In[1]:=
```
SetDirectory[NotebookDirectory[]];
Get["DiscreteToPDE.m "];
```

Bidirectional Pedestrian Flow

```
(* This is the running example  from  our paper. *)
pde = DiscreteToPDE[{
        T[r, {1, 0}] == (1-ρ[1, 0]) * (1+α*r[2, 0]),
        T[r, {0, -1}] == (1-ρ[0, -1]) * (γ0+γ1*b[1, 0]),
        T[r, {0, 1}] == (1-ρ[0, 1]) * (γ0+γ2*b[1, 0]),
        T[b, {-1, 0}] == (1-ρ[-1, 0]) * (1+α*b[-2, 0]),
        T[b, {0, 1}] == (1-ρ[0, 1]) * (γ0+γ1*r[-1, 0]),
        T[b, {0, -1}] == (1-ρ[0, -1]) * (γ0+γ2*r[-1, 0])
      } /. ρ[i_, j_] → r[i, j] + b[i, j], {x, y}]
```

Out[3]=

$$
\begin{aligned}
\{&\partial_x[b]-2b\partial_x[b]-r\partial_x[b]+2b\alpha\partial_x[b]-3b^2\alpha\partial_x[b]-2br\alpha\partial_x[b]-h\alpha\partial_x[b]^2+3bh\alpha\partial_x[b]^2+\\
&hr\alpha\partial_x[b]^2-b\partial_x[r]-b^2\alpha\partial_x[r]+2bh\alpha\partial_x[b]\,\partial_x[r]-r\gamma_1\partial_y[b]+2br\gamma_1\partial_y[b]+\\
&r^2\gamma_1\partial_y[b]+r\gamma_2\partial_y[b]-2br\gamma_2\partial_y[b]-r^2\gamma_2\partial_y[b]+h\gamma_1\partial_x[r]\,\partial_y[b]-2bh\gamma_1\partial_x[r]\,\partial_y[b]-\\
&hr\gamma_1\partial_x[r]\,\partial_y[b]-h\gamma_2\partial_x[r]\,\partial_y[b]+2bh\gamma_2\partial_x[r]\,\partial_y[b]+hr\gamma_2\partial_x[r]\,\partial_y[b]-b\gamma_1\partial_y[r]+\\
&b^2\gamma_1\partial_y[r]+2br\gamma_1\partial_y[r]+b\gamma_2\partial_y[r]-b^2\gamma_2\partial_y[r]-2br\gamma_2\partial_y[r]-bh\gamma_1\partial_x[r]\,\partial_y[r]+\\
&bh\gamma_2\partial_x[r]\,\partial_y[r]+h\gamma_1\partial_y[b]\,\partial_y[r]-bh\gamma_1\partial_y[b]\,\partial_y[r]-hr\gamma_1\partial_y[b]\,\partial_y[r]+\\
&h\gamma_2\partial_y[b]\,\partial_y[r]-bh\gamma_2\partial_y[b]\,\partial_y[r]-hr\gamma_2\partial_y[b]\,\partial_y[r]+\frac{1}{2}h\partial_{xx}[b]-\frac{1}{2}hr\partial_{xx}[b]-\\
&bh\alpha\partial_{xx}[b]+\frac{3}{2}b^2h\alpha\partial_{xx}[b]+bhr\alpha\partial_{xx}[b]+\frac{1}{2}bh\partial_{xx}[r]+\frac{1}{2}b^2h\alpha\partial_{xx}[r]+bh\gamma_1\partial_{xy}[r]-\\
&b^2h\gamma_1\partial_{xy}[r]-bhr\gamma_1\partial_{xy}[r]-bh\gamma_2\partial_{xy}[r]+b^2h\gamma_2\partial_{xy}[r]+bhr\gamma_2\partial_{xy}[r]+h\gamma_0\partial_{yy}[b]-\\
&hr\gamma_0\partial_{yy}[b]+\frac{1}{2}hr\gamma_1\partial_{yy}[b]-\frac{1}{2}hr^2\gamma_1\partial_{yy}[b]+\frac{1}{2}hr\gamma_2\partial_{yy}[b]-\frac{1}{2}hr^2\gamma_2\partial_{yy}[b]+\\
&bh\gamma_0\partial_{yy}[r]+\frac{1}{2}bh\gamma_1\partial_{yy}[r]-\frac{1}{2}b^2h\gamma_1\partial_{yy}[r]+\frac{1}{2}bh\gamma_2\partial_{yy}[r]-\frac{1}{2}b^2h\gamma_2\partial_{yy}[r],\\
&r\partial_x[b]+r^2\alpha\partial_x[b]-\partial_x[r]+b\partial_x[r]+2r\partial_x[r]-2r\alpha\partial_x[r]+2br\alpha\partial_x[r]+3r^2\alpha\partial_x[r]+\\
&2hr\alpha\partial_x[b]\,\partial_x[r]-h\alpha\partial_x[r]^2+bh\alpha\partial_x[r]^2+3hr\alpha\partial_x[r]^2+r\gamma_1\partial_y[b]-2br\gamma_1\partial_y[b]-\\
&r^2\gamma_1\partial_y[b]-r\gamma_2\partial_y[b]+2br\gamma_2\partial_y[b]+r^2\gamma_2\partial_y[b]-hr\gamma_1\partial_x[b]\,\partial_y[b]+hr\gamma_2\partial_x[b]\,\partial_y[b]+\\
&b\gamma_1\partial_y[r]-b^2\gamma_1\partial_y[r]-2br\gamma_1\partial_y[r]-b\gamma_2\partial_y[r]+b^2\gamma_2\partial_y[r]+2br\gamma_2\partial_y[r]+\\
&h\gamma_1\partial_x[b]\,\partial_y[r]-bh\gamma_1\partial_x[b]\,\partial_y[r]-2hr\gamma_1\partial_x[b]\,\partial_y[r]-h\gamma_2\partial_x[b]\,\partial_y[r]+\\
&bh\gamma_2\partial_x[b]\,\partial_y[r]+2hr\gamma_2\partial_x[b]\,\partial_y[r]+h\gamma_1\partial_y[b]\,\partial_y[r]-bh\gamma_1\partial_y[b]\,\partial_y[r]-\\
&hr\gamma_1\partial_y[b]\,\partial_y[r]+h\gamma_2\partial_y[b]\,\partial_y[r]-bh\gamma_2\partial_y[b]\,\partial_y[r]-hr\gamma_2\partial_y[b]\,\partial_y[r]+\frac{1}{2}hr\partial_{xx}[b]+\\
&\frac{1}{2}hr^2\alpha\partial_{xx}[b]+\frac{1}{2}h\partial_{xx}[r]-\frac{1}{2}bh\partial_{xx}[r]-hr\alpha\partial_{xx}[r]+bhr\alpha\partial_{xx}[r]+\frac{3}{2}hr^2\alpha\partial_{xx}[r]+\\
&hr\gamma_1\partial_{xy}[b]-bhr\gamma_1\partial_{xy}[b]-hr^2\gamma_1\partial_{xy}[b]-hr\gamma_2\partial_{xy}[b]+bhr\gamma_2\partial_{xy}[b]+hr^2\gamma_2\partial_{xy}[b]+\\
&hr\gamma_0\partial_{yy}[b]+\frac{1}{2}hr\gamma_1\partial_{yy}[b]-\frac{1}{2}hr^2\gamma_1\partial_{yy}[b]+\frac{1}{2}hr\gamma_2\partial_{yy}[b]-\frac{1}{2}hr^2\gamma_2\partial_{yy}[b]+\\
&h\gamma_0\partial_{yy}[r]-bh\gamma_0\partial_{yy}[r]+\frac{1}{2}bh\gamma_1\partial_{yy}[r]-\frac{1}{2}b^2h\gamma_1\partial_{yy}[r]+\frac{1}{2}bh\gamma_2\partial_{yy}[r]-\frac{1}{2}b^2h\gamma_2\partial_{yy}[r]\}
\end{aligned}
$$

In[4]:=
```
(* This is just some  syntactic sugar. *)
Format [γ12] := γ1+γ2;
Format [γ21] := γ1-γ2;
pde1 = pde /. {γ1 → (γ12+γ21) / 2, γ2 → (γ12-γ21) / 2};
```

```
(* The mean -field PDEs in conservative form  *)
Partial[t][b] == Collect[pde1[[1]], {h, γ12, γ21}, PartialIntegrate[#, {r, b}, {y, x}] &]
```

Out[7]=
$$\partial_t[b] == -\partial_x[b(-1+b+r)(1+b\alpha)] + (\gamma_1-\gamma_2)\,\partial_y[b\,r(-1+b+r)] + h\left(\frac{1}{2}\partial_x\left[2\,b\,\partial_x[r] + \partial_x\left[b\left(1-r-b\alpha+b^2\alpha+b\,r\,\alpha\right)\right]\right] - (\gamma_1-\gamma_2)\,\partial_y[b(-1+b+r)\,\partial_x[r]] - \gamma_0\,\partial_y[\partial_y[b(-1+r)] - 2\,b\,\partial_y[r]] - \frac{1}{2}(\gamma_1+\gamma_2)\,\partial_y[b(b-2\,r)\,\partial_y[r] + \partial_y[b(-1+r)\,r]]\right)$$

In[8]:=
```
Partial[t][r] == Collect[pde1[[2]], {h, γ12, γ21}, PartialIntegrate[#, {b, r}, {y, x}] &]
```

Out[8]=
$$\partial_t[r] == \partial_x[r(-1+b+r)(1+r\alpha)] - (\gamma_1-\gamma_2)\,\partial_y[b\,r(-1+b+r)] + h\left(\frac{1}{2}\partial_x\left[2\,r\,\partial_x[b] + \partial_x\left[r\left(1-b-r\alpha+b\,r\,\alpha+r^2\alpha\right)\right]\right] - (\gamma_1-\gamma_2)\,\partial_y[r(-1+b+r)\,\partial_x[b]] + \gamma_0\,\partial_y[2\,r\,\partial_y[b] - \partial_y[(-1+b)\,r]] + \frac{1}{2}(\gamma_1+\gamma_2)\,\partial_y[(2\,b-r)\,r\,\partial_y[b] - \partial_y[(-1+b)\,b\,r]]\right)$$

Fernando, Landman, Simpson: “Nonlinear diffusion and exclusion processes with contact interactions” (Physical Review E 81, 011903, 2010)

Simplest case

In[9]:=
```
(* This are the transitionprobabilitiesfrom  the simple case (4). *)
(* The result shows that D(C)=1, as claimed in the paper. *)
DiscreteToPDE[{
    T[c, {0, 1}] == p/4*(1-c[0, 1]),
    T[c, {0, -1}] == p/4*(1-c[0, -1]),
    T[c, {1, 0}] == p/4*(1-c[1, 0]),
    T[c, {-1, 0}] == p/4*(1-c[-1, 0])}, {x, y}]
```

Out[9]=
$$\left\{\frac{1}{4}h\,p\,\partial_{xx}[c] + \frac{1}{4}h\,p\,\partial_{yy}[c]\right\}$$

Contact-maintaining interactions

In[10]:=
```
(* Transitionprobabilitiesas given in (6) and (5). *)
(* We use Moore interactingneighbors (defined in neigh). *)
neigh= DeleteCases[Flatten[Table[{i, j}, {i, -1, 1}, {j, -1, 1}], 1], {0, 0}];
maint [v_List] := Intersection[neigh, v+# & /@neigh];
disc= Function[v, T[c, v] ==
      p/4 * (1-c@@v) *With[{P = 1-Times @@ (1-c@@# & /@maint [v])}, u*P+w*(1-P)]] /@
    {{0, 1}, {0, -1}, {1, 0}, {-1, 0}}
```

Out[12]=

$$\Big\{T[c,\{0,1\}] == \frac{1}{4}p\,(1-c[0,1])\,\big(u\,(1-(1-c[-1,0])\,(1-c[-1,1])\,(1-c[1,0])\,(1-c[1,1]))+ w\,(1-c[-1,0])\,(1-c[-1,1])\,(1-c[1,0])\,(1-c[1,1])\big),\; T[c,\{0,-1\}] == \frac{1}{4}p\,(1-c[0,-1])\,\big(u\,(1-(1-c[-1,-1])\,(1-c[-1,0])\,(1-c[1,-1])\,(1-c[1,0]))+ w\,(1-c[-1,-1])\,(1-c[-1,0])\,(1-c[1,-1])\,(1-c[1,0])\big),\; T[c,\{1,0\}] == \frac{1}{4}p\,(1-c[1,0])\,\big(u\,(1-(1-c[0,-1])\,(1-c[0,1])\,(1-c[1,-1])\,(1-c[1,1]))+ w\,(1-c[0,-1])\,(1-c[0,1])\,(1-c[1,-1])\,(1-c[1,1])\big),\; T[c,\{-1,0\}] == \frac{1}{4}p\,(1-c[-1,0])\,\big(u\,(1-(1-c[-1,-1])\,(1-c[-1,1])\,(1-c[0,-1])\,(1-c[0,1]))+ w\,(1-c[-1,-1])\,(1-c[-1,1])\,(1-c[0,-1])\,(1-c[0,1])\big)\Big\}$$

In[13]:=
```
{pde} = DiscreteToPDE[disc, {x, y}]
```

Out[13]=

$$\Big\{hpu\,\partial_x[c]^2-3\,chpu\,\partial_x[c]^2+3\,c^2 hpu\,\partial_x[c]^2-c^3 hpu\,\partial_x[c]^2-hpw\,\partial_x[c]^2+3\,chpw\,\partial_x[c]^2-3\,c^2 hpw\,\partial_x[c]^2+c^3 hpw\,\partial_x[c]^2+hpu\,\partial_y[c]^2-3\,chpu\,\partial_y[c]^2+3\,c^2 hpu\,\partial_y[c]^2-c^3 hpu\,\partial_y[c]^2-hpw\,\partial_y[c]^2+3\,chpw\,\partial_y[c]^2-3\,c^2 hpw\,\partial_y[c]^2+c^3 hpw\,\partial_y[c]^2+chpu\,\partial_{xx}[c]-\frac{3}{2}c^2 hpu\,\partial_{xx}[c]+c^3 hpu\,\partial_{xx}[c]-\frac{1}{4}c^4 hpu\,\partial_{xx}[c]+\frac{1}{4}hpw\,\partial_{xx}[c]-chpw\,\partial_{xx}[c]+\frac{3}{2}c^2 hpw\,\partial_{xx}[c]-c^3 hpw\,\partial_{xx}[c]+\frac{1}{4}c^4 hpw\,\partial_{xx}[c]+chpu\,\partial_{yy}[c]-\frac{3}{2}c^2 hpu\,\partial_{yy}[c]+c^3 hpu\,\partial_{yy}[c]-\frac{1}{4}c^4 hpu\,\partial_{yy}[c]+\frac{1}{4}hpw\,\partial_{yy}[c]-chpw\,\partial_{yy}[c]+\frac{3}{2}c^2 hpw\,\partial_{yy}[c]-c^3 hpw\,\partial_{yy}[c]+\frac{1}{4}c^4 hpw\,\partial_{yy}[c]\Big\}$$

In[14]:=
```
PartialIntegrate[pde, {c}, {x, y}, Depth→1]
```

Out[14]=

$$\frac{1}{4}hp\,\partial_x\big[\left(4\,cu-6\,c^2 u+4\,c^3 u-c^4 u+w-4\,cw+6\,c^2 w-4\,c^3 w+c^4 w\right)\partial_x[c]\big]+\frac{1}{4}hp\,\partial_y\big[\left(4\,cu-6\,c^2 u+4\,c^3 u-c^4 u+w-4\,cw+6\,c^2 w-4\,c^3 w+c^4 w\right)\partial_y[c]\big]$$

```
(* This agrees with the result given in (7). *)
% /. a_Plus /; Length[a] > 2 :→ u+Factor[a-u]
```

Out[14]=

$$\frac{1}{4}hp\,\partial_x\big[\left(u-(-1+c)^4\,(u-w)\right)\partial_x[c]\big]+\frac{1}{4}hp\,\partial_y\big[\left(u-(-1+c)^4\,(u-w)\right)\partial_y[c]\big]$$

In[30]:=
```
(* This is a special case of the general formula implemented in DiscreteModel: *)
Simplify[(Last/@DiscreteModel[{{0, 1}, {0, -1}, {1, 0}, {-1, 0}}, neigh] /.
    {s→1, p→1, q→1, r→1}) - (Last/@disc/. p→1)]
```

Out[30]=
```
{0, 0, 0, 0}
```

Combined contact-forming or contact-breaking and contact-maintaining interactions

In[15]:=
```
(* This are the transitionprobabilitiesas given in (13), *)
(* for 2D square lattice and Moore interactingneighbors. *)
disc= DiscreteModel[{{0, 1}, {0, -1}, {1, 0}, {-1, 0}}, neigh]
```

Out[15]=
```
{T[c, {0, 1}] == 1/4 (1-c[0, 1]) (u (1- (1-c[-1, 0]) (1-c[-1, 1]) (1-c[1, 0]) (1-c[1, 1])) +
        w (1-c[-1, 0]) (1-c[-1, 1]) (1-c[1, 0]) (1-c[1, 1]))
    (r (1- (1-c[-1, -1]) (1-c[0, -1]) (1-c[1, -1]))
          (1- (1-c[-1, 2]) (1-c[0, 2]) (1-c[1, 2])) +p (1-c[-1, -1]) (1-c[0, -1])
          (1-c[1, -1]) (1- (1-c[-1, 2]) (1-c[0, 2]) (1-c[1, 2])) +q (1-c[-1, 2])
          (1-c[0, 2]) (1- (1-c[-1, -1]) (1-c[0, -1]) (1-c[1, -1])) (1-c[1, 2]) +
        s (1-c[-1, -1]) (1-c[-1, 2]) (1-c[0, -1]) (1-c[0, 2]) (1-c[1, -1]) (1-c[1, 2])),
  T[c, {0, -1}] == 1/4 (1-c[0, -1])
    (u (1- (1-c[-1, -1]) (1-c[-1, 0]) (1-c[1, -1]) (1-c[1, 0])) +
        w (1-c[-1, -1]) (1-c[-1, 0]) (1-c[1, -1]) (1-c[1, 0]))
    (r (1- (1-c[-1, -2]) (1-c[0, -2]) (1-c[1, -2]))
          (1- (1-c[-1, 1]) (1-c[0, 1]) (1-c[1, 1])) +q (1-c[-1, -2]) (1-c[0, -2])
          (1-c[1, -2]) (1- (1-c[-1, 1]) (1-c[0, 1]) (1-c[1, 1])) +p (1-c[-1, 1])
          (1-c[0, 1]) (1- (1-c[-1, -2]) (1-c[0, -2]) (1-c[1, -2])) (1-c[1, 1]) +
        s (1-c[-1, -2]) (1-c[-1, 1]) (1-c[0, -2]) (1-c[0, 1]) (1-c[1, -2]) (1-c[1, 1])),
  T[c, {1, 0}] == 1/4 (1-c[1, 0]) (u (1- (1-c[0, -1]) (1-c[0, 1]) (1-c[1, -1]) (1-c[1, 1])) +
        w (1-c[0, -1]) (1-c[0, 1]) (1-c[1, -1]) (1-c[1, 1]))
    (r (1- (1-c[-1, -1]) (1-c[-1, 0]) (1-c[-1, 1]))
          (1- (1-c[2, -1]) (1-c[2, 0]) (1-c[2, 1])) +p (1-c[-1, -1])
          (1-c[-1, 0]) (1-c[-1, 1]) (1- (1-c[2, -1]) (1-c[2, 0]) (1-c[2, 1])) +
        q (1- (1-c[-1, -1]) (1-c[-1, 0]) (1-c[-1, 1])) (1-c[2, -1])
          (1-c[2, 0]) (1-c[2, 1]) +s (1-c[-1, -1]) (1-c[-1, 0]) (1-c[-1, 1])
          (1-c[2, -1]) (1-c[2, 0]) (1-c[2, 1])), T[c, {-1, 0}] ==
   1/4 (1-c[-1, 0]) (u (1- (1-c[-1, -1]) (1-c[-1, 1]) (1-c[0, -1]) (1-c[0, 1])) +
        w (1-c[-1, -1]) (1-c[-1, 1]) (1-c[0, -1]) (1-c[0, 1]))
    (r (1- (1-c[-2, -1]) (1-c[-2, 0]) (1-c[-2, 1]))
          (1- (1-c[1, -1]) (1-c[1, 0]) (1-c[1, 1])) +q (1-c[-2, -1])
          (1-c[-2, 0]) (1-c[-2, 1]) (1- (1-c[1, -1]) (1-c[1, 0]) (1-c[1, 1])) +
        p (1- (1-c[-2, -1]) (1-c[-2, 0]) (1-c[-2, 1]))
          (1-c[1, -1]) (1-c[1, 0]) (1-c[1, 1]) +
        s (1-c[-2, -1]) (1-c[-2, 0]) (1-c[-2, 1]) (1-c[1, -1]) (1-c[1, 0]) (1-c[1, 1]))}
```

In[16]:=
```
Timing[{pde} = DiscreteToPDE[disc, {x, y}]]
```

Out[16]=
```
{1.541000,
  {-12 c h p u ∂x[c]^2 + 72 c^2 h p u ∂x[c]^2 - 146 c^3 h p u ∂x[c]^2 + 225/2 c^4 h p u ∂x[c]^2 + 39/2 c^5 h p u ∂x[c]^2 -
```

$$98\,c^6\,h\,p\,u\,\partial_x[c]^2+72\,c^7\,h\,p\,u\,\partial_x[c]^2-\frac{45}{2}c^8\,h\,p\,u\,\partial_x[c]^2+\frac{5}{2}c^9\,h\,p\,u\,\partial_x[c]^2+24\,c\,h\,q\,u\,\partial_x[c]^2-$$

$$171\,c^2\,h\,q\,u\,\partial_x[c]^2+466\,c^3\,h\,q\,u\,\partial_x[c]^2-675\,c^4\,h\,q\,u\,\partial_x[c]^2+\frac{1173}{2}c^5\,h\,q\,u\,\partial_x[c]^2-$$

$$\frac{637}{2}c^6\,h\,q\,u\,\partial_x[c]^2+108\,c^7\,h\,q\,u\,\partial_x[c]^2-\frac{45}{2}c^8\,h\,q\,u\,\partial_x[c]^2+\frac{5}{2}c^9\,h\,q\,u\,\partial_x[c]^2+27\,c^2\,h\,r\,u\,\partial_x[c]^2-$$

$$126\,c^3\,h\,r\,u\,\partial_x[c]^2+255\,c^4\,h\,r\,u\,\partial_x[c]^2-\frac{585}{2}c^5\,h\,r\,u\,\partial_x[c]^2+\frac{413}{2}c^6\,h\,r\,u\,\partial_x[c]^2-$$

$$90\,c^7\,h\,r\,u\,\partial_x[c]^2+\frac{45}{2}c^8\,h\,r\,u\,\partial_x[c]^2-\frac{5}{2}c^9\,h\,r\,u\,\partial_x[c]^2+h\,s\,u\,\partial_x[c]^2-15\,c\,h\,s\,u\,\partial_x[c]^2+$$

$$75\,c^2\,h\,s\,u\,\partial_x[c]^2-195\,c^3\,h\,s\,u\,\partial_x[c]^2+\frac{615}{2}c^4\,h\,s\,u\,\partial_x[c]^2-\frac{627}{2}c^5\,h\,s\,u\,\partial_x[c]^2+$$

$$210\,c^6\,h\,s\,u\,\partial_x[c]^2-90\,c^7\,h\,s\,u\,\partial_x[c]^2+\frac{45}{2}c^8\,h\,s\,u\,\partial_x[c]^2-\frac{5}{2}c^9\,h\,s\,u\,\partial_x[c]^2-\frac{3}{2}h\,p\,w\,\partial_x[c]^2+$$

$$\frac{39}{2}c\,h\,p\,w\,\partial_x[c]^2-78\,c^2\,h\,p\,w\,\partial_x[c]^2+140\,c^3\,h\,p\,w\,\partial_x[c]^2-105\,c^4\,h\,p\,w\,\partial_x[c]^2-21\,c^5\,h\,p\,w\,\partial_x[c]^2+$$

$$98\,c^6\,h\,p\,w\,\partial_x[c]^2-72\,c^7\,h\,p\,w\,\partial_x[c]^2+\frac{45}{2}c^8\,h\,p\,w\,\partial_x[c]^2-\frac{5}{2}c^9\,h\,p\,w\,\partial_x[c]^2+3\,h\,q\,w\,\partial_x[c]^2-$$

$$\frac{87}{2}c\,h\,q\,w\,\partial_x[c]^2+\frac{411}{2}c^2\,h\,q\,w\,\partial_x[c]^2-490\,c^3\,h\,q\,w\,\partial_x[c]^2+\frac{1365}{2}c^4\,h\,q\,w\,\partial_x[c]^2-$$

$$588\,c^5\,h\,q\,w\,\partial_x[c]^2+\frac{637}{2}c^6\,h\,q\,w\,\partial_x[c]^2-108\,c^7\,h\,q\,w\,\partial_x[c]^2+\frac{45}{2}c^8\,h\,q\,w\,\partial_x[c]^2-$$

$$\frac{5}{2}c^9\,h\,q\,w\,\partial_x[c]^2+\frac{9}{2}c\,h\,r\,w\,\partial_x[c]^2-\frac{81}{2}c^2\,h\,r\,w\,\partial_x[c]^2+141\,c^3\,h\,r\,w\,\partial_x[c]^2-\frac{525}{2}c^4\,h\,r\,w\,\partial_x[c]^2+$$

$$294\,c^5\,h\,r\,w\,\partial_x[c]^2-\frac{413}{2}c^6\,h\,r\,w\,\partial_x[c]^2+90\,c^7\,h\,r\,w\,\partial_x[c]^2-\frac{45}{2}c^8\,h\,r\,w\,\partial_x[c]^2+\frac{5}{2}c^9\,h\,r\,w\,\partial_x[c]^2-$$

$$\frac{5}{2}h\,s\,w\,\partial_x[c]^2+\frac{45}{2}c\,h\,s\,w\,\partial_x[c]^2-90\,c^2\,h\,s\,w\,\partial_x[c]^2+210\,c^3\,h\,s\,w\,\partial_x[c]^2-315\,c^4\,h\,s\,w\,\partial_x[c]^2+$$

$$315\,c^5\,h\,s\,w\,\partial_x[c]^2-210\,c^6\,h\,s\,w\,\partial_x[c]^2+90\,c^7\,h\,s\,w\,\partial_x[c]^2-\frac{45}{2}c^8\,h\,s\,w\,\partial_x[c]^2+\frac{5}{2}c^9\,h\,s\,w\,\partial_x[c]^2-$$

$$12\,c\,h\,p\,u\,\partial_y[c]^2+72\,c^2\,h\,p\,u\,\partial_y[c]^2-146\,c^3\,h\,p\,u\,\partial_y[c]^2+\frac{225}{2}c^4\,h\,p\,u\,\partial_y[c]^2+\frac{39}{2}c^5\,h\,p\,u\,\partial_y[c]^2-$$

$$98\,c^6\,h\,p\,u\,\partial_y[c]^2+72\,c^7\,h\,p\,u\,\partial_y[c]^2-\frac{45}{2}c^8\,h\,p\,u\,\partial_y[c]^2+\frac{5}{2}c^9\,h\,p\,u\,\partial_y[c]^2+24\,c\,h\,q\,u\,\partial_y[c]^2-$$

$$171\,c^2\,h\,q\,u\,\partial_y[c]^2+466\,c^3\,h\,q\,u\,\partial_y[c]^2-675\,c^4\,h\,q\,u\,\partial_y[c]^2+\frac{1173}{2}c^5\,h\,q\,u\,\partial_y[c]^2-$$

$$\frac{637}{2}c^6\,h\,q\,u\,\partial_y[c]^2+108\,c^7\,h\,q\,u\,\partial_y[c]^2-\frac{45}{2}c^8\,h\,q\,u\,\partial_y[c]^2+\frac{5}{2}c^9\,h\,q\,u\,\partial_y[c]^2+27\,c^2\,h\,r\,u\,\partial_y[c]^2-$$

$$126\,c^3\,h\,r\,u\,\partial_y[c]^2+255\,c^4\,h\,r\,u\,\partial_y[c]^2-\frac{585}{2}c^5\,h\,r\,u\,\partial_y[c]^2+\frac{413}{2}c^6\,h\,r\,u\,\partial_y[c]^2-$$

$$90\,c^7\,h\,r\,u\,\partial_y[c]^2+\frac{45}{2}c^8\,h\,r\,u\,\partial_y[c]^2-\frac{5}{2}c^9\,h\,r\,u\,\partial_y[c]^2+h\,s\,u\,\partial_y[c]^2-15\,c\,h\,s\,u\,\partial_y[c]^2+$$

$$75\,c^2\,h\,s\,u\,\partial_y[c]^2-195\,c^3\,h\,s\,u\,\partial_y[c]^2+\frac{615}{2}c^4\,h\,s\,u\,\partial_y[c]^2-\frac{627}{2}c^5\,h\,s\,u\,\partial_y[c]^2+$$

$$210\,c^6\,h\,s\,u\,\partial_y[c]^2-90\,c^7\,h\,s\,u\,\partial_y[c]^2+\frac{45}{2}c^8\,h\,s\,u\,\partial_y[c]^2-\frac{5}{2}c^9\,h\,s\,u\,\partial_y[c]^2-\frac{3}{2}h\,p\,w\,\partial_y[c]^2+$$

$$\frac{39}{2}c\,h\,p\,w\,\partial_y[c]^2-78\,c^2\,h\,p\,w\,\partial_y[c]^2+140\,c^3\,h\,p\,w\,\partial_y[c]^2-105\,c^4\,h\,p\,w\,\partial_y[c]^2-21\,c^5\,h\,p\,w\,\partial_y[c]^2+$$

$$98 c^6 hpw\partial_y[c]^2 - 72 c^7 hpw\partial_y[c]^2 + \frac{45}{2} c^8 hpw\partial_y[c]^2 - \frac{5}{2} c^9 hpw\partial_y[c]^2 + 3 hqw\partial_y[c]^2 -$$
$$\frac{87}{2} chqw\partial_y[c]^2 + \frac{411}{2} c^2 hqw\partial_y[c]^2 - 490 c^3 hqw\partial_y[c]^2 + \frac{1365}{2} c^4 hqw\partial_y[c]^2 -$$
$$588 c^5 hqw\partial_y[c]^2 + \frac{637}{2} c^6 hqw\partial_y[c]^2 - 108 c^7 hqw\partial_y[c]^2 + \frac{45}{2} c^8 hqw\partial_y[c]^2 -$$
$$\frac{5}{2} c^9 hqw\partial_y[c]^2 + \frac{9}{2} chrw\partial_y[c]^2 - \frac{81}{2} c^2 hrw\partial_y[c]^2 + 141 c^3 hrw\partial_y[c]^2 - \frac{525}{2} c^4 hrw\partial_y[c]^2 +$$
$$294 c^5 hrw\partial_y[c]^2 - \frac{413}{2} c^6 hrw\partial_y[c]^2 + 90 c^7 hrw\partial_y[c]^2 - \frac{45}{2} c^8 hrw\partial_y[c]^2 + \frac{5}{2} c^9 hrw\partial_y[c]^2 -$$
$$\frac{5}{2} hsw\partial_y[c]^2 + \frac{45}{2} chsw\partial_y[c]^2 - 90 c^2 hsw\partial_y[c]^2 + 210 c^3 hsw\partial_y[c]^2 - 315 c^4 hsw\partial_y[c]^2 +$$
$$315 c^5 hsw\partial_y[c]^2 - 210 c^6 hsw\partial_y[c]^2 + 90 c^7 hsw\partial_y[c]^2 - \frac{45}{2} c^8 hsw\partial_y[c]^2 + \frac{5}{2} c^9 hsw\partial_y[c]^2 -$$
$$6 c^2 hpu\partial_{xx}[c] + 24 c^3 hpu\partial_{xx}[c] - \frac{73}{2} c^4 hpu\partial_{xx}[c] + \frac{45}{2} c^5 hpu\partial_{xx}[c] + \frac{13}{4} c^6 hpu\partial_{xx}[c] -$$
$$14 c^7 hpu\partial_{xx}[c] + 9 c^8 hpu\partial_{xx}[c] - \frac{5}{2} c^9 hpu\partial_{xx}[c] + \frac{1}{4} c^{10} hpu\partial_{xx}[c] + 12 c^2 hqu\partial_{xx}[c] -$$
$$57 c^3 hqu\partial_{xx}[c] + \frac{233}{2} c^4 hqu\partial_{xx}[c] - 135 c^5 hqu\partial_{xx}[c] + \frac{391}{4} c^6 hqu\partial_{xx}[c] -$$
$$\frac{91}{2} c^7 hqu\partial_{xx}[c] + \frac{27}{2} c^8 hqu\partial_{xx}[c] - \frac{5}{2} c^9 hqu\partial_{xx}[c] + \frac{1}{4} c^{10} hqu\partial_{xx}[c] + 9 c^3 hru\partial_{xx}[c] -$$
$$\frac{63}{2} c^4 hru\partial_{xx}[c] + 51 c^5 hru\partial_{xx}[c] - \frac{195}{4} c^6 hru\partial_{xx}[c] + \frac{59}{2} c^7 hru\partial_{xx}[c] - \frac{45}{4} c^8 hru\partial_{xx}[c] +$$
$$\frac{5}{2} c^9 hru\partial_{xx}[c] - \frac{1}{4} c^{10} hru\partial_{xx}[c] + chsu\partial_{xx}[c] - \frac{15}{2} c^2 hsu\partial_{xx}[c] + 25 c^3 hsu\partial_{xx}[c] -$$
$$\frac{195}{4} c^4 hsu\partial_{xx}[c] + \frac{123}{2} c^5 hsu\partial_{xx}[c] - \frac{209}{4} c^6 hsu\partial_{xx}[c] + 30 c^7 hsu\partial_{xx}[c] -$$
$$\frac{45}{4} c^8 hsu\partial_{xx}[c] + \frac{5}{2} c^9 hsu\partial_{xx}[c] - \frac{1}{4} c^{10} hsu\partial_{xx}[c] - \frac{3}{2} chpw\partial_{xx}[c] + \frac{39}{4} c^2 hpw\partial_{xx}[c] -$$
$$26 c^3 hpw\partial_{xx}[c] + 35 c^4 hpw\partial_{xx}[c] - 21 c^5 hpw\partial_{xx}[c] - \frac{7}{2} c^6 hpw\partial_{xx}[c] + 14 c^7 hpw\partial_{xx}[c] -$$
$$9 c^8 hpw\partial_{xx}[c] + \frac{5}{2} c^9 hpw\partial_{xx}[c] - \frac{1}{4} c^{10} hpw\partial_{xx}[c] + 3 chqw\partial_{xx}[c] - \frac{87}{4} c^2 hqw\partial_{xx}[c] +$$
$$\frac{137}{2} c^3 hqw\partial_{xx}[c] - \frac{245}{2} c^4 hqw\partial_{xx}[c] + \frac{273}{2} c^5 hqw\partial_{xx}[c] - 98 c^6 hqw\partial_{xx}[c] +$$
$$\frac{91}{2} c^7 hqw\partial_{xx}[c] - \frac{27}{2} c^8 hqw\partial_{xx}[c] + \frac{5}{2} c^9 hqw\partial_{xx}[c] - \frac{1}{4} c^{10} hqw\partial_{xx}[c] + \frac{9}{4} c^2 hrw\partial_{xx}[c] -$$
$$\frac{27}{2} c^3 hrw\partial_{xx}[c] + \frac{141}{4} c^4 hrw\partial_{xx}[c] - \frac{105}{2} c^5 hrw\partial_{xx}[c] + 49 c^6 hrw\partial_{xx}[c] -$$
$$\frac{59}{2} c^7 hrw\partial_{xx}[c] + \frac{45}{4} c^8 hrw\partial_{xx}[c] - \frac{5}{2} c^9 hrw\partial_{xx}[c] + \frac{1}{4} c^{10} hrw\partial_{xx}[c] + \frac{1}{4} hsw\partial_{xx}[c] -$$
$$\frac{5}{2} chsw\partial_{xx}[c] + \frac{45}{4} c^2 hsw\partial_{xx}[c] - 30 c^3 hsw\partial_{xx}[c] + \frac{105}{2} c^4 hsw\partial_{xx}[c] - 63 c^5 hsw\partial_{xx}[c] +$$
$$\frac{105}{2} c^6 hsw\partial_{xx}[c] - 30 c^7 hsw\partial_{xx}[c] + \frac{45}{4} c^8 hsw\partial_{xx}[c] - \frac{5}{2} c^9 hsw\partial_{xx}[c] + \frac{1}{4} c^{10} hsw\partial_{xx}[c] -$$
$$6 c^2 hpu\partial_{yy}[c] + 24 c^3 hpu\partial_{yy}[c] - \frac{73}{2} c^4 hpu\partial_{yy}[c] + \frac{45}{2} c^5 hpu\partial_{yy}[c] + \frac{13}{4} c^6 hpu\partial_{yy}[c] -$$

$$14 c^7 h p u \partial_{yy}[c] + 9 c^8 h p u \partial_{yy}[c] - \frac{5}{2} c^9 h p u \partial_{yy}[c] + \frac{1}{4} c^{10} h p u \partial_{yy}[c] + 12 c^2 h q u \partial_{yy}[c] -$$
$$57 c^3 h q u \partial_{yy}[c] + \frac{233}{2} c^4 h q u \partial_{yy}[c] - 135 c^5 h q u \partial_{yy}[c] + \frac{391}{4} c^6 h q u \partial_{yy}[c] -$$
$$\frac{91}{2} c^7 h q u \partial_{yy}[c] + \frac{27}{2} c^8 h q u \partial_{yy}[c] - \frac{5}{2} c^9 h q u \partial_{yy}[c] + \frac{1}{4} c^{10} h q u \partial_{yy}[c] + 9 c^3 h r u \partial_{yy}[c] -$$
$$\frac{63}{2} c^4 h r u \partial_{yy}[c] + 51 c^5 h r u \partial_{yy}[c] - \frac{195}{4} c^6 h r u \partial_{yy}[c] + \frac{59}{2} c^7 h r u \partial_{yy}[c] - \frac{45}{4} c^8 h r u \partial_{yy}[c] +$$
$$\frac{5}{2} c^9 h r u \partial_{yy}[c] - \frac{1}{4} c^{10} h r u \partial_{yy}[c] + c h s u \partial_{yy}[c] - \frac{15}{2} c^2 h s u \partial_{yy}[c] + 25 c^3 h s u \partial_{yy}[c] -$$
$$\frac{195}{4} c^4 h s u \partial_{yy}[c] + \frac{123}{2} c^5 h s u \partial_{yy}[c] - \frac{209}{4} c^6 h s u \partial_{yy}[c] + 30 c^7 h s u \partial_{yy}[c] -$$
$$\frac{45}{4} c^8 h s u \partial_{yy}[c] + \frac{5}{2} c^9 h s u \partial_{yy}[c] - \frac{1}{4} c^{10} h s u \partial_{yy}[c] - \frac{3}{2} c h p w \partial_{yy}[c] + \frac{39}{4} c^2 h p w \partial_{yy}[c] -$$
$$26 c^3 h p w \partial_{yy}[c] + 35 c^4 h p w \partial_{yy}[c] - 21 c^5 h p w \partial_{yy}[c] - \frac{7}{2} c^6 h p w \partial_{yy}[c] + 14 c^7 h p w \partial_{yy}[c] -$$
$$9 c^8 h p w \partial_{yy}[c] + \frac{5}{2} c^9 h p w \partial_{yy}[c] - \frac{1}{4} c^{10} h p w \partial_{yy}[c] + 3 c h q w \partial_{yy}[c] - \frac{87}{4} c^2 h q w \partial_{yy}[c] +$$
$$\frac{137}{2} c^3 h q w \partial_{yy}[c] - \frac{245}{2} c^4 h q w \partial_{yy}[c] + \frac{273}{2} c^5 h q w \partial_{yy}[c] - 98 c^6 h q w \partial_{yy}[c] +$$
$$\frac{91}{2} c^7 h q w \partial_{yy}[c] - \frac{27}{2} c^8 h q w \partial_{yy}[c] + \frac{5}{2} c^9 h q w \partial_{yy}[c] - \frac{1}{4} c^{10} h q w \partial_{yy}[c] + \frac{9}{4} c^2 h r w \partial_{yy}[c] -$$
$$\frac{27}{2} c^3 h r w \partial_{yy}[c] + \frac{141}{4} c^4 h r w \partial_{yy}[c] - \frac{105}{2} c^5 h r w \partial_{yy}[c] + 49 c^6 h r w \partial_{yy}[c] -$$
$$\frac{59}{2} c^7 h r w \partial_{yy}[c] + \frac{45}{4} c^8 h r w \partial_{yy}[c] - \frac{5}{2} c^9 h r w \partial_{yy}[c] + \frac{1}{4} c^{10} h r w \partial_{yy}[c] + \frac{1}{4} h s w \partial_{yy}[c] -$$
$$\frac{5}{2} c h s w \partial_{yy}[c] + \frac{45}{4} c^2 h s w \partial_{yy}[c] - 30 c^3 h s w \partial_{yy}[c] + \frac{105}{2} c^4 h s w \partial_{yy}[c] - 63 c^5 h s w \partial_{yy}[c] +$$
$$\frac{105}{2} c^6 h s w \partial_{yy}[c] - 30 c^7 h s w \partial_{yy}[c] + \frac{45}{4} c^8 h s w \partial_{yy}[c] - \frac{5}{2} c^9 h s w \partial_{yy}[c] + \frac{1}{4} c^{10} h s w \partial_{yy}[c] \}\}$$

In[17]:=
```
PartialIntegrate[pde, {c}, Depth → 1]
```

Out[17]=

$$\frac{1}{4} h \partial_x \Big[\big(-6 c p + 15 c^2 p - 8 c^3 p - 6 c^4 p + 6 c^5 p - c^6 p + 12 c q - 39 c^2 q + 46 c^3 q - 24 c^4 q + 6 c^5 q - c^6 q + 9 c^2 r -$$
$$18 c^3 r + 15 c^4 r - 6 c^5 r + c^6 r + s - 6 c s + 15 c^2 s - 20 c^3 s + 15 c^4 s - 6 c^5 s + c^6 s\big)$$
$$\big(4 c u - 6 c^2 u + 4 c^3 u - c^4 u + w - 4 c w + 6 c^2 w - 4 c^3 w + c^4 w\big) \partial_x[c] \Big] +$$
$$\frac{1}{4} h \partial_y \Big[\big(-6 c p + 15 c^2 p - 8 c^3 p - 6 c^4 p + 6 c^5 p - c^6 p + 12 c q - 39 c^2 q + 46 c^3 q - 24 c^4 q + 6 c^5 q - c^6 q +$$
$$9 c^2 r - 18 c^3 r + 15 c^4 r - 6 c^5 r + c^6 r + s - 6 c s + 15 c^2 s - 20 c^3 s + 15 c^4 s - 6 c^5 s + c^6 s\big)$$
$$\big(4 c u - 6 c^2 u + 4 c^3 u - c^4 u + w - 4 c w + 6 c^2 w - 4 c^3 w + c^4 w\big) \partial_y[c] \Big]$$

```
(* Verify that our result agrees with Equation (14) in the paper. *)
Expand[
  (-6 c p + 15 c^2 p - 8 c^3 p - 6 c^4 p + 6 c^5 p - c^6 p + 12 c q - 39 c^2 q + 46 c^3 q - 24 c^4 q + 6 c^5 q - c^6 q + 9 c^2 r -
      18 c^3 r + 15 c^4 r - 6 c^5 r + c^6 r + s - 6 c s + 15 c^2 s - 20 c^3 s + 15 c^4 s - 6 c^5 s + c^6 s)
    (4 c u - 6 c^2 u + 4 c^3 u - c^4 u + w - 4 c w + 6 c^2 w - 4 c^3 w + c^4 w) -
   (u + (w - u) * (1 - c) ^ 4) * (r + (s - p - q + r) * (1 - c) ^ 6 + (p * (1 - 9 c) + q * (1 + 9 c) - 2 r) * (1 - c) ^ 3)]
```

Out[18]= 0