```
(* Manuel Kauers's Asymptotics package. Download from http://
 www.risc.jku.at/research/combinat/software/Asymptotics/ *)
<< Asymptotics.m
```

```
Asymptotics  Package by Manuel Kauers — © RISC Linz — V 0.8 (2011-06-28)
```

```
(* Christoph Koutschan's HolonomicFunctions package. Download from http://
 www.risc.jku.at/research/combinat/software/HolonomicFunctions/ *)
<< HolonomicFunctions.m
```

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.5.1 (09.08.2011)
⟶ Type ?HolonomicFunctions for help

Some useful procedures

```
(* Given a differential equation for f(z),
compute the recurrence for the partial sums of the Taylor coefficients of f(z). *)
Clear[RecurrenceForReturnProbability];
RecurrenceForReturnProbability[ann_, Der[z_], f_[n_]] :=
  Module[{ann1, ann2, rec},
   (* View f(z) as a bivariate function f(n,z) *)
   ann1 = ToOrePolynomial[Append[ann, S[n] - 1], OreAlgebra[Der[z], S[n]]];
   ann2 = Annihilator[1 / (1 - z) / z ^ (n + 1), {Der[z], S[n]}];
   (* This computes a recurrence for the Taylor
    coefficients of f(z)/(1-z) via Cauchy's integral formula. *)
   rec = FindCreativeTelescoping[DFiniteTimes[ann1, ann2], Der[z]][[1, 1]];
   Return[ApplyOreOperator[Factor[rec], f[n]]];
  ];

(* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] ⧴ a, Infinity]];
   Do[
    AppendTo[vals, Solve[rec /. n → i /. f[i] → x /. f[a_] ⧴ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];

(* Given the first values {f[0],...,f[m]} of
 a sequence f[n] and a basis of its asymptotic solutions,
compute the limit Limit[f[n], n→Infinity]. *)
Clear[MyLimit];
MyLimit[data_List, asym_, n_] :=
  Module[{c, d = Length[asym], pos, ansatz, sol},
   pos = Length[data] + Range[-d, -1];
   ansatz = Array[c, d].asym;
   sol = Solve[((ansatz /. n → #) == data[[# + 1]]) & /@ pos, Array[c, d]][[1]];
   Return[N[c[d] /. sol, 200]];
  ];
```

Example: The two-dimensional lattice

```
ann = Annihilator[1 / (1 - z * x1 * x2) / Sqrt[1 - x1^2] / Sqrt[1 - x2^2], {Der[x1], Der[x2], Der[z]}]
```

$$\{(-1+x1\,x2\,z)\,D_z + x1\,x2,\ (1 - x2^2 - x1\,x2\,z + x1\,x2^3\,z)\,D_{x2} + (-x2 - x1\,z + 2\,x1\,x2^2\,z),\ (1 - x1^2 - x1\,x2\,z + x1^3\,x2\,z)\,D_{x1} + (-x1 - x2\,z + 2\,x1^2\,x2\,z)\}$$

```
FindCreativeTelescoping[ann, {Der[x1], Der[x2]}]
```

$$\left\{\left\{(-z+z^3)\,D_z^2 + (-1+3\,z^2)\,D_z + z\right\},\ \left\{\left\{\frac{x2 - x1^2\,x2}{-1+x1\,x2\,z},\ \frac{x2\,z - x2^3\,z}{-1+x1\,x2\,z}\right\}\right\}\right\}$$

```
OreReduce[%[[1, 1]] + Der[x1] ** %[[2, 1, 1]] + Der[x2] ** %[[2, 1, 2]],
 Together[ann], Extended → True]
```

$$\left\{0,\ 1,\ \left\{\frac{-z+z^3}{-1+x1\,x2\,z}\,D_z + \frac{1 + x1\,x2\,z - 3\,z^2 + x1\,x2\,z^3}{(-1+x1\,x2\,z)^2},\ -\frac{x2\,z}{(-1+x1\,x2\,z)^2},\ -\frac{x2}{(-1+x1\,x2\,z)^2}\right\}\right\}$$

```
FindCreativeTelescoping[ann, Der[x1]]
```

$$\left\{\left\{(-1+x2^2\,z^2)\,D_z + x2^2\,z,\ (1 - x2^2 - x2^2\,z^2 + x2^4\,z^2)\,D_{x2} + (-x2 - x2\,z^2 + 2\,x2^3\,z^2)\right\},\ \left\{\{-x2 + x1^2\,x2\},\ \{z - x1^2\,z - x2^2\,z + x1^2\,x2^2\,z\}\right\}\right\}$$

```
FindCreativeTelescoping[First[%], Der[x2]]
```

$$\left\{\left\{(-z+z^3)\,D_z^2 + (-1+3\,z^2)\,D_z + z\right\},\ \left\{\left\{\frac{x2\,z - x2^3\,z}{-1+x2^2\,z^2}\right\}\right\}\right\}$$

```
ApplyOreOperator[%[[1, 1]], P[z]] == 0
```

$$z\,P[z] + (-1+3\,z^2)\,P'[z] + (-z+z^3)\,P''[z] == 0$$

```
DSolve[{%, P[0] == 1, P'[0] == 0}, P[z], z]
```

$$\left\{\left\{P[z] \to \frac{2\,\text{EllipticK}[z^2]}{\pi}\right\}\right\}$$

The four-dimensional face-centred cubic lattice

Differential equation for the LGF

The lattice Green's function is given by the following four-fold integral:

```
TraditionalForm[
 HoldForm[Integrate[1 / (1 - z / 6 * (Cos[k1] * Cos[k2] + Cos[k1] * Cos[k3] + Cos[k2] * Cos[k3] +
          Cos[k1] * Cos[k4] + Cos[k2] * Cos[k4] + Cos[k3] * Cos[k4])),
   {k1, 0, Pi}, {k2, 0, Pi}, {k3, 0, Pi}, {k4, 0, Pi}]]]
```

$$\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi \frac{1}{1-\frac{1}{6}z\,(\cos(k1)\cos(k2)+\cos(k1)\cos(k3)+\cos(k2)\cos(k3)+\cos(k1)\cos(k4)+\cos(k2)\cos(k4)+\cos(k3)\cos(k4))}\,dk4\,dk3\,dk2\,dk1$$

After the substitutions xi=cos(ki) the integrand transforms to:

```
integrand = 1 / (1 - z / 6 * (x1 * x2 + x1 * x3 + x1 * x4 + x2 * x3 + x2 * x4 + x3 * x4)) /
   (Sqrt[1 - x1^2] Sqrt[1 - x2^2] Sqrt[1 - x3^2] Sqrt[1 - x4^2]);
```

```
ann0 = Annihilator[integrand, {Der[x1], Der[x2], Der[x3], Der[x4], Der[z]}]
```

$$\{(-6 + x1\,x2\,z + x1\,x3\,z + x2\,x3\,z + x1\,x4\,z + x2\,x4\,z + x3\,x4\,z)\,D_z + (x1\,x2 + x1\,x3 + x2\,x3 + x1\,x4 + x2\,x4 + x3\,x4),$$
$$(6 - 6\,x4^2 - x1\,x2\,z - x1\,x3\,z - x2\,x3\,z - x1\,x4\,z - x2\,x4\,z - x3\,x4\,z + x1\,x2\,x4^2\,z + x1\,x3\,x4^2\,z + x2\,x3\,x4^2\,z + x1\,x4^3\,z + x2\,x4^3\,z + x3\,x4^3\,z)\,D_{x4} + (-6\,x4 - x1\,z - x2\,z - x3\,z + x1\,x2\,x4\,z + x1\,x3\,x4\,z + x2\,x3\,x4\,z + 2\,x1\,x4^2\,z + 2\,x2\,x4^2\,z + 2\,x3\,x4^2\,z),$$
$$(6 - 6\,x3^2 - x1\,x2\,z - x1\,x3\,z - x2\,x3\,z + x1\,x2\,x3^2\,z + x1\,x3^3\,z + x2\,x3^3\,z - x1\,x4\,z - x2\,x4\,z - x3\,x4\,z + x1\,x3^2\,x4\,z + x2\,x3^2\,x4\,z + x3^3\,x4\,z)\,D_{x3} + (-6\,x3 - x1\,z - x2\,z + x1\,x2\,x3\,z + 2\,x1\,x3^2\,z + 2\,x2\,x3^2\,z - x4\,z + x1\,x3\,x4\,z + x2\,x3\,x4\,z + 2\,x3^2\,x4\,z),$$
$$(6 - 6\,x2^2 - x1\,x2\,z + x1\,x2^3\,z - x1\,x3\,z - x2\,x3\,z + x1\,x2^2\,x3\,z + x2^3\,x3\,z - x1\,x4\,z - x2\,x4\,z + x1\,x2^2\,x4\,z + x2^3\,x4\,z - x3\,x4\,z + x2^2\,x3\,x4\,z)\,D_{x2} + (-6\,x2 - x1\,z + 2\,x1\,x2^2\,z - x3\,z + x1\,x2\,x3\,z + 2\,x2^2\,x3\,z - x4\,z + x1\,x2\,x4\,z + 2\,x2^2\,x4\,z + x2\,x3\,x4\,z),$$
$$(6 - 6\,x1^2 - x1\,x2\,z + x1^3\,x2\,z - x1\,x3\,z + x1^3\,x3\,z - x2\,x3\,z + x1^2\,x2\,x3\,z - x1\,x4\,z + x1^3\,x4\,z - x2\,x4\,z + x1^2\,x2\,x4\,z - x3\,x4\,z + x1^2\,x3\,x4\,z)\,D_{x1} + (-6\,x1 - x2\,z + 2\,x1^2\,x2\,z - x3\,z + 2\,x1^2\,x3\,z + x1\,x2\,x3\,z - x4\,z + 2\,x1^2\,x4\,z + x1\,x2\,x4\,z + x1\,x3\,x4\,z)\}$$

ann0 is a system of PDEs for the integrand.

```
Timing[
 {ann1, delta1} = CreativeTelescoping[ann0, Der[x1], {Der[x2], Der[x3], Der[x4], Der[z]}];]
```

```
{0.880054, Null}
```

```
(* Test *)
Timing[OreReduce[MapThread[(#1 + Der[x1] ** #2) &, {ann1, delta1}], ann0]]
```

```
{0.364023, {0, 0, 0, 0}}
```

Hence, ann1 is a system of PDEs for the first integral (integration w.r.t. x1).

```
Timing[{ann2, delta2} = CreativeTelescoping[ann1, Der[x2], {Der[x3], Der[x4], Der[z]}];]
```

```
{2.88818, Null}
```

```
(* Test *)
Timing[OreReduce[MapThread[(#1 + Der[x2] ** #2) &, {ann2, delta2}], ann1]]
```

```
{1.45609, {0, 0, 0}}
```

Hence, ann2 is a system of PDEs for the second integral (integration w.r.t. x1 and x2).

```
Timing[{ann3, delta3} = FindCreativeTelescoping[ann2, Der[x3]];]
```

```
{147.629, Null}
```

```
(* Test *)
Timing[OreReduce[MapThread[(#1 + Der[x3] ** First[#2]) &, {ann3, delta3}], ann2]]
```

```
{7.69248, {0, 0, 0}}
```

Hence, ann3 is a system of PDEs for the third integral (integration w.r.t. x1, x2, and x3).

```
Timing[{ann4, delta4} = FindCreativeTelescoping[ann3, Der[x4]];]
```

```
{68.7043, Null}
```

```
(* Test *)
Timing[OreReduce[ann4[[1]] + Der[x4] ** delta4[[1, 1]], ann3]]
```

{2.18014, 0}

Hence, ann4 is an ODE for the original four-fold integral.

```
Factor[ann4]
```

$$\{(-1+z)\,z^3\,(2+z)\,(3+z)\,(6+z)\,(8+z)\,(4+3z)^2\,D_z^4 + 2z^2\,(4+3z)\,(-3456-2304z+3676z^2+4920z^3+2079z^4+356z^5+21z^6)\,D_z^3 + 6z\,(-5376-5248z+11080z^2+25286z^3+19898z^4+7432z^5+1286z^6+81z^7)\,D_z^2 + 12\,(-384+224z+3716z^2+7633z^3+6734z^4+2939z^5+604z^6+45z^7)\,D_z + 12z\,(256+632z+702z^2+382z^3+98z^4+9z^5)\}$$

The same differential equation, using the notation $\theta_x = x\,D_x$:

```
Factor[ChangeOreAlgebra[z ** ann4[[1]], OreAlgebra[Euler[z]]]]
```

$$(-1+z)\,(2+z)\,(3+z)\,(6+z)\,(8+z)\,(4+3z)^2\,\theta_z^4 + 4z\,(4+3z)\,(360+800z+930z^2+459z^3+91z^4+6z^5)\,\theta_z^3 + z\,(3648+16912z+35184z^2+31620z^3+13949z^4+2850z^5+207z^6)\,\theta_z^2 + 2z\,(4+3z)\,(96+1280z+2262z^2+1563z^3+448z^4+42z^5)\,\theta_z + 12z^2\,(256+632z+702z^2+382z^3+98z^4+9z^5)$$

Return probability

The following command computes a recurrence for f(n)=p(0)+p(1)+...+p(n), where p(k) is the probability that a random walk ends at the origin after exactly k steps.

```
Timing[rec4 = RecurrenceForReturnProbability[ann4, Der[z], f[n]]]
```

$$\{10.5887,\ (2+n)\,(3+n)^2\,(4+n)\,(1252+420n+35n^2)\,f[n] + (3+n)\,(4+n)\,(276024+244384n+79874n^2+11375n^3+595n^4)\,f[1+n] + 3\,(4+n)\,(2638272+2769392n+1156976n^2+240253n^3+24780n^4+1015n^5)\,f[2+n] + (71984160+83134488n+39767416n^2+10080600n^3+1427695n^4+107100n^5+3325n^6)\,f[3+n] - 4\,(33590844+40139838n+19973086n^2+5295615n^3+788848n^4+62580n^5+2065n^6)\,f[4+n] - 12\,(24690708+26259960n+11601091n^2+2725632n^3+359282n^4+25200n^5+735n^6)\,f[5+n] + 288\,(6+n)^4\,(867+350n+35n^2)\,f[6+n]\}$$

These are the asymptotic solutions of this recurrence:

```
Asymptotics[rec4, f[n], Order → 3]
```

$$\left\{\frac{\left(-\frac{1}{2}\right)^n\left(1+\frac{1459}{144n^3}+\frac{67}{24n^2}-\frac{5}{6n}\right)}{n^2},\ \frac{\left(-\frac{1}{3}\right)^n\left(1-\frac{143}{8n^3}+\frac{51}{8n^2}-\frac{5}{2n}\right)}{n^2},\ \frac{\left(-\frac{1}{6}\right)^n\left(1-\frac{112407}{5488n^3}+\frac{4633}{392n^2}-\frac{45}{14n}\right)}{n^2},\ \frac{\left(-\frac{1}{8}\right)^n\left(1-\frac{45820}{243n^3}+\frac{812}{27n^2}-\frac{52}{9n}\right)}{n^2},\ \frac{1-\frac{7}{18n^3}+\frac{7}{9n^2}-\frac{1}{n}}{n},\ 1\right\}$$

The number of excursions in the four-dimensional fcc lattice:

```
exc4 = {1, 0, 24, 192, 3384, 51840};
```

```
Timing[data =
   UnrollRecurrence[rec4, f[n], Table[Sum[exc4[[i + 1]] / 24^i, {i, 0, n}], {n, 0, 5}], 5000];]
```

```
{13.1688, Null}
```

```
Timing[lim4 = MyLimit[data, Asymptotics[rec4, f[n], Order → 30], n]]
```

```
{80.677,
 1.105843797921204760182995470885851074439546236638752858364998386825835100010255700965111
  1305690718862272402527009292198664847691123786296334442863147256320053880897444741858617
  595404679726397225392801}
```

```
Timing[lim4a = MyLimit[data, Asymptotics[rec4, f[n], Order → 32], n]]
```

```
{81.5251,
 1.105843797921204760182995470885851074439546236638752858364998386825835100010255700965111
  1305690718862267001229047443820159586573675211842712025872852514448221279799368187027415
  910001510904910331330024}
```

The results agree on more than 100 digits, and thus we can conclude that (at least) the first 100 digits are correct:

```
lim4 - lim4a
```

```
5.4012979618483785052611174485744536224169902947418718326010980765548312016854031688214868
  94062777 × 10^-103
```

The return probability therefore is

```
1 - 1 / lim4a
```

```
0.0957131541725628967353167649012101857007088196380173576877465530968471922641436096599459
 0968474427003288153228800889179034306223093561370340335742933305422318836693229494948820
 17530971756513580243234
```

The five-dimensional face-centred cubic lattice

■ Differential equation for the LGF

```
TraditionalForm[
 HoldForm[Integrate[1 / (1 - z / 10 * (Cos[k1] * Cos[k2] + Cos[k1] * Cos[k3] + Cos[k1] * Cos[k4] +
         Cos[k1] * Cos[k5] + Cos[k2] * Cos[k3] + Cos[k2] * Cos[k4] + Cos[k2] * Cos[k5] +
         Cos[k3] * Cos[k4] + Cos[k3] * Cos[k5] + Cos[k4] * Cos[k5])),
   {k1, 0, Pi}, {k2, 0, Pi}, {k3, 0, Pi}, {k4, 0, Pi}, {k5, 0, Pi}]]]
```

$$\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi 1 \Big/ \left(1 - \frac{1}{10} z\,(\cos(\text{k1})\cos(\text{k2}) + \cos(\text{k1})\cos(\text{k3}) + \cos(\text{k1})\cos(\text{k4}) + \cos(\text{k1})\cos(\text{k5}) + \cos(\text{k2})\cos(\text{k3}) + \cos(\text{k2})\cos(\text{k4}) + \cos(\text{k2})\cos(\text{k5}) + \cos(\text{k3})\cos(\text{k4}) + \cos(\text{k3})\cos(\text{k5}) + \cos(\text{k4})\cos(\text{k5}))\right) d\text{k5}\, d\text{k4}\, d\text{k3}\, d\text{k2}\, d\text{k1}$$

After the substitutions xi = cos (ki) the integrand transforms to :

```
integrand = 1 / (1 -
      z / 10 * (x1 * x2 + x1 * x3 + x1 * x4 + x1 * x5 + x2 * x3 + x2 * x4 + x2 * x5 + x3 * x4 + x3 * x5 + x4 * x5)) /
   (Sqrt[1 - x1^2] * Sqrt[1 - x2^2] * Sqrt[1 - x3^2] * Sqrt[1 - x4^2] * Sqrt[1 - x5^2]);
```

```
ann0 = Annihilator[integrand, {Der[x1], Der[x2], Der[x3], Der[x4], Der[x5], Der[z]}];
```

ann0 is a system of PDEs for the integrand.

```
Timing[{ann1, delta1} =
   CreativeTelescoping[ann0, Der[x1], {Der[x2], Der[x3], Der[x4], Der[x5], Der[z]}];]
```

```
{1.38409, Null}
```

```
(* Test *)
Timing[OreReduce[MapThread[(#1 + Der[x1] ** #2) &, {ann1, delta1}], ann0]]
```

```
{0.840052, {0, 0, 0, 0, 0}}
```

Hence, ann1 is a system of PDEs for the first integral (integration w.r.t. x1).

```
Timing[
 {ann2, delta2} = CreativeTelescoping[ann1, Der[x2], {Der[x3], Der[x4], Der[x5], Der[z]}];]
```

```
{27.8217, Null}
```

```
(* Test *)
Timing[OreReduce[MapThread[(#1 + Der[x2] ** #2) &, {ann2, delta2}], ann1]]
```

```
{8.42453, {0, 0, 0, 0}}
```

Hence, ann2 is a system of PDEs for the second integral (integration w.r.t. x1 and x2).

To compute the following certificates takes some time, and therefore they are provided in the file fcc5_cert.m.

```
{ann3, delta3, ann4, delta4, ann5, delta5} = << "fcc5_cert.m";
```

```
Timing[OreReduce[MapThread[(#1 + Der[x3] ** #2) &, {ann3, delta3}], ann2]]
```

```
{160.558, {0, 0, 0, 0}}
```

Hence, ann3 is a system of PDEs for the third integral (integration w.r.t. x1, x2, and x3).

```
Timing[OreReduce[MapThread[(#1 + Der[x4] ** #2) &, {ann4, delta4}], ann3]]
```

```
{265.957, {0, 0, 0}}
```

Hence, ann4 is a system of PDEs for the fourth integral (integration w.r.t. x1, x2, x3, and x4).

```
Timing[OreReduce[ann5[[1]] + Der[x5] ** delta5[[1]], ann4]]
```

```
{26.7897, 0}
```

Hence, ann5 is an ODE for the original five-fold integral.

```
Factor[ann5]
```

$$\begin{aligned}
\{&16\,(-5+z)\,(-1+z)\,z^4\,(5+z)^2\,(10+z)\,(15+z)\,(5+3z)\\
&\quad(-675000+3465000 z-1053375 z^2+933650 z^3+449735 z^4+144776 z^5+15678 z^6)\,D_z^6+\\
&8 z^3\,(5+z)\,(-354375000000+1774828125000 z-503550000000 z^2-1289447109375 z^3+\\
&\quad 254876515625 z^4-266627903125 z^5-304623830625 z^6-87265479875 z^7-\\
&\quad 4878146975 z^8+3939663705 z^9+1048560285 z^{10}+97471734 z^{11}+3057210 z^{12})\,D_z^5+\\
&10 z^2\,(-5568750000000+23905125000000 z+3393646875000 z^2-39702348750000 z^3-\\
&\quad 7716298734375 z^4-3779011321875 z^5-7801785421250 z^6-\\
&\quad 3351125770500 z^7-382134335775 z^8+148313757125 z^9+\\
&\quad 68439921540 z^{10}+11725276842 z^{11}+923795772 z^{12}+27279720 z^{13})\,D_z^4+\\
&5 z\,(-13162500000000+45343125000000 z+40530375000000 z^2-190176960000000 z^3-\\
&\quad 77498059625000 z^4-3649915059375 z^5-26918293320000 z^6-\\
&\quad 13545524756500 z^7-465440555100 z^8+1350059072325 z^9+\\
&\quad 524857986060 z^{10}+92744995638 z^{11}+7892060544 z^{12}+255864960 z^{13})\,D_z^3+\\
&5\,(-3240000000000+5055750000000 z+44457862500000 z^2-133825053750000 z^3-\\
&\quad 110925736437500 z^4+13367806743750 z^5-6199228765625 z^6-\\
&\quad 8282515456375 z^7+1646226060075 z^8+2287368823475 z^9+\\
&\quad 810956145330 z^{10}+149186684934 z^{11}+13819981248 z^{12}+496679040 z^{13})\,D_z^2+\\
&10\,(-189000000000+4816462500000 z-7268326875000 z^2-21210430812500 z^3+\\
&\quad 2664478321875 z^4+3711617481250 z^5-135661728250 z^6+689643286650 z^7+\\
&\quad 607021304825 z^8+209673119160 z^9+40678130502 z^{10}+4143853440 z^{11}+167064768 z^{12})\,D_z+\\
&30\,(27000000000+84037500000 z-346865625000 z^2-55567000000 z^3+\\
&\quad 187923165625 z^4+36477006875 z^5+21336230625 z^6+19123388575 z^7+\\
&\quad 6925739310 z^8+1443544710 z^9+163913184 z^{10}+7525440 z^{11})\}
\end{aligned}$$

The same differential equation, using the notation $\theta_x = x\,D_x$:

```
Factor[ChangeOreAlgebra[z^2 ** ann5[[1]], OreAlgebra[Euler[z]]]]
```

$$\begin{aligned}
&16\,(-5+z)\,(-1+z)\,(5+z)^2\,(10+z)\,(15+z)\,(5+3z)\\
&\quad(-675000+3465000 z-1053375 z^2+933650 z^3+449735 z^4+144776 z^5+15678 z^6)\,\theta_z^6+\\
&8\,(5+z)\,(25312500000-262828125000 z+277973437500 z^2-623974453125 z^3-\\
&\quad 99906765625 z^4-18813934375 z^5-63011949375 z^6-14233648625 z^7+\\
&\quad 1167216775 z^8+1815664755 z^9+476108775 z^{10}+48225714 z^{11}+1646190 z^{12})\,\theta_z^5+\\
&10 z\,(212625000000+859865625000 z-2723017500000 z^2-2570837484375 z^3+\\
&\quad 838326928125 z^4-83768271250 z^5-174211300500 z^6+43009502225 z^7+\\
&\quad 50490711925 z^8+17584563300 z^9+3149281834 z^{10}+276965244 z^{11}+9218664 z^{12})\,\theta_z^4+\\
&5 z\,(182250000000+3082050000000 z-4379484375000 z^2-12348738562500 z^3+\\
&\quad 2538370303125 z^4+1418257585000 z^5-262779490500 z^6+345631048200 z^7+\\
&\quad 290432449625 z^8+98126818860 z^9+18404776374 z^{10}+1769720304 z^{11}+65847600 z^{12})\,\theta_z^3+\\
&z\,(151875000000+11539968750000 z-3860325000000 z^2-72944256093750 z^3+7977971203125 z^4+\\
&\quad 17729052934375 z^5+857477030625 z^6+2750916871625 z^7+2240908773050 z^8+\\
&\quad 765974536350 z^9+149808033328 z^{10}+15460889736 z^{11}+629503056 z^{12})\,\theta_z^2+\\
&z^2\,(4596750000000+5231250000000 z-43311088125000 z^2-384877781250 z^3+\\
&\quad 16580971915625 z^4+2238968904375 z^5+2109517529625 z^6+1787830780225 z^7+\\
&\quad 628047922110 z^8+127282042530 z^9+13877710128 z^{10}+605797920 z^{11})\,\theta_z+\\
&30 z^2\,(27000000000+84037500000 z-346865625000 z^2-55567000000 z^3+\\
&\quad 187923165625 z^4+36477006875 z^5+21336230625 z^6+19123388575 z^7+\\
&\quad 6925739310 z^8+1443544710 z^9+163913184 z^{10}+7525440 z^{11})
\end{aligned}$$

Return probability

The following command computes a recurrence for f(n)=p(0)+p(1)+...+p(n), where p(k) is the probability that a random walk ends at the origin after exactly k steps.

```
Timing[rec5 = RecurrenceForReturnProbability[ann5, Der[z], f[n]]]
```

$$\{75.4767,\; 24\,(2+n)\,(3+n)\,(4+n)\,(5+n)\,(6+n)\,(7+2n)\,(23\,706\,589\,420 + 21\,322\,381\,039\,n + 7\,959\,774\,759\,n^2 + 1\,578\,465\,524\,n^3 + 175\,364\,444\,n^4 + 10\,348\,800\,n^5 + 253\,440\,n^6)\,f[n] + 4\,(3+n)\,(4+n)\,(5+n)\,(6+n)\,(138\,987\,206\,947\,500 + 193\,212\,535\,985\,983\,n + 116\,459\,870\,018\,385\,n^2 + 39\,751\,367\,294\,114\,n^3 + 8\,404\,610\,264\,384\,n^4 + 1\,127\,322\,902\,136\,n^5 + 93\,700\,520\,944\,n^6 + 4\,413\,573\,120\,n^7 + 90\,224\,640\,n^8)\,f[1+n] + (4+n)\,(5+n)\,(6+n)\,(26\,710\,903\,123\,486\,020 + 41\,305\,602\,189\,359\,877\,n + 28\,251\,939\,787\,235\,239\,n^2 + 11\,215\,264\,920\,324\,246\,n^3 + 2\,847\,175\,232\,103\,428\,n^4 + 479\,290\,776\,087\,896\,n^5 + 53\,496\,284\,395\,104\,n^6 + 3\,817\,402\,535\,872\,n^7 + 158\,023\,557\,120\,n^8 + 2\,891\,243\,520\,n^9)\,f[2+n] - 5\,(5+n)\,(6+n)\,(2\,232\,465\,211\,738\,124 + 7\,802\,599\,557\,838\,407\,n + 8\,847\,396\,323\,941\,916\,n^2 + 5\,226\,658\,104\,645\,789\,n^3 + 1\,888\,683\,847\,536\,752\,n^4 + 447\,334\,429\,518\,188\,n^5 + 71\,254\,623\,078\,544\,n^6 + 7\,594\,518\,908\,896\,n^7 + 520\,809\,706\,304\,n^8 + 20\,815\,872\,000\,n^9 + 369\,008\,640\,n^{10})\,f[3+n] - 25\,(6+n)\,(1\,642\,494\,821\,461\,378\,480 + 2\,990\,687\,642\,172\,062\,696\,n + 2\,468\,085\,541\,356\,043\,223\,n^2 + 1\,218\,615\,893\,860\,632\,386\,n^3 + 400\,007\,711\,917\,219\,547\,n^4 + 91\,658\,298\,351\,560\,748\,n^5 + 14\,961\,187\,466\,706\,772\,n^6 + 1\,739\,655\,580\,442\,624\,n^7 + 141\,220\,760\,846\,368\,n^8 + 7\,622\,345\,533\,376\,n^9 + 246\,194\,995\,200\,n^{10} + 3\,604\,930\,560\,n^{11})\,f[4+n] - 125\,(7\,414\,827\,775\,901\,488\,440 + 14\,111\,791\,195\,530\,452\,898\,n + 12\,273\,613\,444\,295\,099\,931\,n^2 + 6\,450\,917\,553\,965\,772\,615\,n^3 + 2\,282\,093\,767\,250\,079\,933\,n^4 + 572\,479\,000\,108\,414\,179\,n^5 + 104\,425\,619\,207\,535\,936\,n^6 + 13\,956\,414\,715\,818\,052\,n^7 + 1\,356\,434\,255\,302\,944\,n^8 + 93\,499\,824\,186\,720\,n^9 + 4\,339\,051\,513\,280\,n^{10} + 121\,725\,542\,400\,n^{11} + 1\,561\,190\,400\,n^{12})\,f[5+n] + 1250\,(859\,055\,636\,115\,898\,224 + 1\,659\,860\,314\,269\,140\,764\,n + 1\,465\,298\,720\,791\,597\,480\,n^2 + 781\,583\,568\,522\,873\,365\,n^3 + 280\,582\,117\,360\,621\,811\,n^4 + 71\,427\,191\,465\,724\,051\,n^5 + 13\,222\,768\,332\,630\,509\,n^6 + 1\,793\,740\,942\,122\,328\,n^7 + 176\,984\,999\,903\,228\,n^8 + 12\,387\,985\,271\,408\,n^9 + 583\,919\,720\,864\,n^{10} + 16\,643\,235\,840\,n^{11} + 216\,944\,640\,n^{12})\,f[6+n] + 25\,000\,(7+n)\,(18\,109\,125\,128\,518\,296 + 30\,534\,544\,383\,191\,194\,n + 23\,294\,752\,486\,108\,695\,n^2 + 10\,614\,769\,685\,024\,765\,n^3 + 3\,210\,352\,117\,320\,120\,n^4 + 676\,745\,222\,548\,353\,n^5 + 101\,474\,598\,702\,495\,n^6 + 10\,824\,478\,189\,310\,n^7 + 805\,118\,488\,524\,n^8 + 39\,772\,907\,848\,n^9 + 1\,174\,609\,920\,n^{10} + 15\,713\,280\,n^{11})\,f[7+n] - 1\,500\,000\,(7+n)\,(8+n)^5\,(8\,930\,786\,700 + 9\,486\,993\,677\,n + 4\,176\,878\,451\,n^2 + 975\,426\,948\,n^3 + 127\,422\,044\,n^4 + 8\,828\,160\,n^5 + 253\,440\,n^6)\,f[8+n]\}$$

These are the asymptotic solutions of this recurrence:

```
Asymptotics[rec5, f[n], Order → 3]
```

$$\left\{\frac{\left(-\frac{3}{5}\right)^n \left(1+\frac{87\,063}{65\,536\,n^3}+\frac{4533}{2048\,n^2}-\frac{55}{32\,n}\right)}{n^{5/2}},\; \frac{\left(-\frac{1}{5}\right)^n \left(1-\frac{3\,483\,567\,931}{212\,336\,640\,n^3}+\frac{9\,449\,309}{1\,105\,920\,n^2}-\frac{1931}{576\,n}\right)}{n^{9/4}},\; \frac{\left(-\frac{1}{5}\right)^n \left(1-\frac{47\,753\,699}{4\,718\,592\,n^3}+\frac{434\,671}{73\,728\,n^2}-\frac{529}{192\,n}\right)}{n^{7/4}},\; \frac{\left(-\frac{1}{10}\right)^n \left(1-\frac{199\,852\,087}{1\,362\,944\,n^3}+\frac{475\,737}{15\,488\,n^2}-\frac{541}{88\,n}\right)}{n^{5/2}},\; \frac{\left(-\frac{1}{15}\right)^n \left(1-\frac{9\,127\,341}{32\,768\,n^3}+\frac{47\,903}{1024\,n^2}-\frac{249}{32\,n}\right)}{n^{5/2}},\; \frac{5^{-n} \left(1-\frac{854\,831}{8192\,n^3}+\frac{13\,533}{512\,n^2}-\frac{97}{16\,n}\right)}{n^{5/2}},\; \frac{1-\frac{5795}{6144\,n^3}+\frac{2675}{1792\,n^2}-\frac{3}{2\,n}}{n^{3/2}},\; 1\right\}$$

The number of excursions in the five-dimensional fcc lattice:

```
exc5 = {1, 0, 40, 480, 11880, 281280, 7506400, 210268800};
```

```
Timing[data =
   UnrollRecurrence[rec5, f[n], Table[Sum[exc5[[i + 1]] / 40^i, {i, 0, n}], {n, 0, 7}], 5000];]
```

```
{28.1378, Null}
```

```
Timing[lim5 = MyLimit[data, Asymptotics[rec5, f[n], Order → 34], n]]
```

```
{313.48,
 1.04885235135491485162956376369999275945402550465206640313845200422054063668415716918834 1\
  63184734628100841498003429627013743416202875798446192894483917380733940922357991885743 30\
  054328015418809439811542}
```

```
Timing[lim5a = MyLimit[data, Asymptotics[rec5, f[n], Order → 36], n]]
```

```
{395.193,
 1.04885235135491485162956376369999275945402550465206640313845200422054063668415716918834 1\
  63184734628100841489922788161401894597628171916196906344032947268499185571483828496246 38\
  7534450759994684695312999}
```

The results agree on more than 100 digits, and thus we can conclude that (at least) the first 100 digits are correct:

```
lim5 - lim5a
```

```
8.080641465611848818574703882249286550450970112234755350874163389496913008829394193409702 8\
  0243 × 10^-107
```

The return probability therefore is

```
1 - 1 / lim5a
```

```
0.046576957463848024193374420594803291076402397746321129305321681495362551357283064925322 9\
 49340049157638967365573263262385837524051242561169835647178333357126940980051221943813848\
 6050541559428107002231 4
```

The six-dimensional face-centred cubic lattice

Differential equation for the LGF

```
combs = Total[Times @@@ Subsets[Symbol["x" <> ToString[#]] & /@ Range[6], {2}]]
```

$$x1\,x2 + x1\,x3 + x2\,x3 + x1\,x4 + x2\,x4 + x3\,x4 + x1\,x5 + x2\,x5 + x3\,x5 + x4\,x5 + x1\,x6 + x2\,x6 + x3\,x6 + x4\,x6 + x5\,x6$$

```
integrand =
 1 / (1 - z / Length[combs] * combs) / Product[Sqrt[1 - Symbol["x" <> ToString[i]]^2], {i, 6}]
```

$$1\Big/\left(\sqrt{1-x1^2}\,\sqrt{1-x2^2}\,\sqrt{1-x3^2}\,\sqrt{1-x4^2}\,\sqrt{1-x5^2}\,\sqrt{1-x6^2}\,\left(1-\frac{1}{15}\,(x1\,x2+x1\,x3+x2\,x3+x1\,x4+x2\,x4+x3\,x4+x1\,x5+x2\,x5+x3\,x5+x4\,x5+x1\,x6+x2\,x6+x3\,x6+x4\,x6+x5\,x6)\,z\right)\right)$$

```
ann0 = Annihilator[integrand, {Der[x1], Der[x2], Der[x3], Der[x4], Der[x5], Der[x6], Der[z]}];
```

ann0 is a system of PDEs for the integrand.

```
Timing[{ann1, delta1} =
   CreativeTelescoping[ann0, Der[x1], {Der[x2], Der[x3], Der[x4], Der[x5], Der[x6], Der[z]}];]
```

```
{2.67617, Null}
```

```
(* Test *)
Timing[OreReduce[MapThread[(#1 + Der[x1] ** #2) &, {ann1, delta1}], ann0]]
```

```
{1.83211, {0, 0, 0, 0, 0, 0}}
```

Hence, ann1 is a system of PDEs for the first integral (integration w.r.t. x1).

```
Timing[{ann2, delta2} =
   CreativeTelescoping[ann1, Der[x2], {Der[x3], Der[x4], Der[x5], Der[x6], Der[z]}];]
```

```
{691.615, Null}
```

```
(* Test *)
Timing[OreReduce[MapThread[(#1 + Der[x2] ** #2) &, {ann2, delta2}], ann1]]
```

```
{53.7994, {0, 0, 0, 0, 0}}
```

Hence, ann2 is a system of PDEs for the second integral (integration w.r.t. x1 and x2).

```
{ann3, delta3, ann4, delta4, ann5, delta5, ann6, delta6} = << "fcc6_cert.m";
```

```
Timing[OreReduce[MapThread[(#1 + Der[x3] ** #2) &, {ann3, delta3}], ann2]]
```

```
{5253.16, {0, 0, 0, 0, 0}}
```

```
(* A quicker (but non-rigorous) correctness check. *)
subs = {x4 → 92, x5 → 55, x6 → -74, z → 23};
Timing[{ann3a, delta3a} = OrePolynomialSubstitute[#, subs] & /@ {ann3, delta3};
 OreReduce[MapThread[(#1 + Der[x3] ** #2) &, {ann3a, delta3a}],
  ann2, OrePolynomialSubstitute → subs]]
```

```
{12.2168, {0, 0, 0, 0, 0}}
```

Hence, ann3 is a system of PDEs for the third integral (integration w.r.t. x1, x2, and x3).

```
Timing[OreReduce[MapThread[(#1 + Der[x4] ** #2) &, {ann4, delta4}], ann3]]
```

```
{12 665.4, {0, 0, 0, 0}}
```

```
(* A quicker (but non-rigorous) correctness check. *)
subs = {x5 → 55, x6 → -74, z → 23};
Timing[{ann4a, delta4a} = OrePolynomialSubstitute[#, subs] & /@ {ann4, delta4};
 OreReduce[MapThread[(#1 + Der[x4] ** #2) &, {ann4a, delta4a}],
  ann3, OrePolynomialSubstitute → subs]]
```

```
{62.7319, {0, 0, 0, 0}}
```

Hence, ann4 is a system of PDEs for the fourth integral (integration w.r.t. x1, x2, x3, and x4).

```
Timing[OreReduce[MapThread[(#1 + Der[x5] ** #2) &, {ann5, delta5}], ann4]]
```

```
{6709., {0, 0, 0, 0}}
```

```
(* A quicker (but non-rigorous) correctness check. *)
subs = {x6 → -74, z → 23};
Timing[{ann5a, delta5a} = OrePolynomialSubstitute[#, subs] & /@ {ann5, delta5};
 OreReduce[MapThread[(#1 + Der[x5] ** #2) &, {ann5a, delta5a}],
  ann4, OrePolynomialSubstitute → subs]]
```

```
{72.7965, {0, 0, 0, 0}}
```

Hence, ann5 is a system of PDEs for the fifth integral (integration w.r.t. x1, x2, x3, x4, and x5).

```
Timing[OreReduce[ann6[[1]] + Der[x6] ** delta6[[1]], ann5]]
```

```
{1303.25, 0}
```

```
(* A quicker (but non-rigorous) correctness check. *)
subs = {z → 23};
Timing[{ann6a, delta6a} = OrePolynomialSubstitute[#, subs] & /@ {ann6, delta6};
 OreReduce[MapThread[(#1 + Der[x6] ** #2) &, {ann6a, delta6a}],
  ann5, OrePolynomialSubstitute → subs]]
```

```
{24.3855, {0}}
```

Hence, ann6 is an ODE for the original six-fold integral.

```
Factor[ann6]
```

```
{(-3 + z) (-1 + z) z^6 (4 + z) (5 + z) (9 + z) (15 + z)^2 (24 + z) (3 + 2 z) (15 + 2 z) (15 + 4 z)
    (60 + 7 z) (19 280 523 023 769 600 000 000 000 + 242 306 901 961 056 460 800 000 000 z +
      1 348 035 643 913 347 353 600 000 000 z^2 + 2 878 395 143 123 986 146 432 000 000 z^3 +
      3 920 543 674 198 265 211 436 800 000 z^4 + 753 459 769 629 110 696 243 040 000 z^5 -
      5 337 917 399 156 522 389 289 280 000 z^6 - 8 883 487 977 021 576 719 907 033 600 z^7 -
      7 971 869 741 181 425 686 355 371 200 z^8 - 4 872 861 027 995 366 524 279 994 100 z^9 -
      2 157 072 153 972 513 398 276 826 924 z^10 - 693 159 300 555 093 708 939 611 829 z^11 -
      152 346 950 611 719 661 239 440 526 z^12 - 16 970 927 000 980 381 863 663 141 z^13 +
      2 189 507 486 524 206 284 827 296 z^14 + 1 557 656 993 073 750 677 220 582 z^15 +
      412 843 760 981 101 392 072 948 z^16 + 72 864 795 413 899 911 011 922 z^17 +
      9 465 736 161 794 804 567 892 z^18 + 931 032 563 834 500 230 663 z^19 +
      69 321 047 461 074 869 130 z^20 + 3 823 803 744 461 234 343 z^21 + 149 102 740 118 852 712 z^22 +
      3 764 987 488 054 392 z^23 + 51 659 233 261 888 z^24 + 242 161 043 152 z^25) D_z^8 +
  2 z^5 (15 + z) (1 973 392 380 319 656 591 360 000 000 000 000 000 +
      25 084 009 812 063 190 450 176 000 000 000 000 000 z +
      140 360 356 659 888 583 720 114 176 000 000 000 000 z^2 +
      314 413 056 395 938 625 838 510 182 400 000 000 000 z^3 +
      344 718 972 957 157 801 371 250 560 000 000 000 000 z^4 -
      145 021 874 608 394 651 059 638 847 488 000 000 000 z^5 -
      1 074 498 717 874 767 393 664 900 393 675 200 000 000 z^6 -
      1 460 286 146 960 184 444 033 629 739 148 560 000 000 z^7 -
      682 640 121 106 346 995 555 734 719 308 248 000 000 z^8 +
      564 704 048 394 845 939 194 551 470 638 922 400 000 z^9 +
      1 251 150 937 075 501 602 577 084 871 183 562 120 000 z^10 +
      1 138 666 598 560 461 678 104 890 857 545 212 608 000 z^11 +
      661 181 529 544 504 134 786 063 620 152 764 386 400 z^12 +
      253 995 260 187 409 794 081 727 430 934 766 869 450 z^13 +
      51 498 237 061 832 672 183 443 454 747 804 923 575 z^14 -
      7 977 590 414 255 123 112 276 744 122 571 399 783 z^15 -
      11 704 453 530 273 493 922 795 299 130 700 457 200 z^16 -
      5 466 573 829 106 434 312 238 352 307 226 140 764 z^17 -
      1 638 945 569 143 497 023 502 201 509 481 372 411 z^18 -
```

$331\,259\,809\,437\,872\,111\,827\,650\,003\,935\,308\,209\, z^{19} -$
$35\,907\,063\,701\,591\,969\,077\,649\,893\,288\,537\,878\, z^{20} + 3\,221\,036\,141\,212\,186\,087\,856\,769\,990\,927\,054\, z^{21} +$
$2\,620\,577\,206\,027\,992\,337\,931\,632\,885\,352\,217\, z^{22} + 724\,749\,378\,242\,590\,885\,585\,485\,419\,445\,843\, z^{23} +$
$138\,105\,907\,223\,379\,522\,203\,625\,428\,215\,332\, z^{24} + 20\,337\,622\,679\,657\,217\,515\,316\,342\,764\,256\, z^{25} +$
$2\,406\,227\,015\,296\,631\,910\,854\,902\,756\,563\, z^{26} + 232\,115\,671\,681\,854\,334\,221\,586\,338\,585\, z^{27} +$
$18\,309\,889\,884\,984\,684\,630\,822\,323\,370\, z^{28} + 1\,175\,154\,434\,178\,119\,041\,671\,700\,740\, z^{29} +$
$60\,618\,715\,038\,937\,670\,473\,018\,584\, z^{30} + 2\,462\,288\,021\,152\,606\,885\,358\,700\, z^{31} +$
$76\,318\,086\,060\,490\,791\,960\,792\, z^{32} + 1\,719\,342\,411\,627\,828\,757\,728\, z^{33} +$
$25\,996\,840\,572\,204\,888\,512\, z^{34} + 227\,389\,988\,057\,526\,336\, z^{35} + 800\,100\,086\,574\,208\, z^{36}\big)\, D_z^7 +$

$z^4 \big(512\,323\,021\,813\,756\,999\,680\,000\,000\,000\,000\,000\,000 +$
$6\,311\,156\,771\,304\,917\,325\,766\,656\,000\,000\,000\,000\,000\, z +$
$33\,882\,896\,755\,872\,071\,956\,886\,261\,760\,000\,000\,000\,000\, z^2 +$
$72\,511\,610\,277\,412\,390\,990\,839\,363\,072\,000\,000\,000\,000\, z^3 +$
$59\,704\,683\,972\,170\,679\,548\,931\,977\,222\,400\,000\,000\,000\, z^4 -$
$86\,642\,575\,450\,501\,391\,066\,787\,202\,019\,520\,000\,000\,000\, z^5 -$
$327\,383\,462\,755\,042\,385\,949\,747\,691\,240\,824\,000\,000\,000\, z^6 -$
$395\,683\,465\,592\,680\,867\,401\,293\,480\,616\,198\,000\,000\,000\, z^7 -$
$119\,682\,652\,007\,548\,350\,954\,457\,856\,750\,250\,720\,000\,000\, z^8 +$
$287\,121\,363\,379\,312\,616\,871\,562\,346\,484\,465\,378\,000\,000\, z^9 +$
$495\,779\,225\,046\,771\,906\,420\,255\,540\,348\,281\,344\,800\,000\, z^{10} +$
$429\,409\,878\,921\,957\,648\,790\,555\,775\,268\,242\,743\,350\,000\, z^{11} +$
$240\,689\,360\,358\,498\,296\,007\,939\,096\,187\,740\,586\,134\,000\, z^{12} +$
$85\,149\,274\,357\,043\,292\,385\,925\,033\,653\,294\,291\,853\,550\, z^{13} +$
$10\,278\,671\,248\,090\,335\,377\,408\,918\,358\,815\,408\,788\,425\, z^{14} -$
$9\,076\,459\,539\,413\,303\,184\,641\,722\,134\,776\,573\,895\,810\, z^{15} -$
$7\,573\,126\,212\,785\,007\,618\,891\,225\,542\,456\,994\,124\,245\, z^{16} -$
$3\,356\,732\,946\,224\,373\,601\,649\,087\,937\,349\,109\,785\,896\, z^{17} -$
$1\,033\,954\,017\,266\,382\,248\,984\,767\,586\,852\,072\,344\,191\, z^{18} -$
$226\,886\,176\,666\,918\,560\,987\,240\,200\,768\,631\,693\,150\, z^{19} -$
$31\,072\,001\,737\,970\,299\,221\,405\,533\,198\,706\,303\,141\, z^{20} -$
$137\,626\,809\,673\,226\,795\,399\,591\,264\,079\,041\,112\, z^{21} +$
$1\,319\,636\,945\,498\,761\,264\,973\,744\,224\,282\,378\,779\, z^{22} + 441\,055\,376\,229\,095\,921\,513\,357\,130\,918\,811\,338\, z^{23} + 94\,068\,732\,852\,089\,205\,756\,130\,773\,605\,094\,705\, z^{24} +$
$15\,263\,082\,383\,031\,406\,770\,429\,022\,758\,762\,048\, z^{25} + 1\,986\,708\,322\,085\,667\,572\,665\,525\,016\,037\,411\, z^{26} +$
$211\,815\,796\,834\,464\,054\,711\,973\,645\,322\,142\, z^{27} + 18\,631\,082\,892\,630\,536\,824\,222\,949\,409\,585\, z^{28} +$
$1\,350\,855\,094\,398\,006\,902\,682\,870\,922\,050\, z^{29} + 80\,160\,062\,388\,267\,727\,172\,211\,985\,080\, z^{30} +$
$3\,840\,828\,004\,490\,920\,060\,950\,969\,480\, z^{31} + 145\,494\,567\,985\,766\,484\,898\,923\,048\, z^{32} +$
$4\,221\,606\,838\,983\,473\,228\,197\,008\, z^{33} + 89\,393\,980\,129\,433\,032\,096\,320\, z^{34} +$
$1\,276\,532\,600\,942\,212\,775\,168\, z^{35} + 10\,612\,604\,051\,614\,486\,656\, z^{36} + 35\,882\,454\,730\,090\,752\, z^{37}\big)\, D_z^6 +$

$3\, z^3 \big(595\,812\,699\,442\,665\,547\,776\,000\,000\,000\,000\,000\,000 +$
$6\,994\,092\,214\,348\,464\,533\,004\,288\,000\,000\,000\,000\,000\, z +$
$34\,708\,946\,736\,814\,927\,353\,542\,983\,680\,000\,000\,000\,000\, z^2 +$
$64\,135\,781\,486\,584\,141\,753\,707\,277\,824\,000\,000\,000\,000\, z^3 +$
$1\,049\,740\,530\,978\,348\,996\,701\,293\,958\,400\,000\,000\,000\, z^4 -$
$226\,302\,972\,537\,833\,147\,253\,780\,811\,598\,400\,000\,000\,000\, z^5 -$
$518\,937\,227\,107\,573\,341\,964\,843\,985\,332\,680\,000\,000\,000\, z^6 -$
$526\,332\,032\,930\,456\,915\,428\,235\,817\,813\,056\,400\,000\,000\, z^7 -$

$44\,891\,871\,663\,741\,237\,702\,913\,642\,763\,603\,760\,000\,000\,z^8 +$
$586\,378\,944\,861\,718\,695\,144\,037\,906\,690\,882\,422\,000\,000\,z^9 +$
$865\,953\,342\,265\,454\,601\,104\,437\,816\,976\,581\,680\,000\,000\,z^{10} +$
$696\,554\,593\,654\,757\,665\,866\,719\,966\,270\,600\,171\,130\,000\,z^{11} +$
$349\,803\,608\,265\,045\,461\,612\,489\,069\,936\,675\,179\,800\,000\,z^{12} +$
$87\,213\,988\,833\,696\,382\,614\,552\,027\,738\,719\,280\,959\,850\,z^{13} -$
$23\,459\,339\,067\,193\,287\,788\,165\,144\,055\,727\,575\,111\,225\,z^{14} -$
$38\,330\,478\,964\,162\,570\,556\,645\,949\,941\,637\,505\,810\,110\,z^{15} -$
$23\,191\,419\,391\,770\,985\,171\,480\,237\,991\,217\,872\,142\,915\,z^{16} -$
$9\,468\,529\,098\,949\,077\,023\,394\,535\,618\,861\,256\,937\,240\,z^{17} -$
$2\,858\,027\,882\,158\,570\,016\,919\,188\,514\,224\,326\,558\,185\,z^{18} -$
$635\,954\,475\,887\,313\,295\,192\,241\,042\,199\,635\,547\,930\,z^{19} -$
$93\,149\,956\,267\,467\,504\,725\,225\,680\,596\,497\,523\,339\,z^{20} -$
$3\,066\,274\,907\,647\,801\,401\,815\,807\,099\,801\,425\,704\,z^{21} +$
$3\,006\,740\,720\,618\,245\,361\,400\,876\,608\,130\,182\,349\,z^{22} +$
$1\,112\,001\,535\,696\,035\,843\,878\,120\,629\,687\,073\,790\,z^{23} +$
$247\,864\,598\,814\,302\,846\,690\,177\,415\,162\,792\,735\,z^{24} +$
$41\,511\,153\,616\,540\,066\,669\,903\,815\,109\,576\,752\,z^{25} + 5\,552\,646\,100\,941\,335\,755\,747\,908\,121\,811\,397\,z^{26} +$
$607\,255\,705\,204\,278\,811\,351\,245\,801\,585\,018\,z^{27} + 54\,750\,340\,798\,147\,926\,328\,921\,245\,513\,135\,z^{28} +$
$4\,068\,564\,888\,973\,003\,880\,820\,853\,550\,310\,z^{29} + 247\,501\,384\,020\,921\,867\,412\,586\,484\,240\,z^{30} +$
$12\,162\,402\,278\,802\,667\,065\,896\,636\,880\,z^{31} + 472\,739\,613\,103\,493\,977\,658\,692\,800\,z^{32} +$
$14\,079\,224\,644\,087\,925\,329\,523\,520\,z^{33} + 305\,988\,393\,455\,491\,537\,290\,240\,z^{34} +$
$4\,480\,274\,117\,205\,321\,023\,232\,z^{35} + 38\,072\,220\,474\,786\,769\,152\,z^{36} + 130\,240\,020\,872\,181\,248\,z^{37}\Big)\,D_z^5 +$

$15\,z^2\,\Big(161\,818\,175\,186\,211\,840\,491\,520\,000\,000\,000\,000\,000 +$
$1\,776\,029\,394\,112\,720\,931\,570\,319\,360\,000\,000\,000\,000\,z +$
$7\,522\,568\,512\,298\,824\,734\,532\,104\,192\,000\,000\,000\,000\,z^2 +$
$7\,747\,728\,379\,627\,393\,494\,726\,545\,203\,200\,000\,000\,000\,z^3 -$
$36\,772\,706\,828\,360\,958\,944\,274\,523\,883\,520\,000\,000\,000\,z^4 -$
$140\,261\,247\,415\,772\,885\,691\,546\,407\,435\,520\,000\,000\,000\,z^5 -$
$242\,455\,701\,875\,928\,553\,517\,844\,332\,493\,302\,400\,000\,000\,z^6 -$
$214\,965\,129\,809\,120\,690\,827\,282\,902\,731\,468\,640\,000\,000\,z^7 +$
$6\,337\,926\,159\,808\,918\,213\,308\,690\,816\,700\,464\,000\,000\,z^8 +$
$276\,342\,679\,146\,887\,322\,412\,220\,759\,883\,497\,997\,600\,000\,z^9 +$
$379\,975\,092\,805\,467\,869\,163\,550\,626\,412\,993\,759\,200\,000\,z^{10} +$
$279\,266\,241\,080\,334\,469\,793\,315\,941\,614\,102\,969\,564\,000\,z^{11} +$
$107\,413\,528\,041\,921\,729\,529\,347\,960\,434\,391\,761\,302\,800\,z^{12} -$
$10\,222\,760\,436\,927\,155\,616\,364\,669\,208\,395\,729\,054\,260\,z^{13} -$
$47\,185\,211\,186\,009\,106\,848\,535\,876\,331\,178\,061\,122\,490\,z^{14} -$
$38\,476\,335\,393\,060\,119\,379\,820\,741\,759\,126\,402\,451\,166\,z^{15} -$
$20\,284\,887\,219\,829\,242\,010\,855\,806\,602\,752\,336\,703\,097\,z^{16} -$
$7\,949\,778\,754\,688\,875\,639\,594\,299\,226\,888\,542\,864\,672\,z^{17} -$
$2\,396\,582\,727\,922\,965\,009\,354\,571\,656\,000\,074\,347\,578\,z^{18} -$
$548\,617\,946\,604\,162\,829\,617\,617\,348\,998\,523\,187\,024\,z^{19} -$
$87\,288\,636\,539\,051\,237\,531\,541\,938\,169\,181\,610\,997\,z^{20} -$
$5\,626\,714\,951\,506\,760\,337\,684\,784\,884\,293\,147\,302\,z^{21} +$
$1\,820\,210\,924\,970\,374\,403\,477\,059\,898\,368\,292\,414\,z^{22} + 815\,865\,984\,997\,630\,892\,337\,526\,061\,797\,547\,730$
$z^{23} + 194\,225\,784\,819\,376\,433\,418\,854\,177\,036\,400\,765\,z^{24} +$

$33\,925\,520\,928\,056\,707\,379\,949\,042\,245\,154\,948\,z^{25} + 4\,693\,678\,127\,508\,685\,757\,329\,704\,793\,118\,274\,z^{26} +$
$528\,960\,737\,538\,220\,962\,199\,232\,165\,726\,700\,z^{27} + 49\,056\,517\,288\,448\,701\,934\,966\,949\,399\,201\,z^{28} +$
$3\,746\,772\,515\,516\,029\,997\,311\,378\,363\,446\,z^{29} + 234\,205\,994\,182\,438\,943\,769\,949\,245\,108\,z^{30} +$
$11\,827\,310\,475\,440\,684\,698\,801\,079\,376\,z^{31} + 472\,534\,466\,386\,674\,980\,533\,072\,704\,z^{32} +$
$14\,467\,601\,136\,584\,109\,707\,654\,400\,z^{33} + 323\,165\,791\,319\,702\,484\,035\,520\,z^{34} +$
$4\,857\,665\,734\,098\,963\,690\,240\,z^{35} + 42\,232\,680\,898\,487\,251\,200\,z^{36} + 146\,187\,778\,529\,999\,360\,z^{37}\big)\,D_z^4 +$
$90\,z\,\big(11\,486\,155\,649\,552\,872\,980\,480\,000\,000\,000\,000\,000 +$
$114\,230\,678\,131\,481\,922\,666\,823\,680\,000\,000\,000\,000\,z +$
$284\,911\,453\,840\,859\,719\,602\,001\,920\,000\,000\,000\,000\,z^2 -$
$1\,112\,041\,174\,659\,253\,407\,521\,806\,233\,600\,000\,000\,000\,z^3 -$
$9\,932\,878\,926\,912\,153\,370\,258\,947\,363\,840\,000\,000\,000\,z^4 -$
$27\,649\,387\,021\,455\,520\,276\,766\,166\,546\,048\,000\,000\,000\,z^5 -$
$41\,909\,264\,304\,440\,185\,602\,876\,764\,536\,603\,200\,000\,000\,z^6 -$
$34\,653\,454\,861\,369\,485\,847\,062\,964\,251\,845\,520\,000\,000\,z^7 +$
$757\,729\,323\,937\,951\,939\,044\,642\,929\,351\,040\,000\,000\,z^8 +$
$41\,970\,729\,402\,708\,473\,923\,386\,620\,935\,623\,814\,800\,000\,z^9 +$
$54\,660\,627\,321\,107\,405\,540\,934\,107\,870\,983\,869\,840\,000\,z^{10} +$
$32\,573\,268\,392\,371\,003\,654\,841\,290\,966\,684\,606\,314\,000\,z^{11} +$
$340\,763\,873\,540\,255\,131\,808\,343\,067\,503\,063\,454\,800\,z^{12} -$
$18\,559\,051\,142\,634\,901\,231\,618\,230\,067\,011\,245\,261\,730\,z^{13} -$
$20\,395\,042\,168\,164\,862\,736\,248\,341\,991\,799\,243\,143\,275\,z^{14} -$
$13\,791\,392\,258\,782\,895\,819\,955\,453\,998\,955\,102\,517\,548\,z^{15} -$
$6\,879\,647\,707\,640\,439\,013\,900\,747\,488\,611\,335\,523\,490\,z^{16} -$
$2\,671\,193\,766\,306\,193\,321\,259\,081\,077\,503\,739\,718\,922\,z^{17} -$
$818\,596\,118\,205\,128\,605\,985\,330\,478\,856\,111\,679\,058\,z^{18} -$
$195\,183\,178\,990\,057\,349\,643\,272\,275\,435\,126\,736\,340\,z^{19} -$
$33\,929\,658\,665\,256\,259\,408\,812\,784\,354\,866\,385\,557\,z^{20} -$
$3\,212\,526\,847\,572\,548\,623\,801\,062\,566\,839\,102\,968\,z^{21} +$
$335\,162\,333\,006\,577\,190\,998\,078\,624\,832\,466\,745\,z^{22} + 232\,117\,491\,219\,054\,750\,436\,300\,759\,063\,832\,796$
$z^{23} + 61\,070\,425\,289\,478\,623\,056\,319\,494\,081\,223\,364\,z^{24} +$
$11\,283\,714\,208\,962\,998\,257\,330\,503\,635\,013\,918\,z^{25} + 1\,627\,987\,793\,820\,686\,707\,319\,681\,442\,965\,532\,z^{26} +$
$190\,122\,674\,553\,786\,922\,619\,563\,973\,540\,916\,z^{27} + 18\,213\,230\,428\,133\,179\,674\,440\,523\,308\,931\,z^{28} +$
$1\,434\,485\,821\,162\,175\,237\,888\,091\,472\,086\,z^{29} + 92\,390\,999\,114\,814\,905\,907\,317\,974\,392\,z^{30} +$
$4\,805\,890\,762\,274\,729\,535\,435\,673\,296\,z^{31} + 197\,763\,282\,456\,363\,307\,438\,541\,552\,z^{32} +$
$6\,235\,802\,763\,945\,868\,063\,424\,352\,z^{33} + 143\,387\,361\,084\,360\,543\,557\,376\,z^{34} +$
$2\,215\,666\,629\,279\,250\,997\,248\,z^{35} + 19\,728\,125\,958\,978\,028\,032\,z^{36} + 69\,106\,949\,850\,545\,152\,z^{37}\big)\,D_z^3 +$
$45\,\big(1\,619\,193\,747\,954\,590\,023\,680\,000\,000\,000\,000\,000 + 14\,860\,150\,621\,853\,249\,942\,323\,200\,000\,000\,000\,000$
$z - 83\,241\,123\,892\,330\,166\,885\,744\,640\,000\,000\,000\,000\,z^2 -$
$1\,428\,583\,143\,864\,269\,960\,769\,790\,771\,200\,000\,000\,000\,z^3 -$
$7\,784\,392\,307\,839\,726\,168\,650\,555\,924\,480\,000\,000\,000\,z^4 -$
$20\,932\,834\,089\,033\,885\,270\,730\,650\,301\,440\,000\,000\,000\,z^5 -$
$29\,659\,078\,571\,699\,608\,256\,375\,734\,426\,214\,400\,000\,000\,z^6 -$
$20\,656\,761\,408\,545\,661\,580\,810\,751\,146\,327\,680\,000\,000\,z^7 +$
$6\,746\,831\,082\,562\,798\,982\,378\,495\,636\,957\,952\,000\,000\,z^8 +$
$34\,764\,119\,013\,156\,176\,353\,837\,403\,619\,970\,113\,600\,000\,z^9 +$
$37\,297\,341\,452\,565\,155\,702\,787\,810\,516\,361\,533\,600\,000\,z^{10} +$
$9\,719\,645\,940\,829\,530\,820\,988\,532\,518\,598\,953\,424\,000\,z^{11} -$

$24 193 553 263 042 351 259 117 425 539 502 701 518 400\, z^{12} -$
$40 652 966 100 310 576 219 422 839 345 851 085 154 840\, z^{13} -$
$36 747 814 326 347 114 270 377 987 158 311 612 338 260\, z^{14} -$
$23 667 524 905 718 087 319 814 208 022 941 410 083 354\, z^{15} -$
$11 757 721 460 891 217 253 150 507 437 222 976 590 963\, z^{16} -$
$4 646 227 686 063 347 368 140 269 721 102 656 923 194\, z^{17} -$
$1 472 149 779 764 303 912 910 700 825 119 513 125 745\, z^{18} -$
$369 692 934 875 862 692 678 770 756 612 360 457 070\, z^{19} -$
$70 294 647 356 901 524 101 024 740 972 933 056 916\, z^{20} -$
$8 583 686 545 551 708 471 758 291 210 460 691 032\, z^{21} -$
$3 744 645 921 582 101 044 070 547 736 300 950\, z^{22} + 322 041 161 855 435 062 814 533 420 723 282 482$
$z^{23} + 99 771 357 205 875 220 145 109 466 450 106 517\, z^{24} +$
$19 908 118 207 277 143 280 846 917 552 738 638\, z^{25} + 3 028 085 987 873 439 981 041 316 741 040 299\, z^{26} +$
$369 055 333 918 742 878 506 923 895 821 094\, z^{27} + 36 707 414 555 219 468 440 447 241 903 970\, z^{28} +$
$2 993 264 774 540 100 816 050 708 154 540\, z^{29} + 199 288 291 693 600 445 167 066 471 488\, z^{30} +$
$10 707 051 961 496 414 217 407 305 536\, z^{31} + 454 875 015 831 485 400 909 097 248\, z^{32} +$
$14 802 080 405 483 677 823 943 104\, z^{33} + 351 010 067 005 351 488 224 256\, z^{34} +$
$5 584 340 634 105 826 525 184\, z^{35} + 50 980 706 267 636 984 832\, z^{36} + 180 741 253 455 271 936\, z^{37}\big)\, D_z^2 +$

$45\, \big(186 207 281 014 777 852 723 200 000 000 000 000 - 8 434 528 659 189 021 937 434 624 000 000 000 000\, z -$
$122 588 504 883 178 716 188 285 337 600 000 000 000\, z^2 -$
$655 267 817 084 534 423 521 940 643 840 000 000 000\, z^3 -$
$1 863 534 767 021 891 922 131 179 987 968 000 000 000\, z^4 -$
$2 226 964 464 248 713 386 006 518 356 377 600 000 000\, z^5 +$
$401 336 331 886 317 774 107 713 318 790 400 000 000\, z^6 +$
$5 165 781 565 021 067 274 342 996 673 450 656 000 000\, z^7 +$
$8 691 043 975 963 666 049 447 299 379 144 001 600 000\, z^8 +$
$7 349 743 557 503 879 010 410 921 836 212 410 400 000\, z^9 +$
$698 114 077 775 776 671 885 153 675 463 762 080 000\, z^{10} -$
$6 955 035 214 429 661 410 040 236 974 622 315 476 000\, z^{11} -$
$10 758 301 750 323 045 400 708 026 810 527 005 985 400\, z^{12} -$
$9 808 779 912 515 181 085 311 292 716 635 118 617 340\, z^{13} -$
$6 500 636 144 955 681 369 542 005 264 067 707 999 470\, z^{14} -$
$3 351 334 353 377 309 619 203 633 178 809 010 250 269\, z^{15} -$
$1 382 954 753 973 214 192 431 623 770 039 149 437 562\, z^{16} -$
$461 005 100 390 610 028 275 047 960 932 687 009 761\, z^{17} -$
$123 304 322 017 356 000 844 884 963 447 213 004 302\, z^{18} -$
$25 665 990 995 028 381 347 757 284 132 973 790 086\, z^{19} -$
$3 776 626 287 411 277 314 694 612 568 191 478 460\, z^{20} -$
$232 966 958 115 695 319 966 898 071 487 115 550\, z^{21} +$
$65 404 062 287 190 045 292 473 501 882 376 446\, z^{22} +$
$27 828 342 208 285 269 645 811 267 613 975 751\, z^{23} + 6 203 408 988 166 712 509 967 367 951 961 350\, z^{24} +$
$1 010 115 611 151 696 866 102 360 444 043 867\, z^{25} + 129 674 818 596 578 381 841 709 352 363 310\, z^{26} +$
$13 478 285 221 767 374 237 433 813 894 156\, z^{27} + 1 143 508 859 378 085 891 069 139 805 496\, z^{28} +$
$79 010 991 647 695 967 734 365 641 136\, z^{29} + 4 398 883 914 180 352 580 752 205 664\, z^{30} +$
$193 479 386 194 110 772 817 766 720\, z^{31} + 6 513 463 004 865 397 861 819 008\, z^{32} +$
$159 628 611 480 988 435 906 560\, z^{33} + 2 619 357 527 554 007 840 768\, z^{34} +$
$24 549 299 776 964 745 216\, z^{35} + 88 092 375 633 661 952\, z^{36}\big)\, D_z +$

$$90\,(-26\,986\,562\,465\,909\,833\,728\,000\,000\,000\,000\,000 - 578\,659\,365\,675\,271\,609\,712\,640\,000\,000\,000\,000\,z - 3\,932\,207\,868\,973\,120\,630\,810\,214\,400\,000\,000\,000\,z^2 - 12\,270\,310\,453\,108\,287\,668\,341\,923\,840\,000\,000\,000\,z^3 - 9\,698\,100\,095\,942\,063\,765\,846\,249\,472\,000\,000\,000\,z^4 + 52\,113\,850\,317\,609\,070\,332\,668\,882\,227\,200\,000\,000\,z^5 + 165\,979\,815\,868\,291\,791\,006\,070\,607\,462\,400\,000\,000\,z^6 + 252\,029\,928\,377\,053\,385\,449\,407\,192\,172\,320\,000\,000\,z^7 + 234\,855\,990\,648\,514\,674\,287\,291\,744\,222\,356\,800\,000\,z^8 + 98\,749\,247\,882\,439\,137\,822\,044\,179\,686\,396\,640\,000\,z^9 - 83\,930\,464\,288\,781\,215\,080\,378\,386\,513\,083\,200\,000\,z^{10} - 204\,430\,925\,935\,804\,223\,158\,200\,138\,096\,719\,244\,000\,z^{11} - 217\,051\,701\,285\,403\,806\,039\,787\,021\,788\,244\,210\,200\,z^{12} - 158\,672\,230\,290\,697\,625\,052\,364\,901\,820\,833\,352\,540\,z^{13} - 88\,492\,994\,651\,041\,978\,105\,789\,511\,893\,808\,827\,410\,z^{14} - 39\,203\,789\,245\,543\,299\,948\,038\,211\,301\,310\,631\,735\,z^{15} - 14\,017\,460\,872\,371\,123\,201\,967\,056\,591\,950\,292\,270\,z^{16} - 4\,044\,657\,270\,312\,306\,250\,764\,976\,742\,472\,089\,595\,z^{17} - 924\,626\,001\,493\,256\,833\,520\,380\,233\,115\,382\,826\,z^{18} - 158\,195\,048\,236\,903\,725\,948\,800\,257\,698\,582\,066\,z^{19} - 16\,377\,415\,461\,160\,421\,103\,082\,005\,421\,146\,444\,z^{20} + 574\,602\,465\,936\,356\,660\,227\,512\,513\,519\,630\,z^{21} + 717\,575\,244\,018\,720\,111\,969\,771\,948\,822\,450\,z^{22} + 190\,773\,160\,991\,774\,404\,319\,508\,940\,400\,373\,z^{23} + 34\,047\,746\,401\,934\,351\,907\,977\,621\,763\,618\,z^{24} + 4\,663\,284\,432\,121\,091\,702\,260\,620\,852\,777\,z^{25} + 510\,811\,439\,434\,664\,402\,615\,401\,586\,970\,z^{26} + 45\,371\,384\,308\,945\,745\,114\,138\,623\,620\,z^{27} + 3\,269\,391\,489\,631\,666\,671\,425\,989\,920\,z^{28} + 189\,382\,045\,823\,502\,675\,349\,219\,920\,z^{29} + 8\,653\,460\,076\,869\,413\,651\,316\,640\,z^{30} + 302\,276\,251\,598\,295\,683\,586\,240\,z^{31} + 7\,675\,748\,903\,189\,765\,748\,480\,z^{32} + 130\,185\,473\,751\,277\,349\,888\,z^{33} + 1\,254\,502\,960\,824\,572\,928\,z^{34} + 4\,556\,502\,187\,948\,032\,z^{35})\}$$

Return probability

The following command computes a recurrence for f(n)=p(0)+p(1)+...+p(n), where p(k) is the probability that a random walk ends at the origin after exactly k steps.

```
Timing[rec6 = RecurrenceForReturnProbability[ann6, Der[z], f[n]];]
```

```
{3856.13, Null}
```

These are the asymptotic solutions of this recurrence:

```
Asymptotics[rec6, f[n], Order → 3]
```

$$\left\{\frac{\left(-\frac{2}{3}\right)^n\left(1+\frac{44\,767}{30\,375\,n^3}+\frac{673}{225\,n^2}-\frac{97}{45\,n}\right)}{n^3},\ \frac{\left(-\frac{4}{15}\right)^n\left(1+\frac{1\,892\,524\,061}{555\,579\,n^3}+\frac{1\,045\,301}{9747\,n^2}+\frac{697}{171\,n}\right)}{n^3},\right.$$

$$\frac{\left(-\frac{1}{4}\right)^n\left(1-\frac{8\,300\,103}{4000\,n^3}+\frac{31\,441}{400\,n^2}-\frac{87}{20\,n}\right)}{n^3},\ \frac{\left(-\frac{1}{5}\right)^n\left(1-\frac{2887}{18\,n^3}+\frac{611}{24\,n^2}-\frac{4}{n}\right)}{n^3},\ \frac{\left(-\frac{2}{15}\right)^n\left(1+\frac{1\,712\,737}{4913\,n^3}+\frac{12\,441}{289\,n^2}-\frac{45}{17\,n}\right)}{n^3},$$

$$\frac{\left(-\frac{7}{60}\right)^n\left(1+\frac{231\,687\,011\,719}{86\,619\,744\,n^3}+\frac{282\,989\,795}{1\,939\,248\,n^2}-\frac{12\,979}{2412\,n}\right)}{n^3},\ \frac{\left(-\frac{1}{9}\right)^n\left(1-\frac{14\,861\,717}{1000\,n^3}+\frac{81\,077}{200\,n^2}-\frac{91}{5\,n}\right)}{n^3},$$

$$\frac{\left(-\frac{1}{15}\right)^n\left(1-\frac{303\,946\,429}{171\,072\,n^3}+\frac{405\,347}{2376\,n^2}-\frac{168}{11\,n}\right)}{n^4},\ \frac{\left(-\frac{1}{15}\right)^n\left(1+\frac{2\,042\,527\,055}{6\,967\,296\,n^3}-\frac{162\,901}{32\,256\,n}\right)}{n^3},$$

$$\left.\frac{\left(-\frac{1}{24}\right)^n\left(1-\frac{123\,788\,753}{253\,125\,n^3}+\frac{1\,237\,049}{16\,875\,n^2}-\frac{2273}{225\,n}\right)}{n^3},\ \frac{3^{-n}\left(1-\frac{25\,655}{108\,n^3}+\frac{9499}{216\,n^2}-\frac{70}{9\,n}\right)}{n^3},\ \frac{1-\frac{949}{500\,n^3}+\frac{197}{80\,n^2}-\frac{2}{n}}{n^2},\ 1\right\}$$

The number of excursions in the six-dimensional fcc lattice:

```
exc6 = {1, 0, 60, 960, 30 780, 996 480, 36 560 400, 1 430 553 600, 59 089 923 900,
   2 543 035 488 000, 113 129 280 527 760, 5 170 796 720 812 800, 241 741 903 350 301 200};
```

```
Timing[data =
   UnrollRecurrence[rec6, f[n], Table[Sum[exc6[[i + 1]] / 60^i, {i, 0, n}], {n, 0, 12}], 5000];]
```

```
{88.8015, Null}
```

```
Timing[lim6 = MyLimit[data, Asymptotics[rec6, f[n], Order → 34], n]]
```

```
{982.177,
 1.027749100627498839859363679273968502092439909001148724251721657966446062670013475278332⋱
  2144087699974470591504168230717101538561678399979176027548142015365826483813060157747293⋱
  500851937288753329019614}
```

```
Timing[lim6a = MyLimit[data, Asymptotics[rec6, f[n], Order → 36], n]]
```

```
{1069.03,
 1.027749100627498839859363679273968502092439909001148724251721657966446062670013475278332⋱
  2144087699974451809374466354296298616773941265118617224484709751588099359120330373634022⋱
  216665202253203955264516}
```

The results agree on more than 100 digits, and thus we can conclude that (at least) the first 100 digits are correct:

```
lim6 - lim6a
```

```
1.878212970187642080292178773713486055880306343226377772712469272978411327128418673503554 9⋱
  373755099 × 10^-102
```

The return probability therefore is

```
1 - 1 / lim6a
```

```
0.026999878287956124269364175426196380216122626762395014138433829946619912478462848294580 7⋱
 3278915270522072078697337300969134088808584568849293104477658402363045102451996559620440 2⋱
 23701757372588715476199
```