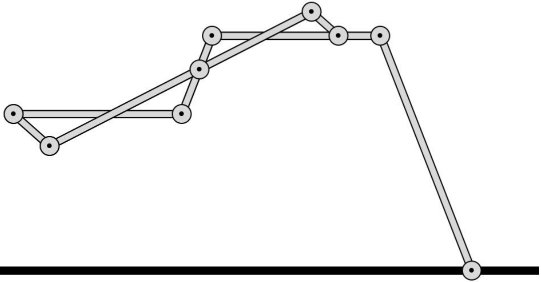

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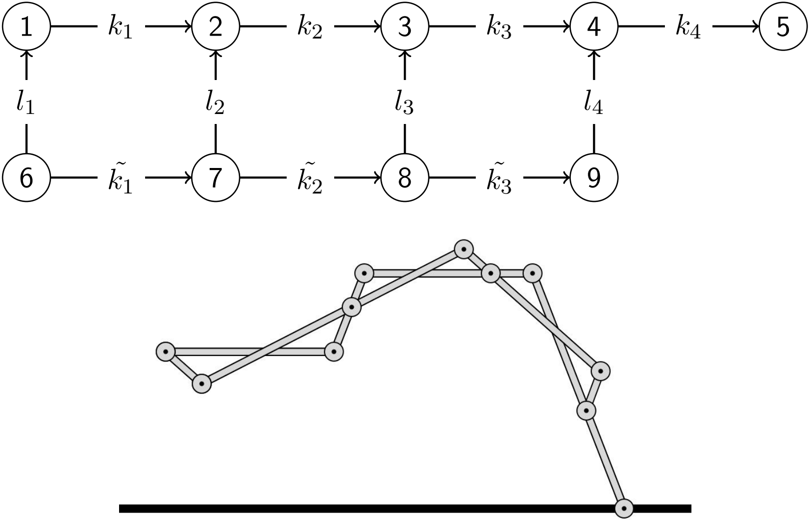

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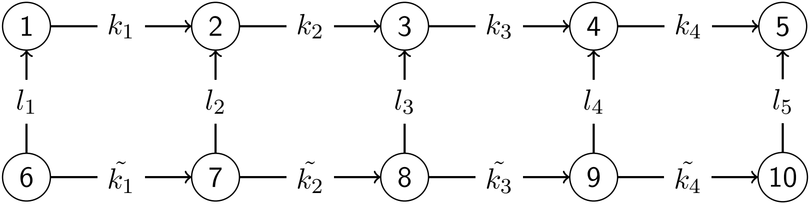

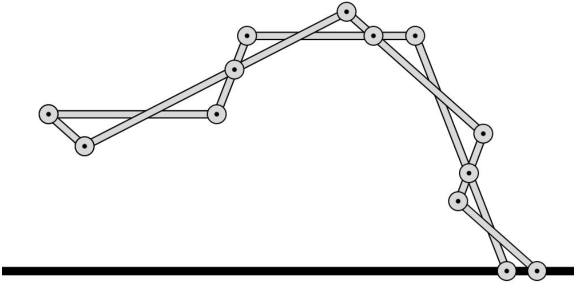

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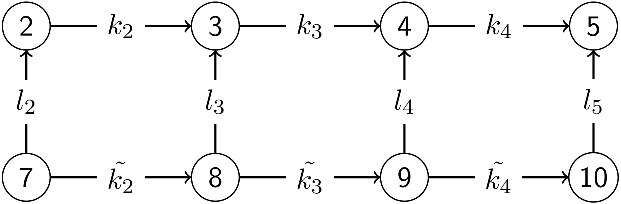

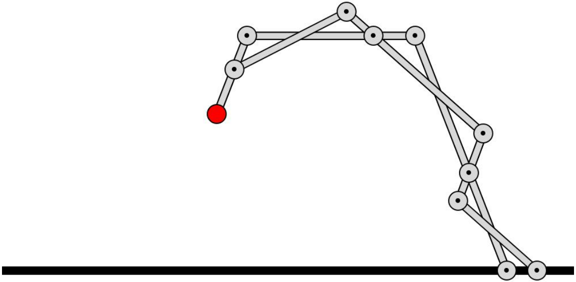

Construction of strong linkages

Theorem. The linkage obtained in this way has mobility one and strongly realizes the motion ϕ.

Corollary. A bounded rational curve given by $(f/h, g/h)$ with $f, g, h \in \mathbb{R}[t]$ such that $\deg h > \max\{\deg f, \deg g\}$ can be drawn by a linkage with at most $4d$ links and $6d-2$ joints ($d = \deg h$).

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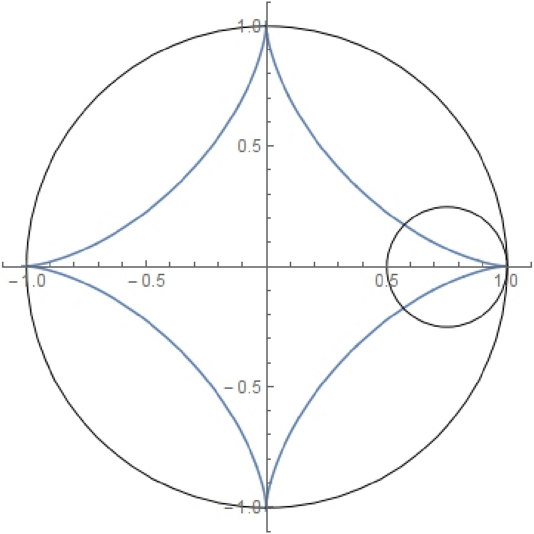
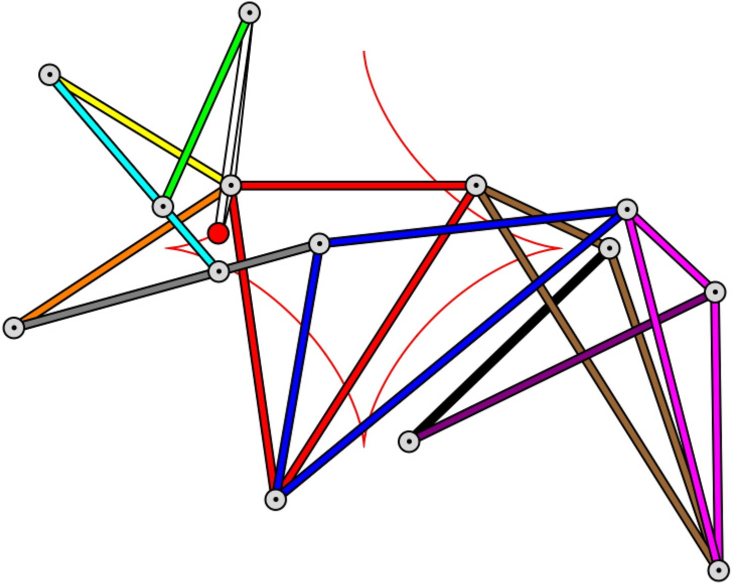
$\longrightarrow$ Show video!

Astroid



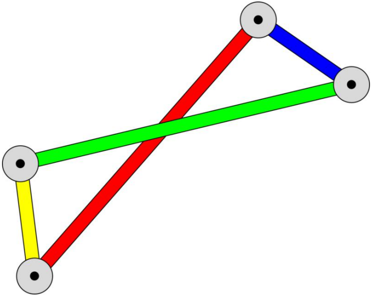
Astroid

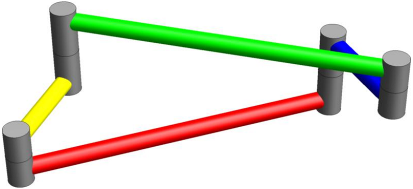


Physical realization

Standard way of realizing planar linkages:

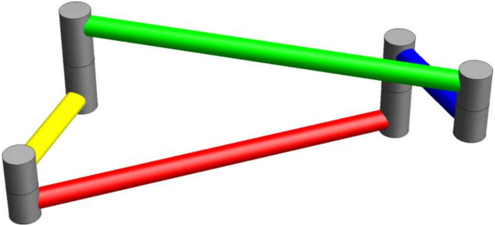
- Each link is realized as a polygon (convex hull of the positions of its joints)
- Each link moves parallel to the horizontal (x, y)-plane at a certain height z.
- The joints are realized as vertical connections between links.





Self-collisions

Vertical arrangement of links is crucial when studying collisions!



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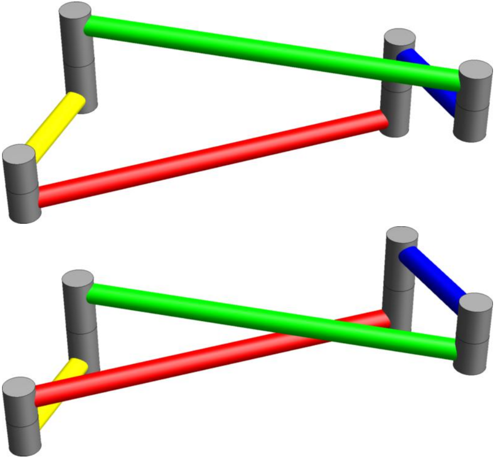

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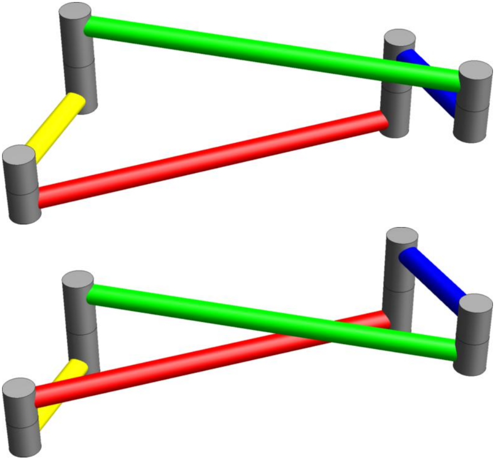

⟶ It suffices to study link–joint collisions.

Self-collisions

W.l.o.g. assume that the link labels $\{1, \ldots, n\}$ correspond to their heights, i.e., their z-coordinates.

Collision:

- links $i < k < j$
- $\big(x_1(t), y_1(t)\big)$ = position of joint (i, j)
- $\big(x_2(t), y_2(t)\big)$ = position of some joint connected to k
- $\big(x_3(t), y_3(t)\big)$ = position of some other joint of k

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- $\big(x_3(t), y_3(t)\big)$ = position of some other joint of k
- a collision happens if

$$\begin{aligned} x_1(t) &= s \cdot x_2(t) + (1 - s) \cdot x_3(t) \\ y_1(t) &= s \cdot y_2(t) + (1 - s) \cdot y_3(t) \end{aligned}$$

for some $t \in \mathbb{R} \cup \{\infty\}$ and $0 \leqslant s \leqslant 1$.

Detect collisions

$$x_1(t) = s \cdot x_2(t) + (1 - s) \cdot x_3(t)$$
$$y_1(t) = s \cdot y_2(t) + (1 - s) \cdot y_3(t)$$

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- Thus the above equations are a bivariate polynomial system.

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- This can be done with reasonably small effort (note that s appears only linearly).

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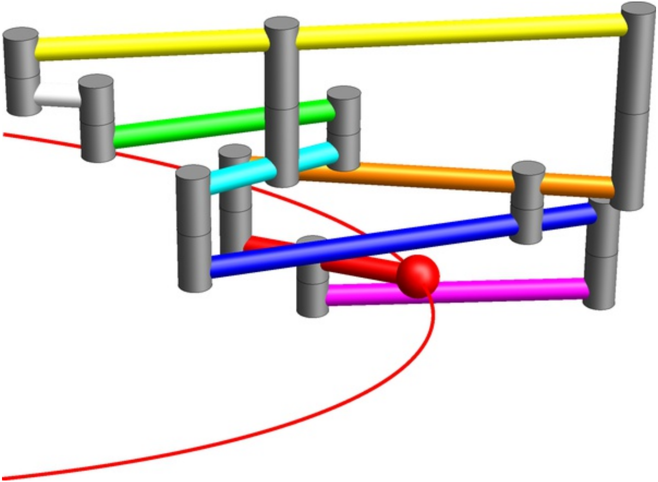
$\longrightarrow$ In contrast to general linkages, our construction allows for a relatively simple collision detection!

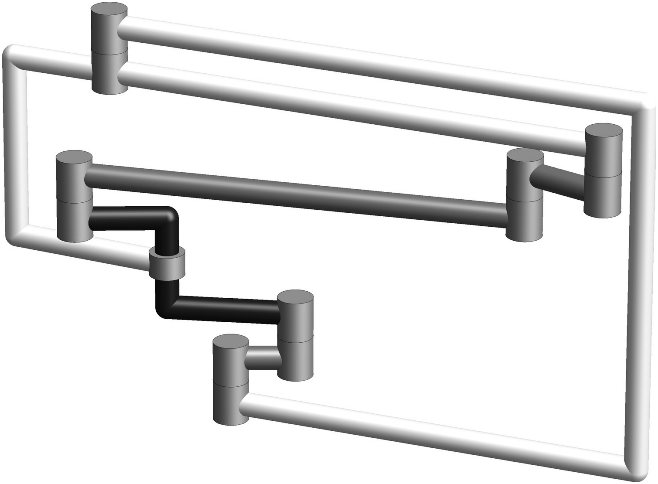
Detect collisions

Example: For our linkage drawing an ellipse we can find a spatial arrangement of the links s.t. only 2 collisions occur (both at $t = \infty$).



Avoid collisions

If we consider links of a more complicated 3-dimensional shape, we can completely avoid collisions:



Collision-free linkage for the ellipse