Counting Embeddings of Laman Graphs

The Symbolic Computation Group at RICAM

Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences

September, 2016
Unusual Configuration Spaces
ICERM, Providence, USA

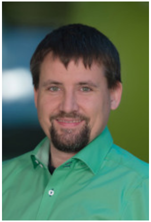
ÖAW ÖSTERREICHISCHE AKADEMIE DER WISSENSCHAFTEN

RICAM JOHANN · RADON · INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

The Symbolic Computation Group at RICAM


Matteo Gallet


Georg Grasegger


C. Koutschan


Niels Lubbes


Clemens Raab


G. Regensburger


Josef Schicho


Nelly Villamizar

Plane Configurations of a Labeled Graph

Notation: Let $G = (V, E)$ be a graph,
and let $\lambda \colon E \to \mathbb{R}_{>0}$ be a labeling of its edges.

We study the set of all configurations[1] of points in the plane corresponding to V, such that

$$||p_u - p_v||^2 = \lambda_{u,v}$$

for every edge $(u, v) \in E$.

Plane Configurations of a Labeled Graph

Notation: Let $G = (V, E)$ be a graph,
and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges.

We study the set of all configurations[1] of points in the plane corresponding to V, such that

$$||p_u - p_v||^2 = \lambda_{u,v}$$

for every edge $(u, v) \in E$.

1) point-tuple modulo rotation and translation

Plane Configurations of a Labeled Graph

Notation: Let $G = (V, E)$ be a graph,
and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges.

We study the set of all configurations[1] of points in the plane corresponding to V, such that

$$||p_u - p_v||^2 = \lambda_{u,v}$$

for every edge $(u, v) \in E$.

1) point-tuple modulo rotation and translation

Non-empty?

Plane Configurations of a Labeled Graph

Notation: Let $G = (V, E)$ be a graph,
and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a labeling of its edges.

We study the set of all configurations[1] of points in the plane corresponding to V, such that

$$||p_u - p_v||^2 = \lambda_{u,v}$$

for every edge $(u, v) \in E$.

1) point-tuple modulo rotation and translation

Non-empty? Finite?

Plane Configurations of a Labeled Graph

Notation: Let $G = (V, E)$ be a graph,
and let $\lambda \colon E \to \mathbb{R}_{>0}$ be a labeling of its edges.

We study the set of all configurations[1] of points in the plane corresponding to V, such that

$$||p_u - p_v||^2 = \lambda_{u,v}$$

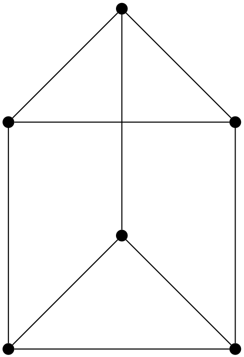
for every edge $(u, v) \in E$.

1) point-tuple modulo rotation and translation

Non-empty? Finite?
If yes, what is the number?

Three-Prism Graph

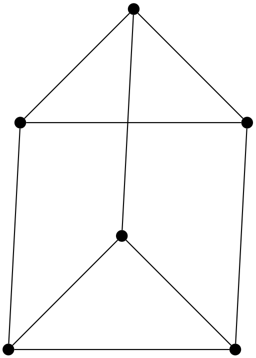
Infinitely many configurations



The number of configuration depends on the labeling
(not just on the graph)

Three-Prism Graph

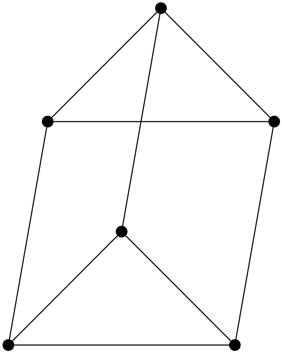
Infinitely many configurations



The number of configuration depends on the labeling
(not just on the graph)

Three-Prism Graph

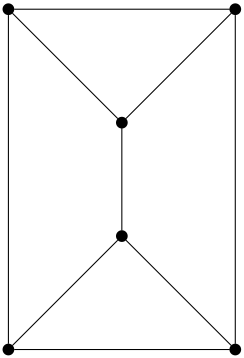
Infinitely many configurations



The number of configuration depends on the labeling
(not just on the graph)

Three-Prism Graph

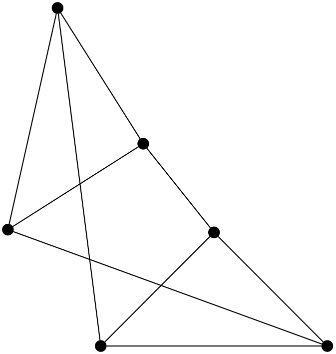
Finitely many configurations



The number of configuration depends on the labeling
(not just on the graph)

Three-Prism Graph

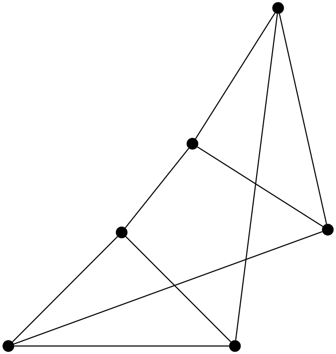
Finitely many configurations



The number of configuration depends on the labeling
(not just on the graph)

Three-Prism Graph

Finitely many configurations



The number of configuration depends on the labeling
(not just on the graph)

Complex & Generic

Proposition: Fix a graph $G = (V, E)$. Then there is a subset $E \subset \mathbb{R}^{|E|}$ of dimension less than $|E|$ and a cardinal Λ_G such that for all labelings outside E, the cardinality of configurations is equal to Λ_G.

Complex & Generic

Proposition: Fix a graph $G = (V, E)$. Then there is a subset $E \subset \mathbb{R}^{|E|}$ of dimension less than $|E|$ and a cardinal Λ_G such that for all labelings outside E, the cardinality of configurations is equal to Λ_G.

In other words: if we count complex configurations and choose the labeling generically, then the number of configurtions depends only on the graph.

Complex & Generic

Proposition: Fix a graph $G = (V, E)$. Then there is a subset $E \subset \mathbb{R}^{|E|}$ of dimension less than $|E|$ and a cardinal Λ_G such that for all labelings outside E, the cardinality of configurations is equal to Λ_G.

In other words: if we count complex configurations and choose the labeling generically, then the number of configurtions depends only on the graph.

Definition: if Λ_G is finite and not zero, then we call G a *Laman graph and* Λ_G *its Laman number.*

Laman Graphs

Question: When can we expect a graph to be Laman?

- # unknowns (coordinates of the vertices): $2 \cdot |V|$
- # equations: $|E|$
- dim(direct isometries): 3

$\longrightarrow$ Hence, $|E| \geqslant 2|V| - 3$ is a necessary condition for rigidity.

Laman Graphs

Question: When can we expect a graph to be Laman?

- $\#$ unknowns (coordinates of the vertices): $2 \cdot |V|$
- $\#$ equations: $|E|$
- dim(direct isometries): 3

$\longrightarrow$ Hence, $|E| \geqslant 2|V| - 3$ is a necessary condition for rigidity.

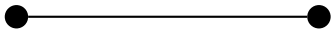
Theorem. (Laman, 1970)
A graph $G = (V, E)$ is Laman if and only if

1. $|E| = 2|V| - 3$,
2. $|E'| \leqslant 2|V'| - 3$ for each subgraph $G' = (V', E')$ of G.

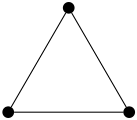
Some Laman Graphs

All Laman graphs with $2 \leqslant n \leqslant 5$ vertices:

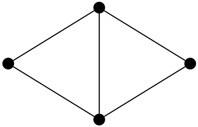
$n = 2$:



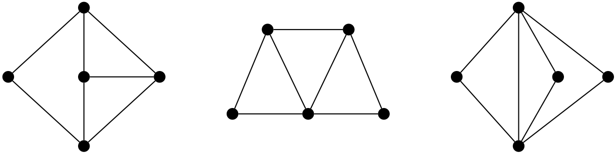
$n = 3$:



$n = 4$:

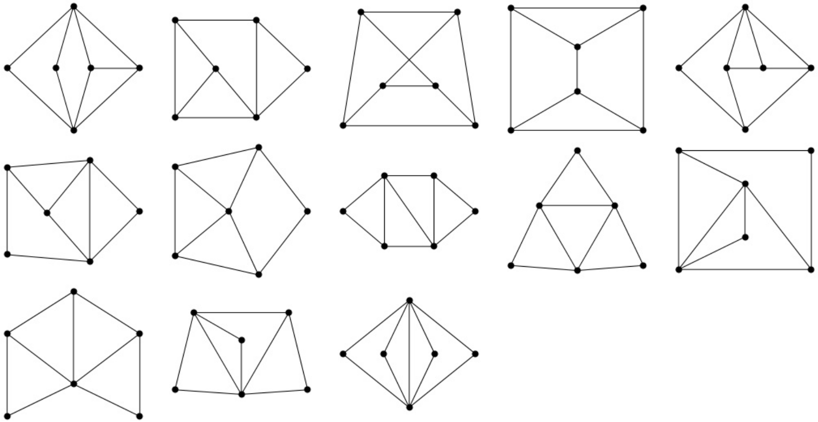


$n = 5$:



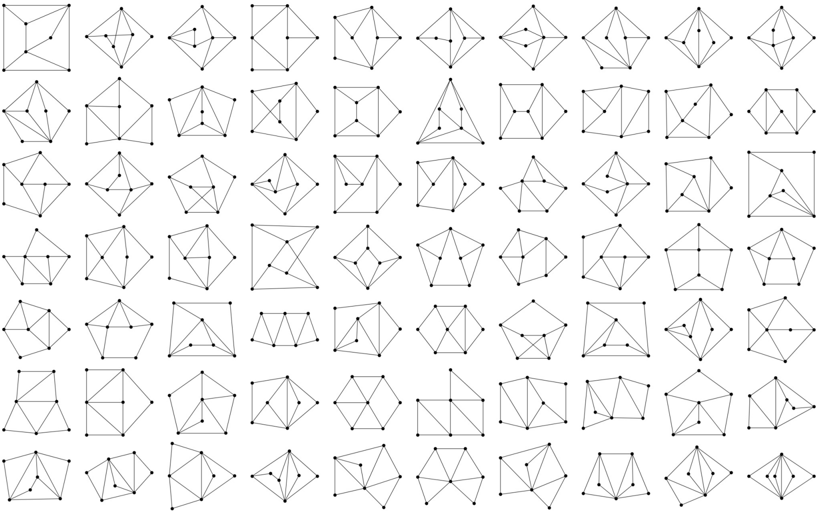
Some Laman Graphs

All Laman graphs with 6 vertices:



Some Laman Graphs

There are 70 Laman graphs with 7 vertices:



Previously Known Facts on Λ

Below, n is always the number of vertices.

Previously Known Facts on Λ

Below, n is always the number of vertices.

- The Laman number is at least 2^{n-2}. This bound is sharp (Steffens, Theobald).

Previously Known Facts on Λ

Below, n is always the number of vertices.

- The Laman number is at least 2^{n-2}. This bound is sharp (Steffens, Theobald).
- For $2 \leq n \leq 5$, the Laman number is 2^{n-2} (easy).

Previously Known Facts on Λ

Below, n is always the number of vertices.

- The Laman number is at least 2^{n-2}. This bound is sharp (Steffens, Theobald).
- For $2 \leq n \leq 5$, the Laman number is 2^{n-2} (easy).
- The only Laman graph with 6 vertices and Laman number > 16 is the three-prism graph. Its Laman number is 24 (classical).

Previously Known Facts on Λ

Below, n is always the number of vertices.

- The Laman number is at least 2^{n-2}. This bound is sharp (Steffens, Theobald).
- For $2 \leq n \leq 5$, the Laman number is 2^{n-2} (easy).
- The only Laman graph with 6 vertices and Laman number > 16 is the three-prism graph. Its Laman number is 24 (classical).
- The maximal Laman number for $n = 7$ is 56 (Emiris, Despotakis).

Previously Known Facts on Λ

Below, n is always the number of vertices.

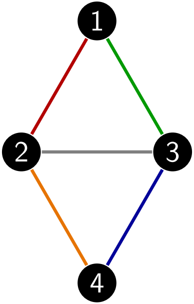
- The Laman number is at least 2^{n-2}. This bound is sharp (Steffens, Theobald).
- For $2 \leq n \leq 5$, the Laman number is 2^{n-2} (easy).
- The only Laman graph with 6 vertices and Laman number > 16 is the three-prism graph. Its Laman number is 24 (classical).
- The maximal Laman number for $n = 7$ is 56 (Emiris, Despotakis).
- There exists a Laman graph with 8 vertices and Laman number 90 (Jackson, Owen).

Previously Known Facts on Λ

Below, n is always the number of vertices.

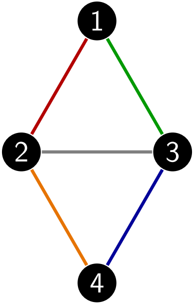
- The Laman number is at least 2^{n-2}. This bound is sharp (Steffens, Theobald).
- For $2 \leq n \leq 5$, the Laman number is 2^{n-2} (easy).
- The only Laman graph with 6 vertices and Laman number > 16 is the three-prism graph. Its Laman number is 24 (classical).
- The maximal Laman number for $n = 7$ is 56 (Emiris, Despotakis).
- There exists a Laman graph with 8 vertices and Laman number 90 (Jackson, Owen).
- The Laman number is at most $\binom{2n-4}{n-2}$ (Borcea, Streinu).

The Equation System for a Laman Graph



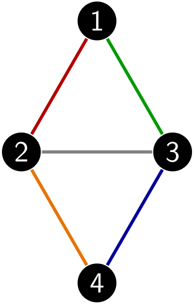
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = \lambda_{\mathbf{r}},$$
$$(x_2 - x_4)^2 + (y_2 - y_4)^2 = \lambda_{\mathbf{o}},$$
$$(x_3 - x_4)^2 + (y_3 - y_4)^2 = \lambda_{\mathbf{b}},$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 = \lambda_{\mathbf{g}},$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = 1, \qquad x_1 = y_1 = 0, \qquad y_2 = 1.$$

The Equation System for a Laman Graph



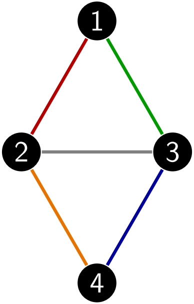
$$x_{12}^2 + y_{12}^2 = \lambda_{\mathbf{r}}, \qquad x_{12} + x_{23} - x_{13} = y_{12} + y_{23} - y_{13} = 0,$$
$$x_{24}^2 + y_{24}^2 = \lambda_{\mathbf{o}}, \qquad x_{24} - x_{34} - x_{23} = y_{24} - y_{34} - y_{23} = 0,$$
$$x_{34}^2 + y_{34}^2 = \lambda_{\mathbf{b}},$$
$$x_{13}^2 + y_{13}^2 = \lambda_{\mathbf{g}},$$
$$x_{23}^2 + y_{23}^2 = 1, \qquad y_{23} = 1.$$

The Equation System for a Laman Graph



$$
\begin{aligned}
&x_{12}\,y_{12} = \lambda_{\mathbf{r}}, && x_{12} + x_{23} - x_{13} = y_{12} + y_{23} - y_{13} = 0,\\
&x_{24}\,y_{24} = \lambda_{\mathbf{o}}, && x_{24} - x_{34} - x_{23} = y_{24} - y_{34} - y_{23} = 0,\\
&x_{34}\,y_{34} = \lambda_{\mathbf{b}}, && \\
&x_{13}\,y_{13} = \lambda_{\mathbf{g}}, && \\
&x_{23}\,y_{23} = 1, && y_{23} = 1.
\end{aligned}
$$

The Equation System for a Laman Graph **over the tropical field** $\mathbb{C}\{\{t\}\}$



$$x_{12}\, y_{12} = \lambda_{\mathbf{r}} t, \qquad x_{12} + x_{23} - x_{13} = y_{12} + y_{23} - y_{13} = 0,$$
$$x_{24}\, y_{24} = \lambda_{\mathbf{o}} t, \qquad x_{24} - x_{34} - x_{23} = y_{24} - y_{34} - y_{23} = 0,$$
$$x_{34}\, y_{34} = \lambda_{\mathbf{b}} t,$$
$$x_{13}\, y_{13} = \lambda_{\mathbf{g}} t,$$
$$x_{23}\, y_{23} = 1, \qquad y_{23} = 1.$$

Bigraphs

A bigraph is a pair of graphs, possible with multiple edges but without self-loops, together with a bijection between the sets of edges. For every bigraph, we can define an equation system, such that the equation systems of Laman graphs appear as special cases.

- biedge equations: $x_{ij}y_{kl} = \lambda_e$ for any biedge $e = (ij|kl)$
- norming equations: $x_{12} = y_{12} = 1$ if $(12|12)$ is the norming biedge
- cocycle equations: $x_{12} + x_{23} + \cdots - x_{1n} = 0$ for cycle $(1, 2, \ldots, n)$

Bigraphs **(tropical case)**

A bigraph is a pair of graphs, possible with multiple edges but without self-loops, together with a bijection between the sets of edges. For every bigraph, we can define an equation system, such that the equation systems of Laman graphs appear as special cases.

- biedge equations: $x_{ij}y_{kl} = \lambda_e t^{w_e}$ for any biedge $e = (ij|kl)$
- norming equations: $x_{12} = y_{12} = 1$ if $(12|12)$ is the norming biedge
- cocycle equations: $x_{12} + x_{23} + \cdots - x_{1n} = 0$ for cycle $(1, 2, \ldots, n)$

Tropical Solutions

Let $B = (G, H)$ be a bigraph.
Let $(x_{ij}, \ldots, y_{kl})$ a solution of the associated system of equations (tropical case).
Let $d_G(i,j) := \mathrm{ord}(x_{ij}), d_H(k,l) := \mathrm{ord}(y_{kl})$ for each biedge $(ij|kl)$. Then:

- biedge conditions: $d_G(i,j) + d_H(k,l) = w_e$ for any biedge $e = (ij|kl)$
- norming conditions: $d_G(1,2) = d_H(1,2) = 0$ if $(12|12)$ is the norming biedge
- cocycle conditions: in every cycle, the minimum of d_G resp. d_H appears at least twice

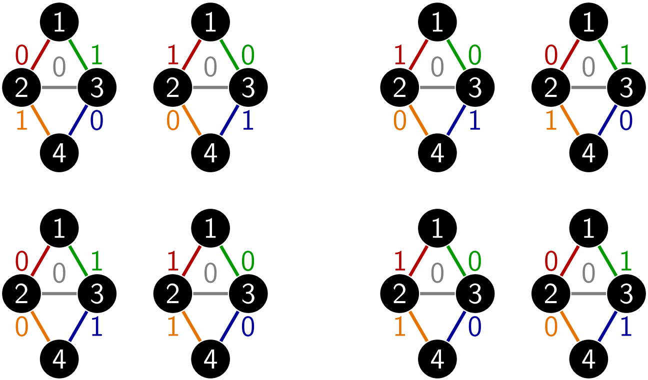
Any labeling of edges of G and H by rational numbers is called a **bidistance** compatible with parameter value vector w.

Bidistances for General Parameter Value Vectors

Theorem: If the parameter value vector w is general, then the number of bidistances is equal to the Laman number.

Bidistances for General Parameter Value Vectors

Theorem: If the parameter value vector w is general, then the number of bidistances is equal to the Laman number.



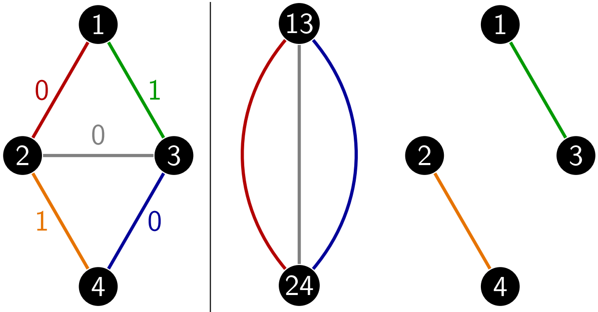
Graph Decompositions

Let $G = (V, E)$ be a graph.
Let $d : E \to \mathbb{Q}$ be a distance (labeling such that the minimum occurs at least twice in each cycle).
Let $a \in \mathrm{im}(d)$.
Then $G_{=a}/G_{>a}$ is defined as follows: two vertices of V are equivalent if they can be connected by edges with labels $> a$. The vertices of $G_{=a}/G_{>a}$ are the classes, and the edges are the edges of G with label a.

Graph Decompositions

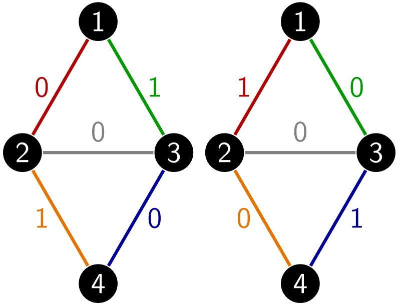
Let $G = (V, E)$ be a graph.
Let $d : E \to \mathbb{Q}$ be a distance (labeling such that the minimum occurs at least twice in each cycle).
Let $a \in \mathrm{im}(d)$.
Then $G_{=a}/G_{>a}$ is defined as follows: two vertices of V are equivalent if they can be connected by edges with labels $> a$. The vertices of $G_{=a}/G_{>a}$ are the classes, and the edges are the edges of G with label a.
The graph G_d is defined as the disjoint union of all $G_{=a}/G_{>a}$, $a \in \mathrm{im}(d)$.

Graph Decompositions



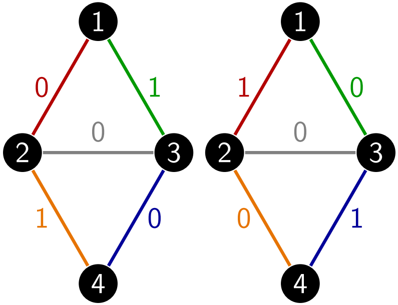
G $G_{=0}/G_{>0}$ $G_{=1}/G_{>1}$

Equations for the Leading Coefficients



$$
\begin{aligned}
\tilde{x}_{12}\,\tilde{y}_{12}t &= \lambda_{\mathbf{r}}t, & \tilde{x}_{12}+\tilde{x}_{23}-\tilde{x}_{13}t &= \tilde{y}_{12}t+\tilde{y}_{23}-\tilde{y}_{13}=0,\\
\tilde{x}_{24}t\,\tilde{y}_{24} &= \lambda_{\mathbf{o}}t, & \tilde{x}_{24}t-\tilde{x}_{34}-\tilde{x}_{23} &= \tilde{y}_{24}-\tilde{y}_{34}t-\tilde{y}_{23}=0,\\
\tilde{x}_{34}\,\tilde{y}_{34}t &= \lambda_{\mathbf{b}}t, & &\\
\tilde{x}_{13}t\,\tilde{y}_{13} &= \lambda_{\mathbf{g}}t, & &\\
\tilde{x}_{23} &= 1, & \tilde{y}_{23} &= 1.
\end{aligned}
$$

Equations for the Leading Coefficients



$$\tilde{x}_{12}\,\tilde{y}_{12} = \lambda_{\mathbf{r}}, \qquad \tilde{x}_{12} + \tilde{x}_{23} - \tilde{x}_{13}t = \tilde{y}_{12}t + \tilde{y}_{23} - \tilde{y}_{13} = 0,$$

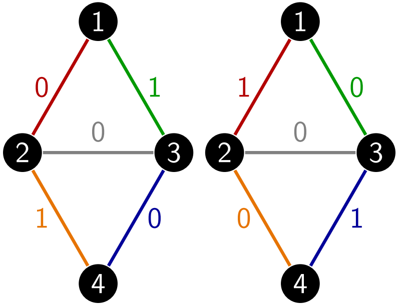
$$\tilde{x}_{24}\,\tilde{y}_{24} = \lambda_{\mathbf{o}}, \qquad \tilde{x}_{24}t - \tilde{x}_{34} - \tilde{x}_{23} = \tilde{y}_{24} - \tilde{y}_{34}t - \tilde{y}_{23} = 0,$$

$$\tilde{x}_{34}\,\tilde{y}_{34} = \lambda_{\mathbf{b}},$$

$$\tilde{x}_{13}\,\tilde{y}_{13} = \lambda_{\mathbf{g}},$$

$$\tilde{x}_{23} = 1, \qquad \tilde{y}_{23} = 1.$$

Equations for the Leading Coefficients



$$\tilde{x}_{12}\,\tilde{y}_{12} = \lambda_{\mathbf{r}}, \qquad \tilde{x}_{12} + \tilde{x}_{23} - \cancel{\tilde{x}_{13} t} = \cancel{\tilde{y}_{12} t} + \tilde{y}_{23} - \tilde{y}_{13} = 0,$$

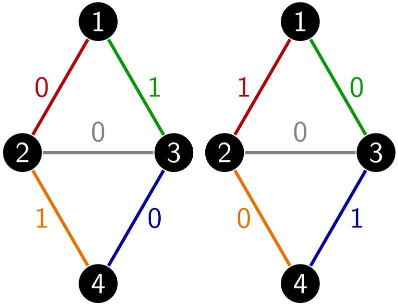
$$\tilde{x}_{24}\,\tilde{y}_{24} = \lambda_{\mathbf{o}}, \qquad \cancel{\tilde{x}_{24} t} - \tilde{x}_{34} - \tilde{x}_{23} = \tilde{y}_{24} - \cancel{\tilde{y}_{34} t} - \tilde{y}_{23} = 0,$$

$$\tilde{x}_{34}\,\tilde{y}_{34} = \lambda_{\mathbf{b}},$$

$$\tilde{x}_{13}\,\tilde{y}_{13} = \lambda_{\mathbf{g}},$$

$$\tilde{x}_{23} = 1, \qquad \tilde{y}_{23} = 1.$$

Equations for the Leading Coefficients



$$\tilde{x}_{12}\,\tilde{y}_{12} = \lambda_{\mathbf{r}}, \qquad \tilde{x}_{12} + \tilde{x}_{23} = \tilde{y}_{23} - \tilde{y}_{13} = 0,$$

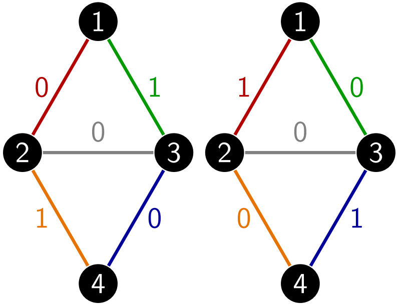
$$\tilde{x}_{24}\,\tilde{y}_{24} = \lambda_{\mathbf{o}}, \qquad -\tilde{x}_{34} - \tilde{x}_{23} = \tilde{y}_{24} - \tilde{y}_{23} = 0,$$

$$\tilde{x}_{34}\,\tilde{y}_{34} = \lambda_{\mathbf{b}},$$

$$\tilde{x}_{13}\,\tilde{y}_{13} = \lambda_{\mathbf{g}},$$

$$\tilde{x}_{23} = 1, \qquad \tilde{y}_{23} = 1.$$

Equations for the Leading Coefficients



$$\tilde{x}_{12}\,\tilde{y}_{12} = \lambda_{\mathbf{r}}, \qquad \tilde{x}_{12} = -\tilde{x}_{23} = \tilde{x}_{34} \qquad ,$$

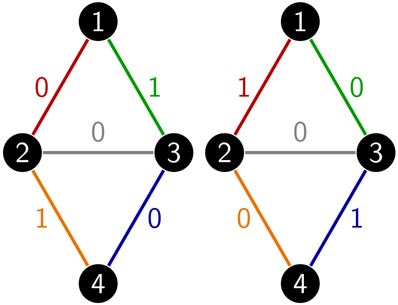
$$\tilde{x}_{24}\,\tilde{y}_{24} = \lambda_{\mathbf{o}}, \qquad \tilde{y}_{13} = \tilde{y}_{23} = \tilde{y}_{24},$$

$$\tilde{x}_{34}\,\tilde{y}_{34} = \lambda_{\mathbf{b}},$$

$$\tilde{x}_{13}\,\tilde{y}_{13} = \lambda_{\mathbf{g}},$$

$$\tilde{x}_{23} = 1, \qquad \tilde{y}_{23} = 1.$$

Equations for the Leading Coefficients



$$\tilde{x}_{12}\,\tilde{y}_{12} = \lambda_{\mathbf{r}}, \qquad \tilde{x}_{12} = -\tilde{x}_{23} = \tilde{x}_{34} = \tilde{x}_{(13)(24)},$$

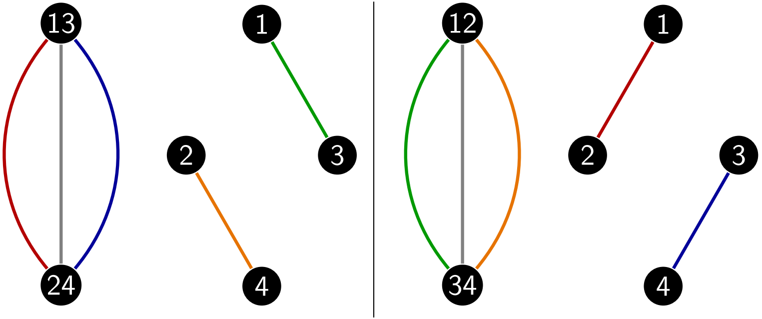
$$\tilde{x}_{24}\,\tilde{y}_{24} = \lambda_{\mathbf{o}}, \qquad \tilde{y}_{13} = \tilde{y}_{23} = \tilde{y}_{24} = \tilde{y}_{(12)(34)},$$

$$\tilde{x}_{34}\,\tilde{y}_{34} = \lambda_{\mathbf{b}},$$

$$\tilde{x}_{13}\,\tilde{y}_{13} = \lambda_{\mathbf{g}},$$

$$\tilde{x}_{23} = 1, \qquad \tilde{y}_{23} = 1.$$

Equations for the Leading Coefficients



$$\tilde{x}_{(13)(24)}\,\tilde{y}_{12} = \lambda_{\mathbf{r}},$$
$$\tilde{x}_{24}\,\tilde{y}_{(12)(34)} = \lambda_{\mathbf{o}},$$
$$\tilde{x}_{(13)(24)}\,\tilde{y}_{34} = \lambda_{\mathbf{b}},$$
$$\tilde{x}_{13}\,\tilde{y}_{(12)(34)} = \lambda_{\mathbf{g}},$$
$$-\tilde{x}_{(13)(24)} = 1, \qquad \tilde{y}_{(12)(34)} = 1.$$

A Recursion Formula

For a bigraph $B = (G, H)$ and bidistance $d = (d_G, d_H)$, we define $B_d := (G_{d_G}, H_{d_H})$.

Theorem: Let B be a bigraph. Let w be a parameter value vector. Then

$$\Lambda(B) = \sum_d \Lambda(B_d),$$

the sum ranging over all bidistances compatible with w.

A Recursion Formula

For a bigraph $B = (G, H)$ and bidistance $d = (d_G, d_H)$, we define $B_d := (G_{d_G}, H_{d_H})$.

Theorem: Let B be a bigraph. Let w be a parameter value vector. Then

$$\Lambda(B) = \sum_{d} \Lambda(B_d),$$

the sum ranging over all bidistances compatible with w.

Proof: The left side is the number of solutions for the tropical field $\mathbb{C}\{\{t\}\}$. Each solution defines a bidistance. For each bidistance d, compute possible leading coefficients. This is equivalent to solving the system for B_d over $\mathbb{C}$. The number of all these solutions is the right side. Then show that any of these solutions lift uniquely to the tropical field $\mathbb{C}\{\{t\}\}$.

A Recursive Algorithm

Lemma: Assume that $w_e = 1$ except for the norming edge (which has value 0). Then all bidistances d such that $\Lambda(B_d) > 0$ have values in $\{0, 1\}$.

This reduces the computation of $\Lambda(B)$ to a finite number of computations of Laman numbers of simpler bigraphs.

A Recursive Algorithm

Lemma: Assume that $w_e = 1$ except for the norming edge (which has value 0). Then all bidistances d such that $\Lambda(B_d) > 0$ have values in $\{0, 1\}$.

This reduces the computation of $\Lambda(B)$ to a finite number of computations of Laman numbers of simpler bigraphs.

Why are they simpler? They have the same number of biedges, and a larger number of vertices...

The reason is that the new graphs have more components, which allows to decompose into blocks that can be solved independently. Afterwards one just multiplies the number of solutions of each block.

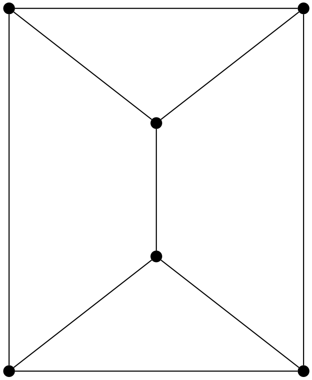
Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

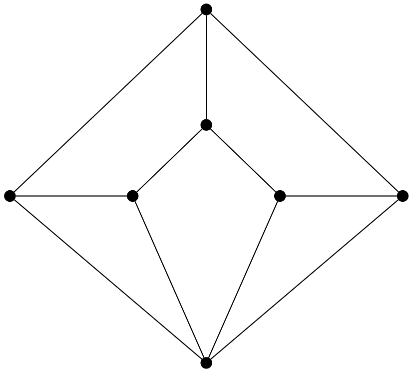
n	6
$\#$	24



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

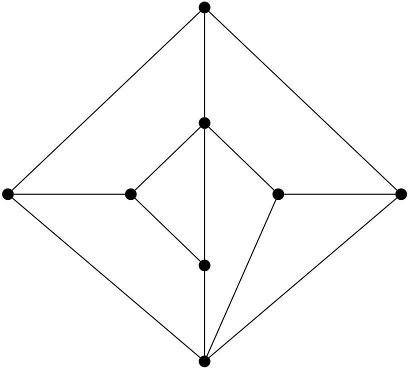
n	6	7
#	24	56



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

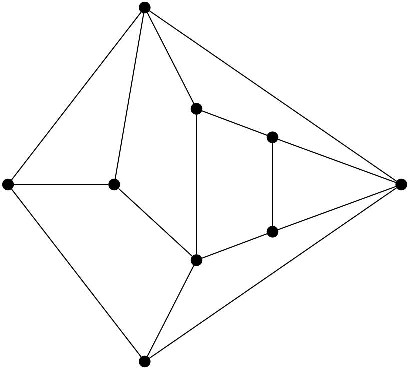
n	6	7	8
#	24	56	136



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

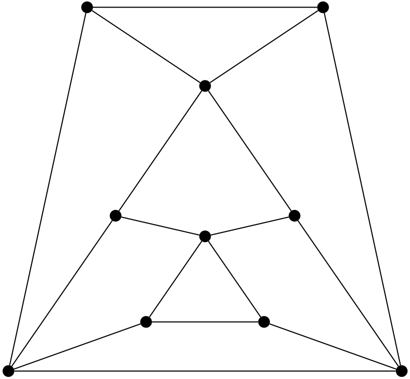
n	6	7	8	9
#	24	56	136	344



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

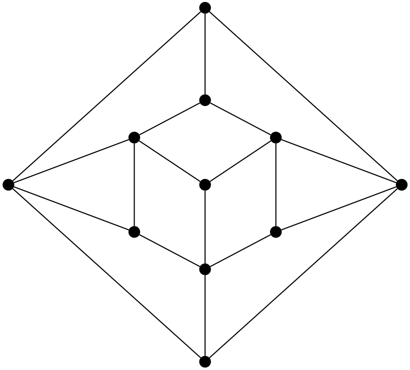
n	6	7	8	9	10
#	24	56	136	344	880



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

n	6	7	8	9	10	11
#	24	56	136	344	880	2288



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

n	6	7	8	9	10	11
#	24	56	136	344	880	2288

There are 44176717 Laman graphs with 12 vertices...

Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

n	6	7	8	9	10	11
#	24	56	136	344	880	2288

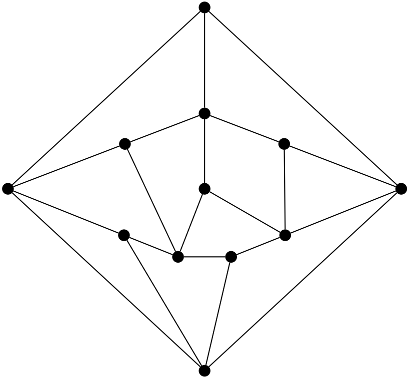
Conjecture: For each $n \geqslant 6$ there is a unique Laman graph $G_n^{\max}$ with n vertices that has the maximal Laman number; moreover $G_n^{\max}$ has the following properties:

- $G_n^{\max}$ is a planar graph.
- $G_n^{\max}$ has exactly 6 vertices with valency 3.
- $G_n^{\max}$ has exactly $n-6$ vertices with valency 4.
- $G_n^{\max}$ has exactly 2 triangles and $n-3$ quadrilaterals.
- The two triangles do not share an edge.

Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

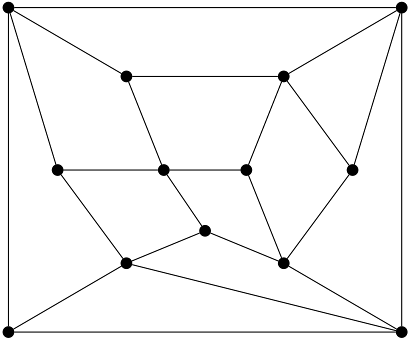
n	6	7	8	9	10	11	12
#	24	56	136	344	880	2288	5952



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

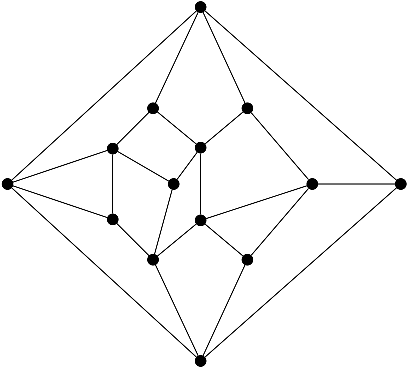
n	6	7	8	9	10	11	12	13
#	24	56	136	344	880	2288	5952	15056



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

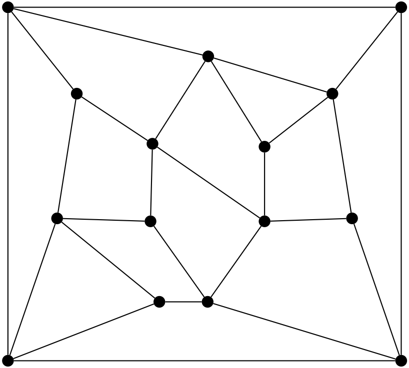
n	6	7	8	9	10	11	12	13	14
#	24	56	136	344	880	2288	5952	15056	39696



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

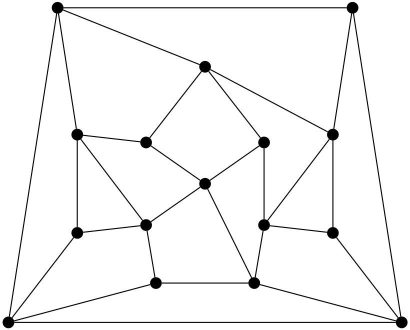
n	6	7	8	9	10	11	12	13	14	15
#	24	56	136	344	880	2288	5952	15056	39696	105384



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

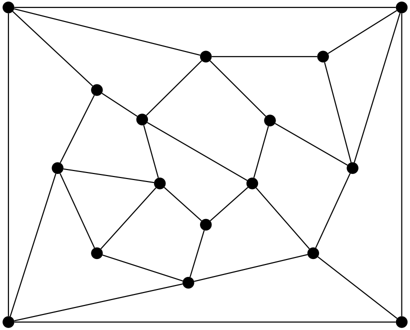
n	6	7	8	9	10	11	12	13	14	15	16
#	24	56	136	344	880	2288	5952	15056	39696	105384	277864



Laman Graphs with Maximal Laman Number

Question: Among all Laman graphs with n vertices, which one has the largest number of configurations?

n	6	7	8	9	10	11	12	13	14	15	16	17
#	24	56	136	344	880	2288	5952	15056	39696	105384	277864	731336



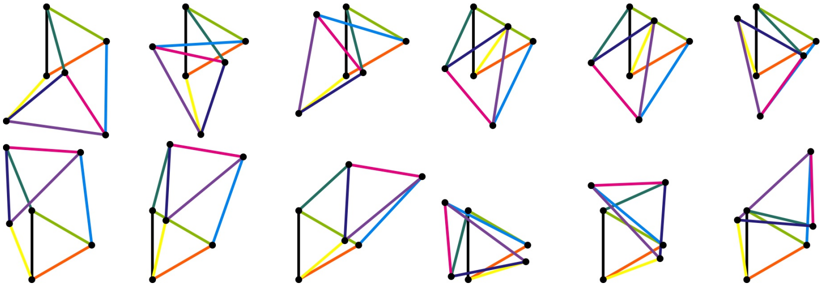
Real configurations

Question: Given a Laman graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $\Lambda(G)$ **real** configurations?

Real configurations

Question: Given a Laman graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $\Lambda(G)$ **real** configurations?

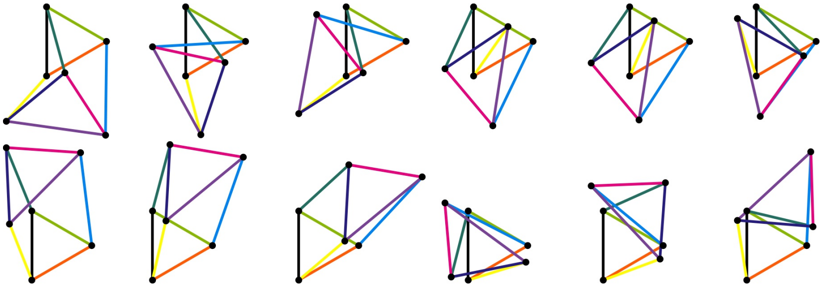
Answer: Maybe. At least sometimes:



Real configurations

Question: Given a Laman graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $\Lambda(G)$ **real** configurations?

Answer: Maybe. At least sometimes:

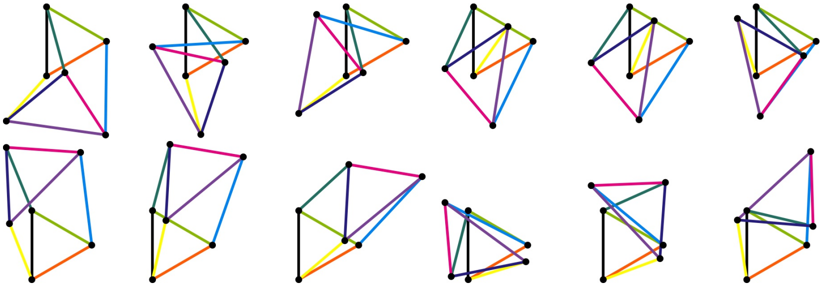


Question: Can we move (in 3D) from any one configuration on to any other, without "disassembling" the graph?

Real configurations

Question: Given a Laman graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $\Lambda(G)$ **real** configurations?

Answer: Maybe. At least sometimes:



Question: Can we move (in 3D) from any one configuration on to any other, without "disassembling" the graph?

Answer: Show movie.