Some D-finite and Some Possibly D-finite Sequences in the OEIS

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The paper has been published in the Journal in Integer Sequences, and can be found at https://cs.uwaterloo.ca/journals/JIS/ or at https://arxiv.org/abs/2303.02793

```
(* Specify directory where the data files are stored. *)
SetDirectory[NotebookDirectory[]];
(* A procedure to load and display relevant information about a sequence. *)
InitializeSeq[num_] :=
  Module[{str, ord},
   str = StringSplit /@ Import["b" <> ToString[num] <> ".txt", "Lines"];
   data = ToExpression /@ Last /@ str;
   Print["Sequence: ",
    StringTake[ToString[Take[data, 10]], {2, -2}] <> ", ..."];
   Print["Length: ", Length[data]];
   Print["Offset: ", offset = ToExpression[str[[1, 1]]]];
   Print["Recurrence: ", rec = Get["rec" <> ToString[num] <> ".m"]];
   ord = Max[Cases[rec, a[n + nn_.] :> nn, Infinity]];
   Print["Check: ", MatchQ[
     Table[rec, {n, offset, offset + Length[data] - ord - 1}] /.
      a[n_] :> data[[n - offset + 1]], {(0) ..}]];
  ];
(* Load RISC packages,
see http://www3.risc.jku.at/research/combinat/software/ergosum/ *)
<< RISC`GeneratingFunctions`;
<< RISC`Guess`;
<< RISC`HolonomicFunctions`;
```

Package GeneratingFunctions version 0.8 written by Christian Mallinger

Guess Package version 0.52
written by Manuel Kauers

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan

--> Type ?HolonomicFunctions for help.

1 Introduction

1.1 A237684

In[400]:= **InitializeSeq[237 684]**

Sequence: 1, 1, 1, 1, 1, 1, 2, 1, 2, 2, ...

Length: 87

Offset: 1

Recurrence: $(-8+n)\,a[n]-2\,(-7+n)\,a[1+n]+2\,(-5+n)\,a[2+n]+(4-n)\,a[3+n]$

Check: True

1.1 A039836

In[394]:= **InitializeSeq["039836"]**

Sequence: 1, 2, 3, 3, 4, 4, 4, 5, 5, 5, ...

Length: 110

Offset: 1

Recurrence: $(-105+n)\,(-86+n)\,(-71+n)\,(-57+n)\,(-45+n)$
$(-34+n)\,(-24+n)\,(-18+n)\,(-12+n)\,(-7+n)\,(-4+n)\,(-2+n)\,(2+n)$
$(-19\,118\,627\,807\,787\,440\,444\,768\,256\,000\,000-5\,921\,314\,870\,322\,072\,839\,937\,679\,360\,000\,n+$
$2\,755\,504\,839\,874\,945\,047\,375\,112\,704\,000\,n^{2}+2\,567\,674\,525\,768\,477\,198\,945\,236\,748\,800\,n^{3}+$
$998\,317\,619\,951\,792\,167\,167\,842\,626\,560\,n^{4}+259\,668\,745\,699\,000\,784\,144\,248\,751\,328\,n^{5}+$
$51\,136\,849\,720\,615\,825\,916\,617\,067\,984\,n^{6}+7\,988\,693\,394\,029\,647\,950\,933\,106\,496\,n^{7}+$
$1\,006\,523\,849\,819\,551\,686\,143\,539\,256\,n^{8}+102\,927\,634\,295\,702\,918\,201\,233\,526\,n^{9}+$
$8\,578\,120\,751\,776\,773\,794\,736\,653\,n^{10}+584\,828\,145\,350\,837\,868\,461\,633\,n^{11}+$
$32\,714\,418\,827\,113\,895\,968\,502\,n^{12}+1\,503\,774\,856\,041\,096\,897\,944\,n^{13}+$
$56\,766\,787\,605\,820\,969\,607\,n^{14}+1\,754\,041\,295\,884\,809\,335\,n^{15}+$
$44\,072\,002\,668\,578\,588\,n^{16}+890\,963\,551\,249\,022\,n^{17}+14\,265\,151\,289\,291\,n^{18}+$
$176\,762\,589\,647\,n^{19}+1\,637\,275\,094\,n^{20}+10\,712\,180\,n^{21}+44\,465\,n^{22}+89\,n^{23})\,a[n]+$
$(666\,663\,376\,686\,599\,988\,070\,665\,595\,010\,141\,493\,657\,600\,000\,000-$
$304\,438\,779\,642\,565\,249\,749\,707\,227\,360\,268\,843\,483\,136\,000\,000\,n-$
$7\,017\,160\,916\,389\,082\,537\,099\,278\,260\,679\,606\,743\,859\,200\,000\,n^{2}+$
$78\,359\,745\,035\,464\,351\,450\,744\,347\,012\,648\,693\,564\,702\,720\,000\,n^{3}-$
$1\,837\,768\,682\,772\,166\,401\,053\,123\,421\,071\,180\,969\,902\,080\,000\,n^{4}-$
$907\,504\,654\,626\,276\,884\,611\,862\,398\,410\,650\,591\,290\,982\,400\,n^{5}-$
$647\,473\,880\,584\,728\,215\,978\,509\,341\,662\,080\,600\,944\,967\,680\,n^{6}-$
$150\,677\,516\,965\,375\,334\,508\,582\,483\,308\,019\,303\,338\,360\,832\,n^{7}-$
$12\,243\,507\,103\,684\,692\,652\,590\,665\,380\,519\,601\,809\,480\,704\,n^{8}+$

```
719 396 583 264 347 812 575 421 152 225 398 804 628 480 n^9 +
449 306 608 651 436 810 361 760 600 822 330 374 892 800 n^10 +
69 306 000 770 773 595 584 625 213 789 528 364 002 304 n^11 +
3 407 632 047 654 584 274 092 204 396 149 918 811 008 n^12 -
307 797 058 939 805 800 433 498 417 974 230 142 080 n^13 -
51 746 599 988 598 230 144 824 084 286 222 529 120 n^14 -
2 399 607 756 180 837 560 535 054 588 627 178 752 n^15 +
44 863 643 657 130 263 185 988 339 225 432 436 n^16 +
10 725 230 617 437 820 400 568 239 458 889 700 n^17 + 489 032 651 805 384 677 167 434 065 053 025 n^18 +
298 412 204 598 453 406 066 198 528 320 n^19 - 945 546 996 043 261 799 896 911 910 785 n^20 -
38 543 944 176 824 988 933 555 123 600 n^21 - 41 846 786 912 388 397 114 583 100 n^22 +
42 294 120 303 548 917 211 957 280 n^23 + 1 083 076 345 166 505 542 368 260 n^24 -
9 223 466 447 958 324 143 400 n^25 - 799 204 753 314 114 398 370 n^26 - 6 764 655 884 175 671 328 n^27 +
215 087 633 489 655 234 n^28 + 3 968 188 821 915 120 n^29 - 13 361 885 003 500 n^30 -
751 735 996 704 n^31 - 3 852 795 408 n^32 + 50 958 180 n^33 + 735 945 n^34 - 288 n^35 - 41 n^36) a[1 + n] -
6 (-104 + n) (-85 + n) (-70 + n) (-56 + n) (-44 + n) (-33 + n) (-23 + n)
(-17 + n) (-11 + n) (-6 + n) (-3 + n) (3 + n)
(-656 133 783 482 188 158 271 488 000 000 + 777 401 272 860 289 728 705 331 200 000 n -
3 294 969 365 791 002 527 839 604 736 000 n^2 - 1 846 073 481 839 903 766 226 069 094 400 n^3 -
510 620 892 744 983 872 109 276 897 280 n^4 - 91 449 798 964 328 921 265 056 240 640 n^5 -
10 832 030 737 237 496 329 693 865 856 n^6 - 578 491 551 413 118 315 861 674 688 n^7 +
82 574 355 195 238 576 161 409 568 n^8 + 27 211 740 058 742 098 381 150 480 n^9 +
4 155 573 338 482 503 924 247 704 n^10 + 438 529 000 931 352 446 633 452 n^11 +
34 911 406 086 783 683 580 518 n^12 + 2 171 441 844 377 729 600 775 n^13 +
107 429 934 352 224 596 924 n^14 + 4 270 017 538 910 405 347 n^15 + 137 010 618 961 991 388 n^16 +
3 549 515 358 645 070 n^17 + 73 849 917 321 464 n^18 + 1 218 207 626 202 n^19 +
15 563 765 798 n^20 + 148 144 315 n^21 + 985 764 n^22 + 4087 n^23 + 8 n^24) a[2 + n]
```

Check: True

1.2 A187990

In[∘]:= **InitializeSeq[187 990]**

Sequence: 117, 181, 260, 355, 467, 597, 746, 915, 1105, 1317, ...

Length: 50

Offset: 1

Recurrence:

$(27 - n)(-26 + n)(702 + 341 n + 42 n^2 + n^3)\, a[n] + (-27 + n)(-26 + n)(402 + 260 n + 39 n^2 + n^3)\, a[1 + n]$

Check: True

In[∘]:= **rec1 = Collect[rec / ((n - 26) (n - 27)), a[_], Factor]**

Out[∘]= $(-702 - 341 n - 42 n^2 - n^3)\, a[n] + (402 + 260 n + 39 n^2 + n^3)\, a[1 + n]$

In[∘]:= `Table[rec1, {n, 1, 49}] /. a[n_] :> data[[n]]`

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4 496 976, 5 298 480, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[∘]:= `RSolve[{rec1 == 0, a[1] == 117}, a[n], n]`

Out[∘]= $\left\{\left\{a[n] \to \frac{1}{6}\left(402 + 260\, n + 39\, n^2 + n^3\right)\right\}\right\}$

In[∘]:= `Table[%[[1, 1, 2]], {n, 27}]`

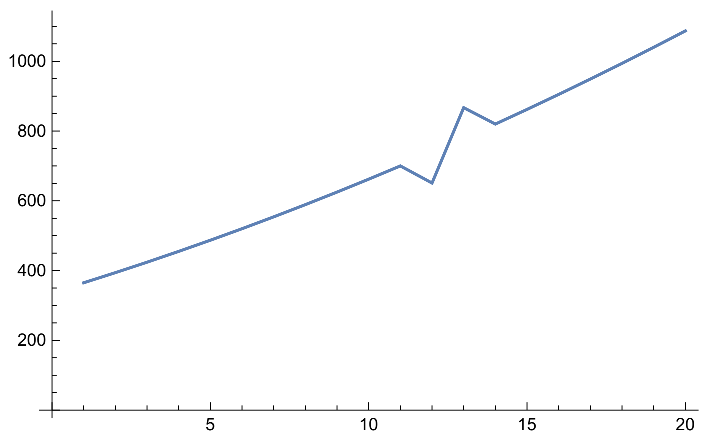
Out[∘]= {117, 181, 260, 355, 467, 597, 746, 915, 1105, 1317, 1552, 1811, 2095, 2405, 2742, 3107, 3501, 3925, 4380, 4867, 5387, 5941, 6530, 7155, 7817, 8517, 9256}

In[∘]:= `Take[data, 27]`

Out[∘]= {117, 181, 260, 355, 467, 597, 746, 915, 1105, 1317, 1552, 1811, 2095, 2405, 2742, 3107, 3501, 3925, 4380, 4867, 5387, 5941, 6530, 7155, 7817, 8517, 9168}

In[∘]:= `ListPlot[Differences[Take[data, {15, 35}]], Joined → True]`

Out[∘]=



In[∘]:=
```
(* Assuming that n denotes the maximal possible value,
substitute n → n+4 to get OEIS sequence. *)
Enumerate187990[n_] :=
  Sort /@ DeleteCases[Join[
     Flatten[
      Table[{-x1, -x2, -x3, x3, x2, x1}, {x1, 1, n}, {x2, x1, n}, {x3, x2, n}], 2],
     Flatten[Table[{-x1, -x2, x2 - 1, x2 - 1, x1 - 1, x1 - 1},
       {x1, 1, n}, {x2, 1, x1}], 1],
     Flatten[Table[If[x1 ≠ x2 + 1, {-x1, -x2, -x2, x2 + 1, x1 - 1, x1 - 1}, {}],
       {x1, 1, n}, {x2, 1, n - 1}], 1],
     Flatten[Table[{-x1, -x1, -x2, -x2, x2 + 1, x1 + 1},
       {x1, 1, n - 1}, {x2, 1, x1}], 1],
     Flatten[Table[If[x1 ≠ x2 + 1, {-x1, -x2, x2, x1 - 2, x1 - 2, x1 - 1}, {}],
       {x1, 2, n}, {x2, 1, n}], 1],
     Flatten[Table[If[x1 ≠ x2, {-x1, -x1 + 1, -x1 + 1, -x2, x2, x1 + 1}, {}],
       {x1, 2, n - 1}, {x2, 1, n}], 1],
     Table[{-x1 - 3, x1, x1, x1, x1, x1 + 2}, {x1, 0, n - 3}],
     Table[{-x1 - 3, x1, x1, x1 + 1, x1 + 1, x1 + 1}, {x1, 0, n - 3}],
     Table[{-x1 - 4, x1, x1, x1 + 1, x1 + 2, x1 + 3}, {x1, 0, n - 4}],
     Table[{-x1 - 2, -x1, -x1, -x1, -x1, x1 + 3}, {x1, 1, n - 3}],
     Table[{-x1 - 1, -x1 - 1, -x1 - 1, -x1, -x1, x1 + 3}, {x1, 1, n - 3}],
     Table[{-x1 - 3, -x1 - 2, -x1 - 1, -x1, -x1, x1 + 4}, {x1, 1, n - 4}]
    ], {}];
Enumerate187990[5]
```

Out[∘]= {{-1, -1, -1, 1, 1, 1}, {-2, -1, -1, 1, 1, 2}, {-3, -1, -1, 1, 1, 3},
{-4, -1, -1, 1, 1, 4}, {-5, -1, -1, 1, 1, 5}, {-2, -2, -1, 1, 2, 2},
{-3, -2, -1, 1, 2, 3}, {-4, -2, -1, 1, 2, 4}, {-5, -2, -1, 1, 2, 5},
{-3, -3, -1, 1, 3, 3}, {-4, -3, -1, 1, 3, 4}, {-5, -3, -1, 1, 3, 5},
{-4, -4, -1, 1, 4, 4}, {-5, -4, -1, 1, 4, 5}, {-5, -5, -1, 1, 5, 5},
{-2, -2, -2, 2, 2, 2}, {-3, -2, -2, 2, 2, 3}, {-4, -2, -2, 2, 2, 4},
{-5, -2, -2, 2, 2, 5}, {-3, -3, -2, 2, 3, 3}, {-4, -3, -2, 2, 3, 4},
{-5, -3, -2, 2, 3, 5}, {-4, -4, -2, 2, 4, 4}, {-5, -4, -2, 2, 4, 5},
{-5, -5, -2, 2, 5, 5}, {-3, -3, -3, 3, 3, 3}, {-4, -3, -3, 3, 3, 4},
{-5, -3, -3, 3, 3, 5}, {-4, -4, -3, 3, 4, 4}, {-5, -4, -3, 3, 4, 5},
{-5, -5, -3, 3, 5, 5}, {-4, -4, -4, 4, 4, 4}, {-5, -4, -4, 4, 4, 5},
{-5, -5, -4, 4, 5, 5}, {-5, -5, -5, 5, 5, 5}, {-1, -1, 0, 0, 0, 0},
{-2, -1, 0, 0, 1, 1}, {-2, -2, 1, 1, 1, 1}, {-3, -1, 0, 0, 2, 2}, {-3, -2, 1, 1, 2, 2},
{-3, -3, 2, 2, 2, 2}, {-4, -1, 0, 0, 3, 3}, {-4, -2, 1, 1, 3, 3}, {-4, -3, 2, 2, 3, 3},
{-4, -4, 3, 3, 3, 3}, {-5, -1, 0, 0, 4, 4}, {-5, -2, 1, 1, 4, 4}, {-5, -3, 2, 2, 4, 4},
{-5, -4, 3, 3, 4, 4}, {-5, -5, 4, 4, 4, 4}, {-1, -1, -1, 0, 0, 2},
{-2, -2, -1, 0, 0, 3}, {-3, -3, -1, 0, 0, 4}, {-4, -4, -1, 0, 0, 5},
{-2, -2, -2, 1, 1, 3}, {-3, -3, -2, 1, 1, 4}, {-4, -4, -2, 1, 1, 5},
{-3, -1, -1, 2, 2, 2}, {-3, -3, -3, 2, 2, 4}, {-4, -4, -3, 2, 2, 5},
{-4, -1, -1, 2, 3, 3}, {-4, -2, -2, 3, 3, 3}, {-4, -4, -4, 3, 3, 5},
{-5, -1, -1, 2, 4, 4}, {-5, -2, -2, 3, 4, 4}, {-5, -3, -3, 4, 4, 4},
{-1, -1, -1, -1, 2, 2}, {-2, -2, -1, -1, 2, 3}, {-2, -2, -2, -2, 3, 3},
{-3, -3, -1, -1, 2, 4}, {-3, -3, -2, -2, 3, 4}, {-3, -3, -3, -3, 4, 4},
{-4, -4, -1, -1, 2, 5}, {-4, -4, -2, -2, 3, 5}, {-4, -4, -3, -3, 4, 5},
{-4, -4, -4, -4, 5, 5}, {-2, -2, 0, 0, 1, 2}, {-3, -2, 0, 0, 1, 3}, {-4, -2, 0, 0, 1, 4},
{-5, -2, 0, 0, 1, 5}, {-3, -1, 1, 1, 1, 2}, {-3, -3, 1, 1, 2, 3}, {-4, -3, 1, 1, 2, 4},
{-5, -3, 1, 1, 2, 5}, {-4, -1, 1, 2, 2, 3}, {-4, -2, 2, 2, 2, 3}, {-4, -4, 2, 2, 3, 4},
{-5, -4, 2, 2, 3, 5}, {-5, -1, 1, 3, 3, 4}, {-5, -2, 2, 3, 3, 4}, {-5, -3, 3, 3, 3, 4},
{-5, -5, 3, 3, 4, 5}, {-2, -1, -1, -1, 1, 3}, {-3, -2, -1, -1, 3, 3},
{-4, -2, -1, -1, 3, 4}, {-5, -2, -1, -1, 3, 5}, {-3, -2, -2, -1, 1, 4},
{-3, -2, -2, -2, 2, 4}, {-4, -3, -2, -2, 4, 4}, {-5, -3, -2, -2, 4, 5},
{-4, -3, -3, -1, 1, 5}, {-4, -3, -3, -2, 2, 5}, {-4, -3, -3, -3, 3, 5},
{-5, -4, -3, -3, 5, 5}, {-3, 0, 0, 0, 0, 2}, {-4, 1, 1, 1, 1, 3}, {-5, 2, 2, 2, 2, 4},
{-3, 0, 0, 1, 1, 1}, {-4, 1, 1, 2, 2, 2}, {-5, 2, 2, 3, 3, 3}, {-4, 0, 0, 1, 2, 3},
{-5, 1, 1, 2, 3, 4}, {-3, -1, -1, -1, -1, 4}, {-4, -2, -2, -2, -2, 5},
{-2, -2, -2, -1, -1, 4}, {-3, -3, -3, -2, -2, 5}, {-4, -3, -2, -1, -1, 5}}

In[∘]:= **Length[%]**

Out[∘]= 117

In[∘]:=
```
(* Count all instances produced above. *)
{Sum[1, {i, n}, {j, i, n}, {k, j, n}],
 Sum[1, {i, n}, {j, i}],
 (n - 1 + Sum[n - 2, {i, 2, n}]),
 Sum[1, {i, n - 1}, {j, i}],
 Sum[n - 1, {i, 2, n}],
 Sum[n - 1, {i, 2, n - 1}],
 n - 3 + 2 * (n - 2),
 n - 4 + 2 * (n - 3)}
```

Out[∘]= $\left\{\frac{1}{6}\left(2 n+3 n^2+n^3\right), \frac{1}{2} n(1+n), -1+(-2+n)(-1+n)+n, \frac{1}{2}(-1+n) n, (-1+n)^2, (-2+n)(-1+n), -3+2(-2+n)+n, -4+2(-3+n)+n\right\}$

In[∘]:=
```
Together[Total[%]]
```

Out[∘]= $\frac{1}{6}\left(-78-4 n+27 n^2+n^3\right)$

```
(* Indeed, we get the same polynomial, that solved the guessed recurrence. *)
Together[% /. n → n + 4]
```

Out[∘]= $\frac{1}{6}\left(402+260 n+39 n^2+n^3\right)$

3 Transfer Matrix Method

Note: all transfer matrices here are transposed, because Mathematica computes M.v more efficiently than v.M.

3.1 A177317

In[∘]:=
```
InitializeSeq[177317]
```

```
Sequence: 1, 2, 48, 2288, 135040, 8956752,
   640160976, 48203722464, 3772321496064, 304100156874800, ...
Length: 29
Offset: 0
Recurrence:
```

$$-3 n^3 (1+n) (1+3 n) (2+3 n) \left(3281160+13324928 n+23607946 n^2+23825758 n^3+14975281 n^4+6000286 n^5+1496236 n^6+212252 n^7+13113 n^8\right) a[n]+(1+n)^2 \left(14722560+163505952 n+822949992 n^2+2464399296 n^3+4847819730 n^4+6543447222 n^5+6186525969 n^6+4125650658 n^7+1929434771 n^8+618883678 n^9+129652375 n^{10}+15978026 n^{11}+878571 n^{12}\right) a[1+n]-2 (2+n)^2 \left(20370096+207973548 n+951883014 n^2+2588508450 n^3+4659341433 n^4+5838584798 n^5+5211702571 n^6+3333874350 n^7+1515722000 n^8+477646252 n^9+99089547 n^{10}+12162378 n^{11}+668763 n^{12}\right) a[2+n]+(2+n)^2 (3+n)^4 \left(10512+90060 n+332910 n^2+697266 n^3+906481 n^4+745834 n^5+377636 n^6+107348 n^7+13113 n^8\right) a[3+n]$$

```
Check: True
```

```
(* All solutions for n=2. *)
Select[Permutations[Join @@ Table[Range[5], {2}]], Max[Abs[Differences[#]]] ≤ 1 &]
```

Out[∘]=
```
{{1, 2, 3, 4, 5, 5, 4, 3, 2, 1}, {1, 2, 1, 2, 3, 4, 5, 5, 4, 3},
 {1, 2, 1, 2, 3, 4, 3, 4, 5, 5}, {1, 2, 1, 2, 3, 3, 4, 5, 4, 5},
 {1, 2, 1, 2, 3, 3, 4, 5, 5, 4}, {1, 2, 1, 2, 3, 3, 4, 4, 5, 5},
 {1, 1, 2, 3, 4, 5, 5, 4, 3, 2}, {1, 1, 2, 3, 2, 3, 4, 5, 4, 5},
 {1, 1, 2, 3, 2, 3, 4, 5, 5, 4}, {1, 1, 2, 3, 2, 3, 4, 4, 5, 5},
 {1, 1, 2, 2, 3, 4, 5, 5, 4, 3}, {1, 1, 2, 2, 3, 4, 3, 4, 5, 5},
 {1, 1, 2, 2, 3, 3, 4, 5, 4, 5}, {1, 1, 2, 2, 3, 3, 4, 5, 5, 4},
 {1, 1, 2, 2, 3, 3, 4, 4, 5, 5}, {2, 1, 1, 2, 3, 4, 5, 5, 4, 3},
 {2, 1, 1, 2, 3, 4, 3, 4, 5, 5}, {2, 1, 1, 2, 3, 3, 4, 5, 4, 5},
 {2, 1, 1, 2, 3, 3, 4, 5, 5, 4}, {2, 1, 1, 2, 3, 3, 4, 4, 5, 5},
 {2, 3, 4, 5, 5, 4, 3, 2, 1, 1}, {3, 2, 1, 1, 2, 3, 4, 5, 4, 5},
 {3, 2, 1, 1, 2, 3, 4, 5, 5, 4}, {3, 2, 1, 1, 2, 3, 4, 4, 5, 5},
 {3, 4, 5, 5, 4, 3, 2, 1, 1, 2}, {3, 4, 5, 5, 4, 3, 2, 1, 2, 1},
 {3, 4, 5, 5, 4, 3, 2, 2, 1, 1}, {4, 3, 2, 1, 1, 2, 3, 4, 5, 5},
 {4, 5, 5, 4, 3, 2, 1, 1, 2, 3}, {4, 5, 5, 4, 3, 2, 3, 2, 1, 1},
 {4, 5, 5, 4, 3, 3, 2, 1, 1, 2}, {4, 5, 5, 4, 3, 3, 2, 1, 2, 1},
 {4, 5, 5, 4, 3, 3, 2, 2, 1, 1}, {5, 4, 3, 2, 1, 1, 2, 3, 4, 5},
 {5, 4, 5, 4, 3, 2, 1, 1, 2, 3}, {5, 4, 5, 4, 3, 2, 3, 2, 1, 1},
 {5, 4, 5, 4, 3, 3, 2, 1, 1, 2}, {5, 4, 5, 4, 3, 3, 2, 1, 2, 1},
 {5, 4, 5, 4, 3, 3, 2, 2, 1, 1}, {5, 5, 4, 3, 2, 1, 1, 2, 3, 4},
 {5, 5, 4, 3, 4, 3, 2, 1, 1, 2}, {5, 5, 4, 3, 4, 3, 2, 1, 2, 1},
 {5, 5, 4, 3, 4, 3, 2, 2, 1, 1}, {5, 5, 4, 4, 3, 2, 1, 1, 2, 3},
 {5, 5, 4, 4, 3, 2, 3, 2, 1, 1}, {5, 5, 4, 4, 3, 3, 2, 1, 1, 2},
 {5, 5, 4, 4, 3, 3, 2, 1, 2, 1}, {5, 5, 4, 4, 3, 3, 2, 2, 1, 1}}
```

In[•]:=
```
Length[%]
```
Out[•]= 48

In[•]:=
```
(* Transfer matrix *)
MatrixForm[tmat = Table[If[Abs[i - j] ≤ 1, x[i], 0], {i, 5}, {j, 5}] /. x[5] → 1]
```
Out[•]//MatrixForm=

$$\begin{pmatrix} x[1] & x[1] & 0 & 0 & 0 \\ x[2] & x[2] & x[2] & 0 & 0 \\ 0 & x[3] & x[3] & x[3] & 0 \\ 0 & 0 & x[4] & x[4] & x[4] \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In[•]:=
```
vec = svec = Append[Array[x, 4], 1];
t0 = AbsoluteTime[];
Table[
  vec = Expand[tmat.vec];
  If[Mod[n5, 5] === 0,
   cf = Coefficient[Total[vec], (Times @@ Array[x, 4]) ^ (n5 / 5)];
   Print["n=", n5 / 5, " (timing = ", - t0 + (t0 = AbsoluteTime[]), "): ", cf];
  ];
  , {n5, 2, 60}];
```
```
n=1 (timing = 0.002628): 2
n=2 (timing = 0.022653): 48
n=3 (timing = 0.162193): 2288
n=4 (timing = 0.414114): 135 040
n=5 (timing = 1.259894): 8 956 752
n=6 (timing = 3.080780): 640 160 976
n=7 (timing = 7.231564): 48 203 722 464
n=8 (timing = 11.449138): 3 772 321 496 064
n=9 (timing = 20.076767): 304 100 156 874 800
n=10 (timing = 36.754489): 25 098 440 923 318 048
n=11 (timing = 54.319901): 2 111 488 538 062 121 088
n=12 (timing = 85.416601): 180 477 438 192 133 215 952
```

In[•]:=
```
ByteCount[vec]
```
Out[•]= 1 539 277 608

```
In[•]:= (* The rational generating function. *)
        mat1 = IdentityMatrix[5] - t * tmat;
        rat = Total[Expand[Table[
              (-1) ^ (i + j) * Det[Delete[#, i] & /@ Delete[mat1, j]], {i, 5}, {j, 5}].svec]] / Det[m
```

Out[•]= $\left(1+x[1]-2\,t\,x[1]+x[2]-2\,t\,x[2]+x[3]-2\,t\,x[3]-2\,t\,x[1]\,x[3]+3\,t^2\,x[1]\,x[3]-t^2\,x[1]\,x[2]\,x[3]+2\,t^3\,x[1]\,x[2]\,x[3]+x[4]-2\,t\,x[1]\,x[4]-2\,t\,x[2]\,x[4]-t^2\,x[3]\,x[4]+2\,t^3\,x[1]\,x[3]\,x[4]-t^2\,x[2]\,x[3]\,x[4]+2\,t^3\,x[2]\,x[3]\,x[4]+2\,t^3\,x[1]\,x[2]\,x[3]\,x[4]\right)\Big/\left(1-t-t\,x[1]+t^2\,x[1]-t\,x[2]+t^2\,x[2]-t\,x[3]+t^2\,x[3]+t^2\,x[1]\,x[3]-t^3\,x[1]\,x[3]+t^3\,x[1]\,x[2]\,x[3]-t^4\,x[1]\,x[2]\,x[3]-t\,x[4]+t^2\,x[1]\,x[4]+t^2\,x[2]\,x[4]+t^3\,x[3]\,x[4]-t^4\,x[1]\,x[3]\,x[4]+t^3\,x[2]\,x[3]\,x[4]-t^4\,x[2]\,x[3]\,x[4]-t^4\,x[1]\,x[2]\,x[3]\,x[4]\right)$

```
        (* Sanity check (by series expansion). *)
        ser = Normal[
           Series[rat, {x[1], 0, 4}, {x[2], 0, 4}, {x[3], 0, 4}, {x[4], 0, 4}, {t, 0, 19}]];
        Table[Coefficient[ser, (x[1] x[2] x[3] x[4]) ^ i * t ^ (5 i - 1)], {i, 4}]

Out[•]= {2, 48, 2288, 135 040}

In[•]:= Timing[ct1 = CreativeTelescoping[rat / ((x[1] x[2] x[3] x[4]) ^ (n + 1) t ^ (5 n)),
            Der[x[1]], {Der[x[2]], Der[x[3]], Der[x[4]], Der[t], S[n]}][[1]];
         #[ct1] & /@ {ByteCount, UnderTheStaircase}
        ]

Out[•]= {3.21323, {671 304, {1}}}

In[•]:= Timing[
         ct2 = CreativeTelescoping[ct1, Der[x[2]]][[1]];
         #[ct2] & /@ {ByteCount, UnderTheStaircase}
        ]
```

Out[•]= $\{46.9576, \{2\,741\,312, \{1, S_n\}\}\}$

```
In[•]:= Timing[
         ct3 = FindCreativeTelescoping[ct2, Der[x[3]]][[1]];
         #[ct3] & /@ {ByteCount, UnderTheStaircase}
        ]
```

Out[•]= $\{324.692, \{1\,609\,520, \{1, S_n, D_t\}\}\}$

```
In[•]:= Timing[
         ct4 = FindCreativeTelescoping[ct3, Der[x[4]]][[1]];
         #[ct4] & /@ {ByteCount, UnderTheStaircase}
        ]
```

Out[•]= $\{215.314, \{207\,448, \{1, S_n, D_t, S_n^2\}\}\}$

In[•]:=
```
Timing[
 ct5 = FindCreativeTelescoping[ct4, Der[t]][[1]];
 #[ct5] & /@ {ByteCount, UnderTheStaircase}
]
```

Out[•]= $\{11.5728, \{9776, \{1, S_n, S_n^2\}\}\}$

In[•]:=
```
ct5
```

Out[•]= $\{(-3405888 - 37126512\,n - 183611448\,n^2 - 547158348\,n^3 - 1098140922\,n^4 - 1568813838\,n^5 - 1643152101\,n^6 - 1280524302\,n^7 - 745726744\,n^8 - 322910078\,n^9 - 102362074\,n^{10} - 23044934\,n^{11} - 3485182\,n^{12} - 317156\,n^{13} - 13113\,n^{14})\,S_n^3 + (162960768 + 1826749152\,n + 9319592688\,n^2 + 28739078808\,n^3 + 59886565092\,n^4 + 89160426748\,n^5 + 97720981818\,n^6 + 80041784964\,n^7 + 49220175942\,n^8 + 22614694716\,n^9 + 7645330392\,n^{10} + 1845307904\,n^{11} + 300828222\,n^{12} + 29674860\,n^{13} + 1337526\,n^{14})\,S_n^2 + (-14722560 - 192951072\,n - 1164684456\,n^2 - 4273805232\,n^3 - 10599568314\,n^4 - 18703485978\,n^5 - 24121240143\,n^6 - 23042149818\,n^7 - 16367262056\,n^8 - 8603403878\,n^9 - 3296854502\,n^{10} - 894166454\,n^{11} - 162486998\,n^{12} - 17735168\,n^{13} - 878571\,n^{14})\,S_n + (19686960\,n^3 + 188227848\,n^4 + 758552940\,n^5 + 1730154198\,n^6 + 2510703840\,n^7 + 2454191463\,n^8 + 1658947494\,n^9 + 778997331\,n^{10} + 249887460\,n^{11} + 52292709\,n^{12} + 6438906\,n^{13} + 354051\,n^{14})\}$

```
(* Compare with the guessed recurrence: *)
Together[ApplyOreOperator[ct5, a[n]] / rec]
```

Out[•]= -1

3.2 A199250

In[•]:=
```
InitializeSeq[199250]
```

```
Sequence: 1, 1, 14, 21, 424, 571, 14160, 18157, 508802, 635901, ...

Length: 56

Offset: 1
```

Recurrence: $-40353607 (-1+n)^3 a[n] + 5764801 (1-3n+3n^2+3n^3) a[1+n] +$
$1647086 (99+137n+297n^2+85n^3) a[2+n] - 117649 (4659+7491n+3705n^2+497n^3) a[3+n] -$
$33614 (130191+152365n+51585n^2+5325n^3) a[4+n] +$
$2401 (2297725+1732049n+403119n^2+29743n^3) a[5+n] +$
$343 (35427745+21821813n+4422867n^2+291335n^3) a[6+n] -$
$196 (43632133+21290377n+3446967n^2+184155n^3) a[7+n] -$
$294 (12216953+7010629n+1251019n^2+69519n^3) a[8+n] +$
$14 (43265483+34953887n+6674673n^2+363745n^3) a[9+n] -$
$56 (56021365+14070412n+999843n^2+16333n^3) a[10+n] +$
$2 (1169837943+288181519n+22203093n^2+523973n^3) a[11+n] +$
$8 (215952499+54188164n+4303491n^2+108253n^3) a[12+n] -$
$2 (319118939+76538839n+5867433n^2+144617n^3) a[13+n] -$
$6 (30198073+7895981n+622603n^2+15351n^3) a[14+n] +$
$4 (-9533493-797233n+5961n^2+1149n^3) a[15+n] +$
$(-47301801-6579653n-291195n^2-4007n^3) a[16+n] +$
$(24399653+3680273n+184143n^2+3055n^3) a[17+n] +$
$2 (6051283+874285n+41961n^2+669n^3) a[18+n] +$
$(-1844963-268035n-12969n^2-209n^3) a[19+n] -$
$2 (389463+53321n+2433n^2+37n^3) a[20+n] +$
$(30425+4221n+195n^2+3n^3) a[21+n] + (23+n)^3 a[22+n]$

```
Check: True
```

In[•]:=
```
(* Produce more terms with the guessed recurrence, and perform two checks:
   values agree with existing data, all new values are integers. *)
re2l = RE2L[Prepend[Table[a[i] == data[[i]], {i, 22}], rec == 0], a[n], {1, 1000}];
{Take[re2l, Length[data]] === data, And @@ (IntegerQ /@ re2l)}
```

... Solve: Equations may not give solutions for all "solve" variables.

Out[•]= {True, True}

In[•]:=
```
(* Guess minimal-order recurrence. *)
mrec = Collect[GuessMinRE[Take[re2l, 98], a[n], StartPoint → 1], a[_], Factor]
```

Out[•]=
$$\frac{1}{129600} 49 (-1+n)^3 (5+n) (23702634187776 + 79047255995136 n + 1739357751264 n^2 - 244115802834588 n^3 - 286729089498510 n^4 - 72870502443527 n^5 + 79677982747601 n^6 + 78318156258530 n^7 + 33349441406776 n^8 + 8488176358713 n^9 + 1391456813397 n^{10} + 148405757400 n^{11} + 9958024800 n^{12} + 381507840 n^{13} + 6350400 n^{14}) a[n] - \frac{1}{8100}$$
$$7 (5+n) (1481414636736 + 496209589488 n - 10268406729426 n^2 + 8166423453660 n^3 + 19104984484953 n^4 - 33639683731502 n^5 - 59866269283749 n^6 - 22062680825770 n^7 + 10213704117127 n^8 + 12095113998222 n^9 + 4989374195639 n^{10} + 1165610322702 n^{11} +$$

$168\,037\,906\,560\,n^{12}+14\,869\,963\,440\,n^{13}+743\,744\,160\,n^{14}+16\,148\,160\,n^{15}\big)\ a[1+n]+$
$\frac{1}{64\,800}\big(-5\,613\,263\,347\,415\,040-5\,038\,133\,571\,495\,360\,n+37\,384\,358\,529\,580\,168\,n^2+$
$32\,882\,332\,130\,461\,220\,n^3-83\,552\,619\,143\,196\,282\,n^4-88\,216\,356\,429\,967\,755\,n^5+$
$52\,279\,870\,500\,708\,546\,n^6+107\,291\,096\,491\,272\,392\,n^7+42\,750\,283\,613\,167\,772\,n^8-$
$14\,761\,916\,567\,817\,262\,n^9-22\,188\,021\,880\,190\,036\,n^{10}-10\,918\,459\,582\,381\,812\,n^{11}-$
$3\,190\,460\,171\,857\,570\,n^{12}-617\,908\,792\,861\,263\,n^{13}-81\,662\,254\,296\,918\,n^{14}-$
$7\,304\,795\,357\,040\,n^{15}-423\,546\,245\,280\,n^{16}-14\,362\,004\,160\,n^{17}-215\,913\,600\,n^{18}\big)\ a[2+n]+$
$\frac{1}{56\,700}\big(2\,704\,458\,831\,758\,784+10\,566\,872\,639\,654\,256\,n+5\,704\,465\,178\,741\,190\,n^2-$
$54\,616\,634\,397\,875\,922\,n^3-130\,655\,225\,960\,023\,839\,n^4-121\,434\,632\,527\,632\,055\,n^5-$
$40\,563\,407\,134\,483\,615\,n^6+18\,713\,270\,947\,940\,379\,n^7+26\,982\,948\,785\,206\,929\,n^8+$
$14\,390\,269\,075\,261\,079\,n^9+4\,684\,592\,155\,538\,857\,n^{10}+1\,027\,235\,529\,159\,959\,n^{11}+$
$155\,681\,917\,330\,254\,n^{12}+16\,158\,016\,079\,664\,n^{13}+1\,099\,826\,970\,480\,n^{14}+$
$44\,321\,191\,200\,n^{15}+802\,509\,120\,n^{16}\big)\ a[3+n]+\frac{1}{3\,175\,200}$
$\big(1\,120\,816\,363\,315\,691\,520-853\,911\,249\,956\,987\,904\,n-7\,165\,025\,504\,055\,768\,320\,n^2+$
$626\,983\,285\,839\,560\,272\,n^3+13\,146\,020\,711\,402\,993\,458\,n^4+8\,279\,991\,011\,898\,391\,732\,n^5-$
$3\,315\,269\,871\,049\,589\,169\,n^6-5\,686\,125\,461\,641\,740\,216\,n^7-2\,053\,265\,641\,740\,346\,236\,n^8+$
$350\,855\,103\,231\,051\,944\,n^9+604\,811\,357\,860\,988\,166\,n^{10}+271\,563\,579\,785\,550\,464\,n^{11}+$
$71\,866\,830\,772\,073\,458\,n^{12}+12\,684\,344\,603\,192\,388\,n^{13}+1\,540\,790\,118\,312\,963\,n^{14}+$
$127\,799\,093\,521\,800\,n^{15}+6\,930\,289\,200\,960\,n^{16}+221\,635\,431\,360\,n^{17}+3\,168\,849\,600\,n^{18}\big)$
$a[4+n]+\frac{1}{396\,900}\big(25\,930\,005\,420\,405\,312-43\,972\,849\,543\,160\,304\,n-$
$206\,524\,275\,362\,456\,006\,n^2+103\,543\,310\,590\,135\,262\,n^3+555\,473\,753\,282\,035\,615\,n^4+$
$467\,862\,427\,787\,573\,761\,n^5+88\,702\,550\,644\,109\,279\,n^6-106\,311\,776\,319\,947\,533\,n^7-$
$95\,783\,569\,282\,936\,225\,n^8-40\,819\,637\,813\,781\,505\,n^9-11\,066\,066\,254\,837\,545\,n^{10}-$
$2\,056\,101\,915\,928\,465\,n^{11}-266\,843\,068\,568\,670\,n^{12}-23\,901\,619\,353\,456\,n^{13}-$
$1\,413\,026\,718\,480\,n^{14}-49\,733\,192\,160\,n^{15}-790\,534\,080\,n^{16}\big)\ a[5+n]+\frac{1}{3\,175\,200}$
$\big(-193\,180\,552\,715\,243\,520+181\,483\,101\,935\,961\,408\,n+1\,292\,623\,911\,318\,588\,264\,n^2-$
$218\,123\,185\,870\,000\,484\,n^3-2\,563\,989\,949\,950\,537\,506\,n^4-1\,801\,322\,322\,700\,168\,181\,n^5+$
$238\,271\,014\,934\,197\,514\,n^6+827\,350\,095\,586\,093\,656\,n^7+387\,378\,068\,705\,902\,732\,n^8+$
$32\,678\,303\,728\,009\,262\,n^9-41\,667\,940\,285\,585\,380\,n^{10}-22\,391\,195\,392\,597\,132\,n^{11}-$
$6\,068\,286\,785\,469\,226\,n^{12}-1\,054\,536\,872\,908\,209\,n^{13}-124\,161\,744\,189\,678\,n^{14}-$
$9\,908\,437\,428\,720\,n^{15}-515\,122\,823\,520\,n^{16}-15\,771\,792\,960\,n^{17}-215\,913\,600\,n^{18}\big)\ a[6+n]+$
$\frac{1}{396\,900}(3+n)\ \big(-249\,895\,672\,773\,312+488\,839\,154\,965\,648\,n+1\,765\,469\,084\,642\,178\,n^2-$
$1\,529\,201\,948\,265\,636\,n^3-4\,471\,316\,116\,023\,649\,n^4-2\,573\,516\,049\,478\,638\,n^5+$
$186\,866\,054\,963\,661\,n^6+912\,322\,435\,603\,526\,n^7+524\,481\,476\,347\,137\,n^8+$
$166\,448\,297\,680\,494\,n^9+34\,167\,867\,860\,593\,n^{10}+4\,735\,855\,255\,806\,n^{11}+$
$442\,457\,887\,872\,n^{12}+26\,789\,636\,880\,n^{13}+951\,268\,320\,n^{14}+15\,059\,520\,n^{15}\big)\ a[7+n]+$
$\frac{1}{6\,350\,400}(3+n)\ (9+n)^3\ \big(4\,125\,677\,184\,000-6\,854\,499\,340\,288\,n-23\,802\,986\,182\,272\,n^2+$
$22\,537\,606\,777\,148\,n^3+44\,503\,203\,953\,242\,n^4+6\,192\,259\,308\,815\,n^5-15\,563\,477\,196\,743\,n^6-$

$8\,157\,826\,508\,610\, n^7 - 458\,247\,252\,656\, n^8 + 805\,224\,030\,543\, n^9 + 313\,468\,626\,957\, n^{10} + 56\,355\,525\,720\, n^{11} + 5\,576\,309\,280\, n^{12} + 292\,602\,240\, n^{13} + 6\,350\,400\, n^{14}\big)$ a[8 + n]

In[•]:=
```
(* Guess recurrence for even indices (ae[n] = a[2n]). *)
rece = Collect[Numerator[Together[
    GuessUnivRE[Take[re2l, {2, 200, 2}], ae[n],
      Order → 4, Degree → 8, StartPoint → 1][[1]]]], ae[_], Factor]
```

Out[•]=

$2401\, n^3 (5 + 2 n) (1119 + 2829 n + 2425 n^2 + 824 n^3 + 96 n^4)\, ae[n] - 98 (23\,232 + 227\,996 n + 960\,783 n^2 + 1\,960\,439 n^3 + 2\,151\,893 n^4 + 1\,338\,307 n^5 + 470\,452 n^6 + 86\,848 n^7 + 6528 n^8)\, ae[1 + n] + 2 (-151\,008 + 3\,194\,000 n + 25\,261\,108 n^2 + 53\,468\,052 n^3 + 53\,319\,121 n^4 + 29\,037\,852 n^5 + 8\,890\,558 n^6 + 1\,438\,672 n^7 + 95\,808 n^8)\, ae[2 + n] - 2 (-47\,232 + 243\,564 n + 2\,728\,691 n^2 + 5\,650\,345 n^3 + 5\,266\,809 n^4 + 2\,637\,037 n^5 + 736\,180 n^6 + 108\,160 n^7 + 6528 n^8)\, ae[3 + n] + (4 + n)^3 (3 + 2 n) (-13 + 67 n + 529 n^2 + 440 n^3 + 96 n^4)\, ae[4 + n]$

In[•]:=
```
(* Guess recurrence for odd indices (ao[n] = a[2n+1]). *)
reco = Collect[Numerator[Together[
    GuessUnivRE[Take[re2l, {1, 200, 2}], ao[n],
      Order → 4, Degree → 10, StartPoint → 0][[1]]]], ao[_], Factor]
```

Out[•]=

$2401\, n^3 (7 + 2 n) (150\,312 + 472\,150 n + 566\,901 n^2 + 331\,908 n^3 + 100\,149 n^4 + 14\,904 n^5 + 864 n^6)\, ao[n] - 49 (7 + 2 n) (1\,228\,128 + 12\,549\,268 n + 55\,318\,177 n^2 + 118\,911\,819 n^3 + 139\,678\,988 n^4 + 95\,529\,783 n^5 + 38\,777\,853 n^6 + 9\,129\,108 n^7 + 1\,143\,936 n^8 + 58\,752 n^9)\, ao[1 + n] + (-156\,900\,576 + 635\,576\,668 n + 9\,349\,986\,451 n^2 + 24\,663\,169\,255 n^3 + 30\,687\,106\,706 n^4 + 21\,910\,345\,387 n^5 + 9\,644\,646\,333 n^6 + 2\,664\,337\,824 n^7 + 450\,289\,356 n^8 + 42\,566\,688 n^9 + 1\,724\,544 n^{10})\, ao[2 + n] - (3 + 2 n) (-5\,010\,656 + 16\,627\,420 n + 251\,763\,403 n^2 + 561\,479\,353 n^3 + 541\,644\,308 n^4 + 281\,844\,117 n^5 + 85\,376\,223 n^6 + 15\,113\,172 n^7 + 1\,453\,248 n^8 + 58\,752 n^9)\, ao[3 + n] + (5 + n)^3 (3 + 2 n) (-736 + 2812 n + 35\,991 n^2 + 63\,072 n^3 + 38\,589 n^4 + 9720 n^5 + 864 n^6)\, ao[4 + n]$

```
(* These are all possible rows: *)
rows = DeleteCases[Tuples[Range[0, 3], 2], {a_, a_}]
```

Out[•]=
```
{{0, 1}, {0, 2}, {0, 3}, {1, 0}, {1, 2},
 {1, 3}, {2, 0}, {2, 1}, {2, 3}, {3, 0}, {3, 1}, {3, 2}}
```

In[•]:=
```
(* Construct the transfer matrix. *)
tmat = Table[With[{r1 = rows[[j]], r2 = rows[[i]]},
    If[r1[[1]] === r2[[1]] || r1[[2]] === r2[[2]],
     0, x^Count[r2, 0] * y^Count[r2, 1] * z^Count[r2, 2]]
   ], {i, 12}, {j, 12}];
MatrixForm[tmat]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & x\,y & x\,y & x\,y & x\,y & 0 & x\,y & x\,y & 0 & x\,y \\ 0 & 0 & 0 & x\,z & 0 & x\,z & x\,z & x\,z & x\,z & x\,z & x\,z & 0 \\ 0 & 0 & 0 & x & x & 0 & x & x & 0 & x & x & x \\ x\,y & x\,y & x\,y & 0 & 0 & 0 & 0 & x\,y & x\,y & 0 & x\,y & x\,y \\ y\,z & 0 & y\,z & 0 & 0 & 0 & y\,z & y\,z & y\,z & y\,z & y\,z & 0 \\ y & y & 0 & 0 & 0 & 0 & y & y & 0 & y & y & y \\ x\,z & x\,z & x\,z & 0 & x\,z & x\,z & 0 & 0 & 0 & 0 & x\,z & x\,z \\ 0 & y\,z & y\,z & y\,z & y\,z & y\,z & 0 & 0 & 0 & y\,z & 0 & y\,z \\ z & z & 0 & z & z & 0 & 0 & 0 & 0 & z & z & z \\ x & x & x & 0 & x & x & 0 & x & x & 0 & 0 & 0 \\ 0 & y & y & y & y & y & y & 0 & y & 0 & 0 & 0 \\ z & 0 & z & z & 0 & z & z & z & z & 0 & 0 & 0 \end{pmatrix}$$

```
(* Compute values with the transfer matrix approach. *)
vec = Prepend[Table[0, {11}], x y];
vvec = First[DeleteCases[#, 0]] & /@ tmat;
tmat1 = tmat /. {x → 1, y → 1, z → 1};
Timing[
 data1 = Table[
    vec = Expand[vvec * (tmat1.vec)];
    val = Table[n / 2, {4}];
    If[OddQ[n], val += {1, 1, -1, -1} / 2];
    val = (Times @@ ({x, y, z, 1}^#)) & /@ Permutations[val];
    val = Coefficient[Total[vec], val];
    Total[val / 2], {n, 2, 100}];
]
```

Out[•]= {1671.23, Null}

```
(* Sanity check. *)
Take[data1, 55] - Rest[data]
```

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:=
```
mat1 = IdentityMatrix[12] - t * tmat;
rat =
 Factor[Sum[t x y * (-1)^(i + 1) * Det[Delete[#, i] & /@ Rest[mat1]], {i, 12}] / Det[mat1]]
```

Out[•]=

$$-\frac{t\,x\,y\,(1+t\,z)}{-1+t\,x+t\,y+t\,x\,y+t\,z+t\,x\,z+t\,y\,z+7\,t^2\,x\,y\,z}$$

```
(* Test coefficient extraction formula. *)
ser = Normal[Series[rat, {t, 0, 10}, {x, 0, 5}, {y, 0, 5}, {z, 0, 5}]];
Table[With[{n = nn / 2}, 1 / 2 * Coefficient[ser, (x y z) ^ n t ^ (2 n)]],
  {nn, 2, 10, 2}] === Take[data, {2, 10, 2}]
Table[With[{n = (nn - 1) / 2},
   1 / 2 * Coefficient[(x + y + x y + z + x z + y z) * ser, (x y z) ^ (n + 1) t ^ (2 n + 1)]],
  {nn, 3, 9, 2}] === Take[data, {3, 9, 2}]
```

Out[•]= True

Out[•]= True

```
In[•]:= Timing[ct1 = CreativeTelescoping[rat / (x y z) ^ (n + 1) / t ^ (2 n + 1),
      Der[t], {Der[x], Der[y], Der[z], S[n]}][[1]];]
```

Out[•]= {1.38218, Null}

```
In[•]:= Timing[ct2 = FindCreativeTelescoping[ct1, Der[x]][[1]];]
```

Out[•]= {4.08319, Null}

```
In[•]:= Timing[ct3 = FindCreativeTelescoping[ct2, Der[y]][[1]];]
```

Out[•]= {14.6397, Null}

```
In[•]:= Timing[ct4 = FindCreativeTelescoping[ct3, Der[z]][[1]];]
```

Out[•]= {33.4088, Null}

```
Exponent[opeven = ct4[[1]], {S[n], n}]
```

Out[•]= {6, 17}

```
(* Sanity check. *)
test = ApplyOreOperator[opeven, f[n]];
Table[test, {n, 0, 100}] /. f[n_] :> re2l[[2 n]]
```

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
In[•]:= (* This operator is a left multiple of the guessed recurrence. *)
OreReduce[opeven, {ToOrePolynomial[rece, ae[n]]}]
```

Out[•]= 0

```
In[•]:= (* For the odd case, modify the rational function accordingly. *)
ratodd =
 Together[Total[(rat / #) & /@ ((t x y z) t ^ (2 n + 1) (x y z) ^ n * {x, y, z, x y, x z, y z})]]
```

Out[•]= $-\frac{t^{-1-2n}\,(x\,y\,z)^{-n}\,(1+t\,z)\,(x+y+x\,y+z+x\,z+y\,z)}{x\,y\,z^2\,\left(-1+t\,x+t\,y+t\,x\,y+t\,z+t\,x\,z+t\,y\,z+7\,t^2\,x\,y\,z\right)}$

```
In[•]:= Timing[
         ct1 = CreativeTelescoping[ratodd, Der[t], {Der[x], Der[y], Der[z], S[n]}][[1]];]

Out[•]= {1.7994, Null}

In[•]:= Timing[ct2 = FindCreativeTelescoping[ct1, Der[x]][[1]];]

Out[•]= {13.2793, Null}

In[•]:= Timing[ct3 = FindCreativeTelescoping[ct2, Der[y]][[1]];]

Out[•]= {43.3427, Null}

In[•]:= Timing[ct4 = FindCreativeTelescoping[ct3, Der[z]][[1]];]

Out[•]= {101.955, Null}

        Exponent[opodd = ct4[[1]], {S[n], n}]

Out[•]= {6, 22}

In[•]:= (* Sanity check. *)
        test = ApplyOreOperator[opodd, f[n]];
        Table[test, {n, 0, 100}] /. f[n_] :> re2l[[2 n + 1]]

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
         0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
         0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
         0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= (* This operator is a left multiple of the guessed recurrence. *)
        OreReduce[opodd, {ToOrePolynomial[reco, ao[n]]}]

Out[•]= 0

In[•]:= (* The two operators can be combined to
         an operator annihilating the whole sequence. *)
        op = First[DFinitePlus[DFiniteSubstitute[{opeven}, {n → n / 2}],
            DFiniteSubstitute[{opodd}, {n → (n - 1) / 2}]]];
        Exponent[
         op,
         {S[
           n], n}]

Out[•]= {24, 79}

In[•]:= (* Verify that it is compatible with the guessed order-8 recurrence. *)
        OreReduce[op, ToOrePolynomial[{mrec}, a[n]]]

Out[•]= 0
```

3.3 A250556

In[•]:=
```
InitializeSeq[250556]
```

Sequence: 8, 60, 302, 1516, 7126, 30780, 127586, 518052, 2085808, 8367220, ...

Length: 47

Offset: 1

Recurrence: -512 (-11 + n) a[n] + 1280 (-10 + n) a[1 + n] - 480 (-103 + 15 n) a[2 + n] + 64 (-752 + 115 n) a[3 + n] + 64 (-1242 + 181 n) a[4 + n] + 8 (1922 + 1353 n) a[5 + n] + 46 (1593 + 443 n) a[6 + n] - 2 (-29924 + 11971 n) a[7 + n] - 12 (18523 + 7120 n) a[8 + n] + (-1350950 - 102033 n) a[9 + n] + (-1120737 - 68327 n) a[10 + n] + 8 (29831 + 9997 n) a[11 + n] + (3717409 + 315386 n) a[12 + n] + (1169167 + 72865 n) a[13 + n] - 113 (2964 + 29 n) a[14 + n] - 38 (29036 + 4017 n) a[15 + n] - 2 (1334424 + 108679 n) a[16 + n] + 7 (33436 + 5599 n) a[17 + n] + (787693 + 52993 n) a[18 + n] + 4 (173490 + 22313 n) a[19 + n] - 13 (-2487 + 1820 n) a[20 + n] + (-359485 - 22129 n) a[21 + n] + 3 (24806 + 1811 n) a[22 + n]

Check: True

```
(* The C-finite recurrence can be guessed by omitting the first 11 terms. *)
GuessMinRE[Take[data, {12, 47}], f[n]]
```

Out[•]= -32 f[n] + 56 f[1 + n] - 28 f[2 + n] + 36 f[3 + n] + 8 f[4 + n] - 84 f[5 + n] + 44 f[6 + n] - 58 f[7 + n] + 73 f[8 + n] + f[9 + n] - 4 f[10 + n] + 8 f[11 + n] - 42 f[12 + n] + 26 f[13 + n] - 12 f[14 + n] + 14 f[15 + n] - 7 f[16 + n] + f[17 + n]

```
(* Produce more terms with the guessed recurrence, and perform two checks:
  values agree with existing data, all new values are integers. *)
re2l = RE2L[Prepend[Table[a[i] == data[[i]], {i, 22}], rec == 0], a[n], {1, 1000}];
{Take[re2l, Length[data]] === data, And @@ (IntegerQ /@ re2l)}
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]= {True, True}

In[∘]:=
```
(* Minimal-order recurrence *)
mrec = Collect[Numerator[
   Together[GuessMinRE[Take[re2l, {12, 61}], a[n], StartPoint → 11]]], a[_], Factor]
```

Out[∘]=
```
16 (-3 083 270 574 676 - 1 595 520 607 766 n - 416 580 358 045 n^2 - 197 130 879 175 n^3 +
     4 907 779 197 n^4 + 39 377 598 475 n^5 + 12 085 401 845 n^6 + 1 429 364 775 n^7 + 61 659 000 n^8)
   a[n] - 4 (-22 121 467 802 484 - 14 120 376 026 418 n - 4 656 962 531 487 n^2 -
     1 077 668 467 325 n^3 + 46 483 174 511 n^4 + 120 912 907 105 n^5 +
     33 843 532 935 n^6 + 4 041 458 325 n^7 + 184 977 000 n^8) a[1 + n] +
  2 (-39 001 451 138 664 - 22 568 942 244 422 n - 4 457 653 439 977 n^2 + 966 665 497 173 n^3 +
     673 382 298 417 n^4 + 148 638 061 155 n^5 + 19 383 522 545 n^6 + 1 552 682 775 n^7 + 61 659 000 n^8)
   a[2 + n] - 4 (-22 154 349 102 155 - 9 282 668 239 659 n - 343 769 712 969 n^2 + 197 793 147 204 n^3 -
     13 316 605 541 n^4 + 30 001 323 425 n^5 + 15 683 631 565 n^6 + 2 396 287 050 n^7 + 123 318 000 n^8)
   a[3 + n] - 2 (23 101 736 749 920 + 1 817 788 220 036 n - 1 873 081 370 598 n^2 - 653 336 396 071 n^3 -
     111 629 571 630 n^4 + 65 695 316 775 n^5 + 36 724 423 300 n^6 + 5 790 325 875 n^7 + 308 295 000 n^8)
   a[4 + n] + (6 271 096 030 604 - 21 785 195 040 043 n - 9 972 963 292 690 n^2 -
     2 352 462 291 340 n^3 - 242 645 844 138 n^4 + 166 533 777 310 n^5 +
     78 196 452 220 n^6 + 12 208 449 525 n^7 + 678 249 000 n^8) a[5 + n] -
  2 (-3 985 026 580 600 - 11 146 769 303 747 n - 3 133 681 450 747 n^2 + 214 202 554 602 n^3 +
     266 043 109 512 n^4 + 61 392 337 945 n^5 + 10 502 567 720 n^6 + 1 213 558 275 n^7 + 61 659 000 n^8)
   a[6 + n] + 2 (-9 823 798 920 054 - 9 663 099 662 063 n - 799 707 335 673 n^2 + 395 798 058 624 n^3 -
     41 577 844 457 n^4 - 555 629 965 n^5 + 17 803 700 330 n^6 + 3 898 518 600 n^7 + 246 636 000 n^8)
   a[7 + n] - 2 (-5 673 964 555 024 - 4 361 241 843 279 n - 147 620 460 542 n^2 + 33 454 143 740 n^3 -
     141 118 084 509 n^4 - 16 105 879 235 n^5 + 11 846 014 635 n^6 + 2 839 107 825 n^7 + 184 977 000 n^8)
   a[8 + n] + (-1 730 951 568 988 - 1 239 495 783 259 n - 36 526 210 688 n^2 - 18 119 184 988 n^3 -
     56 410 822 628 n^4 - 6 571 056 320 n^5 + 3 806 300 420 n^6 + 936 092 775 n^7 + 61 659 000 n^8) a[9 + n]
```

```
(* Naive implementation for enumeration. *)
D2[a_] := Differences[Differences[a]];
MyTest[v_] := MemberQ[v.# & /@ Tuples[{1, -1}, Length[v]], 0];
Do[Print[n, ": ",
  Timing[Length[Select[Tuples[{0, 1, 2, 3}, n + 2], MyTest[D2[#]] &]]]], {n, 7}]
```

```
1: {0.000835, 8}
2: {0.004573, 60}
3: {0.025977, 302}
4: {0.104368, 1516}
5: {0.405649, 7126}
6: {2.38047, 30 780}
7: {14.4953, 127 586}
```

```
(* Derive a global bound for the partial signed sums. *)
(* Example for bnd=18 *)
D2[{3, 0, 3, 0, 3, 1, 3, 0, 1}]
```

Out[∘]= {6, -6, 6, -5, 4, -5, 4}

In[∘]:=
```
(* Example for bnd=19 *)
D2[{1, 3, 0, 2, 0, 3, 0, 3, 0, 3, 1}]
```

Out[∘]= {-5, 5, -4, 5, -6, 6, -6, 6, -5}

In[∘]:=
```
(* Construct the transfer matrix, using the optimal bound B=19. *)
bnd = 19;
Timing[
 (* Possible start configurations (after one step). *)
 s1 = Flatten[Table[{{a2, a3}, Intersection[{1, -1} * (a1 - 2 a2 + a3), Range[0, bnd]]}
    , {a1, 0, 3}, {a2, 0, 3}, {a3, 0, 3}], 2];
 (* Transition function. *)
 NextState[{{a1_Integer, a2_Integer}, set_List}, k_Integer] :=
  {{a2, k}, Intersection[
    Abs[Flatten[Outer[Plus, set, {-1, 1} * (a1 - 2 a2 + k)]]], Range[0, bnd]]};
 (* Construct set of reachable states. *)
 states = FixedPoint[Function[s,
    Union[s, Union @@ (Table[NextState[#, k], {k, 0, 3}] & /@ s)]], Union[s1]];
 (* start vector v_{init} and end vector v_{final}. *)
 svec = Table[0, {Length[states]}];
 (svec[[Position[states, #][[1, 1]]]] += 1) & /@ s1;
 evec = If[MemberQ[#2, 0], 1, 0] & @@@ states;
 (* Build the transfer matrix. *)
 tmat = row0 = SparseArray[{Table[0, {Length[states]}]}];
 Function[s,
   row = row0;
   Do[row[[1, Position[states, NextState[s, k]][[1, 1]]]] = 1, {k, 0, 3}];
   tmat = Join[tmat, row];
  ] /@ states;
 tmat = Transpose[Rest[tmat]];
 {Dimensions[tmat], Count[Normal[tmat], 1, 2], ByteCount /@ {tmat, Normal[tmat]}}
]
```

Out[∘]= {18.3888, {{2484, 2484}, 9936, {180 208, 49 362 200}}}

In[∘]:=
```
dim = Dimensions[tmat][[1]]
```

Out[∘]= 2484

```
In[•]:= (* Compute values using the transfer matrix. *)
vec = svec;
Table[vec = tmat.vec;
 vec.evec, {10}]

Out[•]= {60, 302, 1516, 7126, 30780, 127586, 518052, 2085808, 8367220, 33513408}

In[•]:= (* Computation is very fast. Values agree
 with those produced by the guessed recurrence. *)
Timing[vec = svec;
 Rest[re2l] === Table[vec = tmat.vec;
   vec.evec, {999}]]

Out[•]= {0.582597, True}

(* Number of nonzero entries in v_init and v_final. *)
Count[Normal[#], 1 | 2] & /@ {svec, evec}

Out[•]= {60, 720}

In[•]:= (* Positions of these nonzero entries. *)
epos = Flatten[Position[evec, 1]];
spos = Flatten[Position[svec, 1 | 2]]

Out[•]= {1, 2, 3, 4, 145, 146, 147, 148, 297, 298, 299, 300, 469, 470, 471, 472, 646, 647, 648,
 817, 818, 819, 935, 936, 937, 938, 1094, 1095, 1096, 1097, 1243, 1244, 1245, 1246,
 1392, 1393, 1394, 1395, 1551, 1552, 1553, 1669, 1670, 1671, 1840, 1841, 1842,
 1843, 2017, 2018, 2019, 2020, 2189, 2190, 2191, 2192, 2341, 2342, 2343, 2344}

(* Estimated time when applying the
  determinant formula naively: 60*720*3s = 36h. *)
With[{i = 227, j = 1392}, Timing[
  Factor[(-1) ^ (i + j) * Det[Delete[#, i] & /@ Delete[IdentityMatrix[dim] - t * tmat, j]]]
 ]]
```

Out[•]= $\{3.06684, -(-1+t)^{531}\, t^4\, (1+t)^{36}\, (-1+2t)^{10}\, (1+2t)^2\, (-1+4t)^2\, (1+t^2)^{12}\, (-1+2t^3)^8\, (1+5t-4t^2+19t^3-21t^4+63t^5-87t^6+128t^7-130t^8+118t^9-85t^{10}+54t^{11}-30t^{12}+14t^{13}-5t^{14}+t^{15})\}$

```
(* Do one column at a time. *)
mat1 = IdentityMatrix[dim] - t * tmat;
Timing[
 nums = Function[i, Det[ReplacePart[mat1, i → evec]]] /@ spos;
]

Out[•]= {1752.5, Null}
```

```
(* The rational generating function. *)
rat = t * Together[(svec[[spos]].nums) / Det[mat1]]
```

Out[•]= $-\left(\left(2\,t\,\left(4+2\,t-3\,t^{2}+73\,t^{3}+115\,t^{4}-139\,t^{5}-453\,t^{6}-1231\,t^{7}+38\,t^{8}+406\,t^{9}+3597\,t^{10}+2087\,t^{11}+1666\,t^{12}-3614\,t^{13}-4178\,t^{14}-4504\,t^{15}+903\,t^{16}+1985\,t^{17}+4173\,t^{18}+403\,t^{19}-202\,t^{20}-1324\,t^{21}-1296\,t^{22}+684\,t^{23}-300\,t^{24}+508\,t^{25}-56\,t^{26}+32\,t^{27}\right)\right)\Big/\left(\left(-1+t\right)^{3}\,\left(1+t\right)^{2}\,\left(-1+2\,t\right)\,\left(-1+4\,t\right)\,\left(1+t^{2}\right)^{2}\,\left(-1+2\,t^{3}\right)^{2}\right)\right)$

```
(* Sanity check. *)
Series[rat, {t, 0, 46}] - data.t^Range[47]
```

Out[•]= $O[t]^{47}$

Appendix: show that *B* = 19

```
(* Find all pairs of multisets (S1,S2) with values in {2,...,6} such that
   Total[S1]=Total[S2]=b and there are no nontrivial subsets T1 of S1 and
    T2 of S2 with Total[T1]=Total[T2]. *)
Test0[b_] := Union[Join @@ (
      Test1[b, Append[#, 6], Complement[Range[2, 5], #]] & /@
       Subsets[Range[2, 5], {0, 3}])];
Test1[b_, s1_, s2_] :=
  Module[{sol1, sol2},
   sol1 =
    Solve[Prepend[x[#] ≥ 0 & /@ s1, Total[# * x[#] & /@ s1] == b], x /@ s1, Integers];
   sol2 = Solve[Prepend[x[#] ≥ 0 & /@ s2, Total[# * x[#] & /@ s2] == b],
     x /@ s2, Integers];
   Return[Join @@ Flatten[Outer[Test2, sol1, sol2, 1], 1]];
  ];
Test2[sol1_, sol2_] :=
  Module[{s1 = sol1, s2 = sol2, ps1, ps2},
   {s1, s2} = Flatten /@ ({s1, s2} /. HoldPattern[x[i_] → k_] :→ Table[i, {k}]);
   {ps1, ps2} = Union[Total /@ Subsets[#, {1, Length[#] - 1}]] & /@ {s1, s2};
   Return[If[Intersection[ps1, ps2] === {}, {Sort[{s1, s2}]}, {}]];
  ];
```

In[•]:= `Test0[18]`

Out[•]= {{{6, 6, 6}, {3, 5, 5, 5}}, {{6, 6, 6}, {4, 4, 5, 5}}}

In[•]:= `Join @@ Table[Test0[b], {b, 19, 30}]`

Out[•]= {{{2, 6, 6, 6}, {5, 5, 5, 5}}, {{5, 5, 5, 5}, {4, 4, 4, 4, 4}},
{{6, 6, 6, 6}, {4, 5, 5, 5, 5}}, {{6, 6, 6, 6, 6}, {5, 5, 5, 5, 5, 5}}}

```
In[•]:= (* Test which arrays can be built from the given second differences,
        and whose partial sums are at least b. *)
        PartialSums[v_] := FoldList[Plus, First[v], Rest[v]];
        Diff2Array[d2_List] :=
          Module[{n2 = Length[d2] + 2, sol},
           sol = Solve[
             Join[Thread[D2[Array[x, n2]] == d2], Table[0 ≤ x[i] ≤ 3, {i, n2}]], Integers];
           If[sol =!= {}, Array[x, n2] /. sol, {}]];
        Test3[b_, s1_, s2_] :=
          Module[{perms, svect, diffs},
           perms =
            Abs[Select[Permutations[Join[s1, -s2]], Max[Abs[PartialSums[#]]] ≥ b &]];
           svect = (-1) ^Range[Length[Join[s1, s2]]];
           diffs = Join[#, -#] &[svect * # & /@ perms];
           Return[Union @@ (Diff2Array /@ diffs)];
          ];

In[•]:= Test3[18, {6, 6, 6}, {4, 4, 5, 5}]
Out[•]= {{0, 3, 0, 3, 0, 2, 0, 3, 2}, {1, 0, 3, 1, 3, 0, 3, 0, 3},
         {2, 3, 0, 2, 0, 3, 0, 3, 0}, {3, 0, 3, 0, 3, 1, 3, 0, 1}}

In[•]:= Test3[19, {1, 1, 6, 6, 6}, {5, 5, 5, 5}]
Out[•]= {}

In[•]:= Test3[19, {2, 6, 6, 6}, {5, 5, 5, 5}]
Out[•]= {}

In[•]:= Test3[19, {5, 5, 5, 5}, {4, 4, 4, 4, 4}]
Out[•]= {}

        Test3[19, {6, 6, 6, 6}, {4, 5, 5, 5, 5}]
Out[•]= {{1, 3, 0, 2, 0, 3, 0, 3, 0, 3, 1}, {1, 3, 0, 3, 0, 3, 0, 2, 0, 3, 1},
         {2, 0, 3, 0, 3, 0, 3, 1, 3, 0, 2}, {2, 0, 3, 1, 3, 0, 3, 0, 3, 0, 2}}

In[•]:= Test3[19, {6, 6, 6, 6, 6}, {5, 5, 5, 5, 5}]
Out[•]= {}
```

3.4 A264947

The OEIS lists only 20 terms of this sequence. We computed 80 terms (on compute-purley, crashed probably because out-of-memory), but were not able to guess a recurrence with them.

In[∘]:=
```
(* Compute values with the transfer matrix method. *)
cols = Tuples[{0, 1, 2, 3}, 4];
tmat = Table[If[Or @@ MapThread[SameQ, cols[[{i, j}]]], 0, 1], {i, 256}, {j, 256}];
vec = vvec = ((x^Count[#, 0] * y^Count[#, 1] * z^Count[#, 2]) & /@ cols);
Do[
 Print[n, ": ", Timing[
   vec = Expand[vvec * (tmat.vec)];
   Coefficient[Total[vec], (x y z)^n] / 24
  ]], {n, 2, 10}]
```

```
2: {0.047246, 60}
3: {0.483669, 3201}
4: {2.15627, 184 740}
5: {5.94561, 11 375 145}
6: {12.064, 730 983 420}
7: {29.1079, 48 402 531 561}
8: {40.2387, 3 282 992 503 164}
9: {65.7378, 226 854 309 720 993}
10: {90.0964, 15 915 758 107 113 276}
```

In[∘]:=
```
(* We continue this computation on RICAM's compute server radon1. *)
(* Terms n=2,...,53 were computed on grantley processors. *)
(* Terms n=54,...,80 on purley processors. *)
(* Memory consumption at n=80 was more than 215 GB. *)
str = "2:{0.069961,60}
3:{0.637427,3201}
4:{2.19072,184740}
5:{7.41572,11375145}
6:{12.8547,730983420}
7:{29.5854,48402531561}
8:{42.3991,3282992503164}
9:{83.2498,226854309720993}
10:{110.189,15915758107113276}
11:{186.79,1130694005695927761}
12:{230.252,81177583723495750340}
13:{354.157,5880587303767912833417}
14:{420.631,429300706847441007321756}
15:{613.332,31551853305004056089812729}
16:{714.901,2332682592774096715790438556}
17:{976.644,173364378813531387237639561417}
18:{1112.93,12944676911197784353139378818620}
19:{1443.15,970602277004604478741978783654137}
20:{1608.48,73052030408618314498132204294989636}
```

```
21:{2025.26,5517104052146769136190406619946748625}
22:{2219.31,4179702696932932048269407942927552828 76}
23:{2729.9,3175552717442137085074037908110098177 3217}
24:{2970.76,24189759344127190500642215877948423394 32332}
25:{3594.49,18471092134296866913416317509623685510 5044281}
26: {3886.21, 1413588734840161065931052575566014102580 5396412}
27: {4612.65, 1084057674533418597309164076479502490092 692413401}
28: {4926.76, 8329468729769599648729613336248131914002 5771576036}
29: {5812.04, 6411520652304194600222198527622372784375 30027109633}
30: {6187.07, 4943476126553022557566677947978623937060 04006990449116}
31: {7212.1, 38175540463975303292909694032596913469684 571657360319457}
32: {7641.88, 2952414235489779837568180412502990267046 48884254279811193 2}
33: {8759.51, 2286496207716715015164998603202201540283 1481791032352424 7169}
34: {9268.76, 1773086261223583659125575051443650592726 2075267548129408360380}
35: {10586.9, 1376649781393386397202658291679150213747 66816092801217418204 3073}
36: {11110.7, 1070094112224891453879563203843304680594 42500701300626830913088676}
37: {12585.4,
    832718859351736499502032780332662600922694894669360280448631623997 7}
38: {13175.2,
    648676069852905410796568057694024558259206275558011650142953096934972}
39: {15001.6,
    5058100140431689901896831668158710330825642269991239730671292927586567 3}
40: {15508.6,
    39478056596986859886839297347168723592385155508194914079671424067468685 08}
41: {17342.9,
    308399147222261800676330778658313601219073933946560673207166426335652031041}
42: {18083.1,
    2411244740919676504087850275657317951749281619867227564009058363608003470175 6}
43: {20125.5,
    18867878353197999383798181059280767234111846551040472096003025071589689984214 89
44: {20949.,
    147755130878557648268499773809712039244342634235501019196169297111769580 73\
     5381956}
45: {23201.,
    115793700931288254508259334040802478785560659270686260008073929828173726 18\
     932202793}
46: {24201.8,
    908105385301233478422538418034223400562994578658360475063321050637135075 47\
     6054023516}
47: {26825.1,
    712661818247268239213240406548671671934765231249975845310757120293177527 60\
     708333643257}
48: {28162.4,
    559647327987265340946526283635597413293954141808116363105958104546724629 29\
```

```
     15567223014188}
49: {31032.8,
    43976190024994992770266061360685940411144650645751849371446828606720773219⋱
     2957942294411465}
50: {32360.8,
    34576592592762435155379942361008198747574980142123925550748615338429692566⋱
     552624268045657916}
51: {35625.3,
    27201825117119226805394823353413212762073250585053583448750590300578867107⋱
     20822649964987650873}
52: {36998.4,
    21411917900231018027779094253426110448099933206995711768355031408140744380⋱
     2463255672863982869060}
53: {40546.8,
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     917030005288928030168529}
54: {31914.8,
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     42739915902745466980311804}
55: {35993.5,
    10475871817938573704002675304975252532234771418015434519792833239427065880⋱
     4300441247363834496259033153}
56: {37668.5,
    82628284473369755302725730223693597929243592074062492936372437387187313725⋱
     63460177389371732247647022060}
57: {39518.8,
    65203119965177950488857062350987400038805714156555225158421331052766688895⋱
     9569975204911306830850688058457}
58: {39763.6,
    51475694632693150107072891343755372168964819023491604515873799535111461308⋱
     380490581426802331324769881443772}
59: {45254.8,
    40655912793299319751582354029832957626698361076823936461346585393946924834⋱
     89560663912807432021880226860638521}
60: {43636.8,
    32123781748562906852244522041508739926743167415384159840771607525562989595⋱
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61: {51463.2,
    25392482346456698670646502854863139433321194310353726542797843251102949564⋱
     306389470269197438622890523485826653281}
62: {49365.6,
    20079531231004318547943145400372774085767043454884721261681701641818440488⋱
     50732419071052334349195204253850332170652}
63: {54508.6,
```

```
     15884243647352870751214379247868019414858253080832336294629306588727805176\
      9919703297171455138765693933499384187423009}
64: {55498.3,
     12570106624527408113699615647995077654475021071876168273040747672144021267\
      5033434117932515919395036288646542081911150688}
65: {60498.1,
     99509824389392645843290306057650323095706620999972823140475214827876691328\
      981355597604185816355562847186036251781508099 3}
66: {54860.7,
     78803023470047286558686199017340702508568762207108290549219331990771185871\
      3036187225600690773000552154807062341687286225 24}
67: {66118.2,
     62425966742297130863953570236158915093933267803576384385291827297308313467\
      7376745740245714654440004144578013840752020467 9713}
68: {66369.7,
     49468517553853073375908100921508633391665087717370915037205304063692854831\
      5747879920573161026398793698953496217726153364192292}
69: {70763.5,
     39212963913823620993090949297287060246036209842940053638210655052758868077\
      4375011271222889725785580177932462721119345555 32195529}
70: {70749.7,
     31093077051343513252062833174550590774709687904836504765802967991862963342\
      2292797435989575733361533157508895293061880013 2008315772}
71: {78957.2,
     24661941565883259773841679150256139007212576960019579879621563255848720277\
      1110585021525326346036935805950800163408177257142535721737}
72: {80284.5,
     19566663520666839840882651733210307855863950492933725157905822358646622480\
      5379595314272426807488676794773526185364556114361614128375 00}
73: {86086.6,
     15528475069474784829929350022890331401916877979892931169535679412855709205\
      1650789406794819368588733456993792217107012234912747511981 8753}
74: {88913.2,
     12327075876611576177226012859141088061087453657476881584206929255103251438\
      6675747704698864912873205168595481506871186636814260381915562684}
75: {93262.9,
     97883027893446383947409886302168465881539373497839250693673450654088527700\
      5934676178492067266845625218835145026075705476010909889836 3576113}
76: {100431.,
     77744155411151974089868056547615372951055133931608310943911679473488134467\
      8382582465198276256701446196196172088864748856495178757995718880324}
77: {101555.,
     61764398059971578808332533818681508065912573772099647446045251500133874859\
      9417077232644653344450259553125839797684931616066950343001068 7766761}
```

```
78: {113436.,
    4908129096776091045320075917502905956325325264117748476732078466402607695⋱
     0819923696480419572799794858099346973752918009606460246189186220610818 8}
79: {110954.,
    390120111786328113538562455616786929332168155592273598475569710202503605 37⋱
     2602188863559933755059072571477476112028654125245706976834159866392851033}
80: {123981.,
    310157794713970208299398864694105313271777747990370913169892778350913946 21⋱
     986947957747347675373127854226194498480349096934751135478191634525159660 7⋱
     96}";
data = Prepend[
  ToExpression[Last[StringSplit[#, {",", "}"}]]] & /@ StringSplit[str, "\n"], 1];
timings = Prepend[ToExpression[StringSplit[#, {",", "{"}][[2]]] & /@
   StringSplit[str, "\n"], 0];
```

In[∘]:=
```
(* Total CPU time (in days). *)
Total[timings] / 3600 / 24
```

Out[∘]= 27.373

4 Lattice Walks

4.1 A265234

In[10]:=
```
InitializeSeq[265 234]
```

Sequence: 1, 43, 2592, 184740, 14439456, 1196114464,
103142395392, 9160513923648, 832211576040960, 76971887847571968, ...

Length: 31

Offset: 1

Recurrence: $-34836480\,(1+n)\,(2+n)\,(3+n)\,(4+n)\,(15077+7449\,n+832\,n^2)\,a[n]+9216\,(2+n)\,(3+n)\,(4+n)\,(84692065+74692297\,n+20496944\,n^2+1743872\,n^3)\,a[1+n]-768\,(3+n)\,(4+n)\,(716118600+813878537\,n+333112832\,n^2+58438003\,n^3+3709888\,n^4)\,a[2+n]+128\,(4+n)\,(1242989235+1653171497\,n+841828441\,n^2+207271023\,n^3+24822356\,n^4+1161472\,n^5)\,a[3+n]+16\,(1398239904+2087600280\,n+1294537774\,n^2+426421788\,n^3+78610375\,n^4+7680621\,n^5+310336\,n^6)\,a[4+n]-4\,(5+n)\,(208319000+256228730\,n+119399663\,n^2+26840735\,n^3+2941016\,n^4+126464\,n^5)\,a[5+n]+5\,(5+n)\,(6+n)^3\,(8460+5785\,n+832\,n^2)\,a[6+n]$

Check: True

```
In[6]:= (* Compute values using a transfer matrix. *)
rows = DeleteCases[Tuples[{0, 1, 2, 3}, 4], {___, a_, a_, ___}];
tmat = Table[1, {i, Length[rows]}, {j, Length[rows]}];
vec = vvec = ((x^Count[#, 0] * y^Count[#, 1] * z^Count[#, 2]) & /@ rows);
Prepend[Table[vec = Expand[vvec * (tmat.vec)];
  Coefficient[Total[vec], (x y z)^n] / 24, {n, 2, 6}], 1]

Out[9]= {1, 43, 2592, 184 740, 14 439 456, 1 196 114 464}

In[11]:= (* Define the step-set polynomial. *)
ssp = Total[(x^Count[#, 0] * y^Count[#, 1] * z^Count[#, 2]) & /@ rows]

Out[11]= 2 x^2 + 6 x y + 6 x^2 y + 2 y^2 + 6 x y^2 + 2 x^2 y^2 + 6 x z + 6 x^2 z + 6 y z + 24 x y z +
  6 x^2 y z + 6 y^2 z + 6 x y^2 z + 2 z^2 + 6 x z^2 + 2 x^2 z^2 + 6 y z^2 + 6 x y z^2 + 2 y^2 z^2

In[12]:= rat = 1 / (1 - t * ssp)

Out[12]= 1 /
  (1 - t (2 x^2 + 6 x y + 6 x^2 y + 2 y^2 + 6 x y^2 + 2 x^2 y^2 + 6 x z + 6 x^2 z + 6 y z + 24 x y z + 6 x^2 y z + 6 y^2 z +
        6 x y^2 z + 2 z^2 + 6 x z^2 + 2 x^2 z^2 + 6 y z^2 + 6 x y z^2 + 2 y^2 z^2))

In[13]:= (* Sanity check by series expansion. *)
ser = Normal[Series[rat, {t, 0, 6}]];
Table[Coefficient[ser, (t x y z)^n] / 24, {n, 6}]

Out[14]= {1, 43, 2592, 184 740, 14 439 456, 1 196 114 464}

In[15]:= (* Transformed rational function for diagonal extraction by residue. *)
rat1 = Together[(rat /. {t → t / x, x → x / y, y → y / z}) / (x y z)]

Out[15]= - ((y z) / (2 t x^2 y^2 + 6 t x y^3 + 2 t y^4 + 6 t x^2 y z + 6 t x y^2 z + 6 t x y^3 z +
      6 t y^4 z + 2 t x^2 z^2 + 6 t x^2 y z^2 - x y^2 z^2 + 24 t x y^2 z^2 + 6 t y^3 z^2 + 2 t y^4 z^2 +
      6 t x^2 z^3 + 6 t x y z^3 + 6 t x y^2 z^3 + 6 t y^3 z^3 + 2 t x^2 z^4 + 6 t x y z^4 + 2 t y^2 z^4))

In[16]:= Timing[
  ct1 = CreativeTelescoping[rat1, Der[x], Der /@ {t, y, z}][[1]];
  #[ct1] & /@ {ByteCount, UnderTheStaircase}
]

Out[16]= {0.400951, {41 680, {1}}}

In[17]:= Timing[
  ct2 = CreativeTelescoping[ct1, Der[y]][[1]];
  #[ct2] & /@ {ByteCount, UnderTheStaircase}
]

Out[17]= {1.13876, {90 304, {1, D_z}}}
```

In[18]:=
```
Timing[
 ct3 = CreativeTelescoping[ct2, Der[z]][[1]];
 #[ct3] & /@ {ByteCount, UnderTheStaircase}
]
```

Out[18]= $\{32.3531, \{7440, \{1, D_t, D_t^2, D_t^3\}\}\}$

In[21]:=
```
(* This is the differential equation satisfied by the diagonal. *)
Factor[ct3]
```

Out[21]= $\{t^2 (-1+4t)(-1+12t)^2(1+20t)(-5+28t)(-1+108t)(25-2020t+16416t^2) D_t^4 + t(-1+12t)(625-143900t+7423280t^2+9698176t^3-4618562816t^4+71304299520t^5-400832851968t^6+762500874240t^7) D_t^3 + 4(-125+48100t-2911400t^2-4295904t^3+3955210560t^4-103794289920t^5+1123705377792t^6-5563595870208t^7+10293761802240t^8) D_t^2 + 48(925-44710t-2570232t^2+278201312t^3-7796204544t^4+95601807360t^5-539792750592t^6+1143751311360t^7) D_t + 192(-175-86140t+8645912t^2-292787616t^3+4331094912t^4-28838011392t^5+71484456960t^6)\}$

In[23]:=
```
(* Convert it to a recurrence for its Taylor coefficients. *)
recop = CreativeTelescoping[
   DFiniteTimes[ToOrePolynomial[Append[ct3, S[n] - 1], OreAlgebra[Der[t], S[n]]],
    Annihilator[t^(-n - 1), {Der[t], S[n]}]], Der[t]][[1, 1]]
```

Out[23]= $(45684000+63217800n+34735500n^2+9845010n^3+1531255n^4+124605n^5+4160n^6) S_n^6 + (-4166380000-5957850600n-3412908180n^2-1014413352n^3-166183260n^4-14293344n^5-505856n^6) S_n^5 + (22371838464+33401604480n+20712604384n^2+6822748608n^3+1257766000n^4+122889936n^5+4965376n^6) S_n^4 + (636410488320+1005526428544n+642622113408n^2+213876804224n^3+39239737216n^4+3771935232n^5+148668416n^6) S_n^3 + (-6599749017600-11350558190592n-7995357959424n^2-2954437936896n^3-604183686912n^4-64824744192n^5-2849193984n^6) S_n^2 + (18732529704960+36814314866688n+29455764139008n^2+12272912271360n^3+2806322365440n^4+333543555072n^5+16071524352n^6) S_n + (-12605510615040-32489406996480n-32053498122240n^2-15783886540800n^3-4134637301760n^4-549336453120n^5-28983951360n^6)$

In[24]:=
```
(* Compare with the guessed recurrence. *)
Together[rec / ApplyOreOperator[recop, a[n]]]
```

Out[24]= 1

View it as the extraction of a coefficient from powers of the step-set polynomial:

In[25]:=
```
Timing[Table[Coefficient[ssp^n, (x y z)^n] / 24, {n, 10}]]
```

Out[25]=
```
{11.5274, {1, 43, 2592, 184740, 14439456, 1196114464,
  103142395392, 9160513923648, 832211576040960, 76971887847571968}}
```

In[28]:=
```
Timing[poly = 1;
 Table[poly = Expand[poly * ssp];
  Coefficient[poly, (x y z)^n] / 24, {n, 1, 10}]]
```

Out[28]=
```
{0.197929, {1, 43, 2592, 184740, 14439456, 1196114464,
  103142395392, 9160513923648, 832211576040960, 76971887847571968}}
```

```
Timing[poly = 1;
 tab = Table[poly = Expand[poly * ssp];
   Coefficient[poly, (x y z)^n] / 24, {n, 1, 56}];]
```

Out[•]=
```
{102.474, Null}
```

4.2 A172572

In[29]:=
```
InitializeSeq[172572]
```

```
Sequence: 90, 67950, 90291600, 154700988750, 306407299538340,
   666569141498660400, 1548539246648239560000, 3776577900841430197548750,
   9561215418596022668009737500, 24935177268489106332174087326700, ...
```

```
Length: 33
```

```
Offset: 1
```

```
Recurrence:
```

$$20250\,(1+n)(1+3n)(2+3n)(4+3n)(5+3n)(7+3n)(8+3n)\left(470+341n+62n^2\right)a[n] - 9\,(4+3n)(5+3n)(7+3n)(8+3n)\left(827860+2220988n+2354425n^2+1227805n^3+313720n^4+31372n^5\right)a[1+n] - 9\,(2+n)(7+3n)(8+3n)\left(601185+1653960n+1884032n^2+1137319n^3+383756n^4+68634n^5+5084n^6\right)a[2+n] + (2+n)(3+n)^5(5+2n)\left(191+217n+62n^2\right)a[3+n]$$

```
Check: True
```

In[31]:=
```
(* We get a simpler recurrence by dividing out a hypergeometric factor. *)
Factor[DFiniteTimesHyper[ToOrePolynomial[rec, a[n]], 1 / Binomial[3 n, n]]]
```

Out[31]=

$$(3+n)^4\left(191+217n+62n^2\right)S_n^3 - 6\left(601185+1653960n+1884032n^2+1137319n^3+383756n^4+68634n^5+5084n^6\right)S_n^2 - 4\,(3+2n)\left(827860+2220988n+2354425n^2+1227805n^3+313720n^4+31372n^5\right)S_n + 6000\,(1+n)^2(1+2n)(3+2n)\left(470+341n+62n^2\right)$$

9-fold multinomial sum, reduced to 6-fold by Chu-Vandermonde

In[32]:=
```
(* The entries of the multinomial> *)
expr = Array[c, 15] /.
   Solve[Array[c, 15].Permutations[{1, 1, 0, 0, 0, 0}] == Table[n, {6}]][[1]];
vars = Union[Cases[expr, c[_], Infinity]];
TableForm[expr]
```

Out[34]//TableForm=
```
c[1]
c[2]
c[3]
c[4]
n - c[1] - c[2] - c[3] - c[4]
c[6]
c[7]
c[8]
n - c[1] - c[6] - c[7] - c[8]
c[10]
c[11]
n - c[2] - c[6] - c[10] - c[11]
2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10] - c[11]
-n + c[1] + c[2] + c[4] + c[6] + c[8] + c[11]
-n + c[1] + c[2] + c[3] + c[6] + c[7] + c[10]
```

```
(* Sanity check. *)
Clear[A172572];
A172572[n_] := A172572[n] =
   Sum @@ Prepend[Table[{vars[[i]], 0, n}, {i, 9}], Multinomial @@ expr];
Do[Print[Timing[A172572[n]]], {n, 3}]
```

```
{0.058028, 90}
{0.52683, 67 950}
{6.73317, 90 291 600}
```

```
(* Adapt the ranges of summation for faster computation. *)
Clear[A172572];
A172572[n_] := A172572[n] =
   Sum[Multinomial[c[1], c[2], c[3], n - c[1] - c[2] - c[3] - c[4],
     c[4], c[6], c[7], n - c[1] - c[6] - c[7] - c[8], c[8], c[10],
     -n + c[1] + c[2] + c[3] + c[6] + c[7] + c[10], n - c[2] - c[6] - c[10] - c[11],
     2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10] - c[11],
     c[11], -n + c[1] + c[2] + c[4] + c[6] + c[8] + c[11]],
    {c[1], 0, n},
    {c[2], 0, n - c[1]},
    {c[3], 0, n - c[1] - c[2]},
    {c[4], 0, n - c[1] - c[2] - c[3]},
    {c[6], 0, Min[n - c[1], n - c[2], 2 n - c[1] - c[2] - c[3] - c[4]]},
    {c[7], 0, Min[n - c[1] - c[6], 2 n - c[1] - c[2] - c[3] - c[4] - c[6]]},
    {c[8], 0, Min[n - c[1] - c[6] - c[7], 2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7]]},
    {c[10], Max[0, n - c[1] - c[2] - c[3] - c[6] - c[7]],
     Min[n - c[2] - c[6], 2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8]]},
    {c[11], Max[0, n - c[1] - c[2] - c[4] - c[6] - c[8]], Min[n - c[2] - c[6] - c[10],
      2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10]]}];

Do[Print[n, ": ", Timing[A172572[n]]], {n, 14}]
```

```
1: {0.001436, 90}
2: {0.014768, 67 950}
3: {0.047801, 90 291 600}
4: {0.142761, 154 700 988 750}
5: {0.421648, 306 407 299 538 340}
6: {1.22127, 666 569 141 498 660 400}
7: {3.1872, 1 548 539 246 648 239 560 000}
8: {7.36197, 3 776 577 900 841 430 197 548 750}
9: {16.3483, 9 561 215 418 596 022 668 009 737 500}
10: {35.2818, 24 935 177 268 489 106 332 174 087 326 700}
11: {66.534, 66 616 980 501 713 943 527 764 656 942 096 000}
12: {134.856, 181 567 587 344 159 723 781 226 957 237 357 470 000}
13: {230.619, 503 273 760 207 613 155 429 966 482 419 001 606 580 000}
14: {388.146, 1 415 189 158 639 246 716 651 027 917 944 817 871 202 200 000}
```

```
In[•]:= Take[data, 10] - Table[A172572[n], {n, 10}]
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
(* We can apply the Chu-Vandermonde identity. *)
ChuVandermonde = sum[1/k!/(m-k)!/(r-k)!/(n-r+k)!, {k, 0, r}] ==
   (m+n)!/r!/(m+n-r)!/m!/n!;
```

```
In[•]:= expr1 = FunctionExpand[Multinomial @@ expr] /. Gamma[n_] → (n - 1)!;
        expr2 = Select[expr1, Not[FreeQ[#, c[11]]] &]
```

```
Out[•]= 1/((n - c[2] - c[6] - c[10] - c[11])!
          (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10] - c[11])!
          c[11]! (-n + c[1] + c[2] + c[4] + c[6] + c[8] + c[11])!)
```

```
In[•]:= test = ChuVandermonde /. {k → c[11], r → n - c[2] - c[6] - c[10], m →
           2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10], n → c[1] + c[4] + c[8] - c[10]}
```

```
Out[•]= sum[1/((n - c[2] - c[6] - c[10] - c[11])!
             (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10] - c[11])!
             c[11]! (-n + c[1] + c[2] + c[4] + c[6] + c[8] + c[11])!),
          {c[11], 0, n - c[2] - c[6] - c[10]}] == (2 n - c[2] - c[3] - c[6] - c[7] - 2 c[10])!/
          ((n - c[2] - c[6] - c[10])! (n - c[3] - c[7] - c[10])!
            (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10])!
            (c[1] + c[4] + c[8] - c[10])!)
```

```
In[•]:= expr2 === test[[1, 1]]
```

```
Out[•]= True
```

```
In[•]:= expr1 = expr1/expr2 * test[[2]];
        expr2 = Select[expr1, Not[FreeQ[#, c[8]]] &]
```

```
Out[•]= 1/((n - c[1] - c[6] - c[7] - c[8])! c[8]!
          (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10])! (c[1] + c[4] + c[8] - c[10])!)
```

```
In[•]:= test = ChuVandermonde /. {k → c[8], r → n - c[1] - c[6] - c[7],
          n → n + c[4] - c[6] - c[7] - c[10], m → 2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[10]}
```

```
Out[•]= sum[1/((n - c[1] - c[6] - c[7] - c[8])!
             c[8]! (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[8] - c[10])!
             (c[1] + c[4] + c[8] - c[10])!), {c[8], 0, n - c[1] - c[6] - c[7]}] ==
         (3 n - c[1] - c[2] - c[3] - 2 c[6] - 2 c[7] - 2 c[10])!/
          ((n - c[1] - c[6] - c[7])! (2 n - c[2] - c[3] - c[6] - c[7] - 2 c[10])!
            (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[10])! (n + c[4] - c[6] - c[7] - c[10])!)
```

```
In[•]:= expr2 === test[[1, 1]]
```

```
Out[•]= True
```

```
In[•]:= expr1 = expr1 / expr2 * test[[2]];
        expr2 = Select[expr1, Not[FreeQ[#, c[4]]] &]

Out[•]= 1 / ((n - c[1] - c[2] - c[3] - c[4]) ! c[4] !
          (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[10]) ! (n + c[4] - c[6] - c[7] - c[10]) !)

In[•]:= test = ChuVandermonde /.
          {k → c[4], r → n - c[1] - c[2] - c[3], n → 2 n - c[1] - c[2] - c[3] - c[6] - c[7] - c[10],
           m → 2 n - c[1] - c[2] - c[3] - c[6] - c[7] - c[10]}

Out[•]= sum[1 / ((n - c[1] - c[2] - c[3] - c[4]) !
            c[4] ! (2 n - c[1] - c[2] - c[3] - c[4] - c[6] - c[7] - c[10]) !
            (n + c[4] - c[6] - c[7] - c[10]) !), {c[4], 0, n - c[1] - c[2] - c[3]}] ==
         (4 n - 2 c[1] - 2 c[2] - 2 c[3] - 2 c[6] - 2 c[7] - 2 c[10]) ! /
          ((n - c[1] - c[2] - c[3]) ! (3 n - c[1] - c[2] - c[3] - 2 c[6] - 2 c[7] - 2 c[10]) !
            ((2 n - c[1] - c[2] - c[3] - c[6] - c[7] - c[10]) !)^2)

In[•]:= expr2 === test[[1, 1]]

Out[•]= True

In[•]:= expr1 = expr1 / expr2 * test[[2]]

Out[•]= ((3 n) ! (4 n - 2 c[1] - 2 c[2] - 2 c[3] - 2 c[6] - 2 c[7] - 2 c[10]) !) /
         (c[1] ! c[2] ! (n - c[1] - c[2] - c[3]) ! c[3] ! c[6] !
           (n - c[1] - c[6] - c[7]) ! c[7] ! (n - c[2] - c[6] - c[10]) !
           (n - c[3] - c[7] - c[10]) ! ((2 n - c[1] - c[2] - c[3] - c[6] - c[7] - c[10]) !)^2
          c[10] ! (-n + c[1] + c[2] + c[3] + c[6] + c[7] + c[10]) !)

        (* Hence, we end up with a 6-fold hypergeometric sum. *)
        Clear[A172572];
        A172572[n_] := A172572[n] =
          Sum[((3 n) ! (4 n - 2 c[1] - 2 c[2] - 2 c[3] - 2 c[6] - 2 c[7] - 2 c[10]) !) /
             (c[1] ! c[2] ! (n - c[1] - c[2] - c[3]) ! c[3] ! c[6] ! (n - c[1] - c[6] - c[7]) !
               c[7] ! (n - c[2] - c[6] - c[10]) ! (n - c[3] - c[7] - c[10]) !
               ((2 n - c[1] - c[2] - c[3] - c[6] - c[7] - c[10]) !)^2
               c[10] ! (-n + c[1] + c[2] + c[3] + c[6] + c[7] + c[10]) !),
            {c[1], 0, n},
            {c[2], 0, n - c[1]},
            {c[3], 0, n - c[1] - c[2]},
            {c[6], 0, Min[n - c[1], n - c[2]]},
            {c[7], 0, Min[n - c[1] - c[6], n - c[3]]},
            {c[10], Max[0, n - c[1] - c[2] - c[3] - c[6] - c[7]],
             Min[n - c[2] - c[6], n - c[3] - c[7]]}];

In[•]:= Do[Print[n, ": ", Timing[A172572[n]]], {n, 20}]
```

```
1: {0.000543, 90}
2: {0.002697, 67 950}
3: {0.012514, 90 291 600}
4: {0.021674, 154 700 988 750}
5: {0.057804, 306 407 299 538 340}
6: {0.094693, 666 569 141 498 660 400}
7: {0.15283, 1 548 539 246 648 239 560 000}
8: {0.282622, 3 776 577 900 841 430 197 548 750}
9: {0.543685, 9 561 215 418 596 022 668 009 737 500}
10: {0.983657, 24 935 177 268 489 106 332 174 087 326 700}
11: {1.49047, 66 616 980 501 713 943 527 764 656 942 096 000}
12: {2.10496, 181 567 587 344 159 723 781 226 957 237 357 470 000}
13: {2.99719, 503 273 760 207 613 155 429 966 482 419 001 606 580 000}
14: {4.27969, 1 415 189 158 639 246 716 651 027 917 944 817 871 202 200 000}
15: {6.14906, 4 029 200 036 771 699 577 090 637 149 510 314 768 593 535 481 600}
16: {8.66437, 11 596 575 535 834 069 329 340 945 743 908 684 169 826 573 155 948 750}
17: {12.1983, 33 696 387 989 994 684 099 977 914 632 459 510 820 261 684 112 951 287 500}
18: {17.7246, 98 744 132 720 453 916 340 813 668 495 424 059 077 053 759 331 765 698 612 500}
19: {23.8623, 291 554 992 443 256 507 922 708 018 568 624 838 893 131 441 169 268 577 290 500 000}
20: {31.4352, 866 717 702 837 149 834 068 104 732 178 643 953 342 871 839 288 330 601 416 111 907 500}
```

```
(* By exploiting symmetries, we can make the enumeration more efficient. *)
Clear[A172572];
A172572[n_] := A172572[n] =
  Sum[Switch[Length[Union[c /@ {1, 2, 3}]], 1, 1, 2, 3, 3, 6] *
    Sum[((3 n) ! (4 n - 2 c[1] - 2 c[2] - 2 c[3] - 2 c[6] - 2 c[7] - 2 c[10]) !) /
      (c[1] ! c[2] ! (n - c[1] - c[2] - c[3]) ! c[3] ! c[6] ! (n - c[1] - c[6] - c[7]) !
        c[7] ! (n - c[2] - c[6] - c[10]) ! (n - c[3] - c[7] - c[10]) !
        ((2 n - c[1] - c[2] - c[3] - c[6] - c[7] - c[10]) !)^2
        c[10] ! (-n + c[1] + c[2] + c[3] + c[6] + c[7] + c[10]) !),
     {c[6], 0, Min[n - c[1], n - c[2]]},
     {c[7], 0, Min[n - c[1] - c[6], n - c[3]]},
     {c[10], Max[0, n - c[1] - c[2] - c[3] - c[6] - c[7]],
      Min[n - c[2] - c[6], n - c[3] - c[7]]}],
   {c[1], 0, n},
   {c[2], c[1], n - c[1]},
   {c[3], c[2], n - c[1] - c[2]}];
```

```
In[]:= Timing[Do[Print[n, ": ", Timing[A172572[n]]], {n, 33}]]
```

1: {0.002225, 90}

2: {0.006727, 67 950}

3: {0.012614, 90 291 600}

4: {0.013795, 154 700 988 750}

5: {0.024062, 306 407 299 538 340}

6: {0.031041, 666 569 141 498 660 400}

7: {0.05652, 1 548 539 246 648 239 560 000}

8: {0.097194, 3 776 577 900 841 430 197 548 750}

9: {0.127766, 9 561 215 418 596 022 668 009 737 500}

10: {0.217625, 24 935 177 268 489 106 332 174 087 326 700}

11: {0.308053, 66 616 980 501 713 943 527 764 656 942 096 000}

12: {0.492826, 181 567 587 344 159 723 781 226 957 237 357 470 000}

13: {0.804527, 503 273 760 207 613 155 429 966 482 419 001 606 580 000}

14: {1.01916, 1 415 189 158 639 246 716 651 027 917 944 817 871 202 200 000}

15: {1.41334, 4 029 200 036 771 699 577 090 637 149 510 314 768 593 535 481 600}

16: {1.9107, 11 596 575 535 834 069 329 340 945 743 908 684 169 826 573 155 948 750}

17: {2.64209, 33 696 387 989 994 684 099 977 914 632 459 510 820 261 684 112 951 287 500}

18: {3.52705, 98 744 132 720 453 916 340 813 668 495 424 059 077 053 759 331 765 698 612 500}

19: {5.59454, 291 554 992 443 256 507 922 708 018 568 624 838 893 131 441 169 268 577 290 500 000}

20: {6.08567, 866 717 702 837 149 834 068 104 732 178 643 953 342 871 839 288 330 601 416 111 907 500}

21:
{7.72337, 2 592 375 677 123 857 182 653 401 578 050 802 301 420 115 413 171 876 191 880 257 249 825 000}

22: {10.4051,
7 797 148 109 915 507 609 055 754 351 545 480 754 098 792 860 810 120 924 267 461 803 822 000 000}

23: {13.788,
23 571 007 790 381 387 245 049 287 069 486 362 783 508 965 549 924 700 762 984 477 300 563 816 000 000}

24: {17.6066,
71 587 791 514 183 316 622 788 860 936 837 138 063 449 952 235 586 143 944 755 462 731 861 014 398 750 -
000}

25: {21.8971,
218 350 783 435 552 995 449 496 393 057 184 538 450 144 943 669 181 285 535 176 774 912 778 540 542 192 -
575 840}

26: {26.7694,
668 622 733 894 376 274 452 091 386 795 230 794 487 949 900 497 567 112 116 310 611 908 074 803 410 776 -
920 629 600}

27: {33.495,
2 054 896 548 644 832 758 746 250 300 092 907 511 564 295 504 618 963 836 951 284 533 636 852 722 226 -
246 810 329 408 000}

```
28: {40.0679,
  6 336 755 537 553 926 305 733 442 710 201 369 068 985 123 905 527 876 181 590 940 011 001 361 490 805 -
   315 174 039 138 584 000}
29: {47.9294,
  19 602 424 263 713 066 423 689 388 691 607 583 622 211 106 071 326 265 284 732 748 374 946 116 909 122 -
   474 483 550 678 301 200 000}
30: {57.1059,
  60 817 206 776 894 503 064 587 204 350 967 809 876 439 406 565 283 383 292 406 713 920 337 000 671 148 -
   694 207 101 326 748 202 630 400}
31: {68.7357,
  189 205 611 266 182 683 216 418 223 839 047 431 121 418 138 304 342 562 383 939 035 105 334 268 755 809 -
   136 991 531 477 189 536 559 744 000}
32: {81.563,
  590 143 013 685 262 378 133 336 976 827 142 402 637 846 461 475 184 258 746 627 676 988 460 223 488 290 -
   653 771 960 681 941 201 801 109 868 750}
33: {97.8045,
  1 845 134 365 949 724 967 456 250 719 015 909 535 477 751 783 742 177 307 299 963 905 041 916 204 529 -
   884 419 584 335 933 047 701 259 496 871 987 500}
```

Out[∘]= {549.282, Null}

In[∘]:= **Table[A172572[n], {n, 33}] - data**

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[∘]:= **Timing[Do[Print[n, ": ", Timing[A172572[n]]], {n, 34, 44}]]**

34: {117.18,
5 782 079 048 231 602 751 773 508 193 640 478 233 320 363 502 757 896 221 735 857 368 582 856 842 790 -
186 691 493 575 211 339 703 404 973 485 946 062 500}

35: {136.441,
18 158 022 781 345 117 396 481 801 843 530 601 077 978 530 347 164 537 195 676 330 330 557 274 904 088 -
418 464 974 775 932 490 645 322 881 543 975 821 660 000}

36: {158.594,
57 138 596 982 024 968 239 050 317 687 127 822 571 437 150 169 242 623 507 111 632 191 172 838 496 774 -
695 539 595 612 108 983 437 815 453 124 926 480 782 012 500}

37: {185.018,
180 143 793 173 098 445 191 815 608 483 560 889 523 829 624 190 434 911 567 089 162 728 937 123 317 541 -
715 327 423 951 190 280 979 607 021 165 301 494 371 186 375 000}

38: {215.752,
568 975 783 719 356 374 948 762 176 662 831 790 551 102 160 525 940 761 559 833 364 806 671 666 456 206 -
983 335 298 825 684 882 721 339 591 622 891 133 249 025 262 700 000}

39: {250.356,
1 800 165 576 629 111 849 346 091 635 809 357 229 899 413 563 228 001 400 202 833 343 766 139 457 188 -
176 855 749 625 571 294 894 838 457 273 052 363 771 511 557 727 858 000 000}

40: {287.6,
5 704 765 804 907 623 134 985 227 501 383 837 589 421 039 378 865 892 216 507 936 553 252 128 712 819 -
738 637 685 386 212 171 012 909 808 339 444 580 204 485 050 814 280 052 867 500}

41: {332.288,
18 106 526 980 071 510 440 109 527 082 256 994 905 224 114 718 505 593 465 947 395 459 435 131 756 382 -
912 705 855 848 572 136 960 887 749 683 557 104 083 579 817 203 599 744 074 775 000}

42: {380.794,
57 553 564 992 004 547 942 188 932 030 321 165 066 819 803 096 262 094 875 226 769 370 767 895 582 288 -
361 207 005 940 430 197 024 578 767 710 066 451 602 240 759 255 378 096 530 571 875 000}

43: {434.946,
183 197 303 056 623 744 120 784 770 640 733 925 099 696 260 494 196 110 717 737 230 962 767 623 053 369 -
403 900 505 182 418 148 507 526 221 730 458 738 047 791 675 692 255 057 391 667 600 000 000}

44: {517.943,
583 912 517 577 434 981 138 943 546 894 099 130 064 031 884 261 972 594 307 956 065 683 311 526 544 978 -
060 853 225 595 250 646 704 487 722 648 896 952 732 592 370 140 190 026 535 105 108 750 000 000}

Out[∘]= {3016.91, Null}

Interpretation as 6-dimensional walks

In[•]:=
```
steps = Permutations[{1, 1, 0, 0, 0, 0}];
Timing[With[{n = 20},
  vals = Table @@ Prepend[Table[{c[i], 0, If[i === 1, n, c[i - 1]]}, {i, 6}], 0];
  vals[[1, 1, 1, 1, 1, 1]] = 1;
  Do[
   Do[
     vals[[n1 + 1, n2 + 1, n3 + 1, n4 + 1, n5 + 1, nsum - n1 - n2 - n3 - n4 - n5 + 1]] =
      Total[Function[s,
         pos = Reverse[Sort[{n1, n2, n3, n4, n5, nsum - n1 - n2 - n3 - n4 - n5} - s]];
         If[Min[pos] < 0, 0, vals[[##]] & @@ (pos + 1)]
        ] /@ steps],
     {n1, 0, Min[n, nsum]},
     {n2, 0, Min[n1, nsum - n1]},
     {n3, 0, Min[n2, nsum - n1 - n2]},
     {n4, 0, Min[n3, nsum - n1 - n2 - n3]},
     {n5, Ceiling[(nsum - n1 - n2 - n3 - n4) / 2], Min[n4, nsum - n1 - n2 - n3 - n4]}];
   , {nsum, 2, 6 n, 2}];
  Table[vals[[i + 1, i + 1, i + 1, i + 1, i + 1, i + 1]], {i, n}] - Take[data, n]
 ]]
```

Out[•]=
```
{14.0204, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

In[∘]:=
```
(* This variant is slightly slower, but uses a bit less memory. *)
Clear[Walks6D];
Walks6D[n_Integer, steps_List] := Walks6D[n, steps] =
   Module[{vals = {{{{{{1}}}}}}, seq = {}, n0, n1},
    Do[
     vals = Table[
       n0 = Ceiling[(nsum - n1 - n2 - n3 - n4) / 2];
       Join[
        Table[0, {n0}],
        Table[Total[Function[s,
            pos = Reverse[Sort[{n1, n2, n3, n4, n5, nsum - n1 - n2 - n3 - n4 - n5} - s]];
            If[Min[pos] < 0, 0, vals[[##]] &@@ Most[pos + 1]]
           ] /@ steps],
         {n5, n0, Min[n4, nsum - n1 - n2 - n3 - n4]}]],
       {n1, 0, Min[n, nsum]},
       {n2, 0, Min[n1, nsum - n1]},
       {n3, 0, Min[n2, nsum - n1 - n2]},
       {n4, 0, Min[n3, nsum - n1 - n2 - n3]}];
     If[IntegerQ[n1 = nsum / 6 + 1], AppendTo[seq, vals[[n1, n1, n1, n1, n1]]]];
     , {nsum, 2, 6 n, 2}];
    Return[seq];
   ];
```

In[∘]:=
```
Walks6D[33, Permutations[{1, 1, 0, 0, 0, 0}]] - data
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
Timing[MaxMemoryUsed[Walks6D[44, Permutations[{1, 1, 0, 0, 0, 0}]];]]
```

Out[∘]=
```
{1053.39, 304 520 288}
```

4.2 A172671

In[40]:=
```
InitializeSeq[172 671]
```

```
Sequence:
 90, 202410, 747558000, 3536978063850, 19292117692187340, 115428185943399529200,
   737005538936597762145600, 4937928427617947420104982250,
   34335031273255183438800013252500, 245885257930209910994050195049583660, ...
```

```
Length: 33
```

```
Offset: 1
```

Recurrence: $416745\,(1+n)\,(1+3n)\,(2+3n)\,(4+3n)\,(5+3n)\,(7+3n)\,(8+3n)\,(10+3n)\,(11+3n)\left(344237+450988\,n+219945\,n^2+47300\,n^3+3784\,n^4\right)a[n] + 9\,(4+3n)\,(5+3n)\,(7+3n)\,(8+3n)\,(10+3n)\,(11+3n)\left(9799573455+31639900193\,n+43078657918\,n^2+32010306742\,n^3+14001842392\,n^4+3602458816\,n^5+504588832\,n^6+29681696\,n^7\right)a[1+n] - 9\,(2+n)\,(7+3n)\,(8+3n)\,(10+3n)\,(11+3n)\left(15352797306+51243135187\,n+74160044251\,n^2+60768378830\,n^3+30831383530\,n^4+9916013134\,n^5+1973930222\,n^6+222321352\,n^7+10844944\,n^8\right)a[2+n] - (2+n)\,(3+n)^3\,(10+3n)\,(11+3n)\left(2991586122+8487349821\,n+10141503096\,n^2+6617561702\,n^3+2548427912\,n^4+579689880\,n^5+72183584\,n^6+3799136\,n^7\right)a[3+n] + (2+n)\,(3+n)^3\,(4+n)^5\left(69678+137862\,n+100749\,n^2+32164\,n^3+3784\,n^4\right)a[4+n]$

```
Check: True
```

In[41]:=
```
(* Simpler recurrence by hypergeometric transformation (due to E. Kaltofen). *)
Factor[DFiniteTimesHyper[ToOrePolynomial[rec, a[n]], (n!)^3/(3n)!]]
```

Out[41]=
$$3\,(3+n)\,(4+n)^3\left(69678+137862\,n+100749\,n^2+32164\,n^3+3784\,n^4\right)S_n^4 - (3+n)\left(2991586122+8487349821\,n+10141503096\,n^2+6617561702\,n^3+2548427912\,n^4+579689880\,n^5+72183584\,n^6+3799136\,n^7\right)S_n^3 - 3\left(15352797306+51243135187\,n+74160044251\,n^2+60768378830\,n^3+30831383530\,n^4+9916013134\,n^5+1973930222\,n^6+222321352\,n^7+10844944\,n^8\right)S_n^2 + (2+n)\left(9799573455+31639900193\,n+43078657918\,n^2+32010306742\,n^3+14001842392\,n^4+3602458816\,n^5+504588832\,n^6+29681696\,n^7\right)S_n + 15435\,(1+n)^3\,(2+n)\left(344237+450988\,n+219945\,n^2+47300\,n^3+3784\,n^4\right)$$

15-fold multinomial sum, reduced to 11-fold by Chu-Vandermonde

In[45]:=
```
expr = Array[c, 21] /. Solve[Array[c, 21].Join[Permutations[{2, 0, 0, 0, 0, 0}],
        Permutations[{1, 1, 0, 0, 0, 0}]] == Table[n, {6}]][[1]];
vars = Union[Cases[expr, c[_], Infinity]];
TableForm[expr]
```

Out[47]//TableForm=
```
c[1]
c[2]
c[3]
c[4]
c[5]
c[6]
c[7]
c[8]
c[9]
c[10]
n - 2 c[1] - c[7] - c[8] - c[9] - c[10]
c[12]
c[13]
c[14]
n - 2 c[2] - c[7] - c[12] - c[13] - c[14]
c[16]
c[17]
n - 2 c[3] - c[8] - c[12] - c[16] - c[17]
2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] - c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[14] - c
-n + c[1] + c[2] + c[3] - c[4] + c[5] - c[6] + c[7] + c[8] + c[10] + c[12] + c[14] + c[17]
-n + c[1] + c[2] + c[3] + c[4] - c[5] - c[6] + c[7] + c[8] + c[9] + c[12] + c[13] + c[16]
```

In[48]:=
```
(* We can apply the Chu-Vandermonde identity. *)
ChuVandermonde = sum[1 / k! / (m - k)! / (r - k)! / (n - r + k)!, {k, 0, r}] ==
   (m + n)! / r! / (m + n - r)! / m! / n!;
```

In[49]:=
```
expr1 = FunctionExpand[Multinomial @@ expr] /. Gamma[n_] → (n - 1)!;
expr2 = Select[expr1, Not[FreeQ[#, c[17]]] &]
```

Out[50]=
```
1 / ((n - 2 c[3] - c[8] - c[12] - c[16] - c[17])!
   (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] - c[7] - c[8] -
      c[9] - c[10] - c[12] - c[13] - c[14] - c[16] - c[17])! c[17]!
   (-n + c[1] + c[2] + c[3] - c[4] + c[5] - c[6] + c[7] + c[8] + c[10] + c[12] + c[14] + c[17])!)
```

```
In[•]:= test = ChuVandermonde /.
          {k → c[17], r → n - 2 c[3] - c[8] - c[12] - c[16], m → 2 n - c[1] - c[2] - c[3] -
             c[4] - c[5] + c[6] - c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[14] - c[16],
           n → c[1] + c[2] - c[3] - c[4] + c[5] - c[6] + c[7] + c[10] + c[14] - c[16]}

Out[•]= sum[1/
          ((n - 2 c[3] - c[8] - c[12] - c[16] - c[17])! (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] -
                c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[14] - c[16] - c[17])!
             c[17]! (-n + c[1] + c[2] + c[3] - c[4] + c[5] - c[6] + c[7] + c[8] + c[10] +
                c[12] + c[14] + c[17])!), {c[17], 0, n - 2 c[3] - c[8] - c[12] - c[16]}] ==
        (2 n - 2 c[3] - 2 c[4] - c[8] - c[9] - c[12] - c[13] - 2 c[16])! /
         ((n - 2 c[3] - c[8] - c[12] - c[16])! (n - 2 c[4] - c[9] - c[13] - c[16])!
            (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] - c[7] -
               c[8] - c[9] - c[10] - c[12] - c[13] - c[14] - c[16])!
            (c[1] + c[2] - c[3] - c[4] + c[5] - c[6] + c[7] + c[10] + c[14] - c[16])!)

In[•]:= expr2 === test[[1, 1]]

Out[•]= True

In[•]:= expr1 = expr1 / expr2 * test[[2]];

In[•]:= expr2 = Select[expr1, Not[FreeQ[#, c[14]]] &]

Out[•]= 1 / ((n - 2 c[2] - c[7] - c[12] - c[13] - c[14])!
           c[14]! (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] -
              c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[14] - c[16])!
          (c[1] + c[2] - c[3] - c[4] + c[5] - c[6] + c[7] + c[10] + c[14] - c[16])!)

In[•]:= test =
         ChuVandermonde /. {k → c[14], r → n - 2 c[2] - c[7] - c[12] - c[13], m → 2 n - c[1] - c[2] -
             c[3] - c[4] - c[5] + c[6] - c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[16],
           n → n + c[1] - c[2] - c[3] - c[4] + c[5] - c[6] + c[10] - c[12] - c[13] - c[16]}

Out[•]= sum[1/
          ((n - 2 c[2] - c[7] - c[12] - c[13] - c[14])! c[14]! (2 n - c[1] - c[2] - c[3] - c[4] - c[5] +
                c[6] - c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[14] - c[16])!
             (c[1] + c[2] - c[3] - c[4] + c[5] - c[6] + c[7] + c[10] + c[14] - c[16])!),
         {c[14], 0, n - 2 c[2] - c[7] - c[12] - c[13]}] ==
        (3 n - 2 c[2] - 2 c[3] - 2 c[4] - c[7] - c[8] - c[9] - 2 c[12] - 2 c[13] - 2 c[16])! /
         ((n - 2 c[2] - c[7] - c[12] - c[13])!
            (2 n - 2 c[3] - 2 c[4] - c[8] - c[9] - c[12] - c[13] - 2 c[16])! (2 n - c[1] - c[2] -
               c[3] - c[4] - c[5] + c[6] - c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[16])!
            (n + c[1] - c[2] - c[3] - c[4] + c[5] - c[6] + c[10] - c[12] - c[13] - c[16])!)

In[•]:= expr2 === test[[1, 1]]

Out[•]= True
```

```
In[•]:= expr1 = expr1 / expr2 * test[[2]];

In[•]:= expr2 = Select[expr1, Not[FreeQ[#, c[10]]] &]

Out[•]= 1 / ((n - 2 c[1] - c[7] - c[8] - c[9] - c[10]) !
      c[10] ! (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] -
          c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[16]) !
      (n + c[1] - c[2] - c[3] - c[4] + c[5] - c[6] + c[10] - c[12] - c[13] - c[16]) !)

In[•]:= test = ChuVandermonde /. {k → c[10], r → n - 2 c[1] - c[7] - c[8] - c[9],
      m → 2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16],
      n → 2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[6] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]}

Out[•]= sum[
    1 / ((n - 2 c[1] - c[7] - c[8] - c[9] - c[10]) ! c[10] ! (2 n - c[1] - c[2] - c[3] - c[4] - c[5] +
            c[6] - c[7] - c[8] - c[9] - c[10] - c[12] - c[13] - c[16]) !
        (n + c[1] - c[2] - c[3] - c[4] + c[5] - c[6] + c[10] - c[12] - c[13] - c[16]) !),
    {c[10], 0, n - 2 c[1] - c[7] - c[8] - c[9]}] ==
   (4 n - 2 c[1] - 2 c[2] - 2 c[3] - 2 c[4] - 2 c[7] - 2 c[8] - 2 c[9] -
        2 c[12] - 2 c[13] - 2 c[16]) ! / ((n - 2 c[1] - c[7] - c[8] - c[9]) !
      (3 n - 2 c[2] - 2 c[3] - 2 c[4] - c[7] - c[8] - c[9] - 2 c[12] - 2 c[13] - 2 c[16]) !
      (2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[6] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !
      (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] -
          c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !)

In[•]:= expr2 === test[[1, 1]]

Out[•]= True

In[•]:= expr1 = expr1 / expr2 * test[[2]];

In[•]:= expr2 = Select[expr1, Not[FreeQ[#, c[6]]] &]

Out[•]= 1 / (c[6] !
      (2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[6] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !
      (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !
      (-n + c[1] + c[2] + c[3] + c[4] - c[5] - c[6] + c[7] + c[8] + c[9] + c[12] + c[13] + c[16]) !)
```

```
In[•]:= test = ChuVandermonde /. {k → c[6],
          r → 2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16],
          m → -n + c[1] + c[2] + c[3] + c[4] - c[5] + c[7] + c[8] + c[9] + c[12] + c[13] + c[16], n →
           4 n - 2 c[1] - 2 c[2] - 2 c[3] - 2 c[4] - 2 c[7] - 2 c[8] - 2 c[9] - 2 c[12] - 2 c[13] - 2 c[16]}

Out[•]= sum[
         1 / (c[6] ! (2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[6] - c[7] - c[8] - c[9] - c[12] - c[13] -
                c[16]) ! (2 n - c[1] - c[2] - c[3] - c[4] - c[5] + c[6] - c[7] -
                c[8] - c[9] - c[12] - c[13] - c[16]) ! (-n + c[1] + c[2] + c[3] +
                c[4] - c[5] - c[6] + c[7] + c[8] + c[9] + c[12] + c[13] + c[16]) !),
         {c[6], 0, 2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[7] - c[8] - c[9] -
           c[12] - c[13] - c[16]}] ==
        (3 n - c[1] - c[2] - c[3] - c[4] - c[5] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) ! /
         ((n - 2 c[5]) ! (4 n - 2 c[1] - 2 c[2] - 2 c[3] - 2 c[4] -
              2 c[7] - 2 c[8] - 2 c[9] - 2 c[12] - 2 c[13] - 2 c[16]) !
           (2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !
           (-n + c[1] + c[2] + c[3] + c[4] - c[5] + c[7] + c[8] + c[9] + c[12] + c[13] + c[16]) !)

In[•]:= expr2 === test[[1, 1]]

Out[•]= True

In[•]:= expr1 = expr1 / expr2 * test[[2]];

In[•]:= expr1

Out[•]= ((3 n) ! (3 n - c[1] - c[2] - c[3] - c[4] - c[5] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !) /
        (c[1] ! c[2] ! c[3] ! c[4] ! (n - 2 c[5]) ! c[5] ! c[7] ! c[8] !
          (n - 2 c[1] - c[7] - c[8] - c[9]) ! c[9] ! c[12] ! (n - 2 c[2] - c[7] - c[12] - c[13]) !
          c[13] ! (n - 2 c[3] - c[8] - c[12] - c[16]) ! (n - 2 c[4] - c[9] - c[13] - c[16]) !
          (2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) ! c[16] !
          (-n + c[1] + c[2] + c[3] + c[4] - c[5] + c[7] + c[8] + c[9] + c[12] + c[13] + c[16]) !)
```

In[37]:=
```
Clear[A172671];
A172671[n_] := A172671[n] =
  Sum[((3 n) !
      (3 n - c[1] - c[2] - c[3] - c[4] - c[5] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !) /
    (c[1] ! c[2] ! c[3] ! c[4] ! (n - 2 c[5]) ! c[5] ! c[7] ! c[8] !
      (n - 2 c[1] - c[7] - c[8] - c[9]) ! c[9] ! c[12] ! (n - 2 c[2] - c[7] - c[12] - c[13]) !
      c[13] ! (n - 2 c[3] - c[8] - c[12] - c[16]) ! (n - 2 c[4] - c[9] - c[13] - c[16]) !
      (2 n - c[1] - c[2] - c[3] - c[4] + c[5] - c[7] - c[8] - c[9] - c[12] - c[13] - c[16]) !
      c[16] !
      (-n + c[1] + c[2] + c[3] + c[4] - c[5] + c[7] + c[8] + c[9] + c[12] + c[13] + c[16]) !),
   {c[1], 0, n/2},
   {c[5], 0, n/2},
   {c[2], 0, n/2},
   {c[3], 0, n/2},
   {c[4], 0, n/2},
   {c[7], 0, Min[n - 2 c[1], n - 2 c[2]]},
   {c[8], 0, Min[n - 2 c[1] - c[7], n - 2 c[3]]},
   {c[9], 0, Min[n - 2 c[1] - c[7] - c[8], n - 2 c[4]]},
   {c[12], 0, Min[n - 2 c[2] - c[7], n - 2 c[3] - c[8]]},
   {c[13], 0, Min[n - 2 c[2] - c[7] - c[12], n - 2 c[4] - c[9]]},
   {c[16], Max[0, n - c[1] - c[2] - c[3] - c[4] + c[5] - c[7] - c[8] - c[9] - c[12] - c[13]],
    Min[n - 2 c[3] - c[8] - c[12], n - 2 c[4] - c[9] - c[13]]}]
```

In[•]:=
```
Do[Print[n, ": ", Timing[A172671[n]]], {n, 8}]
```

```
1: {0.00266, 90}
2: {0.028569, 202 410}
3: {0.08967, 747 558 000}
4: {0.446199, 3 536 978 063 850}
5: {2.06879, 19 292 117 692 187 340}
6: {6.86439, 115 428 185 943 399 529 200}
7: {21.413, 737 005 538 936 597 762 145 600}
8: {65.0026, 4 937 928 427 617 947 420 104 982 250}
```

In[•]:=
```
Table[A172671[n], {n, 8}] - Take[data, 8]
```

Out[•]=
```
{0, 0, 0, 0, 0, 0, 0, 0}
```

6-dimensional walks

```
(* Using the same procedure Walks6D from A172572 *)
Timing[MaxMemoryUsed[
  test = Walks6D[33,
    Join[Permutations[{1, 1, 0, 0, 0, 0}], Permutations[{2, 0, 0, 0, 0, 0}]]]
 ]]
```

Out[•]= {298.129, 79 851 544}

In[•]:= `test === data`

Out[•]= True

In[•]:=
```
Timing[MaxMemoryUsed[
  test = Walks6D[40,
    Join[Permutations[{1, 1, 0, 0, 0, 0}], Permutations[{2, 0, 0, 0, 0, 0}]]]
 ]]
```

Out[•]= {862.909, 195 057 416}

4.3 A188818

In[53]:= `InitializeSeq[188 818]`

Sequence: 2, 9, 48, 256, 1360, 7056, 36000, 179776, 884256, 4276624, ...

Length: 32

Offset: 1

Recurrence: $-1024\,(-1+n)\,n\,(1+n)^2\,(-150-235\,n-199\,n^2-95\,n^3-7\,n^4+9\,n^5+2\,n^6)\,a[n]+256\,(1+n)\,(-810-1665\,n-1533\,n^2-476\,n^3+255\,n^4+79\,n^5-121\,n^6-52\,n^7+n^8+2\,n^9)\,a[1+n]+64\,(1566+4509\,n+5364\,n^2+4669\,n^3+2250\,n^4-754\,n^5-1362\,n^6-455\,n^7+17\,n^8+32\,n^9+4\,n^{10})\,a[2+n]-16\,(2+n)\,(-207-399\,n-1955\,n^2-1161\,n^3+509\,n^4+183\,n^5-251\,n^6-91\,n^7+8\,n^8+4\,n^9)\,a[3+n]-4\,(3+n)\,(1314+2619\,n+1792\,n^2+1327\,n^3+622\,n^4-279\,n^5-278\,n^6-24\,n^7+15\,n^8+2\,n^9)\,a[4+n]+(-2+n)\,(3+n)^2\,(4+n)\,(-33-61\,n-16\,n^2-17\,n^3-22\,n^4-3\,n^5+2\,n^6)\,a[5+n]$

Check: True

In[54]:=
```
(* Naive transfer matrix implementation. *)
A188818[n_] :=
  Module[{states, tmat},
   states = Tuples[{0, 1}, n];
   tmat = SparseArray[Outer[
      If[MemberQ[Rest[#1] - Most[#2], 1] || MemberQ[Most[#1] - Rest[#2], 1], 0, 1] &,
      states, states, 1]];
   vec = Table[1, {2^n}];
   Do[vec = tmat.vec, {n - 1}];
   Return[Total[vec]];
  ];
Do[Print[Timing[A188818[n]]], {n, 10}]
```

```
{0.026357, 2}
{0.0003, 9}
{0.000673, 48}
{0.002256, 256}
{0.01121, 1360}
{0.034962, 7056}
{0.091868, 36 000}
{0.257259, 179 776}
{0.918895, 884 256}
{3.70155, 4 276 624}
```

In[56]:=
```
(* Enumerate all even solutions for n=5. *)
pos = Select[Tuples[Range[5], 2], EvenQ[Total[#]] &];
sol = DeleteCases[
   If[#2 ≤ #5 ≤ #8 && #1 ≤ #4 ≤ #7 ≤ #10 ≤ #13 && #6 ≤ #9 ≤ #12 && #6 ≤ #4 ≤ #2 && #11 ≤ #9 ≤ #7 ≤
        #5 ≤ #3 && #12 ≤ #10 ≤ #8, {##}, {}] & @@@ Tuples[{0, 1}, Length[pos]], {}];
(* This is in agreement with e_5=40. *)
Length[sol]
```

Out[58]= 40

In[59]:=
```
tab = Table["", {5}, {5}];
MatrixForm[
 Reverse[Transpose[ReplacePart[tab, MapThread[Rule, {pos, Range[13]}]]]]]
```

Out[60]//MatrixForm=

$$\begin{pmatrix} 3 & & 8 & & 13 \\ & 5 & & 10 & \\ 2 & & 7 & & 12 \\ & 4 & & 9 & \\ 1 & & 6 & & 11 \end{pmatrix}$$

```
(* All even solutions for n=5. *)
MatrixForm[Reverse[Transpose[ReplacePart[tab, MapThread[Rule, {pos, #}]]]]] & /@
 sol
```

Out[•]=

$$\left\{\begin{pmatrix} 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & 0 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & 1 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & 1 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 0 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 0 \\ & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 0 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & 1 & & 0 \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 0 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 0 & & 1 & \\ 0 & & 0 & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 0 & \\ 0 & & 0 & & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 0 & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 1 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 0 & & 1 & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 0 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 0 & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{pmatrix}\right\}$$

Lattice Walks

In[61]:=
```
(* Krattenthaler: Catalan paths (a,b) → (c,d) with x≥y. *)
myL[{a_, b_}, {c_, d_}] := Binomial[c + d - a - b, c - a] - Binomial[c + d - a - b, c - b + 1];
(* myD: Dyck paths from (1,y1) → (n,y2) with y≥1,
and where 1+y1 = n+y2 (mod 2). *)
(* If y1 is odd, we map (x,y) to ((x+y-2)/2,(x-y)/2) and apply myL: *)
Simplify[myL @@ ({(#1 + #2 - 2) / 2, (#1 - #2) / 2} & @@@ {{x1, y1}, {x2, y2}})]
(* If y1 is even, we map (x,y) to ((x+y-1)/2,(x-y+1)/2) and apply myL: *)
Simplify[myL @@ ({(#1 + #2 - 1) / 2, (#1 - #2 + 1) / 2} & @@@ {{x1, y1}, {x2, y2}})]
(* Since both expressions are equal, we define: *)
myD[y1_, y2_, n_] :=
  Binomial[n - 1, (n - y1 + y2 - 1) / 2] - Binomial[n - 1, (n + y1 + y2 - 1) / 2];
```

Out[62]= Binomial$\left[-x1+x2, \frac{1}{2}(-x1+x2-y1+y2)\right]$ - Binomial$\left[-x1+x2, \frac{1}{2}(-x1+x2+y1+y2)\right]$

Out[63]= Binomial$\left[-x1+x2, \frac{1}{2}(-x1+x2-y1+y2)\right]$ - Binomial$\left[-x1+x2, \frac{1}{2}(-x1+x2+y1+y2)\right]$

```
(* We start with even paths (on positions (x,y) with x+y even). *)
```

In[65]:=
```
(* Sum over all Dyck paths with y1=1,3,...,n+2, y2=1,3,...,n+2 (if n odd)
   or y2=2,4,...,n+2 (if n even). *)
(* No path can touch both lower and upper boundary. *)
(* Mirror those whose start and end points are
 s.t. it can potentially touch the upper boundary. *)
Table[Sum[If[y1 + y2 ≤ n + 1, myD[y1, y2, n], myD[n + 3 - y1, n + 3 - y2, n]],
  {y1, 1, n + 2, 2}, {y2, n + 2, 1, -2}], {n, 15}]
```

Out[65]= {2, 3, 8, 16, 40, 84, 200, 424, 976, 2068, 4648, 9816, 21 680, 45 608, 99 408}

In[66]:=
```
(* Combine those paths which appear twice. *)
Table[Sum[If[y1 + y2 ≤ n + 1, 2, 1] * myD[y1, y2, n],
  {y1, 1, n + 2, 2}, {y2, n + 3 - y1, 1, -2}], {n, 15}]
```

Out[66]= {2, 3, 8, 16, 40, 84, 200, 424, 976, 2068, 4648, 9816, 21 680, 45 608, 99 408}

In[67]:=
```
(* Substitute y1=2k+1, y2=n+2-2l. *)
Table[
 Sum[If[l ≥ k + 1, 2, 1] * myD[2 k + 1, n - 2 l + 2, n],
  {k, 0, Floor[(n + 1) / 2]}, {l, k, Floor[(n + 1) / 2]}], {n, 15}]
```

Out[67]= {2, 3, 8, 16, 40, 84, 200, 424, 976, 2068, 4648, 9816, 21 680, 45 608, 99 408}

In[68]:=
```
(* Eliminate the case distinction by separating the case l=k. *)
Table[
 Sum[2 * Sum[myD[2 k + 1, n - 2 l + 2, n], {l, k + 1, Floor[(n + 1) / 2]}] +
   myD[2 k + 1, n - 2 k + 2, n], {k, 0, Floor[(n + 1) / 2]}], {n, 15}]
```

Out[68]= {2, 3, 8, 16, 40, 84, 200, 424, 976, 2068, 4648, 9816, 21 680, 45 608, 99 408}

In[69]:=
```
(* Insert the definition in terms of binomial coefficients. *)
Table[
 Sum[2 * Sum[
     Binomial[n - 1, n - k - l] - Binomial[n - 1, n + k - l + 1], {l, k + 1, Floor[(n + 1) / 2]}] +
   Binomial[n - 1, n - 2 k], {k, 0, Floor[(n + 1) / 2]}], {n, 15}]
```

Out[69]= {2, 3, 8, 16, 40, 84, 200, 424, 976, 2068, 4648, 9816, 21 680, 45 608, 99 408}

In[70]:=
```
(* Simplify: Sum[Binomial[n-1,n-2 k], {k,0,(n+1)/2}] = 2^(n-2) for n≥2. *)
Table[
 2 * Sum[Binomial[n - 1, n - k - l] - Binomial[n - 1, n + k - l + 1],
    {k, 0, Floor[(n + 1) / 2]}, {l, k + 1, Floor[(n + 1) / 2]}] + 2 ^ (n - 2), {n, 2, 15}]
```

Out[70]= {3, 8, 16, 40, 84, 200, 424, 976, 2068, 4648, 9816, 21 680, 45 608, 99 408}

```
(* Now the odd paths (on positions (x,y) with x+y odd). *)
```

In[71]:=
```
(* Sum over all Dyck paths with y1=2,4,...,n+1, y2=2,4,...,n+1 (if n odd)
   or y1=2,4,...,n+2, y2=1,3,...,n+1 (if n even). *)
Table[Sum[If[y1 + y2 ≤ n + 1, myD[y1, y2, n], myD[n + 3 - y1, n + 3 - y2, n]],
  {y1, 2, n + 2, 2}, {y2, n + 1, 1, -2}], {n, 15}]
```

Out[71]= {1, 3, 6, 16, 34, 84, 180, 424, 906, 2068, 4396, 9816, 20 756, 45 608, 95 976}

In[72]:=
```
(* Combine those paths which appear twice. *)
Table[Sum[If[y1 + y2 ≤ n + 1, 2, 1] * myD[y1, y2, n],
  {y1, 2, n + 2, 2}, {y2, n + 3 - y1, 1, -2}], {n, 15}]
```

Out[72]= {1, 3, 6, 16, 34, 84, 180, 424, 906, 2068, 4396, 9816, 20 756, 45 608, 95 976}

In[73]:=
```
(* Substitute y1=2k+2, y2=n+1-2l. *)
Table[Sum[If[l ≥ k + 1, 2, 1] * myD[2 k + 2, n - 2 l + 1, n],
  {k, 0, Floor[n / 2]}, {l, k, Floor[n / 2]}], {n, 15}]
```

Out[73]= {1, 3, 6, 16, 34, 84, 180, 424, 906, 2068, 4396, 9816, 20 756, 45 608, 95 976}

In[74]:=
```
(* Eliminate the case distinction by separating the case l=k. *)
Table[Sum[2 * Sum[myD[2 k + 2, n - 2 l + 1, n], {l, k + 1, Floor[n / 2]}] +
   myD[2 k + 2, n - 2 k + 1, n], {k, 0, Floor[n / 2]}], {n, 15}]
```

Out[74]= {1, 3, 6, 16, 34, 84, 180, 424, 906, 2068, 4396, 9816, 20 756, 45 608, 95 976}

In[75]:=
```
(* Insert the definition in terms of binomial coefficients. *)
Table[Sum[2 * Sum[Binomial[n - 1, n - k - l - 1] - Binomial[n - 1, n + k - l + 1],
    {l, k + 1, Floor[n / 2]}] + Binomial[n - 1, n - 2 k - 1], {k, 0, Floor[n / 2]}], {n, 15}]
```

Out[75]=
```
{1, 3, 6, 16, 34, 84, 180, 424, 906, 2068, 4396, 9816, 20 756, 45 608, 95 976}
```

In[76]:=
```
(* Simplify: Sum[Binomial[n-1,n-2 k-1], {k,0,n/2}] = 2^(n-2) for n≥2. *)
Table[2 * Sum[Binomial[n - 1, n - k - l - 1] - Binomial[n - 1, n + k - l + 1],
   {k, 0, Floor[n / 2]}, {l, k + 1, Floor[n / 2]}] + 2 ^ (n - 2), {n, 2, 15}]
```

Out[76]=
```
{3, 6, 16, 34, 84, 180, 424, 906, 2068, 4396, 9816, 20 756, 45 608, 95 976}
```

In[81]:=
```
(* Putting everything together. *)
Table[
  (2 ^ (n - 2) + 2 * Sum[Binomial[n - 1, n - k - l] - Binomial[n - 1, n + k - l + 1],
      {k, 0, Floor[(n + 1) / 2]}, {l, k + 1, Floor[(n + 1) / 2]}]) *
   (2 ^ (n - 2) + 2 * Sum[Binomial[n - 1, n - k - l - 1] - Binomial[n - 1, n + k - l + 1],
      {k, 0, Floor[n / 2]}, {l, k + 1, Floor[n / 2]}]),
  {n, 2, 32}] === Rest[
  data]
```

Out[81]=
```
True
```

Creative telescoping for o_n

In[∘]:=
```
Sum[Binomial[n - 1, n - k - l - 1] - Binomial[n - 1, n + k - l + 1],
 {k, 0, Floor[n / 2]}, {l, k + 1, Floor[n / 2]}]
```

Out[∘]=
$$\sum_{k=0}^{\text{Floor}\left[\frac{n}{2}\right]} \sum_{l=1+k}^{\text{Floor}\left[\frac{n}{2}\right]} \left(\text{Binomial}[-1+n, -1-k-l+n] - \text{Binomial}[-1+n, 1+k-l+n]\right)$$

In[∘]:=
```
(* n even *)
Sum[Binomial[2 m - 1, 2 m - k - l - 1] - Binomial[2 m - 1, 2 m + k - l + 1],
 {k, 0, m}, {l, k + 1, m}]
```

Out[∘]=
$$\sum_{k=0}^{m} \sum_{l=1+k}^{m} \left(\text{Binomial}[-1+2m, -1-k-l+2m] - \text{Binomial}[-1+2m, 1+k-l+2m]\right)$$

In[∘]:=
```
Timing[
 annoe1 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
        Annihilator[Sum[Binomial[2 m - 1, 2 m - k - l - 1], {l, k + 1, m}], {S[k], S[m]}],
        Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[∘]=
$$\{57.5513, \{(1230 + 4847 m + 7962 m^2 + 6823 m^3 + 3174 m^4 + 756 m^5 + 72 m^6)\, S_m^4 +$$
$$(-11\,708 - 48\,716 m - 84\,061 m^2 - 75\,463 m^3 - 36\,834 m^4 - 9252 m^5 - 936 m^6)\, S_m^3 +$$
$$(34\,032 + 155\,610 m + 291\,378 m^2 + 281\,544 m^3 + 147\,456 m^4 + 39\,744 m^5 + 4320 m^6)\, S_m^2 +$$
$$(-25\,152 - 147\,344 m - 334\,368 m^2 - 376\,816 m^3 - 224\,160 m^4 - 67\,392 m^5 - 8064 m^6)\, S_m +$$
$$(-9472 - 23\,392 m + 17\,056 m^2 + 85\,504 m^3 + 82\,176 m^4 + 32\,256 m^5 + 4608 m^6)\}\}$$

In[∘]:=
```
Timing[
 annoe2 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
       Annihilator[Sum[Binomial[2 m - 1, 2 m + k - l + 1], {l, k + 1, m}], {S[k], S[m]}],
       Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[∘]= $\{0.576823, \{(-1-m)\, S_m^2 + (6+8m)\, S_m + (-8-16m)\}\}$

In[∘]:=
```
annoe = DFinitePlus[annoe1, annoe2]
```

Out[∘]=
$$\{(1230+4847m+7962m^2+6823m^3+3174m^4+756m^5+72m^6)\, S_m^4 + (-11708-48716m-84061m^2-75463m^3-36834m^4-9252m^5-936m^6)\, S_m^3 + (34032+155610m+291378m^2+281544m^3+147456m^4+39744m^5+4320m^6)\, S_m^2 + (-25152-147344m-334368m^2-376816m^3-224160m^4-67392m^5-8064m^6)\, S_m + (-9472-23392m+17056m^2+85504m^3+82176m^4+32256m^5+4608m^6)\}$$

In[∘]:=
```
(* n odd, n = 2m+1 *)
Sum[Binomial[2 m, 2 m - k - l] - Binomial[2 m, 2 m + k - l + 2], {k, 0, m}, {l, k + 1, m}]
```

Out[∘]= $\sum_{k=0}^{m}\sum_{l=1+k}^{m} (\text{Binomial}[2m, -k-l+2m] - \text{Binomial}[2m, 2+k-l+2m])$

In[∘]:=
```
Timing[
 annoo1 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
       Annihilator[Sum[Binomial[2 m, 2 m - k - l], {l, k + 1, m}], {S[k], S[m]}],
       Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[∘]=
$$\{40.6922, \{(1680+6290m+9017m^2+6428m^3+2433m^4+468m^5+36m^6)\, S_m^4 + (-16800-65484m-97624m^2-72563m^3-28731m^4-5796m^5-468m^6)\, S_m^3 + (53760+223416m+354006m^2+279858m^3+117900m^4+25272m^5+2160m^6)\, S_m^2 + (-53760-254336m-454448m^2-401408m^3-186960m^4-43776m^5-4032m^6)\, S_m + (23424m+93280m^2+126368m^3+77376m^4+21888m^5+2304m^6)\}\}$$

In[∘]:=
```
Timing[
 annoo2 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
       Annihilator[Sum[Binomial[2 m, 2 m + k - l + 2], {l, k + 1, m}], {S[k], S[m]}],
       Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[∘]= $\{0.637971, \{(-2m-m^2)\, S_m^2 + (2+14m+8m^2)\, S_m + (-8-24m-16m^2)\}\}$

In[∘]:=
```
annoo = DFinitePlus[annoo1, annoo2]
```

Out[∘]=

$$\{(604\,800 + 2\,143\,920\, m + 3\,503\,932\, m^2 + 3\,350\,778\, m^3 + 2\,023\,367\, m^4 + 792\,027\, m^5 + 200\,253\, m^6 + 31\,527\, m^7 + 2808\, m^8 + 108\, m^9)\, S_m^5 +$$
$$(-7\,257\,600 - 26\,786\,040\, m - 45\,258\,386\, m^2 - 44\,622\,861\, m^3 - 27\,805\,240\, m^4 - 11\,263\,974\, m^5 - 2\,958\,774\, m^6 - 486\,009\, m^7 - 45\,360\, m^8 - 1836\, m^9)\, S_m^4 +$$
$$(30\,844\,800 + 120\,656\,280\, m + 213\,743\,476\, m^2 + 219\,896\,076\, m^3 + 142\,921\,040\, m^4 + 60\,509\,724\, m^5 + 16\,659\,084\, m^6 + 2\,876\,544\, m^7 + 282\,960\, m^8 + 12\,096\, m^9)\, S_m^3 +$$
$$(-53\,222\,400 - 228\,939\,000\, m - 437\,342\,956\, m^2 - 480\,342\,876\, m^3 - 332\,011\,940\, m^4 - 149\,357\,484\, m^5 - 43\,698\,144\, m^6 - 8\,019\,504\, m^7 - 838\,080\, m^8 - 38\,016\, m^9)\, S_m^2 +$$
$$(29\,030\,400 + 153\,019\,200\, m + 340\,170\,656\, m^2 + 422\,187\,936\, m^3 + 324\,117\,280\, m^4 + 160\,175\,904\, m^5 + 51\,071\,424\, m^6 + 10\,145\,664\, m^7 + 1\,140\,480\, m^8 + 55\,296\, m^9)\, S_m +$$
$$(-9\,202\,560\, m - 44\,657\,344\, m^2 - 84\,358\,848\, m^3 - 85\,011\,008\, m^4 - 51\,064\,512\, m^5 - 18\,909\,696\, m^6 - 4\,234\,752\, m^7 - 525\,312\, m^8 - 27\,648\, m^9)\}$$

In[∘]:=
```
anno = DFinitePlus[DFiniteSubstitute[annoe, {m → n / 2}, Algebra → OreAlgebra[S[n]]],
    DFiniteSubstitute[annoo, {m → (n - 1) / 2}, Algebra → OreAlgebra[S[n]]]];
#[anno] & /@ {ByteCount, Support}
```

Out[∘]=

$$\{14\,768, \{\{S_n^{12}, S_n^{10}, S_n^8, S_n^6, S_n^4, S_n^2, 1\}\}\}$$

In[∘]:=
```
anno
```

Out[∘]=

$$\{(6\,043\,847\,040 + 9\,234\,515\,984\, n + 6\,415\,603\,548\, n^2 + 3\,155\,184\,552\, n^3 + 1\,489\,266\,271\, n^4 + 645\,010\,572\, n^5 + 210\,832\,480\, n^6 + 47\,593\,036\, n^7 + 7\,239\,502\, n^8 + 727\,764\, n^9 + 46\,332\, n^{10} + 1692\, n^{11} + 27\, n^{12})\, S_n^{12} +$$
$$(-77\,488\,911\,360 - 122\,906\,338\,368\, n - 89\,249\,100\,040\, n^2 - 45\,875\,175\,954\, n^3 - 22\,186\,586\,737\, n^4 - 9\,739\,701\,846\, n^5 - 3\,251\,835\,190\, n^6 - 757\,828\,770\, n^7 - 120\,082\,776\, n^8 - 12\,672\,306\, n^9 - 852\,930\, n^{10} - 33\,156\, n^{11} - 567\, n^{12})\, S_n^{10} +$$
$$(362\,566\,512\,000 + 605\,212\,018\,176\, n + 466\,255\,131\,416\, n^2 + 254\,088\,906\,348\, n^3 + 127\,230\,140\,276\, n^4 + 57\,016\,573\,944\, n^5 + 19\,579\,305\,272\, n^6 + 4\,747\,874\,928\, n^7 + 790\,399\,824\, n^8 + 88\,308\,360\, n^9 + 6\,333\,552\, n^{10} + 263\,844\, n^{11} + 4860\, n^{12})\, S_n^8 +$$
$$(-732\,732\,134\,400 - 1\,320\,112\,439\,296\, n - 1\,107\,987\,668\,416\, n^2 - 656\,288\,560\,128\, n^3 - 346\,705\,698\,576\, n^4 - 160\,731\,423\,264\, n^5 - 57\,445\,966\,992\, n^6 - 14\,675\,627\,648\, n^7 - 2\,599\,333\,904\, n^8 - 311\,187\,360\, n^9 - 24\,036\,912\, n^{10} - 1\,082\,304\, n^{11} - 21\,600\, n^{12})\, S_n^6 +$$
$$(559\,701\,596\,160 + 1\,154\,094\,882\,816\, n + 1\,118\,510\,759\,296\, n^2 + 759\,028\,102\,848\, n^3 + 440\,518\,937\,536\, n^4 + 217\,489\,460\,544\, n^5 + 82\,836\,684\,352\, n^6 + 22\,790\,586\,048\, n^7 + 4\,384\,330\,944\, n^8 + 572\,852\,160\, n^9 + 48\,382\,272\, n^{10} + 2\,381\,184\, n^{11} + 51\,840\, n^{12})\, S_n^4 +$$
$$(-71\,087\,063\,040 - 218\,391\,724\,032\, n - 304\,611\,780\,608\, n^2 - 282\,362\,628\,096\, n^3 - 204\,414\,824\,960\, n^4 - 117\,176\,888\,832\, n^5 - 50\,594\,922\,752\, n^6 - 15\,741\,330\,432\, n^7 - 3\,422\,522\,112\, n^8 - 503\,594\,496\, n^9 - 47\,589\,120\, n^{10} - 2\,598\,912\, n^{11} - 62\,208\, n^{12})\, S_n^2 +$$
$$(-7\,518\,748\,672\, n - 15\,061\,581\,824\, n^2 - 6\,747\,058\,176\, n^3 + 6\,635\,270\,144\, n^4 + 10\,935\,733\,248\, n^5 + 7\,541\,875\,712\, n^6 + 3\,175\,717\,888\, n^7 + 867\,570\,688\, n^8 + 153\,314\,304\, n^9 + 16\,837\,632\, n^{10} + 1\,041\,408\, n^{11} + 27\,648\, n^{12})\}$$

```
In[•]:= annog = NormalizeCoefficients /@ ToOrePolynomial[GuessUnivRE[Table[
          Sum[Binomial[n - 1, n - k - l - 1] - Binomial[n - 1, n + k - l + 1], {k, 0, Floor[n / 2]},
           {l, k + 1, Floor[n / 2]}], {n, 0, 60}], f[n], Order → 3, Degree → 4], f[n]]
```

Out[•]= $\{(-2+n+n^2)\,S_n^3+(2-4n-2n^2)\,S_n^2+(8-4n^2)\,S_n+(8n+8n^2)\}$

```
In[•]:= OreReduce[anno[[1]], annog]
```

Out[•]= 0

Creative telescoping for e_n

```
In[•]:= Sum[Binomial[n - 1, n - k - l] - Binomial[n - 1, n + k - l + 1],
         {k, 0, Floor[(n + 1) / 2]}, {l, k + 1, Floor[(n + 1) / 2]}]
```

Out[•]= $\sum_{k=0}^{\text{Floor}\left[\frac{1+n}{2}\right]}\sum_{l=1+k}^{\text{Floor}\left[\frac{1+n}{2}\right]}(\text{Binomial}[-1+n,-k-l+n]-\text{Binomial}[-1+n,1+k-l+n])$

```
In[•]:= (* n even, n = 2m *)
        Sum[Binomial[2 m - 1, 2 m - k - l] - Binomial[2 m - 1, 2 m + k - l + 1], {k, 0, m}, {l, k + 1, m}]
```

Out[•]= $\sum_{k=0}^{m}\sum_{l=1+k}^{m}(\text{Binomial}[-1+2m,-k-l+2m]-\text{Binomial}[-1+2m,1+k-l+2m])$

```
In[•]:= Timing[
         annee1 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
                Annihilator[Sum[Binomial[2 m - 1, 2 m - k - l], {l, k + 1, m}], {S[k], S[m]}],
                Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[•]= $\{36.7438,\{(420+3515m+7914m^2+7717m^3+3654m^4+828m^5+72m^6)\,S_m^4+$
$(-4200-34\,760m-81\,333m^2-83\,455m^3-41\,922m^4-10\,116m^5-936m^6)\,S_m^3+$
$(13\,440+108\,690m+269\,730m^2+299\,400m^3+164\,160m^4+43\,200m^5+4320m^6)\,S_m^2+$
$(-13\,440-100\,880m-279\,840m^2-364\,912m^3-236\,448m^4-72\,000m^5-8064m^6)\,S_m+$
$(-10\,720m-16\,992m^2+34\,816m^3+66\,816m^4+32\,256m^5+4608m^6)\}\}$

```
In[•]:= Timing[
         annee2 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
                Annihilator[Sum[Binomial[2 m - 1, 2 m + k - l + 1], {l, k + 1, m}], {S[k], S[m]}],
                Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[•]= $\{0.597442,\{(-1-m)\,S_m^2+(6+8m)\,S_m+(-8-16m)\}\}$

```
In[•]:= annee = DFinitePlus[annee1, annee2]
```

Out[•]= $\{(420+3515m+7914m^2+7717m^3+3654m^4+828m^5+72m^6)\,S_m^4+$
$(-4200-34\,760m-81\,333m^2-83\,455m^3-41\,922m^4-10\,116m^5-936m^6)\,S_m^3+$
$(13\,440+108\,690m+269\,730m^2+299\,400m^3+164\,160m^4+43\,200m^5+4320m^6)\,S_m^2+$
$(-13\,440-100\,880m-279\,840m^2-364\,912m^3-236\,448m^4-72\,000m^5-8064m^6)\,S_m+$
$(-10\,720m-16\,992m^2+34\,816m^3+66\,816m^4+32\,256m^5+4608m^6)\}$

In[∘]:=
```
(* n odd, n = 2m+1 *)
Sum[Binomial[2 m, 2 m - k - l + 1] - Binomial[2 m, 2 m + k - l + 2],
 {k, 0, m + 1}, {l, k + 1, m + 1}]
```

Out[∘]= $\sum_{k=0}^{1+m}\sum_{l=1+k}^{1+m}(\text{Binomial}[2\,m, 1-k-l+2\,m]-\text{Binomial}[2\,m, 2+k-l+2\,m])$

In[∘]:=
```
Timing[
 anneo1 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
       Annihilator[Sum[Binomial[2 m, 2 m - k - l + 1], {l, k + 1, m + 1}], {S[k], S[m]}],
       Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[∘]= $\{55.0976, \{(20160+65232\,m+85060\,m^2+58767\,m^3+23456\,m^4+5445\,m^5+684\,m^6+36\,m^7)\,S_m^4+(-201600-674388\,m-911776\,m^2-654525\,m^3-271736\,m^4-65619\,m^5-8568\,m^6-468\,m^7)\,S_m^3+(645120+2272596\,m+3246438\,m^2+2465316\,m^3+1081902\,m^4+275508\,m^5+37800\,m^6+2160\,m^7)\,S_m^2+(-645120-2509536\,m-3981616\,m^2-3349680\,m^3-1616624\,m^4-448272\,m^5-66240\,m^6-4032\,m^7)\,S_m+(138048\,m+561760\,m^2+798912\,m^3+542432\,m^4+190656\,m^5+33408\,m^6+2304\,m^7)\}\}$

In[∘]:=
```
Timing[
 anneo2 = NormalizeCoefficients /@ (First[FindCreativeTelescoping[DFiniteTimes[
       Annihilator[Sum[Binomial[2 m, 2 m + k - l + 2], {l, k + 1, m + 1}], {S[k], S[m]}],
       Annihilator[Gamma[k + 1 + e] / Gamma[k + 1], {S[k], S[m]}]], S[k] - 1]] /. e → 0)]
```

Out[∘]= $\{0.702761, \{(-1-m)\,S_m^2+(6+8\,m)\,S_m+(-8-16\,m)\}\}$

In[∘]:=
```
anneo = DFinitePlus[anneo1, anneo2]
```

Out[∘]= $\{(20160+65232\,m+85060\,m^2+58767\,m^3+23456\,m^4+5445\,m^5+684\,m^6+36\,m^7)\,S_m^4+(-201600-674388\,m-911776\,m^2-654525\,m^3-271736\,m^4-65619\,m^5-8568\,m^6-468\,m^7)\,S_m^3+(645120+2272596\,m+3246438\,m^2+2465316\,m^3+1081902\,m^4+275508\,m^5+37800\,m^6+2160\,m^7)\,S_m^2+(-645120-2509536\,m-3981616\,m^2-3349680\,m^3-1616624\,m^4-448272\,m^5-66240\,m^6-4032\,m^7)\,S_m+(138048\,m+561760\,m^2+798912\,m^3+542432\,m^4+190656\,m^5+33408\,m^6+2304\,m^7)\}$

In[∘]:=
```
anne = DFinitePlus[DFiniteSubstitute[annee, {m → n / 2}, Algebra → OreAlgebra[S[n]]],
   DFiniteSubstitute[anneo, {m → (n - 1) / 2}, Algebra → OreAlgebra[S[n]]]];
#[anne] & /@ {ByteCount, Support}
```

Out[∘]= $\{10032, \{\{S_n^{10}, S_n^8, S_n^6, S_n^4, S_n^2, 1\}\}\}$

In[∘]:=
```
anne
```

Out[∘]= $\{(11269440 + 25181720\, n + 26870838\, n^2 + 16800665\, n^3 + 6496074\, n^4 + 1586273\, n^5 + 244362\, n^6 + 22995\, n^7 + 1206\, n^8 + 27\, n^9)\, S_n^{10} + (-123169200 - 287509972\, n - 319509936\, n^2 - 207738661\, n^3 - 83746860\, n^4 - 21422629\, n^5 - 3474744\, n^6 - 345879\, n^7 - 19260\, n^8 - 459\, n^9)\, S_n^8 + (454100640 + 1132377992\, n + 1336751796\, n^2 + 920168876\, n^3 + 393437760\, n^4 + 107172224\, n^5 + 18584004\, n^6 + 1982844\, n^7 + 118440\, n^8 + 3024\, n^9)\, S_n^6 + (-605660160 - 1702495232\, n - 2232471936\, n^2 - 1690935296\, n^3 - 793841760\, n^4 - 237537824\, n^5 - 45234624\, n^6 - 5287104\, n^7 - 344160\, n^8 - 9504\, n^9)\, S_n^4 + (154882560 + 635769472\, n + 1095560256\, n^2 + 1035622336\, n^3 + 589943040\, n^4 + 209799424\, n^5 + 46595904\, n^6 + 6231744\, n^7 + 455040\, n^8 + 13824\, n^9)\, S_n^2 + (41125888\, n + 64000512\, n^2 - 1116416\, n^3 - 51104256\, n^4 - 37765376\, n^5 - 12698112\, n^6 - 2237184\, n^7 - 198144\, n^8 - 6912\, n^9)\}$

In[∘]:=
```
anneg = NormalizeCoefficients /@ ToOrePolynomial[
   GuessUnivRE[Table[Sum[Binomial[n - 1, n - k - l] - Binomial[n - 1, n + k - l + 1],
       {k, 0, Floor[(n + 1) / 2]}, {l, k + 1, Floor[(n + 1) / 2]}],
      {n, 0, 60}], f[n], Order → 3, Degree → 4], f[n]]
```

Out[∘]= $\{(-3 - 2\, n + n^2)\, S_n^3 + (6 + 2\, n - 2\, n^2)\, S_n^2 + (4 + 12\, n - 4\, n^2)\, S_n + (-16\, n + 8\, n^2)\}$

In[∘]:=
```
OreReduce[anne[[1]], anneg]
```

Out[∘]= 0

Final recurrence

In[∘]:=
```
GBEqual[anno, DFinitePlus[anno, Annihilator[2^(n - 2), S[n]]]]
```

Out[∘]= True

In[∘]:=
```
GBEqual[anne, DFinitePlus[anne, Annihilator[2^(n - 2), S[n]]]]
```

Out[∘]= True

```
Timing[ann = DFiniteTimes[anno, anne];]
```

Out[∘]= {209.842, Null}

In[∘]:=
```
#[ann] & /@ {ByteCount, Support}
```

Out[∘]= $\{1711520, \{\{S_n^{42}, S_n^{40}, S_n^{38}, S_n^{36}, S_n^{34}, S_n^{32}, S_n^{30}, S_n^{28}, S_n^{26}, S_n^{24}, S_n^{22}, S_n^{20}, S_n^{18}, S_n^{16}, S_n^{14}, S_n^{12}, S_n^{10}, S_n^8, S_n^6, S_n^4, S_n^2, 1\}\}\}$

```
OreReduce[ann[[1]], ToOrePolynomial[rec, a[n]]]
```

Out[∘]= 0

4.4 A306322

```
In[82]:= InitializeSeq[306 322]
Sequence: 1, 0, 0, 25, 386, 4657, 54219, 642815, 7852836, 98755951, ...
Length: 41
Offset: 0
Recurrence: 8 n (-1 + 2 n) (1 + 2 n) (3 + 2 n)
   (58 236 160 + 375 620 224 n + 1 022 652 512 n^2 + 1 572 814 284 n^3 + 1 527 319 428 n^4 + 984 186 117 n^5 +
     427 851 585 n^6 + 124 239 510 n^7 + 23 107 950 n^8 + 2 489 625 n^9 + 118 125 n^10) a[n] -
  2 (3 + 2 n) (-272 670 720 - 3 178 474 112 n - 12 713 618 176 n^2 - 18 788 310 824 n^3 +
     11 547 320 420 n^4 + 86 482 913 102 n^5 + 150 478 534 491 n^6 + 147 951 032 109 n^7 + 93 825 035 775 n^8 +
     39 828 085 965 n^9 + 11 279 217 825 n^10 + 2 048 259 375 n^11 + 215 870 625 n^12 + 10 040 625 n^13) a[1 + n] +
  (-2 257 059 840 - 25 779 317 504 n - 108 128 100 864 n^2 - 188 403 075 920 n^3 +
     14 462 120 192 n^4 + 691 541 238 960 n^5 + 1 471 227 292 164 n^6 + 1 718 884 004 625 n^7 +
     1 311 094 658 043 n^8 + 687 475 711 989 n^9 + 250 621 464 735 n^10 +
     62 559 627 795 n^11 + 10 209 053 025 n^12 + 981 642 375 n^13 + 42 170 625 n^14) a[2 + n] -
  (3 + n) (-221 347 840 - 2 428 639 744 n - 9 566 235 392 n^2 - 14 571 577 344 n^3 + 7 017 979 960 n^4 +
     64 349 576 684 n^5 + 117 919 810 482 n^6 + 121 180 651 809 n^7 + 80 085 358 620 n^8 + 35 325 144 315 n^9 +
     10 361 592 450 n^10 + 1 942 194 375 n^11 + 210 555 000 n^12 + 10 040 625 n^13) a[3 + n] +
  2 (3 + n) (4 + n)^2 (7 + 2 n) (-64 000 - 660 240 n - 2 279 152 n^2 - 2 342 808 n^3 + 4 449 768 n^4 +
     15 986 367 n^5 + 20 875 365 n^6 + 14 827 410 n^7 + 6 016 950 n^8 + 1 308 375 n^9 + 118 125 n^10) a[4 + n]
Check: True

In[84]:= (* Nara[n,k] is the Narayana number N(n+k-1,k). *)
Nara[i_, j_] := 1 / (i + j - 1) * Binomial[i + j - 1, i] * Binomial[i + j - 1, i - 1];
Table[Nara[n, n], {n, 10}]

Out[85]= {1, 3, 20, 175, 1764, 19 404, 226 512, 2 760 615, 34 763 300, 449 141 836}

In[90]:= (* First expression. *)
Table[
  2 * (Sum[Nara[i, j] - Binomial[i + j - 2, i - 1], {i, n}, {j, n}] - Binomial[2 n, n] + 1 +
      Sum[Nara[j - i, n], {i, n - 1}, {j, i + 1, n - 1}]) -
   (Nara[n, n] - 2 * Binomial[2 n, n] + 1), {n, 40}] === Rest[data]

Out[90]= True

In[92]:= (* Simplified expression. *)
Table[2 * Sum[Sum[Nara[i, j], {i, 1, n}] + (n - j - 1) * Nara[j, n], {j, 1, n}] -
   2 * Binomial[2 n, n] + Nara[n, n] + 3, {n, 40}] === Rest[data]

Out[92]= True

In[101]:= ann1 = Annihilator[Sum[Nara[i, j], {i, 1, n}, {j, 1, n}], S[n]];
Exponent[ann1[[1]], {S[n], n}]

Out[102]= {9, 49}
```

In[103]:=
```
ann2 = Annihilator[Sum[(n - j - 1) * Nara[j, n], {j, 1, n}], S[n]]
```

Out[103]=
$$\{(48 + 272 n + 532 n^2 + 464 n^3 + 186 n^4 + 28 n^5) S_n^2 + (-144 - 1040 n - 2846 n^2 - 3634 n^3 - 2123 n^4 - 455 n^5) S_n + (68 n + 360 n^2 + 652 n^3 + 464 n^4 + 112 n^5)\}$$

In[104]:=
```
ann3 = Annihilator[-2 * Binomial[2 n, n] + Nara[n, n] + 3, S[n]]
```

Out[104]=
$$\{(144 + 132 n - 428 n^2 + 439 n^3 + 1742 n^4 + 1352 n^5 + 414 n^6 + 45 n^7) S_n^3 + (-168 + 1820 n + 4146 n^2 - 7967 n^3 - 23\,550 n^4 - 19\,932 n^5 - 7164 n^6 - 945 n^7) S_n^2 + (-1896 - 10\,192 n - 10\,446 n^2 + 28\,360 n^3 + 74\,616 n^4 + 66\,372 n^5 + 26\,046 n^6 + 3780 n^7) S_n + (1920 + 8240 n + 6728 n^2 - 20\,832 n^3 - 52\,808 n^4 - 47\,792 n^5 - 19\,296 n^6 - 2880 n^7)\}$$

In[105]:=
```
ann = DFinitePlus[ann1, ann2, ann3];
Exponent[ann[[1]], {S[n], n}]
```

Out[106]= {12, 87}

In[108]:=
```
OreReduce[ann, ToOrePolynomial[{rec}, a[n]]]
```

Out[108]= {0}

5 Further Examples

5.1 A195806

In[226]:=
```
InitializeSeq[195 806]
```

```
Sequence: 16, 105, 496, 1759, 5052, 12469, 27412, 55059, 102952, 181543, ...
Length: 32
Offset: 1
Recurrence:
```

$$(-51\,397\,272 - 121\,478\,616 n - 130\,865\,136 n^2 - 84\,903\,182 n^3 - 36\,821\,323 n^4 - 11\,171\,853 n^5 - 2\,404\,773 n^6 - 363\,090 n^7 - 36\,855 n^8 - 2275 n^9 - 65 n^{10})\, a[n] + (-75\,965\,148 - 180\,891\,090 n - 193\,887\,621 n^2 - 124\,048\,209 n^3 - 52\,670\,308 n^4 - 15\,537\,123 n^5 - 3\,225\,918 n^6 - 465\,075 n^7 - 44\,460 n^8 - 2535 n^9 - 65 n^{10})\, a[1 + n] - 2 (3 + n) (5\,283\,216 + 11\,369\,562 n + 10\,571\,647 n^2 + 5\,714\,280 n^3 + 1\,989\,000 n^4 + 459\,810 n^5 + 69\,225 n^6 + 6240 n^7 + 260 n^8)\, a[2 + n] + (-3\,525\,012 - 8\,501\,490 n - 8\,381\,853 n^2 - 4\,215\,417 n^3 - 881\,222 n^4 + 202\,905 n^5 + 203\,808 n^6 + 68\,445 n^7 + 12\,870 n^8 + 1365 n^9 + 65 n^{10})\, a[3 + n] + (3\,606\,408 + 10\,869\,840 n + 14\,764\,668 n^2 + 12\,068\,530 n^3 + 6\,632\,353 n^4 + 2\,574\,495 n^5 + 717\,633 n^6 + 142\,350 n^7 + 19\,305 n^8 + 1625 n^9 + 65 n^{10})\, a[4 + n]$$

```
Check: True
```

In[227]:=
```
(* Also a C-finite recurrence can be found. *)
crec = GuessMinRE[data, a[n]]
```

Out[227]= $-a[n] + 3a[1+n] - 2a[2+n] + a[3+n] - 6a[4+n] + 5a[5+n] + 3a[6+n] - 3a[8+n] - 5a[9+n] + 6a[10+n] - a[11+n] + 2a[12+n] - 3a[13+n] + a[14+n]$

In[229]:=
```
(* It is compatible with the guessed minimal-order recurrence. *)
OreReduce @@ (ToOrePolynomial[#, a[n]] & /@ {crec, {rec}})
```

Out[229]= 0

In[116]:=
```
(* Compute more terms with the guessed recurrence. *)
re2l = RE2L[Prepend[Table[a[i] == data[[i]], {i, 4}], rec == 0], a[n], {1, 1000}];
{Take[re2l, Length[data]] === data, And @@ (IntegerQ /@ re2l)}
```

Solve: Equations may not give solutions for all "solve" variables.

Out[117]= {True, True}

In[118]:=
```
(* Interpolate the constituents of the quasi-polynomial. *)
Table[
 Expand[InterpolatingPolynomial[Table[{n, re2l[[n]]}, {n, s, 200, 6}], n]], {s, 6}]
```

Out[118]=
$$\left\{\frac{2051}{1296} + \frac{739\,n}{162} + \frac{79\,n^2}{18} + \frac{325\,n^3}{108} + \frac{2275\,n^4}{1296} + \frac{65\,n^5}{108} + \frac{65\,n^6}{648},\right.$$
$$\frac{119}{81} + \frac{683\,n}{162} + \frac{233\,n^2}{54} + \frac{325\,n^3}{108} + \frac{2275\,n^4}{1296} + \frac{65\,n^5}{108} + \frac{65\,n^6}{648},$$
$$\frac{19}{16} + \frac{25\,n}{6} + \frac{79\,n^2}{18} + \frac{325\,n^3}{108} + \frac{2275\,n^4}{1296} + \frac{65\,n^5}{108} + \frac{65\,n^6}{648},$$
$$\frac{113}{81} + \frac{739\,n}{162} + \frac{79\,n^2}{18} + \frac{325\,n^3}{108} + \frac{2275\,n^4}{1296} + \frac{65\,n^5}{108} + \frac{65\,n^6}{648},$$
$$\frac{2147}{1296} + \frac{683\,n}{162} + \frac{233\,n^2}{54} + \frac{325\,n^3}{108} + \frac{2275\,n^4}{1296} + \frac{65\,n^5}{108} + \frac{65\,n^6}{648},$$
$$\left.1 + \frac{25\,n}{6} + \frac{79\,n^2}{18} + \frac{325\,n^3}{108} + \frac{2275\,n^4}{1296} + \frac{65\,n^5}{108} + \frac{65\,n^6}{648}\right\}$$

In[121]:=
```
(* Sanity check. *)
A195806[n_] := Switch[Mod[n, 6],
   1, (2051 + 5912 n + 5688 n^2 + 3900 n^3 + 2275 n^4 + 780 n^5 + 130 n^6) / 1296,
   2, (1904 + 5464 n + 5592 n^2 + 3900 n^3 + 2275 n^4 + 780 n^5 + 130 n^6) / 1296,
   3, (1539 + 5400 n + 5688 n^2 + 3900 n^3 + 2275 n^4 + 780 n^5 + 130 n^6) / 1296,
   4, (1808 + 5912 n + 5688 n^2 + 3900 n^3 + 2275 n^4 + 780 n^5 + 130 n^6) / 1296,
   5, ((1 + n) ^2 (2147 + 1170 n + 1105 n^2 + 520 n^3 + 130 n^4)) / 1296,
   0, (1296 + 5400 n + 5688 n^2 + 3900 n^3 + 2275 n^4 + 780 n^5 + 130 n^6) / 1296];
Table[A195806[n], {n, 1000}] === re2l
```

Out[122]= True

In[123]:=
```
(* Different representation *)
(a[6 n + #1] == Factor[#2 /. n → 6 n + #1]) & @@@
 Partition[Rest[List @@ DownValues[A195806][[1, 2]]], 2]
```

Out[123]=
```
{a[1 + 6 n] == 16 + 198 n + 1133 n^2 + 3900 n^3 + 8125 n^4 + 9360 n^5 + 4680 n^6,
 a[2 + 6 n] == 105 + 1087 n + 4922 n^2 + 12 350 n^3 + 17 875 n^4 + 14 040 n^5 + 4680 n^6,
 a[3 + 6 n] == 496 + 4148 n + 14 783 n^2 + 28 600 n^3 + 31 525 n^4 + 18 720 n^5 + 4680 n^6,
 a[4 + 6 n] == 1759 + 12 121 n + 35 258 n^2 + 55 250 n^3 + 49 075 n^4 + 23 400 n^5 + 4680 n^6,
 a[5 + 6 n] == (1 + n)^2 (5052 + 19 370 n + 28 405 n^2 + 18 720 n^3 + 4680 n^4),
 a[6 n] == 1 + 25 n + 158 n^2 + 650 n^3 + 2275 n^4 + 4680 n^5 + 4680 n^6}
```

In[124]:=
```
% /. n → 1
```

Out[124]=
```
{a[7] == 27 412, a[8] == 55 059, a[9] == 102 952,
 a[10] == 181 543, a[11] == 304 908, a[6] == 12 469}
```

In[•]:=
```
TableForm[arr = Table[Symbol["a" <> ToString[i] <> ToString[j]], {i, 5}, {j, i}] /.
   {a11 → 0, a51 → 0, a55 → 0}]
```

Out[•]//TableForm=
```
0
a21  a22
a31  a32  a33
a41  a42  a43  a44
0    a52  a53  a54  0
```

In[•]:=
```
TableForm[eqns = Table[Total[arr[[i]]] ==
    Sum[arr[[5 - i + j, 6 - i]], {j, i}] == Sum[arr[[5 - i + j, j]], {j, i}], {i, 2, 5}]]
```

Out[•]//TableForm=
```
a21 + a22 == a44 + a54 == a41 + a52
a31 + a32 + a33 == a33 + a43 + a53 == a31 + a42 + a53
a41 + a42 + a43 + a44 == a22 + a32 + a42 + a52 == a21 + a32 + a43 + a54
a52 + a53 + a54 == a21 + a31 + a41 == a22 + a33 + a44
```

In[•]:=
```
ineq =
 Join[Thread[0 ≤ #], Thread[# ≤ n]] &[Variables[List @@@ eqns]] /. First[Solve[eqns]]
```

Out[•]=
```
{0 ≤ a21, 0 ≤ a22, 0 ≤ a31, 0 ≤ a32, 0 ≤ a33, 0 ≤ a41, 0 ≤ 3 a22 - 2 a31 + a32 + 2 a33 - 3 a41,
 0 ≤ 3 a22 - a31 + a32 + a33 - 3 a41, 0 ≤ a21 - a22 + a31 - a33 + a41, 0 ≤ a21 + a22 - a41,
 0 ≤ -3 a22 + 2 a31 - a33 + 3 a41, 0 ≤ 2 a22 - a31 + a33 - a41, a21 ≤ n, a22 ≤ n,
 a31 ≤ n, a32 ≤ n, a33 ≤ n, a41 ≤ n, 3 a22 - 2 a31 + a32 + 2 a33 - 3 a41 ≤ n,
 3 a22 - a31 + a32 + a33 - 3 a41 ≤ n, a21 - a22 + a31 - a33 + a41 ≤ n,
 a21 + a22 - a41 ≤ n, -3 a22 + 2 a31 - a33 + 3 a41 ≤ n, 2 a22 - a31 + a33 - a41 ≤ n}
```

```
(* Such systems of linear inequalities can
 (in principle) be treated by the Omega package. *)
<< RISC`Omega`
```

Omega Package version 2.49
written by Axel Riese
(in cooperation with George E. Andrews and Peter Paule)

In[•]:=
```
CrudeGF = OSum[q^n*z^(a21 + a22 + a31 + a32 + a33 + a41), ineq, λ]
```

Assuming n ≥ 0

Out[•]=

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14},\lambda_{15},\lambda_{16},\lambda_{17},\lambda_{18}}{\underset{\geq}{\Omega}} \left(1 \Big/ \left(\left(1-\frac{z\,\lambda_1\,\lambda_2}{\lambda_{10}\,\lambda_{13}\,\lambda_{14}}\right)\left(1-\frac{z\,\lambda_3\,\lambda_4}{\lambda_7\,\lambda_{15}\,\lambda_{16}}\right)\left(1-\frac{z\,\lambda_1^3\,\lambda_2^3\,\lambda_4\,\lambda_6^2\,\lambda_{15}\,\lambda_{17}^3}{\lambda_3\,\lambda_5^3\,\lambda_8\,\lambda_{13}^3\,\lambda_{14}^3\,\lambda_{16}\,\lambda_{18}^2}\right)\left(1-\frac{z\,\lambda_1^2\,\lambda_2\,\lambda_6\,\lambda_{15}\,\lambda_{17}}{\lambda_3\,\lambda_5\,\lambda_{11}\,\lambda_{13}^2\,\lambda_{14}\,\lambda_{18}}\right)\left(1-\frac{z\,\lambda_3\,\lambda_5^3\,\lambda_{13}^3\,\lambda_{14}^3\,\lambda_{16}\,\lambda_{18}}{\lambda_1^3\,\lambda_2^3\,\lambda_4\,\lambda_6\,\lambda_{12}\,\lambda_{15}\,\lambda_{17}^3}\right)\left(1-\frac{z\,\lambda_3\,\lambda_5^2\,\lambda_{13}^2\,\lambda_{14}\,\lambda_{18}}{\lambda_1^2\,\lambda_2\,\lambda_6\,\lambda_9\,\lambda_{15}\,\lambda_{17}^2}\right)(1-q\,\lambda_7\,\lambda_8\,\lambda_9\,\lambda_{10}\,\lambda_{11}\,\lambda_{12}\,\lambda_{13}\,\lambda_{14}\,\lambda_{15}\,\lambda_{16}\,\lambda_{17}\,\lambda_{18})\right)\right)$$

In[•]:=
```
Timing[OR[CrudeGF]]
```

Eliminating λ_{12}...

Eliminating λ_4...

Eliminating λ_7...

Eliminating λ_8...

Eliminating λ_{16}...

Eliminating λ_9...

Eliminating λ_{10}...

Mathematica crashed during the elimination of the 7th slack variable.

In[•]:=
```
OR[CrudeGF[[1]], λ12]
```

Out[•]=

$$1 \Big/ \left(\left(1-\frac{z\,\lambda_1\,\lambda_2}{\lambda_{10}\,\lambda_{13}\,\lambda_{14}}\right)\left(1-\frac{z\,\lambda_3\,\lambda_4}{\lambda_7\,\lambda_{15}\,\lambda_{16}}\right)\left(1-\frac{z\,\lambda_1^3\,\lambda_2^3\,\lambda_4\,\lambda_6^2\,\lambda_{15}\,\lambda_{17}^3}{\lambda_3\,\lambda_5^3\,\lambda_8\,\lambda_{13}^3\,\lambda_{14}^3\,\lambda_{16}\,\lambda_{18}^2}\right)\left(1-\frac{z\,\lambda_1^2\,\lambda_2\,\lambda_6\,\lambda_{15}\,\lambda_{17}}{\lambda_3\,\lambda_5\,\lambda_{11}\,\lambda_{13}^2\,\lambda_{14}\,\lambda_{18}}\right)\left(1-\frac{z\,\lambda_3\,\lambda_5^2\,\lambda_{13}^2\,\lambda_{14}\,\lambda_{18}}{\lambda_1^2\,\lambda_2\,\lambda_6\,\lambda_9\,\lambda_{15}\,\lambda_{17}^2}\right)(1-q\,\lambda_7\,\lambda_8\,\lambda_9\,\lambda_{10}\,\lambda_{11}\,\lambda_{13}\,\lambda_{14}\,\lambda_{15}\,\lambda_{16}\,\lambda_{17}\,\lambda_{18})\left(1-\frac{q\,z\,\lambda_3\,\lambda_5^3\,\lambda_7\,\lambda_8\,\lambda_9\,\lambda_{10}\,\lambda_{11}\,\lambda_{13}^4\,\lambda_{14}^4\,\lambda_{16}^2\,\lambda_{18}^2}{\lambda_1^3\,\lambda_2^3\,\lambda_4\,\lambda_6\,\lambda_{17}^2}\right)\right)$$

5.1 A216940

In[125]:=
```
InitializeSeq[216 940]
```

```
Sequence: 260, 27768, 1664244, 64697626, 1783839948, 37112483200,
   609829326268, 8196058134921, 92610036317488, 899427798281439, ...

Length: 37

Offset: 1

Recurrence:
```

$$-(8+n)(9+n)(14+n)\left(163\,783\,572\,502\,070\,416\,896\,000\,000+373\,537\,859\,888\,412\,982\,135\,296\,000\,n+409\,372\,299\,530\,242\,678\,774\,022\,400\,n^{2}+286\,252\,164\,962\,189\,491\,933\,877\,760\,n^{3}+143\,046\,324\,509\,506\,890\,024\,137\,904\,n^{4}+54\,240\,993\,934\,327\,573\,105\,412\,736\,n^{5}+16\,176\,998\,037\,775\,972\,989\,013\,560\,n^{6}+3\,882\,559\,668\,757\,849\,029\,669\,568\,n^{7}+761\,043\,846\,075\,942\,239\,596\,747\,n^{8}+122\,962\,525\,840\,318\,764\,255\,456\,n^{9}+16\,457\,246\,508\,233\,909\,290\,422\,n^{10}+1\,827\,011\,229\,237\,292\,366\,464\,n^{11}+167\,885\,420\,065\,560\,943\,260\,n^{12}+12\,696\,365\,052\,033\,491\,200\,n^{13}+782\,307\,152\,834\,582\,896\,n^{14}+38\,662\,756\,030\,232\,448\,n^{15}+1\,496\,456\,904\,306\,543\,n^{16}+43\,713\,765\,103\,008\,n^{17}+906\,575\,290\,122\,n^{18}+11\,901\,463\,360\,n^{19}+74\,384\,146\,n^{20}\right)a[n]+(1+n)(6+n)(7+n)\left(15\,118\,483\,615\,575\,730\,790\,400\,000+37\,557\,333\,457\,279\,933\,473\,792\,000\,n+45\,137\,854\,540\,680\,193\,956\,153\,600\,n^{2}+34\,829\,846\,371\,335\,010\,335\,540\,480\,n^{3}+19\,314\,394\,347\,459\,920\,710\,102\,704\,n^{4}+8\,166\,353\,315\,859\,794\,719\,296\,864\,n^{5}+2\,726\,904\,840\,964\,417\,033\,376\,520\,n^{6}+735\,273\,283\,907\,306\,553\,706\,472\,n^{7}+162\,382\,123\,713\,323\,392\,711\,687\,n^{8}+29\,630\,015\,361\,661\,371\,290\,844\,n^{9}+4\,487\,557\,575\,514\,810\,132\,362\,n^{10}+564\,694\,034\,848\,365\,996\,336\,n^{11}+58\,900\,361\,433\,618\,244\,860\,n^{12}+5\,062\,226\,797\,216\,352\,960\,n^{13}+354\,853\,893\,929\,158\,096\,n^{14}+19\,969\,728\,998\,781\,072\,n^{15}+880\,856\,790\,135\,603\,n^{16}+29\,345\,762\,188\,932\,n^{17}+694\,580\,474\,022\,n^{18}+10\,413\,780\,440\,n^{19}+74\,384\,146\,n^{20}\right)a[1+n]$$

```
Check: True
```

```
(* Compute more terms using the guessed recurrence relation. *)
re2l = RE2L[{a[1] == data[[1]], rec == 0}, a[n], {1, 1000}];
{Take[re2l, Length[data]] === data, And @@ (IntegerQ /@ re2l)}
```

Solve: Equations may not give solutions for all "solve" variables.

Out[127]= {True, True}

```
(* It seems that the n-th sequence term is given by a degree-37 polynomial. *)
Factor[InterpolatingPolynomial[Take[re2l, 100], n]]
```

Out[128]=

$$\Big((1+n)\,(2+n)\,(3+n)\,(4+n)\,(5+n)\,(6+n)^2\,(7+n)^3\,(8+n)^2\,(9+n)\,(10+n)\,(11+n)\,(12+n)\,(13+n)\,\big(15\,118\,483\,615\,575\,730\,790\,400\,000 + 37\,557\,333\,457\,279\,933\,473\,792\,000\,n + 45\,137\,854\,540\,680\,193\,956\,153\,600\,n^2 + 34\,829\,846\,371\,335\,010\,335\,540\,480\,n^3 + 19\,314\,394\,347\,459\,920\,710\,102\,704\,n^4 + 8\,166\,353\,315\,859\,794\,719\,296\,864\,n^5 + 2\,726\,904\,840\,964\,417\,033\,376\,520\,n^6 + 735\,273\,283\,907\,306\,553\,706\,472\,n^7 + 162\,382\,123\,713\,323\,392\,711\,687\,n^8 + 29\,630\,015\,361\,661\,371\,290\,844\,n^9 + 4\,487\,557\,575\,514\,810\,132\,362\,n^{10} + 564\,694\,034\,848\,365\,996\,336\,n^{11} + 58\,900\,361\,433\,618\,244\,860\,n^{12} + 5\,062\,226\,797\,216\,352\,960\,n^{13} + 354\,853\,893\,929\,158\,096\,n^{14} + 19\,969\,728\,998\,781\,072\,n^{15} + 880\,856\,790\,135\,603\,n^{16} + 29\,345\,762\,188\,932\,n^{17} + 694\,580\,474\,022\,n^{18} + 10\,413\,780\,440\,n^{19} + 74\,384\,146\,n^{20}\big)\Big) \Big/ 221\,424\,599\,279\,703\,105\,635\,713\,957\,232\,640\,000\,000$$

In[130]:=

```
(* The hexagonal array with side length 4. *)
TableForm[hex = Table[If[Abs[3 - i] ≤ j ≤ 12 - Abs[3 - i] && OddQ[i + j], Symbol[
    "a" <> ToString[i] <> ToString[(j - Abs[3 - i]) / 2]], ""], {i, 0, 6}, {j, 0, 12}]]
```

Out[130]//TableForm=

```
            a00         a01         a02         a03
      a10         a11         a12         a13         a14
   a20      a21         a22         a23         a24         a25
a30      a31         a32         a33         a34         a35         a36
   a40      a41         a42         a43         a44         a45
      a50         a51         a52         a53         a54
            a60         a61         a62         a63
```

In[131]:=

```
(* We only consider E and SW because SE is implied by these *)
ineq = Select[Flatten[Join[
    Table[hex[[i, j]] ≤ hex[[i, j + 2]], {i, 7}, {j, 11}],
    Table[hex[[i, j + 1]] ≤ hex[[i + 1, j]], {i, 6}, {j, 12}]
   ]], FreeQ[#, ""] &]
```

Out[131]=

```
{a00 ≤ a01, a01 ≤ a02, a02 ≤ a03, a10 ≤ a11, a11 ≤ a12, a12 ≤ a13, a13 ≤ a14, a20 ≤ a21,
 a21 ≤ a22, a22 ≤ a23, a23 ≤ a24, a24 ≤ a25, a30 ≤ a31, a31 ≤ a32, a32 ≤ a33, a33 ≤ a34,
 a34 ≤ a35, a35 ≤ a36, a40 ≤ a41, a41 ≤ a42, a42 ≤ a43, a43 ≤ a44, a44 ≤ a45, a50 ≤ a51,
 a51 ≤ a52, a52 ≤ a53, a53 ≤ a54, a60 ≤ a61, a61 ≤ a62, a62 ≤ a63, a00 ≤ a10, a01 ≤ a11,
 a02 ≤ a12, a03 ≤ a13, a10 ≤ a20, a11 ≤ a21, a12 ≤ a22, a13 ≤ a23, a14 ≤ a24,
 a20 ≤ a30, a21 ≤ a31, a22 ≤ a32, a23 ≤ a33, a24 ≤ a34, a25 ≤ a35, a31 ≤ a40,
 a32 ≤ a41, a33 ≤ a42, a34 ≤ a43, a35 ≤ a44, a36 ≤ a45, a41 ≤ a50, a42 ≤ a51,
 a43 ≤ a52, a44 ≤ a53, a45 ≤ a54, a51 ≤ a60, a52 ≤ a61, a53 ≤ a62, a54 ≤ a63}
```

In[132]:=

```
Length[%]
```

Out[132]= 60

```
(* Again, one can try to solve this system of inequalities with Omega. *)
<< RISC`Omega`
```

Omega Package version 2.49
written by Axel Riese

In[∘]:=
```
CrudeGF = OSum[q^a63 * z^Total[Most[Variables[List @@@ ineq]]], ineq, λ]
```

```
Assuming a00 ≥ 0
Assuming a01 ≥ 0
Assuming a02 ≥ 0
Assuming a03 ≥ 0
Assuming a10 ≥ 0
Assuming a11 ≥ 0
Assuming a12 ≥ 0
Assuming a13 ≥ 0
Assuming a14 ≥ 0
Assuming a20 ≥ 0
Assuming a21 ≥ 0
Assuming a22 ≥ 0
Assuming a23 ≥ 0
Assuming a24 ≥ 0
Assuming a25 ≥ 0
Assuming a30 ≥ 0
Assuming a31 ≥ 0
Assuming a32 ≥ 0
Assuming a33 ≥ 0
Assuming a34 ≥ 0
Assuming a35 ≥ 0
Assuming a36 ≥ 0
Assuming a40 ≥ 0
Assuming a41 ≥ 0
Assuming a42 ≥ 0
Assuming a43 ≥ 0
Assuming a44 ≥ 0
Assuming a45 ≥ 0
Assuming a50 ≥ 0
Assuming a51 ≥ 0
Assuming a52 ≥ 0
Assuming a53 ≥ 0
```

```
Assuming a54 ≥ 0
Assuming a60 ≥ 0
Assuming a61 ≥ 0
Assuming a62 ≥ 0
Assuming a63 ≥ 0
```

Out[•]=

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14},\lambda_{15},\lambda_{16},\lambda_{17},\lambda_{18},\lambda_{19},\lambda_{20},\lambda_{21},\lambda_{22},\lambda_{23},\lambda_{24},\lambda_{25},\lambda_{26},\lambda_{27},\lambda_{28},\lambda_{29},\lambda_{30},\lambda_{31},\lambda_{32},\lambda_{33},\lambda_{34},\lambda_{35},\ldots}{\overset{\Omega}{\geq}}$$

$$\left(1\Big/\left(\left(1-\frac{z}{\lambda_1\,\lambda_{31}}\right)\left(1-\frac{z\,\lambda_1}{\lambda_2\,\lambda_{32}}\right)\left(1-\frac{z\,\lambda_2}{\lambda_3\,\lambda_{33}}\right)\left(1-\frac{z\,\lambda_3}{\lambda_{34}}\right)\left(1-\frac{z\,\lambda_{31}}{\lambda_4\,\lambda_{35}}\right)\left(1-\frac{z\,\lambda_4\,\lambda_{32}}{\lambda_5\,\lambda_{36}}\right)\left(1-\frac{z\,\lambda_5\,\lambda_{33}}{\lambda_6\,\lambda_{37}}\right)\right.\right.$$
$$\left(1-\frac{z\,\lambda_6\,\lambda_{34}}{\lambda_7\,\lambda_{38}}\right)\left(1-\frac{z\,\lambda_7}{\lambda_{39}}\right)\left(1-\frac{z\,\lambda_{35}}{\lambda_8\,\lambda_{40}}\right)\left(1-\frac{z\,\lambda_{40}}{\lambda_{13}}\right)\left(1-\frac{z\,\lambda_8\,\lambda_{36}}{\lambda_9\,\lambda_{41}}\right)\left(1-\frac{z\,\lambda_9\,\lambda_{37}}{\lambda_{10}\,\lambda_{42}}\right)$$
$$\left(1-\frac{z\,\lambda_{10}\,\lambda_{38}}{\lambda_{11}\,\lambda_{43}}\right)\left(1-\frac{z\,\lambda_{11}\,\lambda_{39}}{\lambda_{12}\,\lambda_{44}}\right)\left(1-\frac{z\,\lambda_{12}}{\lambda_{45}}\right)\left(1-\frac{z\,\lambda_{13}\,\lambda_{41}}{\lambda_{14}\,\lambda_{46}}\right)\left(1-\frac{z\,\lambda_{46}}{\lambda_{19}}\right)\left(1-\frac{z\,\lambda_{14}\,\lambda_{42}}{\lambda_{15}\,\lambda_{47}}\right)$$
$$\left(1-\frac{z\,\lambda_{15}\,\lambda_{43}}{\lambda_{16}\,\lambda_{48}}\right)\left(1-\frac{z\,\lambda_{16}\,\lambda_{44}}{\lambda_{17}\,\lambda_{49}}\right)\left(1-\frac{z\,\lambda_{17}\,\lambda_{45}}{\lambda_{18}\,\lambda_{50}}\right)\left(1-\frac{z\,\lambda_{18}}{\lambda_{51}}\right)\left(1-\frac{z\,\lambda_{19}\,\lambda_{47}}{\lambda_{20}\,\lambda_{52}}\right)\left(1-\frac{z\,\lambda_{52}}{\lambda_{24}}\right)$$
$$\left(1-\frac{z\,\lambda_{20}\,\lambda_{48}}{\lambda_{21}\,\lambda_{53}}\right)\left(1-\frac{z\,\lambda_{21}\,\lambda_{49}}{\lambda_{22}\,\lambda_{54}}\right)\left(1-\frac{z\,\lambda_{22}\,\lambda_{50}}{\lambda_{23}\,\lambda_{55}}\right)\left(1-\frac{z\,\lambda_{23}\,\lambda_{51}}{\lambda_{56}}\right)\left(1-\frac{z\,\lambda_{24}\,\lambda_{53}}{\lambda_{25}\,\lambda_{57}}\right)\left(1-\frac{z\,\lambda_{57}}{\lambda_{28}}\right)$$
$$\left.\left.\left(1-\frac{z\,\lambda_{25}\,\lambda_{54}}{\lambda_{26}\,\lambda_{58}}\right)\left(1-\frac{z\,\lambda_{28}\,\lambda_{58}}{\lambda_{29}}\right)\left(1-\frac{z\,\lambda_{26}\,\lambda_{55}}{\lambda_{27}\,\lambda_{59}}\right)\left(1-\frac{z\,\lambda_{29}\,\lambda_{59}}{\lambda_{30}}\right)\left(1-\frac{z\,\lambda_{27}\,\lambda_{56}}{\lambda_{60}}\right)(1-q\,\lambda_{30}\,\lambda_{60})\right)\right)$$

In[•]:= **Timing[OR[CrudeGF]]**

Eliminating λ_{60}...

Eliminating λ_{59}...

Eliminating λ_{58}...

Eliminating λ_{57}...

Eliminating λ_{56}...

Eliminating λ_{55}...

Eliminating λ_{54}...

Eliminating λ_{53}...

Eliminating λ_{52}...

Eliminating λ_{51}...

Eliminating λ_{50}...

Eliminating λ_{49}...

Eliminating λ_{48}...

Eliminating λ_{47}...

Eliminating λ_{46}...

Eliminating λ_{45}...

Eliminating λ_{44}...

Eliminating λ_{43}...

Eliminating λ_{42}...

Eliminating λ_{41}...

Eliminating λ_{40}...

Eliminating λ_{39}...

Eliminating λ_{38}...

Eliminating λ_{37}...

Eliminating λ_{36}...

Eliminating λ_{35}...

Eliminating λ_{34}...

Eliminating λ_{33}...

Eliminating λ_{32}...

Eliminating λ_{31}...

Eliminating λ_{18}...

Eliminating λ_{12}...

Eliminating λ_{7}...

Eliminating λ_{3}...

Eliminating λ_{2}...

Eliminating λ_{1}...

Eliminating λ_{4}...

Eliminating λ_{6}...

Eliminating λ_{11}...

Eliminating λ_{17}...

Eliminating λ_{23}...

Eliminating λ_{8}...

Eliminating λ_{27}...

Eliminating λ_{10}...

Eliminating λ_{16}...

Eliminating λ_{22}...

Eliminating λ_{30}...

Eliminating λ_{13}...

Out[∘]= $Aborted

```
(* Unfortunately, this computation didn't finish in reasonable time. *)
```

5.2 A194478

In[133]:=
```
InitializeSeq[194 478]
```

```
Sequence: 0, 0, 0, 1, 337, 8733, 96478, 668028, 3413828, 14054915, ...

Length: 31

Offset: 1

Recurrence:
```

$$-(2+n)\,(383136-1293160\,n+1253272\,n^2-482042\,n^3+168104\,n^4-91561\,n^5-11948\,n^6+31682\,n^7-7700\,n^8-581\,n^9+224\,n^{10}+14\,n^{11})\,a[n]+(-3854400+10210192\,n-7286000\,n^2+1719696\,n^3+85056\,n^4-1113309\,n^5+1123664\,n^6-354550\,n^7+4662\,n^8+13377\,n^9-966\,n^{10}-126\,n^{11})\,a[1+n]+(-2+n)\,(3620736-6028304\,n+3593716\,n^2-364412\,n^3-1085555\,n^4+953463\,n^5-359946\,n^6+50106\,n^7+5299\,n^8-2051\,n^9+70\,n^{10}+14\,n^{11})\,a[2+n]$$

```
Check: True
```

In[145]:=

```
Binomial[n + 1, 2] // FunctionExpand
```

Out[145]=

$$\frac{1}{2}\,n\,(1+n)$$

In[150]:=

```
(* The closed form. *)
TraditionalForm[cf = HoldForm[Binomial[Binomial[n + 1, 2], 6] -
   (3 Sum[Binomial[i, j] Binomial[x - i, 6 - j], {j, 3, 6}, {i, 1, n}] /.
      x → n (n + 1) / 2) + (3 Sum[Binomial[i, 3] Binomial[j, 3], {i, 1, n}, {j, 1, i - 1}] +
      3 Sum[Binomial[i, 3] Binomial[j, 3], {i, 1, n}, {j, 1, n - i}] +
      3 Sum[Binomial[i - 1, 2] Binomial[j - 1, 2] (Binomial[n + 1, 2] - i - j + 1) +
         Binomial[i - 1, 3] Binomial[j - 1, 2] +
         Binomial[i - 1, 2] Binomial[j - 1, 3], {i, 1, n}, {j, n - i + 1, n}] +
      3 Sum[Binomial[i - 1, 3] Binomial[j - 1, 3], {i, 1, n}, {j, n - i + 1, n}]) -
   Sum[(i - 2) (j - 2) (Sum[l - 2, {l, n - Min[i, j] + 1, 2 n - (i + j)}] +
        Sum[l - 2, {l, 2 n + 2 - (i + j), n}]), {i, 3, n}, {j, n - i + 1, n}]]]
```

Out[150]//TraditionalForm=

$$\binom{\binom{n+1}{2}}{6}-\left(3\sum_{j=3}^{6}\sum_{i=1}^{n}\binom{i}{j}\binom{x-i}{6-j}\;/.\;x\to\frac{1}{2}\,n\,(n+1)\right)+\left(3\sum_{i=1}^{n}\sum_{j=1}^{i-1}\binom{i}{3}\binom{j}{3}+3\sum_{i=1}^{n}\sum_{j=1}^{n-i}\binom{i}{3}\binom{j}{3}+3\sum_{i=1}^{n}\sum_{j=n-i+1}^{n}\left(\binom{i-1}{2}\binom{j-1}{2}\left(\binom{n+1}{2}-i-j+1\right)+\binom{i-1}{3}\binom{j-1}{2}+\binom{i-1}{2}\binom{j-1}{3}\right)+3\sum_{i=1}^{n}\sum_{j=n-i+1}^{n}\binom{i-1}{3}\binom{j-1}{3}\right)-\sum_{i=3}^{n}\sum_{j=n-i+1}^{n}(i-2)\,(j-2)\left(\sum_{l=n-\min(i,j)+1}^{2\,n-(i+j)}(l-2)+\sum_{l=2\,n+2-(i+j)}^{n}(l-2)\right)$$

In[151]:= `Expand[FunctionExpand[ReleaseHold[cf]]]`

Out[151]=
$$\frac{11 n}{840}-\frac{17 n^2}{72}-\frac{3377 n^3}{5760}-\frac{9007 n^4}{11520}+\frac{22313 n^5}{23040}+\frac{8663 n^6}{9216}-\frac{12881 n^7}{32256}+\frac{3727 n^8}{46080}+\frac{91 n^9}{23040}-\frac{3 n^{10}}{1024}+\frac{n^{11}}{7680}+\frac{n^{12}}{46080}+\frac{n}{4\left(-1-\frac{1}{2} n (1+n)\right)}-\frac{3 n^2}{8\left(-1-\frac{1}{2} n (1+n)\right)}-\frac{21 n^3}{16\left(-1-\frac{1}{2} n (1+n)\right)}-\frac{21 n^4}{32\left(-1-\frac{1}{2} n (1+n)\right)}+\frac{7 n^5}{16\left(-1-\frac{1}{2} n (1+n)\right)}+\frac{7 n^6}{8\left(-1-\frac{1}{2} n (1+n)\right)}+\frac{5 n^7}{8\left(-1-\frac{1}{2} n (1+n)\right)}+\frac{5 n^8}{32\left(-1-\frac{1}{2} n (1+n)\right)}-$$

$$\left(\begin{array}{l} \frac{1}{24}\left(44 n-108 n^2+95 n^3-3\right. \\ \frac{1}{240}\left(-960+2272 n-1794 n\right. \\ \frac{1}{240}\left(21840-17788 n+475\right. \\ \frac{1}{240}\left(212 n-284 n^2-85 n^3\right. \\ \quad 19 n^6+188 \text{Floor}\left[\frac{1+n}{2}\right] \\ \quad 530 n^3 \text{Floor}\left[\frac{1+n}{2}\right]-75 \\ \quad 775 n^2 \text{Floor}\left[\frac{1+n}{2}\right]^2+1 \\ \quad 140 n \text{Floor}\left[\frac{1+n}{2}\right]^3+30 \\ \quad 390 n \text{Floor}\left[\frac{1+n}{2}\right]^4+90 \end{array}\right.$$

In[152]:= `Simplify[%, Assumptions → n > 5] /. Floor[(n + 1) / 2] → (n + (1 - (-1) ^n) / 2) / 2`

Out[152]=
$$\frac{1}{480}\left(\frac{1}{2}\left(1+(-1)^{1+n}\right)+n\right)^5(-7+2 n)-\frac{1}{128}\left(\frac{1}{2}\left(1+(-1)^{1+n}\right)+n\right)^4\left(9-13 n+3 n^2\right)+\frac{1}{96}\left(\frac{1}{2}\left(1+(-1)^{1+n}\right)+n\right)^3\left(39-7 n-15 n^2+4 n^3\right)-\frac{1}{192}\left(\frac{1}{2}\left(1+(-1)^{1+n}\right)+n\right)^2\left(-150+352 n-155 n^2+2 n^3+5 n^4\right)+\frac{1}{480}\left(\frac{1}{2}\left(1+(-1)^{1+n}\right)+n\right)\left(-188-582 n+1125 n^2-530 n^3+75 n^4\right)+\frac{1}{322560} n\left(-361344+466816 n+308168 n^2-668836 n^3+180670 n^4+176869 n^5-128810 n^6+26089 n^7+1274 n^8-945 n^9+42 n^{10}+7 n^{11}\right)$$

In[153]:= `Expand[%] /. (-1) ^a_ :> (-1) ^ (a /. n → 0) * (-1) ^ (Mod[Coefficient[a, n], 2] * n) /.`
`a_ * (-1) ^ (n + 1) :> (-a) * (-1) ^n`

Out[153]=
$$\frac{91}{256}-\frac{91(-1)^n}{256}-\frac{9703 n}{5376}+\frac{75}{256}(-1)^n n+\frac{12649 n^2}{11520}-\frac{21}{256}(-1)^n n^2+\frac{5369 n^3}{2880}+\frac{1}{128}(-1)^n n^3-\frac{28957 n^4}{11520}+\frac{14669 n^5}{23040}+\frac{25099 n^6}{46080}-\frac{12881 n^7}{32256}+\frac{3727 n^8}{46080}+\frac{91 n^9}{23040}-\frac{3 n^{10}}{1024}+\frac{n^{11}}{7680}+\frac{n^{12}}{46080}$$

In[154]:=
```
Collect[%, (-1)^n, Factor]
```

Out[154]=
$$\frac{1}{256}(-1)^n(-7+2n)(13-7n+n^2)+\frac{1}{322560}(114660-582180n+354172n^2+601328n^3-810796n^4+205366n^5+175693n^6-128810n^7+26089n^8+1274n^9-945n^{10}+42n^{11}+7n^{12})$$

In[155]:=
```
(* Sanity check *)
Table[%, {n, 31}] === data
```

Out[155]= True

6 Conjectures

6.1 A215570

In[156]:=
```
InitializeSeq[215570]
```

Sequence: 1, 35, 18720, 19369350, 27032968200, 44776592395920, 82881380383401600, 165850226337286576800, 351597937025844947295000, 779279938350147159519336600, ...

Length: 48

Offset: 0

Recurrence:
$$-250(1+5n)(2+5n)(3+5n)(4+5n)(6+5n)(7+5n)(8+5n)(9+5n)(11+5n)(12+5n)(13+5n)(14+5n)(1740+1772n+593n^2+65n^3)a[n]+25(1+n)(2+n)(6+5n)(7+5n)(8+5n)(9+5n)(11+5n)(12+5n)(13+5n)(14+5n)(118368+281088n+258294n^2+114387n^3+24428n^4+2015n^5)a[1+n]-20(1+n)(2+n)^3(3+n)^2(11+5n)(12+5n)(13+5n)(14+5n)(58980+134365n+119686n^2+52047n^3+11032n^4+910n^5)a[2+n]+3(1+n)(2+n)^3(3+n)^4(4+n)^2(8+3n)(10+3n)(496+781n+398n^2+65n^3)a[3+n]$$

Check: True

In[157]:=
```
(* Simpler recurrence for transformed sequence. *)
Factor[DFiniteTimesHyper[ToOrePolynomial[rec, a[n]], (n!)^3 * ((n+1)!)^2 / (5n)!]]
```

Out[157]=
$$3(8+3n)(10+3n)(496+781n+398n^2+65n^3)S_n^3-4(58980+134365n+119686n^2+52047n^3+11032n^4+910n^5)S_n^2+(118368+281088n+258294n^2+114387n^3+24428n^4+2015n^5)S_n-2(1+n)(2+n)(1740+1772n+593n^2+65n^3)$$

```
(* Code from OEIS, slightly simplified. *)
(* Computed 51 terms in about 2.5h on radon1 (memory usage 60GB). *)
Clear[mya, myb];
Timing[
 myb[l_] := myb[l] =
   Module[{m = Length[l], n = Total[l], g},
    g = l.Range[m] - (m + 1) (n - 1) / 2;
    If[n < 2, 1,
     Sum[If[l[[i]] > 0, myb[ReplacePart[l, i → l[[i]] - 1]], 0], {i, 1, Min[g, m]}]]];
 mya[k_] := myb[Array[k &, 5]];
 Table[mya[n], {n, 0, 12}]
]
```

Out[∘]= {6.12811, {1, 35, 18 720, 19 369 350, 27 032 968 200, 44 776 592 395 920, 82 881 380 383 401 600, 165 850 226 337 286 576 800, 351 597 937 025 844 947 295 000, 779 279 938 350 147 159 519 336 600, 1 789 294 251 011 628 021 153 241 548 800, 4 228 135 363 283 244 543 270 651 711 564 000, 10 232 120 200 642 411 474 243 152 429 724 152 000}}

```
(* Code from OEIS, a bit more simplified. *)
Clear[mya, myb];
Timing[
 myb[l_] := myb[l] =
   With[{n = Total[l]},
    If[n < 2, 1, Sum[If[l[[i]] > 0, myb[ReplacePart[l, i → l[[i]] - 1]], 0],
      {i, 1, Min[l.Range[5] - 3 * (n - 1), 5]}]]];
 mya[k_] := myb[Array[k &, 5]];
 Table[mya[n], {n, 0, 12}]
]
```

Out[∘]= {4.38714, {1, 35, 18 720, 19 369 350, 27 032 968 200, 44 776 592 395 920, 82 881 380 383 401 600, 165 850 226 337 286 576 800, 351 597 937 025 844 947 295 000, 779 279 938 350 147 159 519 336 600, 1 789 294 251 011 628 021 153 241 548 800, 4 228 135 363 283 244 543 270 651 711 564 000, 10 232 120 200 642 411 474 243 152 429 724 152 000}}

In[∘]:= `Length[DownValues[myb]]`

Out[∘]= 191 647

```
(* Transfer matrix method is slower, but more memory-efficient. *)
(* Computed 51 terms in 21h (memory about 2 GB) *)
Timing[With[{n = 12},
  seq = {1};
  tmat = Table[If[Abs[j - i] ≤ 2, x[j - i + 3], 0], {i, 3 n + 1}, {j, 3 n + 1}] /. x[5] → 1;
  vec = Prepend[Table[0, {3 n}], 1];
  Do[
   vec = PolynomialMod[Expand[tmat.vec], Array[x, 4] ^ (n + 1)];
   If[Mod[i, 5] === 0,
    AppendTo[seq, Coefficient[Total[vec], (Times @@ Array[x, 4]) ^ (i / 5)]]];
   , {i, 5 n}];
  seq
 ]]
```

Out[∘]= {39.6239, {1, 35, 18 720, 19 369 350, 27 032 968 200,
  44 776 592 395 920, 82 881 380 383 401 600, 165 850 226 337 286 576 800,
  351 597 937 025 844 947 295 000, 779 279 938 350 147 159 519 336 600,
  1 789 294 251 011 628 021 153 241 548 800, 4 228 135 363 283 244 543 270 651 711 564 000,
  10 232 120 200 642 411 474 243 152 429 724 152 000}}

In[161]:=
```
(* The additional three terms that we found. *)
data = Join[data,
   {109 077 635 701 149 385 673 307 834 354 313 488 656 611 420 675 978 722 550 093 654 767 156 \
     150 247 798 646 452 158 745 836 073 766 066 836 510 193 294 858 821 035 415 499 345 237 177 \
     075 859 552 482 287 257 000 000,
    320 518 781 996 956 464 254 305 073 090 595 172 642 716 936 717 597 758 539 843 844 836 406 \
     214 884 389 287 005 904 094 666 281 078 845 230 748 750 512 417 310 938 341 310 184 558 733 \
     989 330 751 809 185 813 089 240 000,
    942 993 553 387 261 719 839 432 368 142 529 421 305 313 544 889 687 234 217 568 261 481 733 \
     850 928 388 393 453 200 584 613 854 491 160 136 599 571 045 529 542 095 321 859 090 899 240 \
     127 873 178 680 468 809 691 523 430 400}];
```

In[166]:=
```
(* Test: they also satisfy the guessed recurrence. *)
Table[rec, {n, 44, 47}] /. a[n_] :→ data[[n + 1]]
```

Out[166]= {0, 0, 0, 0}

6.2 A339987

In[405]:=
```
InitializeSeq[339 987]
```

```
Sequence: 1, 4, 90, 8400, 1426950, 366153480, 134292027870,
   67095690261600, 43893900947947050, 36441011093916429000, ...
Length: 40
Offset: 1
Recurrence: 32 (1 + n) (2 + n) (1 + 2 n) (3 + 2 n) (5 + 2 n) (7 + 2 n) (9 + 2 n)
    (11 589 + 10 844 n + 3300 n^2 + 328 n^3) a[n] - 8 (2 + n) (3 + 2 n) (5 + 2 n) (7 + 2 n) (9 + 2 n)
    (148 119 + 232 328 n + 129 460 n^2 + 30 664 n^3 + 2624 n^4) a[1 + n] - 16 (3 + n) (5 + 2 n) (7 + 2 n)
    (9 + 2 n) (341 634 + 712 135 n + 569 267 n^2 + 219 308 n^3 + 40 852 n^4 + 2952 n^5) a[2 + n] + 8 (4 + n)
    (7 + 2 n) (9 + 2 n) (527 520 + 1 057 879 n + 818 282 n^2 + 306 380 n^3 + 55 672 n^4 + 3936 n^5) a[3 + n] -
  2 (5 + n) (9 + 2 n) (601 452 + 1 117 119 n + 786 236 n^2 + 264 028 n^3 + 42 472 n^4 + 2624 n^5) a[4 + n] +
  3 (4 + n) (6 + n) (3717 + 5228 n + 2316 n^2 + 328 n^3) a[5 + n]
Check: True
```

In[409]:=
```
(* Simpler recurrence for transformed sequence. *)
Factor[
 DFiniteTimesHyper[ToOrePolynomial[rec, a[n]], (n + 1) / Pochhammer[5 / 2, n - 2]]]
```

Out[409]=

$$3\,(4+n)\,(3717+5228\,n+2316\,n^2+328\,n^3)\,S_n^5 - 4\,(601\,452+1\,117\,119\,n+786\,236\,n^2+264\,028\,n^3+42\,472\,n^4+2624\,n^5)\,S_n^4 + 32\,(527\,520+1\,057\,879\,n+818\,282\,n^2+306\,380\,n^3+55\,672\,n^4+3936\,n^5)\,S_n^3 - 128\,(341\,634+712\,135\,n+569\,267\,n^2+219\,308\,n^3+40\,852\,n^4+2952\,n^5)\,S_n^2 - 128\,(148\,119+232\,328\,n+129\,460\,n^2+30\,664\,n^3+2624\,n^4)\,S_n + 1024\,(2+n)\,(11\,589+10\,844\,n+3300\,n^2+328\,n^3)$$

6.3 A269021

In[447]:=
```
InitializeSeq[269 021]
```

```
Sequence: 1, 2, 23, 588, 24553, 1438112,
   108469917, 9996042284, 1086997811325, 136102249609224, ...
Length: 42
Offset: 0
Recurrence:
```

$$64\,(1+n)^2\,(2+n)^2\,(3+n)\,(1+2n)^2\,(3+2n)^2\,(5+2n)^2\,(549\,760+3\,266\,000\,n+7\,264\,534\,n^2+8\,663\,374\,n^3+6\,333\,869\,n^4+3\,012\,795\,n^5+952\,323\,n^6+198\,469\,n^7+26\,156\,n^8+1968\,n^9+64\,n^{10})\,a[n]-16\,(2+n)^2\,(3+n)\,(3+2n)^2\,(5+2n)^2\,(-5\,543\,040-487\,964\,n+78\,563\,984\,n^2+229\,526\,554\,n^3+325\,846\,005\,n^4+284\,054\,698\,n^5+165\,789\,363\,n^6+67\,385\,886\,n^7+19\,359\,535\,n^8+3\,917\,758\,n^9+545\,913\,n^{10}+49\,788\,n^{11}+2672\,n^{12}+64\,n^{13})\,a[1+n]+4\,(3+n)\,(5+2n)^2\,(-250\,467\,360-946\,158\,512\,n-562\,569\,136\,n^2+3\,271\,711\,596\,n^3+9\,313\,604\,242\,n^4+12\,741\,784\,568\,n^5+11\,118\,771\,121\,n^6+6\,753\,094\,929\,n^7+2\,966\,908\,118\,n^8+959\,068\,672\,n^9+228\,837\,227\,n^{10}+39\,928\,763\,n^{11}+4\,969\,164\,n^{12}+419\,248\,n^{13}+21\,568\,n^{14}+512\,n^{15})\,a[2+n]-2\,(-1\,300\,242\,000-6\,360\,099\,840\,n-10\,812\,498\,240\,n^2-778\,719\,870\,n^3+27\,672\,902\,302\,n^4+55\,047\,935\,941\,n^5+60\,004\,107\,039\,n^6+43\,766\,004\,538\,n^7+22\,820\,793\,074\,n^8+8\,747\,566\,435\,n^9+2\,488\,583\,381\,n^{10}+523\,718\,876\,n^{11}+80\,260\,596\,n^{12}+8\,677\,944\,n^{13}+624\,800\,n^{14}+26\,752\,n^{15}+512\,n^{16})\,a[3+n]+3\,(4+n)\,(8+3n)\,(10+3n)\,(-15\,900-61\,278\,n-68\,928\,n^2+48\,699\,n^3+204\,716\,n^4+233\,810\,n^5+143\,536\,n^6+52\,389\,n^7+11\,324\,n^8+1328\,n^9+64\,n^{10})\,a[4+n]$$

```
Check: True
```

In[453]:=
```
(* Simpler recurrence for transformed sequence. *)
Factor[DFiniteTimesHyper[ToOrePolynomial[rec, a[n]], 1/((2 n)!)^2]]
```

Out[453]=

$$12\,(3+n)\,(4+n)^3\,(7+2n)^2\,(8+3n)\,(10+3n)\,(-15\,900-61\,278\,n-68\,928\,n^2+48\,699\,n^3+204\,716\,n^4+233\,810\,n^5+143\,536\,n^6+52\,389\,n^7+11\,324\,n^8+1328\,n^9+64\,n^{10})\,S_n^4-2\,(3+n)\,(-1\,300\,242\,000-6\,360\,099\,840\,n-10\,812\,498\,240\,n^2-778\,719\,870\,n^3+27\,672\,902\,302\,n^4+55\,047\,935\,941\,n^5+60\,004\,107\,039\,n^6+43\,766\,004\,538\,n^7+22\,820\,793\,074\,n^8+8\,747\,566\,435\,n^9+2\,488\,583\,381\,n^{10}+523\,718\,876\,n^{11}+80\,260\,596\,n^{12}+8\,677\,944\,n^{13}+624\,800\,n^{14}+26\,752\,n^{15}+512\,n^{16})\,S_n^3+(-250\,467\,360-946\,158\,512\,n-562\,569\,136\,n^2+3\,271\,711\,596\,n^3+9\,313\,604\,242\,n^4+12\,741\,784\,568\,n^5+11\,118\,771\,121\,n^6+6\,753\,094\,929\,n^7+2\,966\,908\,118\,n^8+959\,068\,672\,n^9+228\,837\,227\,n^{10}+39\,928\,763\,n^{11}+4\,969\,164\,n^{12}+419\,248\,n^{13}+21\,568\,n^{14}+512\,n^{15})\,S_n^2+(5\,543\,040+487\,964\,n-78\,563\,984\,n^2-229\,526\,554\,n^3-325\,846\,005\,n^4-284\,054\,698\,n^5-165\,789\,363\,n^6-67\,385\,886\,n^7-19\,359\,535\,n^8-3\,917\,758\,n^9-545\,913\,n^{10}-49\,788\,n^{11}-2672\,n^{12}-64\,n^{13})\,S_n+(549\,760+3\,266\,000\,n+7\,264\,534\,n^2+8\,663\,374\,n^3+6\,333\,869\,n^4+3\,012\,795\,n^5+952\,323\,n^6+198\,469\,n^7+26\,156\,n^8+1968\,n^9+64\,n^{10})$$

6.4 A181198

In[169]:=
```
InitializeSeq[181198]
```

```
Sequence:
 1, 1, 8, 169, 6392, 352184, 25097600, 2152061145, 212012802584, 23263015359672, ...
Length: 27
Offset: 1
Recurrence: -64 (-1 + 2 n)^2 (1 + 2 n) (-1 + 4 n) (1 + 4 n) (3 + 4 n) (5 + 4 n) (6 + 14 n + 7 n^2) a[n] -
   8 (1 + n) (1 + 2 n) (3 + 4 n) (5 + 4 n) (54 + 157 n - 534 n^2 - 1025 n^3 + 84 n^4 + 364 n^5) a[1 + n] +
   3 (1 + n) (2 + n)^3 (3 + 2 n) (4 + 3 n) (5 + 3 n) (-1 + 7 n^2) a[2 + n]
Check: True
```

```
(* Compute more terms of the sequence. For each n,
timing and number of variables is displayed. *)
NextPartitions[n1_, n2_, n3_, n4_] :=
  If[n1 < n, f[n1 + 1, n2, n3, n4], 0] +
   If[n2 < n1 - 1 || n2 === n - 1, f[n1, n2 + 1, n3, n4], 0] +
   If[n3 < n2 - 1 || n3 === n - 1 === n2 - 1, f[n1, n2, n3 + 1, n4], 0] +
   If[n4 < n3 - 1, f[n1, n2, n3, n4 + 1], 0];
Timing[
 seq181198 = Table[
   pp = f[1, 0, 0, 0];
   Print[n, ": ", Timing[Max[Table[pp = Expand[pp /. f[ns__] :> NextPartitions[ns]];
        Length[pp], {4 n - 2}]]]];
   pp /. f[n, n, n, n - 1] → 1, {n, 100}];
 Take[seq181198, 10]
]
```

```
1: {0.000077, 4}
2: {0.000113, 4}
3: {0.000268, 4}
4: {0.000499, 4}
5: {0.001211, 4}
6: {0.001604, 6}
7: {0.004741, 9}
8: {0.007744, 13}
9: {0.00646, 19}
10: {0.009246, 25}
11: {0.012531, 34}
12: {0.014919, 44}
13: {0.018492, 56}
14: {0.051532, 70}
15: {0.043592, 87}
16: {0.051571, 105}
```

17: {0.062367, 127}

18: {0.082342, 151}

19: {0.095208, 178}

20: {0.101695, 208}

21: {0.11786, 242}

22: {0.144615, 278}

23: {0.171029, 319}

24: {0.214849, 363}

25: {0.279882, 411}

26: {0.306942, 463}

27: {0.392212, 520}

28: {0.895769, 580}

29: {0.630256, 646}

30: {0.563033, 716}

31: {0.608588, 791}

32: {0.648001, 871}

33: {1.22421, 957}

34: {0.957592, 1047}

35: {0.903376, 1144}

36: {1.24864, 1246}

37: {1.40333, 1354}

38: {1.55336, 1468}

39: {1.39934, 1589}

40: {1.98304, 1715}

41: {2.10155, 1849}

42: {1.95636, 1989}

43: {2.19775, 2136}

44: {2.7125, 2290}

45: {2.66881, 2452}

46: {2.94088, 2620}

47: {3.24415, 2797}

48: {3.73189, 2981}

49: {5.85501, 3173}

50: {4.38233, 3373}

51: {4.68264, 3582}

52: {4.91523, 3798}

53: {5.30442, 4024}

54: {5.44595, 4258}

55: {6.37037, 4501}

56: {6.68667, 4753}

57: {7.66979, 5015}

58: {7.68994, 5285}

59: {8.01896, 5566}

60: {8.72684, 5856}

61: {11.6099, 6156}

62: {19.4616, 6466}

63: {10.5952, 6787}

64: {11.3732, 7117}

65: {12.5536, 7459}

66: {13.0855, 7811}

67: {13.6407, 8174}

68: {15.2087, 8548}

69: {15.434, 8934}

70: {16.3187, 9330}

71: {17.6447, 9739}

72: {23.0594, 10 159}

73: {19.4574, 10 591}

74: {24.4285, 11 035}

75: {22.0807, 11 492}

76: {23.4311, 11 960}

77: {24.7756, 12 442}

78: {26.69, 12 936}

79: {29.0007, 13 443}

80: {30.6782, 13 963}

81: {34.6215, 14 497}

82: {35.4403, 15 043}

83: {35.8558, 15 604}

84: {38.6934, 16 178}

85: {43.5737, 16 766}

86: {39.993, 17 368}

87: {41.7022, 17 985}

88: {43.1855, 18 615}

```
89: {46.3875, 19 261}
90: {47.6759, 19 921}
91: {50.8079, 20 596}
92: {52.8968, 21 286}
93: {55.6748, 21 992}
94: {58.908, 22 712}
95: {61.6161, 23 449}
96: {65.3662, 24 201}
97: {71.3483, 24 969}
98: {77.4399, 25 753}
99: {78.0968, 26 554}
100: {79.5438, 27 370}
```

Out[•]= {1552.88,
 {1, 1, 8, 169, 6392, 352 184, 25 097 600, 2 152 061 145, 212 012 802 584, 23 263 015 359 672}}

```
In[178]:= (* The additional values also satisfy the guessed recurrence. *)
Union[Table[rec, {n, 98}] /. a[n_] :→ seq181198[[n]]]
```

Out[178]= {0}

```
(* Mathematica code from OEIS A181196
 (bivariate sequence T(n,k), our A181198 is T(4,k)). *)
Clear[myb];
myb[l_List] := myb[l] =
  With[{n = Length[l]}, If[Union[l] == {0}, 1,
    Sum[If[(i == 1 || l[[i - 1]] ≤ l[[i]]) && l[[i]] > If[i == n, 0, l[[i + 1]]],
      myb[ReplacePart[l, i → l[[i]] - 1]], 0], {i, 1, n}]]];
myT[n_, k_] := myb[Array[n &, k]];
Timing[Table[myT[4, n], {n, 30}] === Take[seq181198, 30]]
```

Out[•]= {17.0821, True}

```
In[•]:= Length[DownValues[myb]]
```

Out[•]= 211 387

```
In[179]:= (* Conjectured closed form. *)
Table[(-64)^n * (n - 1) * Pochhammer[-1/2, 2 n] * Pochhammer[1/2, n] / (4 (3 n)!) *
   (-1 + 3 * Sum[(-4)^k (7 k^2 - 1) / ((k - 1) k (k + 1)^2 (2 k - 1)^2 (2 k + 1)^3) *
        Binomial[3 k, 2 k] * Binomial[k + 1/2, k],
       {k, 2, n - 1}]), {n, 2, 100}] === Rest[seq181198]
```

Out[179]= True

6.4 A181199

In[291]:= **InitializeSeq[181199]**

```
Sequence: 1, 1, 16, 985, 141696, 36372976,
   14083834704, 7372392431849, 4848332563899256, 3808369342900073856, ...
Length: 26
Offset: 1
Recurrence:
```

$$25\,(-1+2n)^2\,(1+2n)\,(-1+4n)\,(1+4n)\,(1+5n)\,(2+5n)\,(3+5n)\,(4+5n)\,(6+5n)\,(7+5n)\,(8+5n)\,(9+5n)\,\big(1\,077\,753\,600+23\,068\,181\,160\,n+128\,320\,688\,700\,n^2+354\,332\,674\,386\,n^3+588\,363\,536\,007\,n^4+638\,861\,237\,719\,n^5+471\,293\,180\,347\,n^6+238\,738\,633\,847\,n^7+81\,945\,774\,178\,n^8+18\,238\,806\,776\,n^9+2\,377\,384\,288\,n^{10}+137\,855\,872\,n^{11}\big)\,a[n]-$$

$$15\,(1+2n)\,(6+5n)\,(7+5n)\,(8+5n)\,(9+5n)\,\big(257\,403\,484\,800+4\,014\,999\,151\,560\,n+21\,777\,752\,002\,716\,n^2+50\,327\,965\,592\,874\,n^3+42\,448\,206\,362\,469\,n^4-54\,713\,722\,756\,179\,n^5-793\,918\,042\,456\,325\,n^6-4\,487\,751\,704\,725\,767\,n^7-12\,634\,296\,322\,883\,951\,n^8-17\,410\,620\,603\,191\,989\,n^9-3\,351\,043\,407\,516\,635\,n^{10}+30\,740\,208\,891\,967\,721\,n^{11}+61\,042\,725\,728\,660\,150\,n^{12}+64\,302\,800\,289\,940\,820\,n^{13}+44\,061\,606\,220\,301\,336\,n^{14}+20\,586\,994\,844\,739\,808\,n^{15}+6\,541\,981\,632\,511\,232\,n^{16}+1\,357\,324\,488\,921\,088\,n^{17}+166\,164\,038\,596\,608\,n^{18}+9\,113\,927\,409\,664\,n^{19}\big)\,a[1+n]-$$

$$(2+n)^2\,(3+2n)\,\big(-1\,104\,708\,217\,536\,000-22\,603\,977\,271\,411\,200\,n-180\,089\,041\,888\,657\,440\,n^2-685\,802\,053\,101\,717\,288\,n^3-850\,259\,283\,025\,327\,524\,n^4+3\,499\,276\,436\,019\,940\,446\,n^5+19\,628\,106\,098\,287\,909\,191\,n^6+47\,097\,079\,083\,848\,116\,511\,n^7+65\,692\,918\,122\,110\,754\,573\,n^8+48\,833\,460\,779\,191\,449\,763\,n^9-1\,215\,620\,047\,630\,796\,049\,n^{10}-46\,634\,308\,960\,957\,698\,567\,n^{11}-54\,791\,308\,745\,183\,340\,309\,n^{12}-31\,771\,892\,184\,486\,994\,333\,n^{13}-6\,080\,171\,043\,591\,548\,610\,n^{14}+5\,462\,710\,847\,171\,622\,988\,n^{15}+5\,556\,799\,141\,672\,247\,640\,n^{16}+2\,616\,700\,630\,350\,746\,208\,n^{17}+760\,770\,273\,998\,231\,808\,n^{18}+139\,665\,528\,127\,686\,656\,n^{19}+14\,962\,008\,250\,398\,720\,n^{20}+717\,088\,749\,346\,816\,n^{21}\big)\,a[2+n]+$$

$$3\,(2+n)^2\,(3+n)^4\,(3+2n)\,(5+2n)^2\,(7+3n)\,(8+3n)\,(9+4n)\,(11+4n)\,\big(927\,360+13\,865\,604\,n+62\,751\,860\,n^2+36\,004\,789\,n^3-524\,330\,624\,n^4-1\,671\,661\,480\,n^5-1\,894\,061\,166\,n^6-24\,192\,441\,n^7+2\,032\,587\,274\,n^8+2\,047\,036\,856\,n^9+860\,969\,696\,n^{10}+137\,855\,872\,n^{11}\big)\,a[3+n]$$

```
Check: True
```

```
(* Compute more terms of the sequence. For each n,
timing and number of variables is displayed. *)
(* Computing the first 100 terms on radon1-grantley took 49660s. *)
NextPartitions[n1_, n2_, n3_, n4_, n5_] :=
  If[n1 < n, f[n1 + 1, n2, n3, n4, n5], 0] +
   If[n2 < n1 - 1 || n2 === n - 1, f[n1, n2 + 1, n3, n4, n5], 0] +
   If[n3 < n2 - 1 || n3 === n - 1 === n2 - 1, f[n1, n2, n3 + 1, n4, n5], 0] +
   If[n4 < n3 - 1 || n4 === n - 1 === n3 - 1, f[n1, n2, n3, n4 + 1, n5], 0] +
   If[n5 < n4 - 1, f[n1, n2, n3, n4, n5 + 1], 0];
Timing[
 seq181199 = Table[
   pp = f[1, 0, 0, 0, 0];
   Print[n, ": ", Timing[Max[Table[pp = Expand[pp /. f[ns__] :> NextPartitions[ns]];
        Length[pp], {5 n - 2}]]]];
   pp /. f[n, n, n, n, n - 1] → 1, {n, 60}];
 Take[seq181199, 10]
]
```

```
1: {0.000156, 5}
2: {0.000233, 5}
3: {0.000378, 5}
4: {0.000692, 5}
5: {0.001685, 5}
6: {0.004306, 6}
7: {0.007431, 9}
8: {0.010739, 15}
9: {0.0166, 24}
10: {0.01729, 37}
11: {0.030127, 55}
12: {0.055153, 80}
13: {0.070867, 110}
14: {0.074449, 150}
15: {0.094442, 200}
16: {0.127548, 263}
17: {0.179097, 338}
18: {0.269347, 429}
19: {0.346152, 537}
20: {0.611531, 666}
21: {0.664791, 814}
22: {0.626202, 988}
```

23: {0.808509, 1188}

24: {1.01398, 1419}

25: {1.24509, 1678}

26: {1.57483, 1973}

27: {1.91173, 2306}

28: {2.25883, 2682}

29: {2.76617, 3097}

30: {3.25494, 3562}

31: {3.901, 4076}

32: {4.64906, 4647}

33: {5.57191, 5271}

34: {6.43764, 5959}

35: {7.5807, 6712}

36: {8.94392, 7538}

37: {10.5401, 8430}

38: {12.0509, 9404}

39: {13.8857, 10 459}

40: {16.154, 11 605}

41: {18.3514, 12 835}

42: {21.3339, 14 165}

43: {28.7275, 15 594}

44: {33.9516, 17 134}

45: {33.9498, 18 776}

46: {35.3743, 20 540}

47: {42.5231, 22 423}

48: {52.2228, 24 439}

49: {59.1358, 26 577}

50: {55.7727, 28 859}

51: {60.0378, 31 283}

52: {67.11, 33 864}

53: {75.4682, 36 589}

54: {83.0579, 39 484}

55: {90.8055, 42 543}

56: {104.542, 45 785}

57: {125.33, 49 196}

58: {120.675, 52 803}

```
59: {133.445, 56 601}

60: {146.197, 60 610}

Out[•]= {1495.78, {1, 1, 16, 985, 141 696, 36 372 976, 14 083 834 704,
   7 372 392 431 849, 4 848 332 563 899 256, 3 808 369 342 900 073 856}}

In[198]:= (* Conjectured closed form. *)
Table[
  1 - 27 / 4 *
    Sum[(-1)^k * (25 216 k^8 + 9888 k^7 - 14 496 k^6 + 11 208 k^5 + 23 832 k^4 + 7383 k^3 -
          1522 k^2 - 939 k - 90) / ((k + 1) (2 k - 1) (2 k + 1) (3 k + 1) (3 k + 2) (3 k + 3)
          (4 k - 1) (4 k + 1) (4 k + 3)) * (5 k) ! / (3 k) ! / (k !)^2 * Sum[(-1)^i *
        ((3 i + 1) (3 i + 2) (4 i + 3) (137 855 872 i^11 + 860 969 696 i^10 + 2 047 036 856 i^9 +
              2 032 587 274 i^8 - 24 192 441 i^7 - 1 894 061 166 i^6 - 1 671 661 480 i^5 -
              524 330 624 i^4 + 36 004 789 i^3 + 62 751 860 i^2 + 13 865 604 i + 927 360)) /
          ((i + 1)^2 (i + 2)^2 (2 i - 1) (2 i + 1) (2 i + 3) (25 216 i^8 + 9888 i^7 -
              14 496 i^6 + 11 208 i^5 + 23 832 i^4 + 7383 i^3 - 1522 i^2 - 939 i - 90)
            (25 216 i^8 + 211 616 i^7 + 760 768 i^6 + 1 543 976 i^5 + 1 973 632 i^4 +
              1 683 047 i^3 + 971 955 i^2 + 353 502 i + 60 480)) * (3 i) ! / (i !)^3,
       {i, 1, k - 1}], {k, 1, n - 1}], {n, 1, 60}] === seq181199

Out[198]= True
```

6.5 A181280

```
In[230]:= InitializeSeq[181 280]

Sequence: 0, 0, 0, 58, 1629, 28924, 507052, 8211776, 133693904, 2140571200, ...

Length: 27

Offset: 1

Recurrence: -22 912 660 668 416 (-3 + n) a[n] +
   4 194 304 (-10 203 879 + 4 419 089 n) a[1 + n] + 458 752 (-107 094 289 + 9 499 785 n) a[2 + n] -
   8192 (-5 570 332 304 + 890 967 049 n) a[3 + n] + 1024 (-7 089 379 615 + 1 488 027 923 n) a[4 + n] +
   6528 (-431 457 568 + 44 221 759 n) a[5 + n] - 32 (-30 850 821 681 + 3 992 176 883 n) a[6 + n] +
   8 (-7 506 938 166 + 1 107 194 741 n) a[7 + n] + 2 (-5 203 998 205 + 610 453 317 n) a[8 + n] +
   (1 455 280 568 - 182 139 823 n) a[9 + n] + (-47 739 905 + 6 061 186 n) a[10 + n]

Check: True

In[231]:= (* Compute more terms using the guessed recurrence. *)
re2l = RE2L[Prepend[Table[a[i] == data[[i]], {i, 10}], rec == 0], a[n], {1, 1000}];
{Take[re2l, Length[data]] === data, And @@ (IntegerQ /@ re2l)}

••• Solve: Equations may not give solutions for all "solve" variables.

Out[232]= {True, True}
```

In[233]:=
```
(* Find the minimal-order recurrence. *)
mrec = GuessUnivRE[Take[re2l, {4, 51}], a[n], Order → 6, Degree → 5, StartPoint → 4];
mrec = Collect[Numerator[Together[mrec[[1]]]], a[_], Expand]
```

Out[234]=

$$(73417942962667520 - 55638161308188672\, n + 6881078879059968\, n^2 + 319516849373184\, n^3 - 68258253864960\, n^4 + 1739037081600\, n^5)\, a[n] + (-49190583825408000 + 30220668909785088\, n - 3195154816376832\, n^2 - 238620979316736\, n^3 + 39916925245440\, n^4 - 978208358400\, n^5)\, a[1+n] + (-1384707508073472 + 2810657269458432\, n - 412596804575232\, n^2 - 12572113794048\, n^3 + 3131674513920\, n^4 - 81517363200\, n^5)\, a[2+n] + (4847877499145856 - 2720920678761024\, n + 254128789663488\, n^2 + 22964726080800\, n^3 - 3266260005600\, n^4 + 76422528000\, n^5)\, a[3+n] + (-282633754613904 + 109484370807768\, n - 5804275885920\, n^2 - 1144869424344\, n^3 + 119942447760\, n^4 - 2547417600\, n^5)\, a[4+n] + (-66439161783912 + 35972827162920\, n - 3170392606314\, n^2 - 309568287024\, n^3 + 41528789910\, n^4 - 955281600\, n^5)\, a[5+n] + (4136586578662 - 2080079355387\, n + 167712049467\, n^2 + 18613898793\, n^3 - 2348432595\, n^4 + 53071200\, n^5)\, a[6+n]$$

```
(* The originally guessed recurrence
  is a left multiple of the minimal-order one. *)
OreReduce @@ (ToOrePolynomial[#, a[n]] & /@ {rec, {mrec}})
```

Out[235]= 0

In[237]:=
```
(* Conjectured closed form. *)
Table[(1/3) * 2^(2 * n - 11) * (6 * n^2 - 219 * n + 820) -
   (1/9) * 2^(n - 5) * (3 * n + 32) - (113/3) * (-1)^n * 2^(3 * n - 14) +
   2^(4 * n - 9) - (1/3) * (-1)^n * 2^(2 * n - 11) * (13 * n - 164) +
   (1/9) * 2^(3 * n - 14) * (288 * n - 3473), {n, 4, 1000}] === Drop[re2l, 3]
```

Out[237]= True

6.6 A253217

In[238]:=
```
InitializeSeq[253217]
```

Sequence: 0, 0, 1, 19, 268, 3568, 47698, 649712, 9023385, 127419681, ...

Length: 37

Offset: 1

Recurrence:

$$32 (1+n) (1+2n)^2 (161046+465785 n+551943 n^2+343020 n^3+117954 n^4+21285 n^5+1575 n^6) a[n] - 8 (4443102+33718283 n+105734340 n^2+180574335 n^3+186866686 n^4+122556360 n^5+51280818 n^6+13267683 n^7+1933470 n^8+121275 n^9) a[1+n] + 2 (12137328+91378536 n+283626704 n^2+478464380 n^3+488415476 n^4+315713355 n^5+130145646 n^6+33170868 n^7+4763070 n^8+294525 n^9) a[2+n] + (10688508+80866406 n+252913504 n^2+431097970 n^3+445804136 n^4+292620525 n^5+122735586 n^6+31877118 n^7+4668570 n^8+294525 n^9) a[3+n] + (-4877748-36871922 n-114948300 n^2-194784258 n^3-199650088 n^4-129484209 n^5-53503836 n^6-13655808 n^7-1961820 n^8-121275 n^9) a[4+n] + 2 (3+n)^2 (7+2n) (2428+16118 n+41382 n^2+52554 n^3+35154 n^4+11835 n^5+1575 n^6) a[5+n]$$

Check: True

In[271]:=
```
(* Conjectured differential equation. *)
deq = -2 (-144 - 9472 x + 59 516 x^2 + 293 099 x^3 - 6 118 786 x^4 + 20 605 760 x^5 + 8 447 840 x^6 -
      8 859 568 x^7 + 576 032 x^8 - 3 834 112 x^9 - 672 768 x^10 + 43 008 x^11) f[x] -
   2 x (136 + 8840 x - 56 698 x^2 - 266 263 x^3 + 5 562 098 x^4 - 18 100 720 x^5 - 9 643 156 x^6 -
      2 227 000 x^7 + 37 415 840 x^8 - 41 803 520 x^9 - 10 291 200 x^10 + 1 118 208 x^11) f'[x] -
   x^2 (-128 - 8208 x + 50 520 x^2 + 156 939 x^3 - 5 838 066 x^4 + 25 114 104 x^5 + 8 254 632 x^6 -
      85 240 800 x^7 + 269 354 016 x^8 - 71 598 976 x^9 - 35 864 064 x^10 + 4 171 776 x^11) f''[x] -
   x^3 (40 + 3032 x - 13 122 x^2 - 215 313 x^3 + 1 094 157 x^4 - 3 361 026 x^5 + 3 620 772 x^6 -
      71 338 632 x^7 + 134 615 328 x^8 - 4 944 256 x^9 - 15 910 400 x^10 + 1 720 320 x^11) f^(3)[x] -
   (-4 + x) (-1 + x) x^4 (1 + 2 x) (-1 + 4 x) (-1 + 16 x)
    (-4 - 202 x + 257 x^2 + 4672 x^3 - 26 492 x^4 - 7520 x^5 + 1344 x^6) f^(4)[x];
```

In[272]:=
```
(* Sanity check. *)
PolynomialMod[deq /. f → Function @@ {x, Sum[data[[n]] * x^n, {n, 37}]}, x^38]
```

Out[272]= 0

```
(* Factorization of the differential operator, computed by Maple. *)
{L1, L2, L3} =
 Factor[ToOrePolynomial[{(-172 032 * x^15 + 1 790 464 * x^14 - 1 744 000 * x^13 -
       14 355 936 * x^12 + 14 262 792 * x^11 + 2 806 428 * x^10 - 3 103 506 * x^9 +
       490 737 * x^8 + 40 371 * x^7 - 15 834 * x^6 + 500 * x^5 + 16 * x^4) **
     (Der[x]^2 + ((7 225 344 * x^11 - 68 551 168 * x^10 - 38 001 536 * x^9 +
             655 096 032 * x^8 - 257 000 136 * x^7 - 120 067 572 * x^6 + 26 515 998 * x^5 +
             181 023 * x^4 - 1 528 359 * x^3 - 207 288 * x^2 + 53 464 * x + 3248) / (7 * x + 2) /
          (172 032 * x^11 - 1 790 464 * x^10 + 1 744 000 * x^9 + 14 355 936 * x^8 -
            14 262 792 * x^7 - 2 806 428 * x^6 + 3 103 506 * x^5 - 490 737 * x^4 -
            40 371 * x^3 + 15 834 * x^2 - 500 * x - 16)) ** Der[x] +
       (52 684 800 * x^12 - 473 285 120 * x^11 - 1 237 651 072 * x^10 + 3 741 714 912 * x^9 +
            709 591 920 * x^8 - 282 597 864 * x^7 + 133 579 476 * x^6 + 114 181 086 * x^5 -
            23 095 209 * x^4 - 8 060 712 * x^3 + 114 296 * x^2 + 1168 * x + 384) / x /
         (7 * x + 2)^2 / (172 032 * x^11 - 1 790 464 * x^10 + 1 744 000 * x^9 +
           14 355 936 * x^8 - 14 262 792 * x^7 - 2 806 428 * x^6 + 3 103 506 * x^5 -
           490 737 * x^4 - 40 371 * x^3 + 15 834 * x^2 - 500 * x - 16)),
    Der[x] + (168 * x^5 - 356 * x^4 - 133 * x^3 + 18 * x^2 + 56 * x + 4) / (7 * x + 2) / (x - 4) / x /
      (x - 1) / (2 * x + 1) / (4 * x - 1),
    Der[x] + (8 * x^3 + 3 * x - 2) / (x - 1) / x / (2 * x + 1) / (4 * x - 1)}]]
```

Out[281]=

$$\Big\{-(-4+x)(-1+x)x^4(1+2x)(-1+4x)(-1+16x)(-4-202x+257x^2+4672x^3-26492x^4-7520x^5+1344x^6)D_x^2-\frac{1}{2+7x}x^4(3248+53464x-207288x^2-1528359x^3+181023x^4+26515998x^5-120067572x^6-257000136x^7+655096032x^8-38001536x^9-68551168x^{10}+7225344x^{11})D_x-\frac{1}{(2+7x)^2}x^3(384+1168x+114296x^2-8060712x^3-23095209x^4+114181086x^5+133579476x^6-282597864x^7+709591920x^8+3741714912x^9-1237651072x^{10}-473285120x^{11}+52684800x^{12}),$$

$$D_x+\frac{4+56x+18x^2-133x^3-356x^4+168x^5}{(-4+x)(-1+x)x(1+2x)(-1+4x)(2+7x)},$$

$$D_x+\frac{-2+3x+8x^3}{(-1+x)x(1+2x)(-1+4x)}\Big\}$$

In[282]:=
```
L1 ** L2 ** L3
```

Out[282]=

$$
\begin{aligned}
&-(-4+x)\,(-1+x)\,x^4\,(1+2x)\,(-1+4x)\,(-1+16x)\\
&\quad(-4-202x+257x^2+4672x^3-26\,492x^4-7520x^5+1344x^6)\,D_x^4-\\
&x^3\,(40+3032x-13\,122x^2-215\,313x^3+1\,094\,157x^4-3\,361\,026x^5+3\,620\,772x^6-\\
&\quad 71\,338\,632x^7+134\,615\,328x^8-4\,944\,256x^9-15\,910\,400x^{10}+1\,720\,320x^{11})\,D_x^3-\\
&x^2\,(-128-8208x+50\,520x^2+156\,939x^3-5\,838\,066x^4+25\,114\,104x^5+8\,254\,632x^6-\\
&\quad 85\,240\,800x^7+269\,354\,016x^8-71\,598\,976x^9-35\,864\,064x^{10}+4\,171\,776x^{11})\,D_x^2-\\
&2x\,(136+8840x-56\,698x^2-266\,263x^3+5\,562\,098x^4-18\,100\,720x^5-9\,643\,156x^6-\\
&\quad 2\,227\,000x^7+37\,415\,840x^8-41\,803\,520x^9-10\,291\,200x^{10}+1\,118\,208x^{11})\,D_x-\\
&2\,(-144-9472x+59\,516x^2+293\,099x^3-6\,118\,786x^4+20\,605\,760x^5+8\,447\,840x^6-\\
&\quad 8\,859\,568x^7+576\,032x^8-3\,834\,112x^9-672\,768x^{10}+43\,008x^{11})
\end{aligned}
$$

```
(* Check correctness of factorization. *)
Together[ApplyOreOperator[%, f[x]] / deq]
```

Out[283]= 1

6.7 A098926

In[295]:=
```
InitializeSeq["098926"]
```

Sequence: 0, 2, 12, 90, 556, 5242, 42380, 479306, 4817484, 63779034, ...

Length: 34

Offset: 3

Recurrence:

$$
\begin{aligned}
&n\,(1+n)\,(14\,025+14\,907n+5983n^2+1113n^3+95n^4+3n^5)\,a[n]-\\
&(1+n)\,(16\,865+13\,424n+3717n^2+388n^3+13n^4)\,a[1+n]+\\
&(-81\,720-157\,554n-146\,304n^2-76\,591n^3-22\,722n^4-3677n^5-294n^6-9n^7)\,a[2+n]+\\
&(32\,845+38\,283n+14\,395n^2+2125n^3+103n^4-n^5)\,a[3+n]+\\
&(35\,600+186\,788n+219\,268n^2+113\,879n^3+30\,672n^4+4409n^5+318n^6+9n^7)\,a[4+n]+\\
&(1765+24\,893n+17\,161n^2+4253n^3+445n^4+17n^5)\,a[5+n]+\\
&(-211\,080-364\,438n-252\,920n^2-90\,333n^3-18\,038n^4-2039n^5-122n^6-3n^7)\,a[6+n]+\\
&(-4465-6967n-3663n^2-833n^3-83n^4-3n^5)\,a[7+n]+\\
&(4080+5915n+3184n^2+763n^3+80n^4+3n^5)\,a[8+n]
\end{aligned}
$$

Check: True

In[∘]:=
```
re2l = RE2L[Prepend[Table[f[i - 1] == data[[i]], {i, LeadingExponent[op][[1]]}],
    ApplyOreOperator[op, f[n]] == 0], f[n], 1000];
And @@ (IntegerQ /@ re2l)
```

Out[∘]= True

In[308]:=
```
(* Generating function: ODE and closed form. *)
f1x = (x^2 - x - 2) / (x (x - 1)) * Exp[(x + 1) / (x (x - 1))];
f2y = (y^5 - 3 y^4 + 2 y^3 - 2 y^2 - y + 1) / (y (y + 1) ^4 (y - 2) ^2) *
    Exp[- (2 y^2 + 2) / (y (y - 1) (y + 1))];
f3z = (z^2 (z - 2) (z^8 - 2 z^7 - 12 z^6 + 28 z^5 - 10 z^4 - 22 z^3 + 4 z^2 + 4 z + 1)) /
     ((z - 1) ^2 (z^5 - 3 z^4 + 2 z^3 - 2 z^2 - z + 1) ^2) * Exp[(z - 1) / (z (z + 1))];
ode =
  DFiniteTimes[
    Annihilator[f1x, Der[x]],
    {DFiniteTimes[
          Annihilator[f2y, Der[y]],
          {Annihilator[f3z, Der[z]][[1]] ** Der[z] /. z → y}
        ][[1]] ** Der[y] /. y → x}
   ][[1]];
Factor[ode]
```

Out[312]=
$$(-1+x)^3 x^5 (1+x)^3 \left(1+4x+4x^2-22x^3-10x^4+28x^5-12x^6-2x^7+x^8\right) D_x^3 + (-1+x) x^4 (1+x) \left(3+8x-4x^2-24x^3+7x^4+36x^5+80x^6-156x^7-143x^8+212x^9-76x^{10}-12x^{11}+5x^{12}\right) D_x^2 + (-1+x) x (1+x) \left(-1-3x+10x^3-53x^4+121x^5+252x^6-336x^7-75x^8-29x^9-288x^{10}+342x^{11}-95x^{12}-9x^{13}+4x^{14}\right) D_x + 2\left(-2-9x-4x^2+104x^3+16x^4-283x^5+218x^6-24x^7-410x^8+513x^9+40x^{10}-128x^{11}+12x^{12}-13x^{13}+2x^{14}\right)$$

In[314]:=
```
(* Sanity check. *)
PolynomialMod[ApplyOreOperator[ode, Sum[data[[n]] * x^(n + 2), {n, 34}]], x^36]
```

Out[314]= 0

6.8 A164735

In[315]:=
```
InitializeSeq[164 735]
```

```
Sequence: 0, 0, 0, 0, 0, 0, 0, 1, 0, 4, ...
Length: 70
Offset: 1
```

Recurrence:

$$
\begin{aligned}
&(-49\,633\,257\,600 - 44\,068\,202\,208\, n - 23\,627\,303\,952\, n^2 - 7\,818\,567\,588\, n^3 - 1\,793\,760\,488\, n^4 - \\
&\quad 275\,719\,619\, n^5 - 28\,072\,440\, n^6 - 1\,990\,671\, n^7 - 103\,992\, n^8 - 3753\, n^9 - 88\, n^{10} - n^{11})\, a[n] - \\
&48\,(31\,572\,000 + 70\,175\,472\, n + 9\,988\,894\, n^2 - 2\,811\,459\, n^3 - 1\,132\,007\, n^4 - \\
&\quad 153\,012\, n^5 - 5759\, n^6 + 99\, n^7 + 12\, n^8)\, a[1+n] - \\
&8\,(1\,448\,748\,720 + 2\,459\,387\,916\, n + 1\,228\,218\,234\, n^2 + 430\,916\,388\, n^3 + 87\,856\,754\, n^4 + \\
&\quad 11\,258\,109\, n^5 + 990\,945\, n^6 + 62\,712\, n^7 + 2706\, n^8 + 75\, n^9 + n^{10})\, a[2+n] + \\
&(-63\,198\,973\,440 - 56\,461\,698\,432\, n - 28\,956\,124\,800\, n^2 - 9\,170\,417\,952\, n^3 - 2\,051\,690\,228\, n^4 - \\
&\quad 310\,134\,980\, n^5 - 31\,151\,019\, n^6 - 2\,199\,057\, n^7 - 112\,902\, n^8 - 3978\, n^9 - 91\, n^{10} - n^{11})\, a[3+n] - \\
&8\,(2\,022\,554\,880 + 3\,011\,749\,632\, n + 1\,337\,298\,240\, n^2 + 411\,673\,668\, n^3 + 81\,719\,732\, n^4 + \\
&\quad 10\,362\,303\, n^5 + 961\,089\, n^6 + 63\,522\, n^7 + 2778\, n^8 + 75\, n^9 + n^{10})\, a[4+n] - \\
&8\,(1\,253\,806\,560 + 2\,749\,131\,864\, n + 1\,239\,004\,956\, n^2 + 416\,421\,600\, n^3 + 80\,409\,692\, n^4 + \\
&\quad 10\,317\,771\, n^5 + 951\,693\, n^6 + 63\,090\, n^7 + 2778\, n^8 + 75\, n^9 + n^{10})\, a[5+n] - \\
&2\,(20\,140\,479\,360 + 21\,072\,072\,096\, n + 10\,198\,546\,896\, n^2 + 3\,133\,495\,068\, n^3 + 639\,145\,004\, n^4 + \\
&\quad 83\,209\,149\, n^5 + 7\,199\,367\, n^6 + 457\,722\, n^7 + 19\,446\, n^8 + 525\, n^9 + 7\, n^{10})\, a[6+n] - \\
&8\,(2\,022\,554\,880 + 3\,011\,749\,632\, n + 1\,337\,298\,240\, n^2 + 411\,673\,668\, n^3 + 81\,719\,732\, n^4 + \\
&\quad 10\,362\,303\, n^5 + 961\,089\, n^6 + 63\,522\, n^7 + 2778\, n^8 + 75\, n^9 + n^{10})\, a[7+n] - \\
&8\,(1\,253\,806\,560 + 2\,749\,131\,864\, n + 1\,239\,004\,956\, n^2 + 416\,421\,600\, n^3 + 80\,409\,692\, n^4 + \\
&\quad 10\,317\,771\, n^5 + 951\,693\, n^6 + 63\,090\, n^7 + 2778\, n^8 + 75\, n^9 + n^{10})\, a[8+n] + \\
&(9\,352\,298\,880 + 1\,924\,058\,016\, n + 3\,230\,210\,160\, n^2 + 1\,551\,577\,452\, n^3 + 515\,470\,480\, n^4 + \\
&\quad 109\,301\,321\, n^5 + 13\,673\,706\, n^6 + 1\,075\,227\, n^7 + 65\,100\, n^8 + 2703\, n^9 + 74\, n^{10} + n^{11})\, a[9+n] - \\
&8\,(1\,833\,122\,880 + 2\,590\,696\,800\, n + 1\,277\,364\,876\, n^2 + 428\,542\,422\, n^3 + 88\,511\,774\, n^4 + \\
&\quad 11\,280\,375\, n^5 + 995\,643\, n^6 + 62\,928\, n^7 + 2706\, n^8 + 75\, n^9 + n^{10})\, a[10+n] - \\
&48\,(-32\,490\,360 + 48\,290\,658\, n + 1\,797\,787\, n^2 - 2\,415\,798\, n^3 - 1\,241\,177\, n^4 - \\
&\quad 156\,723\, n^5 - 6542\, n^6 + 63\, n^7 + 12\, n^8)\, a[11+n] + \\
&(22\,918\,014\,720 + 14\,317\,554\,240\, n + 8\,559\,031\,008\, n^2 + 2\,903\,427\,816\, n^3 + 773\,400\,220\, n^4 + \\
&\quad 143\,716\,682\, n^5 + 16\,752\,285\, n^6 + 1\,283\,613\, n^7 + 74\,010\, n^8 + 2928\, n^9 + 77\, n^{10} + n^{11})\, a[12+n]
\end{aligned}
$$

```
Check: False
```

In[319]:=
```
(* Recurrence is only valid for n ≥ 3. *)
Table[rec, {n, 3, 58}] /. a[n_] :→ data[[n]]
```

Out[319]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

In[322]:=
```
(* Compute more terms,
using the guessed recurrence (n=1,2 have to be omitted). *)
re2l = RE2L[Prepend[Table[a[i] == data[[i]], {i, 3, 14}], rec == 0], a[n], {1, 1000}];
{Take[re2l, {3, Length[data]}] === Drop[data, 2], And @@ (IntegerQ /@ re2l)}
```

Solve: Equations may not give solutions for all "solve" variables.

Out[323]=
```
{True, True}
```

In[328]:=
```
(* With period 18, we get first order recurrence, with polynomial solutions. *)
Collect[GuessMinRE[Take[re2l, {8, 400, 18}], a[n]], a[_], Factor]
```

Out[328]=
$$\frac{1}{729}\left(-11440-33636\,n-38820\,n^2-22335\,n^3-6480\,n^4-729\,n^5\right) a[n] + \frac{1}{729}\left(40+726\,n+3405\,n^2+3705\,n^3+2835\,n^4+729\,n^5\right) a[1+n]$$

In[327]:=
```
RSolve[% == 0, a[n], n]
```

Out[327]=
$$\left\{\left\{a[n] \to \frac{1}{40}\left(40+726\,n+3405\,n^2+3705\,n^3+2835\,n^4+729\,n^5\right) C[1]\right\}\right\}$$

In[329]:=
```
(* Compute all 18 polynomials by interpolation. *)
pols = Table[Factor[
    40 * InterpolatingPolynomial[Take[re2l, {s, 800, 18}], k] /. k → k + 1], {s, 18}];
pols = Prepend[Most[pols], Factor[Last[pols] /. k → k - 1]]
```

Out[330]=
$$\{3\left(40-318\,k+395\,k^2+35\,k^3+405\,k^4+243\,k^5\right),\ k\left(106+225\,k-615\,k^2-405\,k^3+729\,k^4\right),$$
$$40-684\,k+1320\,k^2+735\,k^3+1620\,k^4+729\,k^5,\ k\left(136-705\,k^2+729\,k^4\right),$$
$$3\,k\left(-118+565\,k+515\,k^2+675\,k^3+243\,k^4\right),\ k\left(106-225\,k-615\,k^2+405\,k^3+729\,k^4\right),$$
$$3\,k\left(32+790\,k+845\,k^2+810\,k^3+243\,k^4\right),\ 3\,k\,(1+k)\left(12-142\,k+27\,k^2+243\,k^3\right),$$
$$40+726\,k+3405\,k^2+3705\,k^3+2835\,k^4+729\,k^5,\ 3\,k\,(1+k)\left(-18-127\,k+162\,k^2+243\,k^3\right),$$
$$160+1636\,k+4860\,k^2+5055\,k^3+3240\,k^4+729\,k^5,\ 3\,k\,(1+k)\left(-48-52\,k+297\,k^2+243\,k^3\right),$$
$$400+2926\,k+6795\,k^2+6585\,k^3+3645\,k^4+729\,k^5,\ 3\,k\,(1+k)\left(-58+83\,k+432\,k^2+243\,k^3\right),$$
$$800+4696\,k+9270\,k^2+8295\,k^3+4050\,k^4+729\,k^5,\ 3\,k\,(1+k)\left(-28+278\,k+567\,k^2+243\,k^3\right),$$
$$3\,(3+k)\left(160+734\,k+1127\,k^2+756\,k^3+243\,k^4\right),\ 3\,k\,(1+k)\left(62+533\,k+702\,k^2+243\,k^3\right)\}$$

In[332]:=
```
(* Sanity check. *)
Table[{kk, r} = QuotientRemainder[n, 18];
  pols[[r + 1]] / 40 /. k → kk, {n, 3, 70}] === Drop[data, 2]
```

Out[332]= True

In[335]:=
```
(* Define the Kaprekar map. *)
Kaprekar[n_Integer] :=
  #1 - #2 & @@ (FromDigits /@ ({Reverse[#], #} &[Sort[IntegerDigits[n]]]));
NestList[Kaprekar, 86 526 432, 3]
```

Out[336]= {86 526 432, 64 308 654, 83 208 762, 86 526 432}

In[337]:=
```
(* There are (at least) two different
  patterns that constitute Kaprekar 3-cycles *)
Clear[myZ1, myZ2]
myZ1[k_ /; k ≥ 0, x1_ /; x1 ≥ 0,
   x3_ /; x3 ≥ 0, x5_ /; x5 ≥ 1, x7_ /; x7 ≥ 1, x9_ /; x9 ≥ 1] :=
  FromDigits[Flatten[{
     MapThread[Table[#1, {#2}] &,
      {Reverse[Range[9]], Riffle[{x9, x7, x5, x3, x1}, k]}], 0,
     MapThread[Table[#1, {#2}] &, {Reverse[Range[0, 10]],
       Riffle[{0, x1 + 1, x3, x5, x7, x9 - 1}, k]}], 1
    }]];
myZ2[x1_ /; x1 ≥ 0, x3_ /; x3 ≥ 1, x5_ /; x5 ≥ 0] :=
  FromDigits[Flatten[{
     6, Table[5, {x5}], 4, Table[3, {x3}], Table[1, {x1}],
     0, 8, Table[8, {x1}], Table[6, {x3}], 5, Table[4, {x5}], 4}]];
myZ1[3, 1, 0, 1, 1, 1]
```

Out[340]= 9 888 766 654 442 221 099 988 777 555 433 321 111

In[341]:=
```
(* Create all 3-
 cycles according to our conjecture (that there are no other patterns). *)
Clear[My3Cycles];
My3Cycles[n_] :=
  Module[{s},
   res = Flatten[Table[
      s = (n - 9 k - 2) / 2;
      Table[
       If[x1 + 1 == x5 == x7, {}, myZ1[k, x1, x3, x5, x7, s - x1 - x3 - x5 - x7]]
       , {x1, 0, s - 3}
       , {x3, 0, s - 3 - x1}
       , {x5, 1, s - 2 - x1 - x3}
       , {x7, 1, s - 1 - x1 - x3 - x5}]
      , {k, If[OddQ[n], 1, 0], Floor[(n - 8) / 9], 2}]];
   If[EvenQ[n],
    res = Join[res,
       Flatten[
        Table[myZ2[x1, x3, n / 2 - 3 - x1 - x3], {x1, 0, n / 2 - 3}, {x3, 1, n / 2 - 3 - x1}]]];
   ];
   Return[Union[Sort[NestList[Kaprekar, #, 2]] & /@ res]];
  ];
My3Cycles[12]
```

Out[343]=
```
{{643 110 888 654, 865 552 644 432, 877 320 876 222},
 {643 310 886 654, 865 532 664 432, 873 320 876 622},
 {643 330 866 654, 833 320 876 662, 865 332 666 432},
 {654 310 886 544, 873 210 887 622, 876 552 644 322},
 {654 330 866 544, 833 210 887 662, 876 532 664 322},
 {655 430 865 444, 832 110 888 762, 877 652 643 222},
 {975 110 888 421, 975 550 844 421, 977 750 842 221},
 {975 310 886 421, 975 530 864 421, 977 530 864 221},
 {975 510 884 421, 977 510 884 221, 977 550 844 221},
 {997 510 884 201, 997 550 844 201, 997 750 842 201}}
```

In[344]:=
```
Table[Length[My3Cycles[n]], {n, 70}] === data
```

Out[344]=
```
True
```

In[345]:=
```
(* Now count these 3-cycles. *)
Table[
  s1 = Sum[1, {k, If[OddQ[n], 1, 0], Floor[(n - 8) / 9], 2}
    , {x1, 0, (n - 9 k - 8) / 2}
    , {x3, 0, (n - 9 k - 8) / 2 - x1}
    , {x5, 1, (n - 9 k - 6) / 2 - x1 - x3}
    , {x7, 1, (n - 9 k - 4) / 2 - x1 - x3 - x5}];
  s2 = Sum[1, {k, If[OddQ[n], 1, 0], Floor[(n - 8) / 9], 2}
    , {x1, 0, (n - 9 k - 8) / 2}
    , {x3, 0, (n - 9 k - 8) / 2 - 3 x1}];
  s3 = Sum[1, {x1, 0, n / 2 - 3}, {x3, 1, n / 2 - 3 - x1}];
  1 / 3 * (s1 - s2) + (1 - Mod[n, 2]) * s3, {n, 70}] === data
```

Out[345]= True

In[367]:=
```
(* Derive closed-form expressions for even n (mod 18). *)
expr = 1 / 3 * (sum[1, {k, 0, Floor[(n - 8) / 18]}
        , {x1, 0, (n - 18 k - 8) / 2}
        , {x3, 0, (n - 18 k - 8) / 2 - x1}
        , {x5, 1, (n - 18 k - 6) / 2 - x1 - x3}
        , {x7, 1, (n - 18 k - 4) / 2 - x1 - x3 - x5}]
      - sum[1, {k, 0, Floor[(n - 8) / 18]}
        , {x1, 0, Floor[(n - 18 k - 8) / 6]}
        , {x3, 0, (n - 18 k - 8) / 2 - 3 x1}]) +
   sum[1, {x1, 0, n / 2 - 3}, {x3, 1, n / 2 - 3 - x1}];
expr = Table[expr /. n → 18 m + 2 l, {l, 0, 8}] /.
    e_Floor :→ FullSimplify[e, Element[m, Integers]] /. sum → Sum;
Factor[40 * expr /. m → k] // TableForm
```

Out[369]//TableForm=

$3\ (40 - 318\ k + 395\ k^2 + 35\ k^3 + 405\ k^4 + 243\ k^5)$

$40 - 684\ k + 1320\ k^2 + 735\ k^3 + 1620\ k^4 + 729\ k^5$

$3\ k\ (-118 + 565\ k + 515\ k^2 + 675\ k^3 + 243\ k^4)$

$3\ k\ (32 + 790\ k + 845\ k^2 + 810\ k^3 + 243\ k^4)$

$40 + 726\ k + 3405\ k^2 + 3705\ k^3 + 2835\ k^4 + 729\ k^5$

$160 + 1636\ k + 4860\ k^2 + 5055\ k^3 + 3240\ k^4 + 729\ k^5$

$400 + 2926\ k + 6795\ k^2 + 6585\ k^3 + 3645\ k^4 + 729\ k^5$

$800 + 4696\ k + 9270\ k^2 + 8295\ k^3 + 4050\ k^4 + 729\ k^5$

$3\ (3 + k)\ (160 + 734\ k + 1127\ k^2 + 756\ k^3 + 243\ k^4)$

In[370]:=
```
(* They agree with the interpolated polynomials from above. *)
% === Take[pols, {1, 18, 2}]
```

Out[370]= True

In[363]:=
```
(* Derive closed-form expressions for odd n (mod 18). *)
expr = 1/3 * (sum[1, {k, 0, Floor[(n - 17)/18]}
       , {x1, 0, (n - 18 k - 17)/2}
       , {x3, 0, (n - 18 k - 17)/2 - x1}
       , {x5, 1, (n - 18 k - 15)/2 - x1 - x3}
       , {x7, 1, (n - 18 k - 13)/2 - x1 - x3 - x5}] -
     sum[1, {k, 0, Floor[(n - 17)/18]}
       , {x1, 0, Floor[(n - 18 k - 17)/6]}
       , {x3, 0, (n - 18 k - 17)/2 - 3 x1}]);
expr = Table[expr /. n → 18 m + 2 l + 1, {l, 0, 8}] /.
    e_Floor :→ FullSimplify[e, Element[m, Integers]] /. sum → Sum;
Factor[40 * expr /. m → k] // TableForm
```

Out[365]//TableForm=

$k\,(106 + 225\,k - 615\,k^2 - 405\,k^3 + 729\,k^4)$

$k\,(136 - 705\,k^2 + 729\,k^4)$

$k\,(106 - 225\,k - 615\,k^2 + 405\,k^3 + 729\,k^4)$

$3\,k\,(1 + k)\,(12 - 142\,k + 27\,k^2 + 243\,k^3)$

$3\,k\,(1 + k)\,(-18 - 127\,k + 162\,k^2 + 243\,k^3)$

$3\,k\,(1 + k)\,(-48 - 52\,k + 297\,k^2 + 243\,k^3)$

$3\,k\,(1 + k)\,(-58 + 83\,k + 432\,k^2 + 243\,k^3)$

$3\,k\,(1 + k)\,(-28 + 278\,k + 567\,k^2 + 243\,k^3)$

$3\,k\,(1 + k)\,(62 + 533\,k + 702\,k^2 + 243\,k^3)$

```
(* They agree with the interpolated polynomials from above. *)
% === Take[pols, {2, 18, 2}]
```

Out[366]= True