

```
In[1]:= << "LamanGraphs .m ";
```

LamanGraphs — Copyright © 2016 Christoph Koutschan

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Generate and Count Laman Graphs

```
In[2]:= TableForm [Table [Timing [Length [LamanGraphs [n]]], {n, 3, 10}]]
```

Out[2]//TableForm=

0.	1
0.009000	1
0.008000	3
0.052000	13
0.202000	70
1.702000	608
22.217000	7222
459.611000	110132

```
In[3]:= TableForm [Table [Timing [Length [H1LamanGraphs [n]]], {n, 3, 10}]]
```

Out[3]//TableForm=

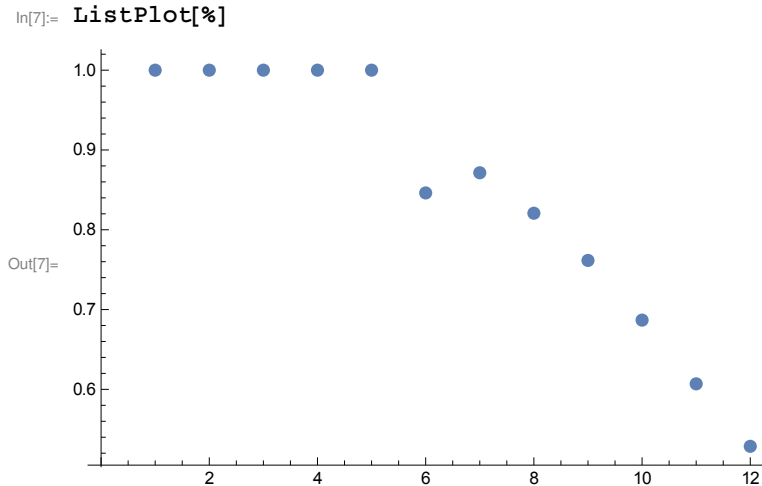
0.	1
0.001000	1
0.002000	3
0.010000	11
0.052000	61
0.424000	499
4.902000	5500
66.256000	75635

```
In[4]:= (* Number of Laman graphs of size n *)  
numLG = {1, 1, 1, 1, 3, 13, 70, 608, 7222, 110132, 2039273, 44176717};
```

```
In[5]:= (* Number of Laman graphs that use only H1 moves *)  
numH1 = {1, 1, 1, 1, 3, 11, 61, 499, 5500, 75635, 1237670, 23352425};
```

```
In[6]:= N[numH1 / numLG ]
```

```
Out[6]= {1., 1., 1., 1., 1., 0.846154, 0.871429,  
0.820724, 0.761562, 0.686767, 0.606917, 0.528614}
```



Number of Embeddings

```
In[8]:= (* Number of embeddings - computed with Groebner basis approach *)
Do[Print["Embs ["<>ToString[n]<>"] = ",
      NumberOfEmbeddings /@LamanGraphs [n]], {n, 3, 6}];

Embs [3] = {2}
Embs [4] = {4}
Embs [5] = {8, 8, 8}
Embs [6] = {16, 16, 16, 24, 16, 16, 16, 16, 16, 16, 16, 16}

In[9]:= (* Number of embeddings - computed with our algorithm *)
Do[Print["Embs ["<>ToString[n]<>"] = ",
      Embs [n] = LamanNumber /@LamanGraphs [n]], {n, 3, 6}];

Embs [3] = {2}
Embs [4] = {4}
Embs [5] = {8, 8, 8}
Embs [6] = {16, 16, 16, 24, 16, 16, 16, 16, 16, 16, 16, 16}

TableForm [Table[Timing[Max[Embs [n] = LamanNumber /@LamanGraphs [n]], {n, 3, 10}]]]
0.001000      2
0.001000      4
0.002000      8
0.072000     24
0.312000     56
6.280000    136
167.964000  344
6449.586000 880

Sort[Tally[Embs [7]]]
{{32, 64}, {44, 1}, {48, 4}, {56, 1}}
```

```
Sort[Tally[Embs [8]]]
```

```
{{64, 525}, {88, 12}, {90, 1}, {96, 49},
 {104, 2}, {108, 1}, {112, 15}, {120, 1}, {128, 1}, {136, 1}}
```

```
Sort[Tally[Embs [9]]]
```

```
{{128, 5826}, {176, 192}, {180, 6}, {192, 719}, {204, 1}, {206, 2}, {208, 27},
 {212, 1}, {216, 25}, {224, 281}, {230, 1}, {232, 6}, {236, 3}, {240, 28},
 {244, 1}, {248, 5}, {252, 2}, {256, 30}, {260, 2}, {264, 2}, {268, 1}, {272, 29},
 {280, 5}, {288, 12}, {296, 2}, {304, 5}, {312, 4}, {320, 2}, {336, 1}, {344, 1}}
```

```
Sort[Tally[Embs [10]]]
```

```
{{256, 80912}, {352, 3370}, {360, 83}, {384, 13215}, {408, 23}, {412, 27}, {416, 580},
 {424, 27}, {432, 552}, {440, 3}, {448, 6012}, {460, 21}, {464, 218}, {468, 2}, {472, 94},
 {476, 1}, {480, 825}, {488, 41}, {492, 3}, {496, 160}, {498, 3}, {500, 1}, {502, 1},
 {504, 31}, {508, 3}, {512, 843}, {516, 2}, {520, 70}, {524, 2}, {528, 114}, {532, 1},
 {536, 55}, {538, 1}, {540, 2}, {544, 968}, {548, 3}, {552, 11}, {556, 5}, {560, 182},
 {564, 3}, {568, 11}, {572, 1}, {574, 2}, {576, 514}, {580, 6}, {584, 21}, {586, 2},
 {588, 1}, {592, 101}, {596, 4}, {600, 14}, {602, 2}, {604, 4}, {608, 212}, {612, 5},
 {616, 19}, {620, 4}, {624, 191}, {628, 5}, {630, 1}, {632, 15}, {636, 4}, {640, 118},
 {644, 4}, {648, 8}, {652, 4}, {656, 33}, {660, 5}, {664, 12}, {668, 1}, {672, 89},
 {680, 12}, {684, 3}, {688, 55}, {692, 1}, {696, 8}, {700, 2}, {704, 28}, {708, 1},
 {710, 1}, {712, 8}, {716, 1}, {720, 23}, {724, 2}, {728, 4}, {732, 1}, {736, 19},
 {744, 7}, {748, 2}, {752, 13}, {756, 1}, {760, 4}, {764, 1}, {768, 19}, {776, 3},
 {784, 12}, {792, 3}, {800, 4}, {808, 4}, {816, 1}, {824, 1}, {840, 4}, {880, 1}}
```

Graph with Maximal Embedding Number

```
In[19]:= Do[
  MaxEmbGraph [n] = LamanGraphs [n][[Position[Embs [n], Max[Embs [n]]][[1, 1]]]];
  Print["MaxEmbGraph [", n, " ] = ", G2Edges[MaxEmbGraph [n]]];
  , {n, 6, 10}];

MaxEmbGraph [6] = {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 5}, {3, 6}, {4, 5}, {4, 6}, {5, 6}}
MaxEmbGraph [7] =
  {{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 6}, {3, 7}, {4, 5}, {4, 6}, {5, 7}, {6, 7}}
MaxEmbGraph [8] = {{1, 3}, {1, 4}, {1, 5}, {1, 6},
  {2, 3}, {2, 4}, {2, 7}, {2, 8}, {3, 5}, {4, 7}, {5, 8}, {6, 7}, {6, 8}}
MaxEmbGraph [9] = {{1, 2}, {1, 4}, {1, 5}, {1, 8}, {2, 6},
  {2, 7}, {2, 9}, {3, 4}, {3, 5}, {3, 6}, {3, 7}, {4, 8}, {5, 9}, {6, 8}, {7, 9}}
MaxEmbGraph [10] = {{1, 2}, {1, 5}, {1, 7}, {1, 8}, {2, 6}, {2, 9}, {2, 10},
  {3, 5}, {3, 6}, {3, 7}, {3, 9}, {4, 5}, {4, 6}, {4, 8}, {4, 10}, {7, 9}, {8, 10}}

In[21]:= (* Compare runtime of combinatorial algorithm with Groebner approach, *)
(* using, for each n, the graph with the largest number of embeddings. *)
Do[
  Print["n = "<>ToString[n]<>":""];
  Print[" Algorithm: ", Timing[LamanNumber [MaxEmbGraph [n]]]];
  Print[" Groebner: ", Timing[NumberOfEmbeddings [MaxEmbGraph [n]]]];
  , {n, 6, 9}];
```

```

n = 6:
  Algorithm: {0.035000, 24}
  Groebner: {0.035000, 24}
n = 7:
  Algorithm: {0.071000, 56}
  Groebner: {0.199000, 56}
n = 8:
  Algorithm: {0.170000, 136}
  Groebner: {3.043000, 136}
n = 9:
  Algorithm: {0.575000, 344}
  Groebner: {79.407000, 344}

```

```
In[33]:= Timing[LamanNumber [MaxEmbGraph [10]]]
```

```
Out[33]= {1.814000, 880}
```

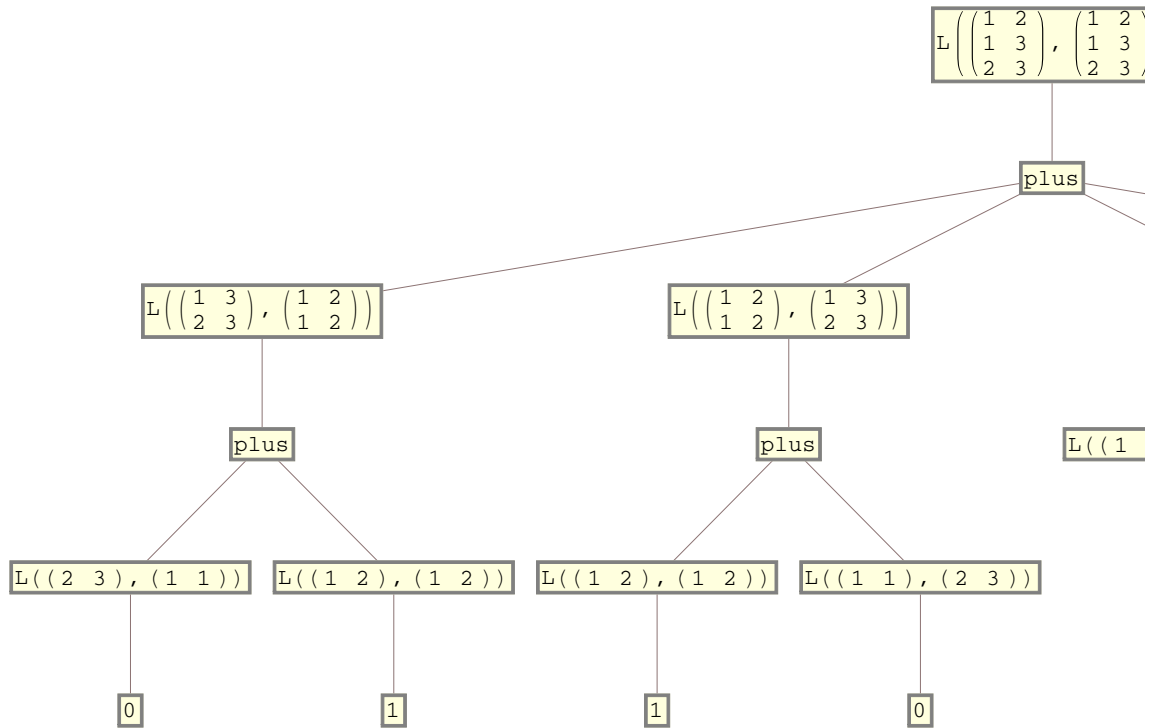
Computation Trees

```
In[10]:= comp3 = LN[G2Edges[7]]
```

```
Out[10]= L[{{1, 2}, {1, 3}, {2, 3}}, {{1, 2}, {1, 3}, {2, 3}}][
  plus[L[{{1, 3}, {2, 3}}, {{1, 2}, {1, 2}}][plus[L[{{2, 3}}, {{1, 1}}][0],
    L[{{1, 2}}, {{1, 2}}][1]]], L[{{1, 2}, {1, 2}}, {{1, 3}, {2, 3}}][
  plus[L[{{1, 2}}, {{1, 2}}][1], L[{{1, 1}}, {{2, 3}}][0]]],
  times[L[{{1, 1}}, {{1, 3}}][0], L[{{2, 3}}, {{1, 1}}][0]],
  times[L[{{1, 1}}, {{2, 3}}][0], L[{{1, 3}}, {{1, 1}}][0]]]
```

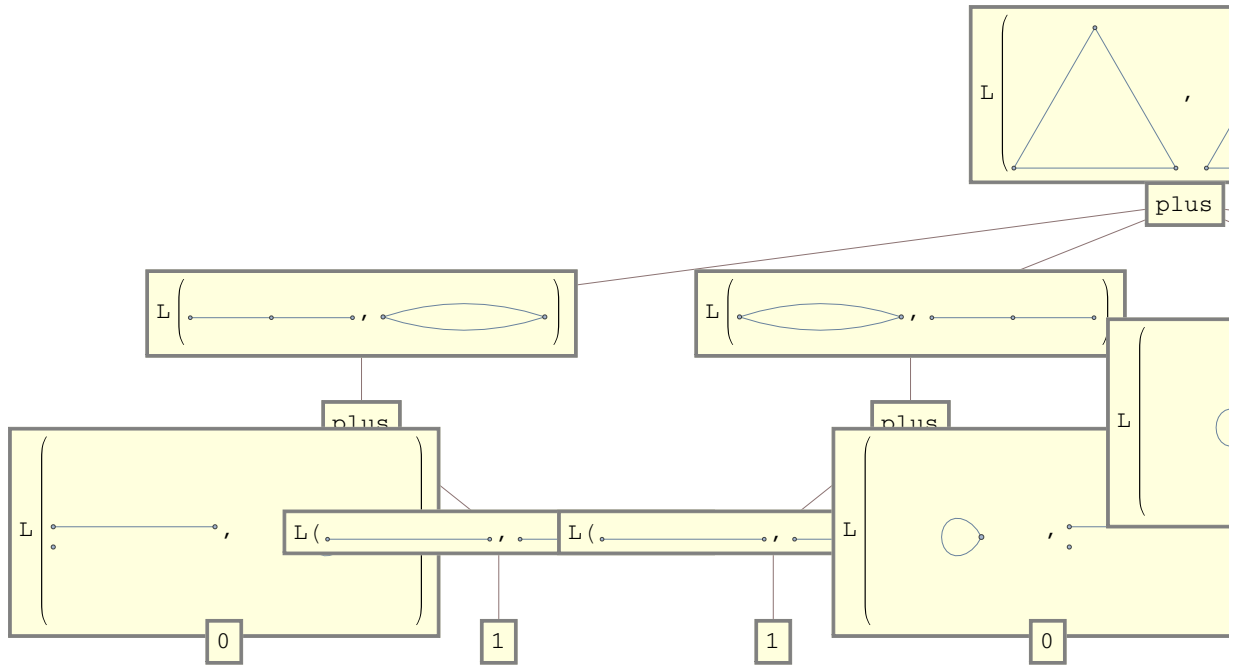
In[32]:= **TreeForm** [comp3 , ImageSize → 1200]

Out[32]/TreeForm=



In[29]:= **TreeForm** [comp3 /. L[G1_, G2_] → L[Graph[G1], Graph[G2]]]

Out[29]/TreeForm=



In[13]:= **LN**[G2Edges[LamanGraphs[4][[1]]]]

```
Out[13]= L[{{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}}, {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}}][
  plus[L[{{1, 3}, {1, 4}, {2, 3}, {2, 4}}, {{1, 2}, {1, 3}, {1, 2}, {1, 3}}][
    plus[L[{{1, 4}, {2, 3}, {2, 4}}, {{1, 2}, {1, 1}, {1, 2}}][0],
      L[{{1, 3}, {1, 2}, {2, 3}}, {{1, 3}, {1, 2}, {1, 3}}][plus[
        L[{{1, 2}, {2, 3}}, {{1, 2}, {1, 1}}][0], L[{{1, 2}, {1, 2}}, {{1, 2}, {1, 3}}][
          plus[L[{{1, 2}}, {{1, 2}}][1], L[{{1, 1}}, {{1, 3}}][0]]],
          times[L[{{1, 1}}, {{1, 2}}][0], L[{{2, 3}}, {{1, 1}}][0]],
          times[L[{{1, 1}}, {{1, 3}}][0], L[{{1, 2}}, {{1, 2}}][1]]],
        times[L[{{1, 1}}, {{1, 3}}][0], L[{{2, 3}, {2, 4}}, {{1, 1}, {1, 1}}][0]],
        times[L[{{1, 1}}, {{1, 2}}][0], L[{{1, 4}, {2, 4}}, {{1, 2}, {1, 2}}][
          plus[L[{{2, 4}}, {{1, 1}}][0], L[{{1, 2}}, {{1, 2}}][1]]],
        times[L[{{1, 1}}, {{1, 3}}][0], L[{{1, 4}, {2, 3}}, {{1, 1}, {1, 1}}][0]],
        times[L[{{1, 2}, {1, 2}}, {{1, 3}, {1, 2}}][plus[L[{{1, 2}}, {{1, 2}}][1],
          L[{{1, 1}}, {{1, 2}}][0]], L[{{2, 4}}, {{1, 1}}][0]],
        times[L[{{1, 2}, {1, 2}}, {{1, 3}, {1, 3}}][0], L[{{2, 3}}, {{1, 1}}][0]],
        times[L[{{1, 2}, {1, 2}}, {{1, 2}, {1, 3}}][plus[L[{{1, 2}}, {{1, 2}}][1],
          L[{{1, 1}}, {{1, 3}}][0]], L[{{1, 4}}, {{1, 1}}][0]]],
      L[{{1, 2}, {1, 3}, {1, 2}, {1, 3}}, {{1, 3}, {1, 4}, {2, 3}, {2, 4}}][
        plus[L[{{1, 3}, {1, 2}, {1, 3}}, {{1, 3}, {1, 2}, {2, 3}}][
          plus[L[{{1, 2}, {1, 3}}, {{1, 2}, {1, 2}}][plus[L[{{1, 3}}, {{1, 1}}][0],
            L[{{1, 2}}, {{1, 2}}][1]], L[{{1, 2}, {1, 1}}, {{1, 2}, {2, 3}}][0],
            times[L[{{1, 2}}, {{1, 2}}][1], L[{{1, 3}}, {{1, 1}}][0]],
            times[L[{{1, 1}}, {{2, 3}}][0], L[{{1, 2}}, {{1, 1}}][0]]],
          L[{{1, 2}, {1, 1}, {1, 2}}, {{1, 4}, {2, 3}, {2, 4}}][0],
          times[L[{{1, 1}}, {{1, 4}}][0], L[{{1, 2}, {1, 3}}, {{1, 2}, {1, 2}}][
            plus[L[{{1, 3}}, {{1, 1}}][0], L[{{1, 2}}, {{1, 2}}][1]]],
            times[L[{{1, 1}}, {{2, 3}}][0], L[{{1, 3}, {1, 3}}, {{1, 2}, {1, 2}}][0]],
            times[L[{{1, 1}}, {{2, 4}}][0], L[{{1, 3}, {1, 2}}, {{1, 2}, {1, 2}}][
```

```

plus[L[{{1, 2}}, {{1, 1}}][0], L[{{1, 2}}, {{1, 2}}][1]]],
times [L[{{1, 1}, {1, 1}}, {{1, 4}, {2, 3}}][0], L[{{1, 3}}, {{1, 1}}][0]],
times [L[{{1, 2}, {1, 2}}, {{1, 4}, {2, 4}}][plus[L[{{1, 2}}, {{1, 2}}][1],
L[{{1, 1}}, {{2, 4}}][0]]], L[{{1, 2}}, {{1, 1}}][0]],
times [L[{{1, 1}, {1, 1}}, {{2, 3}, {2, 4}}][0], L[{{1, 3}}, {{1, 1}}][0]]],
times [L[{{1, 1}}, {{1, 3}}][0],
L[{{1, 4}, {2, 3}, {2, 4}}, {{1, 2}, {1, 1}, {1, 2}}][0]],
times [L[{{1, 1}}, {{1, 4}}][0],
L[{{1, 3}, {2, 3}, {2, 4}}, {{1, 2}, {1, 2}, {1, 1}}][0]],
times [L[{{1, 1}}, {{2, 3}}][0],
L[{{1, 3}, {1, 4}, {2, 4}}, {{1, 1}, {1, 2}, {1, 2}}][0]],
times [L[{{1, 1}}, {{2, 4}}][0],
L[{{1, 3}, {1, 4}, {2, 3}}, {{1, 2}, {1, 1}, {1, 2}}][0]],
times [L[{{1, 1}, {1, 1}}, {{1, 3}, {1, 4}}][0],
L[{{2, 3}, {2, 4}}, {{1, 1}, {1, 1}}][0]],
times [L[{{1, 2}, {1, 2}}, {{1, 3}, {2, 3}}][
plus[L[{{1, 2}}, {{1, 2}}][1], L[{{1, 1}}, {{2, 3}}][0]]],
L[{{1, 4}, {2, 4}}, {{1, 2}, {1, 2}}][
plus[L[{{2, 4}}, {{1, 1}}][0], L[{{1, 2}}, {{1, 2}}][1]]]],
times [L[{{1, 1}, {1, 1}}, {{1, 3}, {2, 4}}][0],
L[{{1, 4}, {2, 3}}, {{1, 1}, {1, 1}}][0]],
times [L[{{1, 1}, {1, 1}}, {{1, 4}, {2, 3}}][0],
L[{{1, 3}, {2, 4}}, {{1, 1}, {1, 1}}][0]],
times [L[{{1, 2}, {1, 2}}, {{1, 4}, {2, 4}}][
plus[L[{{1, 2}}, {{1, 2}}][1], L[{{1, 1}}, {{2, 4}}][0]]],
L[{{1, 3}, {2, 3}}, {{1, 2}, {1, 2}}][
plus[L[{{2, 3}}, {{1, 1}}][0], L[{{1, 2}}, {{1, 2}}][1]]]],
times [L[{{1, 1}, {1, 1}, {1, 1}}, {{2, 3}, {2, 4}}][0],
L[{{1, 3}, {1, 4}}, {{1, 1}, {1, 1}}][0]],
times [L[{{1, 2}, {1, 1}, {1, 2}}, {{1, 3}, {1, 4}, {2, 3}}][0],
L[{{2, 4}}, {{1, 1}}][0]],
times [L[{{1, 1}, {1, 2}, {1, 2}}, {{1, 3}, {1, 4}, {2, 4}}][0],
L[{{2, 3}}, {{1, 1}}][0]],
times [L[{{1, 2}, {1, 2}, {1, 1}}, {{1, 3}, {2, 3}, {2, 4}}][0],
L[{{1, 4}}, {{1, 1}}][0]],
times [L[{{1, 2}, {1, 1}, {1, 2}}, {{1, 4}, {2, 3}, {2, 4}}][0],
L[{{1, 3}}, {{1, 1}}][0]]]]

```

In[14]:= TreeForm [%]

Out[14]/TreeForm=

