In[155]:=

```
(* These packages have to be downloaded from
  http://www.risc.jku.at/research/combinat/software/ergosum/ *)
<< RISC`HolonomicFunctions`;
<< RISC`Guess`;
SetDirectory[NotebookDirectory[]];
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan


```
--> Type  ?HolonomicFunctions  for help.
```

Guess Package version 0.52
written by Manuel Kauers

Code

In[145]:=

```
(* Display all relevant information about an annihilator ideal. *)
AnnInfo[ann_] := With[{vars = First /@ OreAlgebra[ann][[1]]}, Print[
    "ByteCount: ", ByteCount[ann],
    "\nSupport: ", Support[ann],
    "\ndegree " <> ToString[vars] <> ": ", Exponent[#, vars] & /@ ann,
    "\nStandard Monomials: ", UnderTheStaircase[ann],
    "\nHolonomic Rank: ", Length[UnderTheStaircase[ann]]
   ]];
```

In[146]:=

```
InitializeDeterminantProof[f_, e_, {a_, b_, c_, d_}, cf_] :=
  InitializeDeterminantProof[f, e, {a, b, c, d}, cf, "det" <> ToString[f] <>
    StringJoin @@ Riffle[ToString /@ {e, a, b, c, d}, "_"] <> "/"];
InitializeDeterminantProof[f_, e_, {a_, b_, c_, d_}, cf_, path_] := (
  Clear[mya, mya1, mya2, myb, myc, datac, annc, id2ct1, id2ct2, id3ct1, id3ct2];
  (* Define the entries a_{i,j} of the matrix A_n. *)
  mya1[i_, j_] := e^(i + b) * Binomial[i + f j + c, f j + a];
  mya2[i_, j_] := Binomial[-i + f j + d, f j + a];
  mya[i_, j_] := mya1[i, j] + mya2[i, j];
  mya[i_Integer, j_Integer] := FunctionExpand[mya1[i, j] + mya2[i, j]];
  (* Define the conjectured evaluation b_n of det(A_n). *)
  myb[0] = 1;
  SetDelayed @@ (Hold[myb[n_],
      If[IntegerQ[n], FunctionExpand[C /. prod → Product], C]] /. C → cf);
  (* Compute c_{n,j} for concrete integers n and j. *)
  datac[n_Integer] := datac[n] =
    With[{ns = NullSpace[Table[mya[i, j], {i, 0, n - 2}, {j, 0, n - 1}]][[1]]},
     Together[ns / Last[ns]]];
  myc[n_Integer, j_Integer] := Which[n == 1 && j == 0, 1,
    j ≥ n, 0, True, datac[n][[j + 1]]];
  (* Load the precomputed proof certificates. *)
  {annc, id2ct1, id2ct2, id3ct1, id3ct2} =
   Get[path <> # <> ".m"] & /@ {"annc", "id2_ct1", "id2_ct2", "id3_ct1", "id3_ct2"};
  Print["We are going to prove the following determinant evaluation:"];
  start = CurrentDate[];
  TraditionalForm[
   HoldForm @@ {Subscript[det, 0 ≤ i, j < n][mya[i, j]] == myb[n]} /. prod → Product]
 )
```

In[148]:=

```
(* A straight-forward implementation of
  reduction modulo a left ideal in the shift algebra. *)
(* Reason: the built-in procedure "OreReduce" in
   the HolonomicFunctions package sometimes
   causes Mathematica to crash. *)
SortLex[m1_, m2_] := With[{f1 = First[m1], f2 = First[m2]},
   If[f1 =!= f2 || Length[m1] === 1, f1 > f2, SortLex[Rest[m1], Rest[m2]]]];
SortDLex[m1_, m2_] := With[{w1 = Plus @@ m1, w2 = Plus @@ m2},
   If[w1 === w2, SortLex[m1, m2], w1 > w2]];
Add[p1_List, p2_List] :=
  Module[{p = {}, c, i1 = 1, i2 = 1, l1 = Length[p1], l2 = Length[p2], e1, e2},
   While[i1 ≤ l1 && i2 ≤ l2,
    {e1, e2} = {p1[[i1, 2]], p2[[i2, 2]]};
```

```
    Which[
     e1 === e2, If[(c = p1[[i1, 1]] + p2[[i2, 1]]) =!= 0, AppendTo[p, {c, e1}]];
     i1++; i2++;
     ,
     SortDLex[e1, e2], AppendTo[p, p1[[i1]]]; i1++;
     ,
     SortDLex[e2, e1], AppendTo[p, p2[[i2]]]; i2++;
    ];
   ];
   If[i1 ≤ l1, p = Join[p, Take[p1, {i1, l1}]]];
   If[i2 ≤ l2, p = Join[p, Take[p2, {i2, l2}]]];
   Return[p];
  ];
ScalarMult[s_, p_List] := {Expand[Together[s * #1]], #2} & @@@ p;
OreReduce1[p_List, g_List] := OreReduce1[#, g] & /@ p;
OreReduce1[p1_OrePolynomial, g1 : {(_OrePolynomial) ..}] :=
  Module[{p = p1, g = g1, v, e, f, f1, r = {}, k, gk, gcd},
   v = First /@ OreAlgebra[p][[1]];
   {p, g} = {First[p], First /@ g};
   f = PolynomialLCM @@ (Denominator[First[#]] & /@ p);
   p = ScalarMult[f, p];
   While[p =!= {},
    k = 1;
    While[Min[e = (p[[1, 2]] - g[[k, 1, 2]])] < 0, k++];
    If[k > Length[g],
     AppendTo[r, p[[1]]];
     p = Rest[p];
     ,
     gk = {Expand[#1 /. Thread[v → (v + e)]], #2 + e} & @@@ g[[k]];
     gcd = PolynomialGCD[p[[1, 1]], gk[[1, 1]]];
     f *= (f1 = Together[gk[[1, 1]] / gcd]);
     gk = ScalarMult[Together[-p[[1, 1]] / gcd], Rest[gk]];
     p = Add[ScalarMult[f1, Rest[p]], gk];
    ];
   ];
   Return[OrePolynomial[{Together[#1 / f], #2} & @@@ r, p1[[2]], p1[[3]]]];
  ];
```

In[154]:=
```
prodsimp = {prod[a_, {i_, b_}] → prod[a, {i, 1, b}],
   prod[a_, {i_, b0_, b1_}] / prod[a_, {i_, b0_, b2_}] /; IntegerQ[Expand[b1 - b2]] :→
    If[Expand[b1 - b2] ≥ 0, Product[a, {i, b2 + 1, b1}],
     1 / Product[a, {i, b1 + 1, b2}]],
   prod[a1_, b_] ^ e1_. * prod[a2_, b_] ^ e2_. :→
    prod[FunctionExpand[a1 ^ e1 * a2 ^ e2], b]};
```

det1a: A warmup exercise

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+j+x}{j}+a^i\binom{-1+i+j+x}{j}\right) = 2\,(-1+a)^{\frac{1}{2}(-1+n)\,n}$$

In[∘]:=
```
(* Define the matrix entries a_{i,j} and display the matrix A_4. *)
mya[i_, j_] := FunctionExpand[a ^ i * Binomial[x + i + j - 1, j] + Binomial[x - i + j - 1, j]];
TableForm[Table[mya[i, j], {i, 0, 3}, {j, 0, 3}]]
```

Out[∘]//TableForm=

2	$2\,x$	$x\,(1+x)$	$\frac{1}{3}\,x\,(1+x)\,(2+x)$
$1+a$	$-1+x+a\,(1+x)$	$\frac{1}{2}\,(-1+x)\,x+\frac{1}{2}\,a\,(1+x)\,(2+x)$	$\frac{1}{6}\,(-1+x)\,x\,(1+x)+\frac{1}{6}$
$1+a^2$	$-2+x+a^2\,(2+x)$	$\frac{1}{2}\,(-2+x)\,(-1+x)+\frac{1}{2}\,a^2\,(2+x)\,(3+x)$	$\frac{1}{6}\,(-2+x)\,(-1+x)\,x+$
$1+a^3$	$-3+x+a^3\,(3+x)$	$\frac{1}{2}\,(-3+x)\,(-2+x)+\frac{1}{2}\,a^3\,(3+x)\,(4+x)$	$\frac{1}{6}\,(-3+x)\,(-2+x)\,(-1$

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Together[
  Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / (2 (a - 1) ^Binomial[n, 2])], {n, 8}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* Define and display the normalized cofactors c_{n,j}. *)
datac[n_Integer] := datac[n] =
   With[{ns = NullSpace[Table[mya[i, j], {i, 0, n - 2}, {j, 0, n - 1}]][[1]]},
    Together[ns / Last[ns]]];
myc[n_Integer, j_Integer] := Which[n == 1 && j == 0, 1, j ≥ n, 0, True, datac[n][[j + 1]]];
TableForm[Table[myc[n, j], {n, 4}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1			
$-x$	1		
$\frac{x+a\,x-x^2+a\,x^2}{2\,(-1+a)}$	$\frac{-a+x-a\,x}{-1+a}$	1	
$\frac{-2\,x-8\,a\,x-2\,a^2\,x+3\,x^2-3\,a^2\,x^2-x^3+2\,a\,x^3-a^2\,x^3}{6\,(-1+a)^2}$	$\frac{2\,a+2\,a^2-x-2\,a\,x+3\,a^2\,x+x^2-2\,a\,x^2+a^2\,x^2}{2\,(-1+a)^2}$	$\frac{-2\,a+x-a\,x}{-1+a}$	1

In[•]:=
```
Table[myc[n, j], {n, 4}, {j, 0, 3}]
```

Out[•]=

$$\left\{\{1,0,0,0\},\{-x,1,0,0\},\left\{\frac{x+a\,x-x^2+a\,x^2}{2\,(-1+a)},\frac{-a+x-a\,x}{-1+a},1,0\right\},\left\{\frac{-2\,x-8\,a\,x-2\,a^2\,x+3\,x^2-3\,a^2\,x^2-x^3+2\,a\,x^3-a^2\,x^3}{6\,(-1+a)^2},\frac{2\,a+2\,a^2-x-2\,a\,x+3\,a^2\,x+x^2-2\,a\,x^2+a^2\,x^2}{2\,(-1+a)^2},\frac{-2\,a+x-a\,x}{-1+a},1\right\}\right\}$$

In[•]:=
```
guess = GuessMultRE[Table[myc[n, j], {n, 10}, {j, 0, 9}],
  {c[n, j], c[n, j + 1], c[n + 1, j], c[n + 1, j + 1]},
  {n, j}, 2, StartPoint → {1, 0}, Constraints → (j < n)]
```

Out[•]=

$$\left\{\frac{1}{(-1+a)\,a}\left(-3\,a-a^2-5\,a\,j-a^2\,j-2\,a\,j^2+3\,a\,n+a^2\,n+3\,a\,j\,n+a^2\,j\,n-2\,x-6\,a\,x-j\,x-4\,a\,j\,x+a^2\,j\,x+n\,x+6\,a\,n\,x+a^2\,n\,x-x^2+a^2\,x^2\right)c[n,j]+\frac{(2+j-n+x)\,(2\,a\,j+x+3\,a\,x)\,c[n,1+j]}{(-1+a)\,a}-\frac{(j-n)\,(a+a\,j+x+a\,x)\,c[1+n,j]}{a}-\frac{\left(a\,j+a\,j^2+j\,x+a\,j\,x-n\,x+a\,n\,x\right)c[1+n,1+j]}{(-1+a)\,a},\right.$$

$$\left.\frac{1}{a}\left(2\,a+3\,a\,j+a\,j^2-2\,a\,n-2\,a\,j\,n+2\,x+2\,a\,x+j\,x+a\,j\,x-n\,x-3\,a\,n\,x+x^2-a\,x^2\right)c[n,j]-\frac{(2+j-n+x)\,(a\,j+x+a\,x)\,c[n,1+j]}{a}+\frac{(-1+a)\,(j-n)\,x\,c[1+n,j]}{a}+\frac{(a\,j\,n+j\,x-n\,x+a\,n\,x)\,c[1+n,1+j]}{a}\right\}$$

In[•]:=
```
annc = OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[guess, c[n, j]]];
Factor[annc]
```

Out[•]=

$$\left\{-(-1+a)\,(j-n)\,n\,\left(a\,j+a\,j^2-x+a\,x+2\,a\,j\,x-x^2+a\,x^2\right)S_n-(2+j-n+x)\left(a\,j^2+a\,j^3-2\,a\,j^2\,n+j\,x+a\,j\,x+2\,a\,j^2\,x-4\,a\,j\,n\,x-j\,x^2+a\,j\,x^2+2\,n\,x^2-2\,a\,n\,x^2\right)S_j+\left(2\,a\,j+5\,a\,j^2+4\,a\,j^3+a\,j^4-5\,a\,j\,n-a^2\,j\,n-9\,a\,j^2\,n-a^2\,j^2\,n-4\,a\,j^3\,n+3\,a\,j\,n^2+a^2\,j\,n^2+3\,a\,j^2\,n^2+a^2\,j^2\,n^2+j\,x+3\,a\,j\,x+j^2\,x+5\,a\,j^2\,x+2\,a\,j^3\,x+n\,x-a^2\,n\,x-11\,a\,j\,n\,x-a^2\,j\,n\,x-9\,a\,j^2\,n\,x+a^2\,j^2\,n\,x-n^2\,x+a^2\,n^2\,x+6\,a\,j\,n^2\,x+2\,a^2\,j\,n^2\,x-j\,x^2+a\,j\,x^2-j^2\,x^2+a\,j^2\,x^2+4\,n\,x^2-4\,a\,n\,x^2+4\,j\,n\,x^2-6\,a\,j\,n\,x^2+2\,a^2\,j\,n\,x^2-3\,n^2\,x^2+2\,a\,n^2\,x^2+a^2\,n^2\,x^2+n\,x^3-2\,a\,n\,x^3+a^2\,n\,x^3\right),\right.$$

$$\left.-(3+j-n+x)\left(a\,j+a\,j^2-x+a\,x+2\,a\,j\,x-x^2+a\,x^2\right)S_j^2+\left(8\,a\,j-2\,a^2\,j+8\,a\,j^2-3\,a^2\,j^2+2\,a\,j^3-a^2\,j^3-4\,a\,j\,n-2\,a\,j^2\,n-4\,x+6\,a\,x-2\,a^2\,x-2\,j\,x+14\,a\,j\,x-6\,a^2\,j\,x+5\,a\,j^2\,x-3\,a^2\,j^2\,x+2\,n\,x-2\,a\,n\,x-4\,a\,j\,n\,x-5\,x^2+8\,a\,x^2-3\,a^2\,x^2-2\,j\,x^2+5\,a\,j\,x^2-3\,a^2\,j\,x^2+2\,n\,x^2-2\,a\,n\,x^2-x^3+2\,a\,x^3-a^2\,x^3\right)S_j+(-1+a)\,(1+j-n)\left(2\,a+3\,a\,j+a\,j^2-x+3\,a\,x+2\,a\,j\,x-x^2+a\,x^2\right)\right\}$$

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```
```
ByteCount: 32784
Support: {{Sn, Sj, 1}, {Sj^2, Sj, 1}}
degree {n, j}: {{2, 4}, {1, 3}}
Standard Monomials: {1, Sj}
Holonomic Rank: 2
```

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[Together[test], {n, 6}, {j, 0, n - 1}]]]]
```
Out[∘]=
```
{0}
```

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
Select[AnnihilatorSingularities[annc, {1, 0}], FreeQ[#, x | a] &]
```
Out[∘]=
```
{{{j → 0, n → 1}, True}, {{j → 1, n → 1}, True}, {{j → 1, n → 2}, True}}
```

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]=
```
{Sn^2, Sn, 1}
```

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[∘]=
```
0
```

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]=
```
{-1}
```

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, a^i * Binomial[x + i + j - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annci, Binomial[x - i + j - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {0.690554, {111 344, 111 392}}

In[∘]:=
```
Timing[ByteCount /@ (id2ct1 = FindCreativeTelescoping[annSmnd1, S[j] - 1])]
```
Out[∘]= {15.9733, {5440, 93 768}}

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]= {0.516128, {0, 0}}

In[∘]:=
```
Timing[ByteCount /@ (id2ct2 = FindCreativeTelescoping[annSmnd2, S[j] - 1])]
```
Out[∘]= {22.1118, {32 016, 408 592}}

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]= {1.26129, {0, 0}}

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```
ByteCount: 32 016

Support: $\{\{S_n, S_i, 1\}, \{S_i^3, S_i^2, S_i, 1\}\}$

degree {n, i}: {{1, 2}, {1, 6}}

Standard Monomials: $\{1, S_i, S_i^2\}$

Holonomic Rank: 3

In[∘]:=
```
Factor[id2ann]
```
Out[∘]=

$$\Big\{(-1+a)\,(1+i)\,n\,S_n - 2\,i\,(2+i-n)\,S_i + (1+a)\,(1+i)\,(-1+i+n),$$
$$2\,i\,(1+i)\,(4+i-n)\,(-2\,a+a\,i+a\,i^2-x-x^2)\,S_i^3 -$$
$$i\,(-12\,a-12\,a^2-19\,a\,i-10\,a^2\,i+4\,a\,i^2+10\,a^2\,i^2+18\,a\,i^3+10\,a^2\,i^3+8\,a\,i^4+2\,a^2\,i^4+a\,i^5+$$
$$3\,a\,i\,n+a\,i^2\,n-3\,a\,i^3\,n-a\,i^4\,n-9\,x-6\,a\,x-15\,i\,x+a\,i\,x-7\,i^2\,x-5\,a\,i^2\,x-i^3\,x-$$
$$5\,a\,i^3\,x-a\,i^4\,x+n\,x+2\,i\,n\,x-3\,a\,i\,n\,x+i^2\,n\,x+2\,a\,i^2\,n\,x+a\,i^3\,n\,x-6\,x^2-6\,a\,x^2-11\,i\,x^2-$$
$$8\,a\,i\,x^2-6\,i^2\,x^2-2\,a\,i^2\,x^2-i^3\,x^2+i\,n\,x^2+i^2\,n\,x^2+3\,x^3+4\,i\,x^3+i^2\,x^3-n\,x^3-i\,n\,x^3)\,S_i^2+$$
$$i\,(-6\,a\,i-18\,a^2\,i-5\,a\,i^2-3\,a^2\,i^2+5\,a\,i^3+13\,a^2\,i^3+5\,a\,i^4+7\,a^2\,i^4+a\,i^5+a^2\,i^5-$$
$$12\,a^2\,n+2\,a^2\,i\,n+8\,a^2\,i^2\,n+2\,a^2\,i^3\,n-2\,x-2\,a\,x-5\,i\,x-5\,a\,i\,x+6\,a^2\,i\,x-$$
$$4\,i^2\,x-7\,a\,i^2\,x-a^2\,i^2\,x-i^3\,x-5\,a\,i^3\,x-4\,a^2\,i^3\,x-a\,i^4\,x-a^2\,i^4\,x-$$
$$6\,a\,n\,x-2\,a\,i\,n\,x-2\,i\,x^2-8\,a\,i\,x^2-3\,i^2\,x^2-5\,a\,i^2\,x^2-i^3\,x^2-a\,i^3\,x^2-$$
$$6\,a\,n\,x^2-2\,a\,i\,n\,x^2+2\,x^3+2\,a\,x^3+3\,i\,x^3+3\,a\,i\,x^3+i^2\,x^3+a\,i^2\,x^3)\,S_i-$$
$$a\,(2+i)\,(-1+i+n)\,(1+i-x)\,(-3\,a\,i+2\,a\,i^2+a\,i^3-x-i\,x-x^2-i\,x^2)\Big\}$$

In[•]:=
```
(* We are required to check initial values at the following indices: *)
Select[AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1],
 FreeQ[#, x | a] &]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 3, n → 5}, True}, {{i → 3, n → 6}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Together[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}]], {n, 6}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, a^(n - 1) * Binomial[x + n + j - 2, j]];
 annSmnd2 = DFiniteTimesHyper[annc, Binomial[x - n + j, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.471342, {92 056, 122 488}}
```

In[•]:=
```
Timing[ByteCount /@ (id3ct1 = FindCreativeTelescoping[annSmnd1, S[j] - 1])]
```

Out[•]=
```
{10.3727, {2376, 324 368}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{1.42081, 0}
```

In[•]:=
```
Timing[ByteCount /@ (id3ct2 = FindCreativeTelescoping[annSmnd2, S[j] - 1])]
```

Out[•]=
```
{23.5796, {2376, 381 224}}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{2.2226, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 2520

Support: $\{\{S_n^2, S_n, 1\}\}$

degree {n}: {{1}}

Standard Monomials: $\{1, S_n\}$

Holonomic Rank: 2

In[∘]:=
```
Factor[id3ann]
```
Out[∘]= $(-1+a)\,n\,S_n^2 + \left(-2\,a - n + 6\,a\,n - a^2\,n\right)S_n - 2\,(-1+a)\,a\,(-1+2\,n)$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
ApplyOreOperator[id3ann, (a - 1) ^ (n - 1)] // Simplify
```
Out[∘]= 0

In[∘]:=
```
(* Alternatively: *)
OreReduce[id3ann, Annihilator[(a - 1) ^ (n - 1), S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Together[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] / (a - 1) ^ (n - 1)], {n, 2, 3}]
```
Out[∘]= {1, 1}

detdf: Di Francesco's determinant

In[∘]:=
```
InitializeDeterminantProof[2, 2,
 {1, 0, 1, 1}, 2 prod[2^(i-1) (4 i - 2) ! / (n + 2 i - 1) !, {i, 1, n}], "detdf/"]
```
We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{1-i+2j}{1+2j}+2^i\binom{1+i+2j}{1+2j}\right) = 2\prod_{i=1}^{n}\frac{2^{-1+i}\,(-2+4\,i)!}{(-1+2\,i+n)!}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[∘]//TableForm=

2	2	2	2	2	2
4	8	12	16	20	24
11	40	84	144	220	312
30	160	448	960	1760	2912
77	559	2016	5280	11 440	21 840
188	1788	8064	25 344	64 064	139 776

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-1	1				
1	-2	1			
$-\frac{16}{15}$	$\frac{47}{15}$	$-\frac{46}{15}$	1		
$\frac{16}{13}$	$-\frac{60}{13}$	$\frac{85}{13}$	$-\frac{54}{13}$	1	
$-\frac{20}{13}$	$\frac{88}{13}$	$-\frac{633}{52}$	$\frac{291}{26}$	$-\frac{21}{4}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 163 736

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 10}, {5, 8}, {11, 11}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True}, {{j → 1, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[∘]= {-4}

```
In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]

Out[•]= {1.94118, {579 032, 578 960}}

In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {57.5673, {0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {74.4942, {0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
        id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
        AnnInfo[id2ann]

        ByteCount: 1 382 336
        Support: {{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1}, {S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1},
          {S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1}, {S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1}}
        degree {n, i}: {{14, 18}, {9, 14}, {14, 19}, {22, 21}}
        Standard Monomials: {1, S_i, S_n, S_i^2, S_n S_i, S_n^2}
        Holonomic Rank: 6

In[•]:= (* We are required to check initial values at the following indices: *)
        AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
         {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
         {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
         {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
         {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True},
         {{i → 2, n → 8}, True}, {{i → 3, n → 5}, True}, {{i → 4, n → 6}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{0.652719, {244 432, 263 440}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{10.8745, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]=
```
{10.8507, 0}
```

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 72 040

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{52}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]=
```
{-7}
```

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]=
$$\frac{\text{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\ \text{Gamma}[-1+4\,n]}{\text{Gamma}[3\,n]\ \text{Gamma}\left[-\frac{1}{2}+\frac{3\,n}{2}\right]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```
Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[∘]= 2.57956 min

det22: Variations I (Theorem 10)

{0,1,-5,-7}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {0, 1, -5, -7},
 -168 prod[Gamma[1/2 (-1 + i)] Gamma[-9 + 4 i] / (Gamma[-7/2 + 3 i/2] Gamma[-6 + 3 i]), {i, 3, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-7-i+2j}{2j}+2^{1+i}\binom{-5+i+2j}{2j}\right) = -168\prod_{i=3}^{n}\frac{\Gamma\left(\frac{1}{2}(-1+i)\right)\Gamma(-9+4i)}{\Gamma\left(-\frac{7}{2}+\frac{3i}{2}\right)\Gamma(-6+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[∘]//TableForm=

3	27	17	1	0	0
5	33	35	7	0	0
9	36	70	28	1	0
17	36	126	84	9	0
33	45	210	210	45	1
65	119	394	526	229	75

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[∘]= $\left\{-\frac{1}{56}, \frac{3}{14}, -\frac{9}{7}, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-9	1				
$-\frac{32}{3}$	$\frac{5}{9}$	1			
$-\frac{119}{9}$	$\frac{49}{54}$	$\frac{5}{6}$	1		
-15	1	1	1	1	
1	$-\frac{1}{15}$	$-\frac{1}{15}$	$-\frac{1}{15}$	$-\frac{1}{15}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 166 976

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 10}, {5, 8}, {11, 11}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 5, 15}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {5, 0}]
```

Out[∘]= {{{j → 0, n → 5}, True}, {{j → 0, n → 6}, True},
 {{j → 1, n → 5}, True}, {{j → 5, n → 5}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[•]:=
```
(* Check that the leading coefficient
 does not have any positive integer roots. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-2}

In[•]:=
```
(* Look at the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {2.58151, {590376, 590376}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]= {66.3762, {0, 0, 0, 0}}

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]= {65.7758, {0, 0, 0, 0}}

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```
ByteCount: 1658840

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 20}, {9, 16}, {14, 21}, {22, 24}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {5, 0}, Assumptions → i < n - 1]
```
Out[•]= {{{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True}, {{i → 2, n → 5}, True}}

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.664858, {253272, 269488}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{10.3486, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{11.006, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 74552

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{54}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Check that the leading coefficient
 does not have any positive integer roots. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]=
```
{-5}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=
$$\frac{\text{Gamma}\left[-\frac{1}{2}+\frac{n}{2}\right]\ \text{Gamma}[-9+4n]}{\text{Gamma}\left[-\frac{7}{2}+\frac{3n}{2}\right]\ \text{Gamma}[-6+3n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 5, 4 + LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 2.70862 min

{0,1,-4,-6}

In[•]:=
```
InitializeDeterminantProof[2, 2, {0, 1, -4, -6},
 40 prod[Gamma[i/2] Gamma[-8 + 4 i] / (Gamma[-2 + 3 i/2] Gamma[-6 + 3 i]), {i, 3, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-6-i+2j}{2j}+2^{1+i}\binom{-4+i+2j}{2j}\right)=40\prod_{i=3}^{n}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma(-8+4i)}{\Gamma\left(-2+\frac{3i}{2}\right)\Gamma(-6+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

3	16	5	0	0	0
5	19	15	1	0	0
9	21	35	7	0	0
17	28	70	28	1	0
33	68	158	116	41	32
65	237	530	658	621	705

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= $\left\{\frac{3}{40}, -\frac{23}{40}, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
$-\frac{16}{3}$	1				
$-\frac{145}{23}$	$\frac{20}{23}$	1			
−7	1	1	1		
1	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	1	
$-\frac{14}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$-\frac{14}{11}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 165 736

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 9}, {5, 7}, {12, 11}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 4, 12}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {4, 0}]
```

Out[•]= {{{j → 0, n → 4}, True}, {{j → 0, n → 5}, True},
 {{j → 1, n → 4}, True}, {{j → 4, n → 4}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[•]= {-2, 3}

```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {2.19071, {581 216, 584 288}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```

Out[•]= {59.9382, {0, 0, 0, 0}}

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```

Out[•]= {65.765, {0, 0, 0, 0}}

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 595 776

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{15, 18}, {10, 14}, {15, 19}, {24, 22}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {4, 0}, Assumptions → i < n - 1]
```

Out[•]= {{{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True}}

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.651539, {249120, 265392}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{10.526, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{10.6083, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 73296

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{53}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]=
```
{-5}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=
$$\frac{\text{Gamma}\left[\frac{n}{2}\right]\,\text{Gamma}[-8+4n]}{\text{Gamma}\left[-2+\frac{3n}{2}\right]\,\text{Gamma}[-6+3n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 4, 3 + LeadingExponent[id3ann][[1]]}]
```
Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[∘]= 2.51069 min

{0,1,-3,-5}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {0, 1, -3, -5},
 -10 prod[((-3 + 2 i) Gamma[1/2 (-1 + i)] Gamma[-7 + 4 i]) / (Gamma[1/2 (-5 + 3 i)] Gamma[-4 + 3 i]), {i, 3, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-5-i+2j}{2j}+2^{1+i}\binom{-3+i+2j}{2j}\right) = -10\prod_{i=3}^{n}\frac{(-3+2i)\,\Gamma\left(\frac{1}{2}(-1+i)\right)\Gamma(-7+4i)}{\Gamma\left(\frac{1}{2}(-5+3i)\right)\Gamma(-4+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[∘]//TableForm=

3	8	1	0	0	0
5	10	5	0	0	0
9	15	15	1	0	0
17	37	51	23	16	16
33	124	230	252	289	352
65	420	1086	1876	2889	4224

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[∘]= $\left\{-\frac{3}{10}, 1, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{8}{3}$	1				
-3	1	1			
1	$-\frac{1}{3}$	$-\frac{1}{3}$	1		
$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	1	
$\frac{27}{11}$	$-\frac{9}{11}$	$-\frac{9}{11}$	$\frac{27}{11}$	$-\frac{29}{11}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 86 616

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{5, 7}, {5, 4}, {10, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 2, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 0, n → 4}, True}, {{j → 1, n → 1}, True}, {{j → 1, n → 3}, True},
 {{j → 2, n → 1}, True}, {{j → 2, n → 2}, True}, {{j → 3, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

```
In[•]:= (* Check that the leading coefficient
         does not have any positive integer roots. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]

Out[•]= {-2}

In[•]:= (* Look at the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]

Out[•]= {1.49067, {332 584, 334 608}}

In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {20.6016, {0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {21.9631, {0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
        id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
        AnnInfo[id2ann]
```

ByteCount: 865 920

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{13, 12}, {9, 9}, {13, 13}, {21, 17}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True},
 {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True}, {{i → 3, n → 5}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.548863, {148616, 162288}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{6.56251, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{6.76446, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 66632

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{48}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Check that the leading coefficient
 does not have any positive integer roots. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-5}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $\dfrac{(-3+2n)\,\text{Gamma}\left[-\frac{1}{2}+\frac{n}{2}\right]\,\text{Gamma}[-7+4n]}{\text{Gamma}\left[-\frac{5}{2}+\frac{3n}{2}\right]\,\text{Gamma}[-4+3n]}$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 3, 2 + LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 59.3375 s

{0,1,-2,-4}

In[•]:=
```
InitializeDeterminantProof[2, 2, {0, 1, -2, -4},
 3 prod[Gamma[i/2] Gamma[-5 + 4 i] / (Gamma[3 (-1 + i)] Gamma[1/2 (-4 + 3 i)]), {i, 2, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-4-i+2j}{2j}+2^{1+i}\binom{-2+i+2j}{2j}\right)=3\prod_{i=2}^{n}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma(-5+4i)}{\Gamma(3(-1+i))\,\Gamma\left(\frac{1}{2}(-4+3i)\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

3	3	0	0	0	0
5	6	1	0	0	0
9	18	13	8	8	8
17	63	95	113	144	176
33	213	515	903	1440	2112
65	668	2310	5404	10 561	18 304

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-1	1				
1	-1	1			
-2	2	-2	1		
$\frac{62}{15}$	$-\frac{62}{15}$	$\frac{62}{15}$	$-\frac{46}{15}$	1	
$-\frac{114}{13}$	$\frac{114}{13}$	$-\frac{114}{13}$	$\frac{98}{13}$	$-\frac{54}{13}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 97 224

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 7}, {5, 4}, {11, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]=
```
{{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 0, n → 3}, True}, {{j → 1, n → 1}, True}, {{j → 1, n → 2}, True},
 {{j → 2, n → 1}, True}, {{j → 2, n → 2}, True}, {{j → 3, n → 1}, True}}
```

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]= {-2}

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {1.39841, {370 696, 376 776}}

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]= {20.4831, {0, 0, 0, 0}}

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]= {22.5058, {0, 0, 0, 0}}

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 927 904

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 12}, {9, 9}, {14, 13}, {22, 17}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
 {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {0.499155, {163 904, 177 464}}

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]= {6.64748, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {6.60184, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

```
ByteCount: 65 088
Support: {{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1}}
degree {n}: {{47}}
Standard Monomials: {1, S_n, S_n^2, S_n^3, S_n^4, S_n^5}
Holonomic Rank: 6
```

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-5}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```

Out[•]= $\dfrac{\text{Gamma}\left[\frac{n}{2}\right]\,\text{Gamma}[-5+4\,n]}{\text{Gamma}\left[-2+\frac{3\,n}{2}\right]\,\text{Gamma}[-3+3\,n]}$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 58.3039 s

{0,1,1,-1}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {0, 1, 1, -1},
  3 prod[Gamma[1/2 (-1 + i)] Gamma[4 i] / (Gamma[3/2 (-1 + i)] Gamma[1 + 3 i]), {i, 2, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+2j}{2j}+2^{1+i}\binom{1+i+2j}{2j}\right)=3\prod_{i=2}^{n}\frac{\Gamma\left(\frac{1}{2}(-1+i)\right)\Gamma(4i)}{\Gamma\left(\frac{3}{2}(-1+i)\right)\Gamma(1+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

3	6	10	14	18	22
5	24	60	112	180	264
9	81	280	672	1320	2288
17	243	1120	3360	7920	16 016
33	678	4033	14 784	41 184	96 096
65	1802	13 445	59 136	192 192	512 512

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-2	1				
$\frac{20}{7}$	$-\frac{65}{21}$	1			
-4	$\frac{20}{3}$	$-\frac{21}{5}$	1		
$\frac{852}{143}$	$-\frac{1836}{143}$	$\frac{8256}{715}$	$-\frac{69}{13}$	1	
$-\frac{125}{13}$	$\frac{937}{39}$	$-\frac{349}{13}$	$\frac{6413}{364}$	$-\frac{77}{12}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 110 720

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 7}, {6, 4}, {12, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```
Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```
Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True}, {{j → 1, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]= {-2}

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {1.58404, {420 544, 411 664}}

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]= {25.4465, {0, 0, 0, 0}}

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]=
```
{27.0052, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 008 320

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{15, 12}, {10, 9}, {15, 13}, {23, 17}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```
Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
 {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```
Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]=
```
{0.570148, {182 136, 174 288}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```
Out[∘]=
```
{6.57022, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```
Out[∘]=
```
{7.18557, 0}
```

```
In[•]:= (* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 63 704
Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$
degree {n}: {{46}}
Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$
Holonomic Rank: 6

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-6, -5}

```
In[•]:= (* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $\dfrac{\text{Gamma}\left[-\frac{1}{2}+\frac{n}{2}\right]\,\text{Gamma}[4\,n]}{\text{Gamma}\left[-\frac{3}{2}+\frac{3\,n}{2}\right]\,\text{Gamma}[1+3\,n]}$

```
In[•]:= (* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

```
In[•]:= (* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

```
In[•]:= (* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 1.15048 min

{0,2,-5,-9}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {0, 2, -5, -9},
  -3696 prod[Gamma[1/2 (-1 + i)] Gamma[-9 + 4 i] / (Gamma[3 (-2 + i)] Gamma[1/2 (-7 + 3 i)]), {i, 3, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-9-i+2j}{2j}+2^{2+i}\binom{-5+i+2j}{2j}\right)=-3696\prod_{i=3}^{n}\frac{\Gamma\left(\frac{1}{2}(-1+i)\right)\Gamma(-9+4i)}{\Gamma(3(-2+i))\Gamma\left(\frac{1}{2}(-7+3i)\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

5	52	74	28	1	0
9	60	126	84	9	0
17	61	210	210	45	1
33	55	330	462	165	11
65	66	495	924	495	66
129	206	843	1844	1415	414

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{-\frac{5}{3696}, \frac{1}{22}, -\frac{235}{308}, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{52}{5}$	1				
$-\frac{88}{7}$	$-\frac{3}{14}$	1			
$-\frac{3584}{235}$	$\frac{238}{235}$	$-\frac{14}{235}$	1		
-15	0	1	0	1	
0	1	$-\frac{16}{15}$	1	$-\frac{16}{15}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 115 536

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 7}, {5, 5}, {11, 9}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

```
In[∘]:= (* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
        With[{test = ApplyOreOperator[annc, myc[n, j]]},
         Union[Flatten[Table[test, {n, 5, 15}, {j, 0, n - 1}]]]]

Out[∘]= {0}

In[∘]:= (* The values at these indices have to be given as initial conditions,
        in order to uniquely define c_{n,j} via the recurrences in annc. *)
        AnnihilatorSingularities[annc, {5, 0}]

Out[∘]= {{{j → 0, n → 5}, True}, {{j → 0, n → 6}, True},
         {{j → 0, n → 7}, True}, {{j → 1, n → 5}, True},
         {{j → 1, n → 6}, True}, {{j → 2, n → 5}, True}, {{j → 6, n → 5}, True}}
```

Proof of Identity (H1)

```
In[∘]:= (* Compute a recurrence for c_{n,n-1} *)
        Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

```
In[∘]:= (* Verify that this recurrence admits a constant sequence as solution. *)
        OreReduce[cnn1, Annihilator[1, S[n]]]

Out[∘]= 0

In[∘]:= (* Look at the integer roots of the leading coefficient. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]

Out[∘]= {-2, 6}

In[∘]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[∘]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[∘]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]

Out[∘]= {1.88769, {432768, 432848}}
```

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{32.6572, {0, 0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]=
```
{37.3799, {0, 0, 0, 0}}
```

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 074 944

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 14}, {9, 10}, {14, 15}, {22, 18}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {5, 0}, Assumptions → i < n - 1]
```
Out[•]=
```
{{{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True},
 {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True}, {{i → 2, n → 5}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```
Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{0.607074, {186 800, 201 104}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```
Out[•]=
```
{9.33065, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```
Out[∘]= {8.00168, 0}

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```
ByteCount: 71 920

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{52}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[∘]= {-5, 3}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[∘]= $\dfrac{\text{Gamma}\left[-\frac{1}{2}+\frac{n}{2}\right]\ \text{Gamma}[-9+4n]}{\text{Gamma}\left[-\frac{7}{2}+\frac{3n}{2}\right]\ \text{Gamma}[-6+3n]}$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 5, 4 + LeadingExponent[id3ann][[1]]}]
```
Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[∘]= 1.49967 min

{0,2,3,-1}

In[•]:=
```
InitializeDeterminantProof[2, 2, {0, 2, 3, -1},
  prod[(3 (-1 + 2 i) Gamma[(1+i)/2] Gamma[3 + 4 i])/(4 (2 + i) Gamma[1 + 3 i] Gamma[1/2 (5 + 3 i)]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+2j}{2j}+2^{2+i}\binom{3+i+2j}{2j}\right)=\prod_{i=1}^{n}\frac{3(-1+2i)\,\Gamma\left(\frac{1+i}{2}\right)\Gamma(3+4i)}{4(2+i)\,\Gamma(1+3i)\,\Gamma\left(\frac{1}{2}(5+3i)\right)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

5	40	140	336	660	1144
9	120	560	1680	3960	8008
17	337	2016	7392	20 592	48 048
33	899	6720	29 568	96 096	256 256
65	2310	21 121	109 824	411 840	1 244 672
129	5770	63 365	384 384	1 647 360	5 601 024

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
-8	1				
$\frac{70}{3}$	$-\frac{77}{12}$	1			
$-\frac{560}{11}$	$\frac{112}{5}$	$-\frac{384}{55}$	1		
$\frac{1320}{13}$	$-\frac{792}{13}$	$\frac{363}{13}$	$-\frac{55}{7}$	1	
$-\frac{10\,252}{51}$	$\frac{2508}{17}$	$-\frac{30\,932}{357}$	$\frac{152\,009}{4284}$	$-\frac{1352}{153}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 123 024

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{5, 9}, {7, 5}, {12, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```
Out[•]=
```
{0}
```

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```
Out[•]=
```
{{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True}, {{j → 1, n → 1}, True}}
```

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[•]=
```
0
```

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]=
```
{-3}
```

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{1.67248, {417 024, 408 056}}
```

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{20.0672, {0, 0, 0, 0}}
```

```
In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {22.5374, {0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 907 600

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{15, 12}, {11, 9}, {15, 12}, {23, 15}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

```
In[•]:= (* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
 {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True}}

In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

```
In[•]:= (* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]

Out[•]= {0.565368, {184 016, 169 480}}

In[•]:= (* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]

Out[•]= {7.64726, 0}

In[•]:= (* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]

Out[•]= {6.67648, 0}
```

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 73 768

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{52}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-8, -7, -6}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]= $$\frac{3\,(-1+2\,n)\,\text{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\,\text{Gamma}[3+4\,n]}{4\,(2+n)\,\text{Gamma}\left[\frac{5}{2}+\frac{3\,n}{2}\right]\,\text{Gamma}[1+3\,n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 1.00629 min

{1,1,-2,-4}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {1, 1, -2, -4},
  -32 prod[(Gamma[i/2] Gamma[-5 + 4 i])/(Gamma[3 (-1 + i)] Gamma[1/2 (-4 + 3 i)]), {i, 2, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-4-i+2j}{1+2j}+2^{1+i}\binom{-2+i+2j}{1+2j}\right) = -32\prod_{i=2}^{n}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma(-5+4i)}{\Gamma(3(-1+i))\,\Gamma\left(\frac{1}{2}(-4+3i)\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-8	-4	0	0	0	0
-9	-10	-1	0	0	0
-6	-20	-6	0	0	0
9	-19	-5	15	16	16
56	72	136	248	320	384
183	556	1218	2268	3519	4992

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{\frac{1}{4}, -\frac{11}{8}, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{1}{2}$	1				
$\frac{1}{11}$	$-\frac{2}{11}$	1			
0	0	0	1		
0	0	0	$-\frac{16}{15}$	1	
0	0	0	$\frac{16}{13}$	$-\frac{28}{13}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 88 384

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 7}, {5, 4}, {11, 7}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 4, 14}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {4, 0}]
```

Out[•]=
```
{{{j → 0, n → 4}, True}, {{j → 0, n → 5}, True}, {{j → 1, n → 4}, True},
 {{j → 3, n → 4}, True}, {{j → 3, n → 5}, True}, {{j → 4, n → 4}, True}}
```

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[•]= {-2, 1, 2}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {2.30225, {313 936, 319 240}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{18.6365, {0, 0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]=
```
{16.8013, {0, 0, 0, 0}}
```

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 760 336

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 10}, {9, 8}, {14, 11}, {22, 15}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {4, 0}, Assumptions → i < n - 1]
```
Out[•]=
```
{{{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True},
 {{i → 3, n → 5}, True}, {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```
Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{0.679458, {139 040, 152 056}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```
Out[•]=
```
{7.8688, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {7.07733, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 70752

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{51}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-4}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $\dfrac{\text{Gamma}\left[\frac{n}{2}\right]\ \text{Gamma}[-5+4\,n]}{\text{Gamma}\left[-2+\frac{3\,n}{2}\right]\ \text{Gamma}[-3+3\,n]}$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 4, 3 + LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 55.6491 s

{1,1,-1,-3}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {1, 1, -1, -3},
  8 prod[Gamma[1/2 (-1 + i)] Gamma[-4 + 4 i] / (3 Gamma[3/2 (-1 + i)] Gamma[-3 + 3 i]), {i, 2, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-3-i+2j}{1+2j}+2^{1+i}\binom{-1+i+2j}{1+2j}\right)=8\prod_{i=2}^{n}\frac{\Gamma\left(\frac{1}{2}(-1+i)\right)\Gamma(-4+4i)}{3\,\Gamma\left(\frac{3}{2}(-1+i)\right)\Gamma(-3+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-5	-1	0	0	0	0
-4	-4	0	0	0	0
3	-2	7	8	8	8
26	44	90	128	160	192
89	285	651	1151	1760	2496
248	1224	3528	7672	14 080	23 296

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{-\frac{5}{8}, 1, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{1}{5}$	1				
0	0	1			
0	0	$-\frac{8}{7}$	1		
0	0	$\frac{16}{11}$	$-\frac{25}{11}$	1	
0	0	$-\frac{2032}{1001}$	$\frac{4176}{1001}$	$-\frac{309}{91}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 103 064

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 8}, {5, 4}, {12, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 3, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {3, 0}]
```

Out[•]= {{{j → 0}, n ≥ 3}, {{j → 0, n → 3}, True}, {{j → 0, n → 4}, True}, {{j → 1, n → 3}, True}, {{j → 2, n → 3}, True}, {{j → 2, n → 4}, True}, {{j → 3, n → 3}, True}}

From the previous output it seems that annc cannot be used to compute c_{n,0}. However, this is not true, by the following consideration: although the leading coefficient of the recurrence annc[[3]] vanishes, it can still be used to compute c[n,0].

In[•]:=
```
ApplyOreOperator[Factor[annc[[3]]], c[n, j]] /. j → 0
```

Out[•]= $-4\,(-1+n)\,(2+3n)\,(-3+4n)\,(-1+4n)\,(-2n^2-3n^3+17n^4+27n^5+9n^6)\,c[n,0]-4\,(1-n)\,n\,(-1+3n)\,(1+3n)\,(2+3n)\,(-3+4n)\,(-1+4n)\,(2n+3n^2+n^3)\,c[n,1]$

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Check that the leading coefficient
 does not have any positive integer roots. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[•]= {-2, 0}

In[•]:=
```
(* Look at the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{1.55694, {359 168, 365 792}}
```

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```

Out[∘]=
```
{21.7283, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```

Out[∘]=
```
{23.7935, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 020 848

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{15, 12}, {10, 10}, {15, 13}, {24, 17}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {3, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
 {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {0.495878, {156 392, 170 064}}

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]= {7.32931, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {6.77854, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 70 488

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{51}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Check that the leading coefficient
 does not have any positive integer roots. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-5}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $$\frac{\text{Gamma}\left[-\frac{1}{2}+\frac{n}{2}\right]\,\text{Gamma}[-4+4\,n]}{3\,\text{Gamma}\left[-\frac{3}{2}+\frac{3\,n}{2}\right]\,\text{Gamma}[-3+3\,n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 3, 2 + LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 1.03987 min

{1,1,0,-2}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {1, 1, 0, -2},
 -2 prod[((-1 + 2 i) Gamma[i/2] Gamma[-3 + 4 i])/(2 Gamma[3 i/2] Gamma[-2 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-2-i+2j}{1+2j}+2^{1+i}\binom{i+2j}{1+2j}\right) = -2\prod_{i=1}^{n}\frac{(-1+2i)\,\Gamma\left(\frac{i}{2}\right)\Gamma(-3+4i)}{2\,\Gamma\left(\frac{3i}{2}\right)\Gamma(-2+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-2	0	0	0	0	0
1	3	4	4	4	4
12	28	48	64	80	96
43	150	335	576	880	1248
122	620	1786	3840	7040	11 648
313	2205	8043	21 119	45 760	87 360

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
0	1				
0	$-\frac{4}{3}$	1			
0	2	$-\frac{5}{2}$	1		
0	$-\frac{36}{11}$	$\frac{56}{11}$	$-\frac{40}{11}$	1	
0	$\frac{40}{7}$	$-\frac{208}{21}$	$\frac{197}{21}$	$-\frac{100}{21}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 91112

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{5, 8}, {5, 5}, {10, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 1, n → 2}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]=
```
{-2}
```

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]=
```
{1.50278, {346040, 336176}}
```

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]=
```
{20.8796, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]=
```
{20.2597, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 749720

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{13, 12}, {9, 9}, {13, 12}, {21, 15}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.476947, {149 624, 161 616}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{7.38297, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{6.55921, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 63 752

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{46}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-5}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]= $\frac{(-1+2\,n)\,\mathrm{Gamma}\left[\frac{n}{2}\right]\,\mathrm{Gamma}[-3+4\,n]}{2\,\mathrm{Gamma}\left[\frac{3\,n}{2}\right]\,\mathrm{Gamma}[-2+3\,n]}$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 58.6119 s

{1,1,1,-1}

In[∘]:=
```
InitializeDeterminantProof[2, 2,
 {1, 1, 1, -1}, prod[Gamma[(1+i)/2] Gamma[-1 + 4 i] / (Gamma[3 i] Gamma[-1/2 + 3 i/2]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+2\,j}{1+2\,j}+2^{1+i}\binom{1+i+2\,j}{1+2\,j}\right)=\prod_{i=1}^{n}\frac{\Gamma\left(\frac{1+i}{2}\right)\Gamma(-1+4\,i)}{\Gamma(3\,i)\,\Gamma\left(-\frac{1}{2}+\frac{3\,i}{2}\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

1	2	2	2	2	2
6	16	24	32	40	48
21	79	168	288	440	624
60	316	896	1920	3520	5824
155	1110	4031	10 560	22 880	43 680
378	3564	16 122	50 688	128 128	279 552

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
-2	1				
4	-3	1			
$-\frac{124}{15}$	$\frac{36}{5}$	$-\frac{61}{15}$	1		
$\frac{228}{13}$	$-\frac{212}{13}$	$\frac{152}{13}$	$-\frac{67}{13}$	1	
$-\frac{493}{13}$	$\frac{473}{13}$	$-\frac{385}{13}$	$\frac{907}{52}$	$-\frac{25}{4}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 102 352

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 7}, {5, 5}, {11, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True}, {{j → 1, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]

Out[•]= {-2}

In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]

Out[•]= {2.27374, {382368, 374520}}

In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {26.7379, {0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {29.5234, {0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
        id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
        AnnInfo[id2ann]
```

ByteCount: 985304

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 13}, {9, 10}, {14, 14}, {22, 17}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

```
In[•]:= (* We are required to check initial values at the following indices: *)
        AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
         {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 1, n → 3}, True},
         {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.538713, {161744, 179656}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{7.79715, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{7.20629, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 62528

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{45}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]=
```
{-5}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```

Out[•]=

$$\frac{\text{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\ \text{Gamma}[-1+4n]}{\text{Gamma}[3n]\ \text{Gamma}\left[-\frac{1}{2}+\frac{3n}{2}\right]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 1.25192 min

{1,2,-5,-9}

In[∘]:=
```
InitializeDeterminantProof[2, 2, {1, 2, -5, -9},
 -337920 prod[Gamma[1/2 (-1 + i)] Gamma[-9 + 4 i] / (Gamma[-7/2 + 3 i/2] Gamma[-6 + 3 i]), {i, 3, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-9-i+2j}{1+2j}+2^{2+i}\binom{-5+i+2j}{1+2j}\right)=-337920\prod_{i=3}^{n}\frac{\Gamma\left(\frac{1}{2}(-1+i)\right)\Gamma(-9+4i)}{\Gamma\left(-\frac{7}{2}+\frac{3i}{2}\right)\Gamma(-6+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-29	-124	-130	-36	-1	0
-42	-152	-252	-120	-10	0
-59	-181	-462	-330	-55	-1
-76	-220	-792	-792	-220	-12
-77	-286	-1287	-1716	-715	-78
-14	-364	-2002	-3432	-2002	-364

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{\frac{29}{337920}, \frac{5}{2112}, -\frac{3287}{42240}, -\frac{1333}{880}, -\frac{359}{400}, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{124}{29}$	1				
$-\frac{359}{25}$	$\frac{231}{100}$	1			
$\frac{78264}{3287}$	$-\frac{10722}{3287}$	$-\frac{8142}{3287}$	1		
$\frac{11561}{1333}$	$-\frac{1848}{1333}$	$-\frac{773}{1333}$	$-\frac{580}{3999}$	1	
$\frac{13832}{1077}$	$-\frac{723}{359}$	$-\frac{10948}{11847}$	$-\frac{3770}{35541}$	$\frac{14588}{11847}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 161 136

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 9}, {5, 8}, {11, 11}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 7, 19}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {7, 0}]
```

Out[∘]= {{{j → 0, n → 7}, True}, {{j → 0, n → 8}, True},
 {{j → 1, n → 7}, True}, {{j → 7, n → 7}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]=
```
{-2, 6}
```

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]=
```
{1.98104, {566728, 566728}}
```

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]=
```
{80.579, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]=
```
{76.7008, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1693320

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 20}, {9, 16}, {14, 21}, {22, 24}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {7, 0}, Assumptions -> i < n - 1]
```
Out[∘]=
```
{{{i -> 0, n -> 7}, True}, {{i -> 0, n -> 8}, True}, {{i -> 0, n -> 9}, True},
 {{i -> 1, n -> 7}, True}, {{i -> 1, n -> 8}, True}, {{i -> 2, n -> 7}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.682202, {245 256, 261 344}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{12.3569, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{10.3918, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 78 672

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{57}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]=
```
{-5}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=

$$\frac{\text{Gamma}\left[-\frac{1}{2}+\frac{n}{2}\right]\ \text{Gamma}[-9+4\,n]}{\text{Gamma}\left[-\frac{7}{2}+\frac{3\,n}{2}\right]\ \text{Gamma}[-6+3\,n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 7, 6 + LeadingExponent[id3ann][[1]]}]
```
Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[•]= 3.18004 min

{1,2,-4,-8}

In[•]:=
```
InitializeDeterminantProof[2, 2, {1, 2, -4, -8},
 -24576 prod[Gamma[i/2] Gamma[-8 + 4 i] / (Gamma[-2 + 3 i/2] Gamma[-6 + 3 i]), {i, 3, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\leq i,j<n}\left(\binom{-8-i+2j}{1+2j}+2^{2+i}\binom{-4+i+2j}{1+2j}\right)=-24576\prod_{i=3}^{n}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma(-8+4i)}{\Gamma\left(-2+\frac{3i}{2}\right)\Gamma(-6+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[•]//TableForm=

-24	-72	-56	-8	0	0
-33	-92	-126	-36	-1	0
-42	-120	-252	-120	-10	0
-43	-165	-462	-330	-55	-1
-12	-220	-792	-792	-220	-12
115	-158	-1159	-1588	-587	50

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[•]= $\left\{\frac{1}{1024}, \frac{7}{1024}, -\frac{49}{128}, -\frac{67}{64}, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
-3	1				
$-\frac{70}{3}$	7	1			
$\frac{46}{7}$	$-\frac{12}{7}$	$-\frac{37}{49}$	1		
$\frac{330}{67}$	$-\frac{89}{67}$	$-\frac{1345}{2814}$	$\frac{211}{402}$	1	
-1	$\frac{3}{11}$	$\frac{1}{11}$	$-\frac{1}{11}$	$-\frac{3}{11}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 165 736

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 9}, {5, 7}, {12, 11}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 6, 16}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {6, 0}]
```

Out[•]= {{{j → 0, n → 6}, True}, {{j → 0, n → 7}, True},
 {{j → 1, n → 6}, True}, {{j → 6, n → 6}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-2, 5}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {3.08958, {581136, 583912}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]= {77.1844, {0, 0, 0, 0}}

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]= {73.6565, {0, 0, 0, 0}}

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1625776

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{15, 18}, {10, 14}, {15, 19}, {24, 22}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {6, 0}, Assumptions → i < n - 1]
```
Out[•]= {{{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True}, {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 6}, True}}

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```
Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{0.762562, {249 264, 264 056}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```
Out[•]=
```
{12.007, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```
Out[•]=
```
{11.9582, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 77 400

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{56}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[•]=
```
{-5}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[•]=

$$\frac{\text{Gamma}\left[\frac{n}{2}\right]\ \text{Gamma}[-8+4n]}{\text{Gamma}\left[-2+\frac{3n}{2}\right]\ \text{Gamma}[-6+3n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 6, 5 + LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 3.00021 min

{1,2,-3,-7}

In[•]:=
```
InitializeDeterminantProof[2, 2, {1, 2, -3, -7},
 -2016 prod[((-3 + 2 i) Gamma[1/2 (-1 + i)] Gamma[-7 + 4 i]) / (Gamma[1/2 (-5 + 3 i)] Gamma[-4 + 3 i]), {i, 3, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-7-i+2j}{1+2j}+2^{2+i}\binom{-3+i+2j}{1+2j}\right)=-2016\prod_{i=3}^{n}\frac{(-3+2i)\,\Gamma\left(\frac{1}{2}(-1+i)\right)\Gamma(-7+4i)}{\Gamma\left(\frac{1}{2}(-5+3i)\right)\Gamma(-4+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

-19	-39	-21	-1	0	0
-24	-56	-56	-8	0	0
-25	-84	-126	-36	-1	0
-10	-120	-252	-120	-10	0
53	-101	-398	-266	9	63
244	292	-24	232	1060	1524

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= $\left\{\frac{19}{2016}, -\frac{4}{63}, -\frac{17}{18}, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
$-\frac{39}{19}$	1				
$\frac{63}{8}$	$-\frac{35}{8}$	1			
$\frac{40}{17}$	$-\frac{61}{51}$	$\frac{16}{357}$	1		
-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	
$\frac{18}{11}$	$-\frac{9}{11}$	0	$\frac{9}{11}$	$-\frac{18}{11}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 106 088
Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$
degree {n, j}: {{5, 7}, {5, 6}, {10, 9}}
Standard Monomials: $\{1, S_j, S_n\}$
Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 5, 15}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {5, 0}]
```

Out[•]= {{{j → 0, n → 5}, True}, {{j → 0, n → 6}, True},
 {{j → 1, n → 5}, True}, {{j → 5, n → 5}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]=
```
{-2, 4}
```

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]=
```
{1.76931, {382368, 385144}}
```

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]=
```
{40.7574, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]=
```
{44.3117, {0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1174400

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{13, 16}, {9, 12}, {13, 17}, {21, 20}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {5, 0}, Assumptions → i < n - 1]
```
Out[∘]=
```
{{{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True},
 {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True}, {{i → 2, n → 5}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.535335, {173 736, 187 000}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{8.69695, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{9.41073, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 70 744

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{51}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]=
```
{-5}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=

$$\frac{(-3 + 2 n)\,\text{Gamma}\left[-\frac{1}{2} + \frac{n}{2}\right]\,\text{Gamma}[-7 + 4 n]}{\text{Gamma}\left[-\frac{5}{2} + \frac{3 n}{2}\right]\,\text{Gamma}[-4 + 3 n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 5, 4 + LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 1.8598 min

{1,2,-2,-6}

In[•]:=
```
InitializeDeterminantProof[2, 2, {1, 2, -2, -6},
 -224 prod[Gamma[i/2] Gamma[-5 + 4 i] / (Gamma[3 (-1 + i)] Gamma[1/2 (-4 + 3 i)]), {i, 2, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-6-i+2j}{1+2j}+2^{2+i}\binom{-2+i+2j}{1+2j}\right) = -224\prod_{i=2}^{n}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma(-5+4i)}{\Gamma(3(-1+i))\,\Gamma\left(\frac{1}{2}(-4+3i)\right)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

-14	-20	-6	0	0	0
-15	-35	-21	-1	0	0
-8	-56	-56	-8	0	0
23	-52	-94	-4	31	32
118	136	132	392	630	768
373	1115	2226	4278	6985	9983

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= $\left\{\frac{1}{16}, -\frac{95}{112}, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{10}{7}$	1				
$\frac{21}{19}$	$-\frac{102}{95}$	1			
-1	1	-1	1		
$\frac{31}{15}$	$-\frac{31}{15}$	$\frac{31}{15}$	$-\frac{31}{15}$	1	
$-\frac{57}{13}$	$\frac{57}{13}$	$-\frac{57}{13}$	$\frac{57}{13}$	$-\frac{41}{13}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 114 608

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 7}, {5, 5}, {11, 9}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 4, 15}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {4, 0}]
```

Out[∘]= $\{\{\{j \to 0, n \to 4\}, \text{True}\}, \{\{j \to 0, n \to 5\}, \text{True}\}, \{\{j \to 1, n \to 4\}, \text{True}\}, \{\{j \to 4, n \to 4\}, \text{True}\}\}$

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]=
```
{-2, 3}
```
In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```
In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{1.60222, {418 480, 428 000}}
```
In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{31.2867, {0, 0, 0, 0}}
```
In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]=
```
{37.2182, {0, 0, 0, 0}}
```
In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 068 504

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 14}, {9, 10}, {14, 15}, {22, 18}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {4, 0}, Assumptions → i < n - 1]
```
Out[•]=
```
{{{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 2, n → 4}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.810103, {184 432, 199 744}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{10.5755, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{9.12039, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 69 240

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{50}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]=
```
{-5}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=
$$\frac{\text{Gamma}\left[\frac{n}{2}\right]\,\text{Gamma}[-5+4n]}{\text{Gamma}\left[-2+\frac{3n}{2}\right]\,\text{Gamma}[-3+3n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 4, 3 + LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 1.50379 min

{2,1,1,-1}

In[•]:=
```
InitializeDeterminantProof[2, 2,
 {2, 1, 1, -1}, prod[Gamma[(1+i)/2] Gamma[-1 + 4 i] / (Gamma[3 i] Gamma[-1/2 + 3 i/2]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+2j}{2+2j}+2^{1+i}\binom{1+i+2j}{2+2j}\right)=\prod_{i=1}^{n}\frac{\Gamma\left(\frac{1+i}{2}\right)\Gamma(-1+4i)}{\Gamma(3i)\,\Gamma\left(-\frac{1}{2}+\frac{3i}{2}\right)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

1	0	0	0	0	0
7	4	4	4	4	4
30	41	56	72	88	104
106	245	448	720	1056	1456
335	1135	2689	5280	9152	14 560
981	4515	13 447	31 680	64 064	116 480

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
0	1				
0	-1	1			
0	$\frac{16}{15}$	$-\frac{31}{15}$	1		
0	$-\frac{16}{13}$	$\frac{44}{13}$	$-\frac{41}{13}$	1	
0	$\frac{20}{13}$	$-\frac{68}{13}$	$\frac{361}{52}$	$-\frac{17}{4}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 109 816
Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$
degree {n, j}: {{6, 9}, {5, 5}, {11, 8}}
Standard Monomials: $\{1, S_j, S_n\}$
Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 1, n → 2}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[•]= {-3}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```

Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{1.80329, {391 792, 383 440}}
```

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```

Out[•]=
```
{18.546, {0, 0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```

Out[•]=
```
{25.4991, {0, 0, 0, 0}}
```

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 814 072

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{14, 12}, {9, 9}, {14, 12}, {22, 15}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}
```

```
In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

```
In[•]:= (* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]

Out[•]= {0.583216, {163 712, 176 736}}

In[•]:= (* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]

Out[•]= {7.3662, 0}

In[•]:= (* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]

Out[•]= {7.50753, 0}

In[•]:= (* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 66 560
Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$
degree {n}: {{48}}
Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$
Holonomic Rank: 6

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]

Out[•]= {-5}

In[•]:= (* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $$\frac{\text{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\,\text{Gamma}[-1+4n]}{\text{Gamma}[3n]\,\text{Gamma}\left[-\frac{1}{2}+\frac{3n}{2}\right]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 1.04477 min

{2,1,2,0}

In[•]:=
```
InitializeDeterminantProof[2, 2,
 {2, 1, 2, 0}, prod[Gamma[1 + i/2] Gamma[4 i] / (Gamma[3 i] Gamma[1 + 3 i/2]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-i+2j}{2+2j}+2^{1+i}\binom{2+i+2j}{2+2j}\right)=\prod_{i=1}^{n}\frac{\Gamma\left(1+\frac{i}{2}\right)\Gamma(4i)}{\Gamma(3i)\,\Gamma\left(1+\frac{3i}{2}\right)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

2	2	2	2	2	2
13	20	28	36	44	52
51	120	224	360	528	728
166	561	1344	2640	4576	7280
490	2245	6720	15 840	32 032	58 240
1359	8079	29 569	82 368	192 192	396 032

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
-1	1				
$\frac{8}{7}$	$-\frac{15}{7}$	1			
$-\frac{16}{11}$	$\frac{41}{11}$	$-\frac{36}{11}$	1		
$\frac{2032}{1001}$	$-\frac{6208}{1001}$	$\frac{7575}{1001}$	$-\frac{400}{91}$	1	
$-\frac{52}{17}$	$\frac{176}{17}$	$-\frac{1049}{68}$	$\frac{215}{17}$	$-\frac{375}{68}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 117 920

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 8}, {5, 5}, {12, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True}, {{j → 1, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[•]= {-3}

```
In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]

Out[•]= {2.648, {359 216, 359 192}}

In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {24.5984, {0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {30.4218, {0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
        id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
        AnnInfo[id2ann]

        ByteCount: 1 230 744
        Support: {{Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1}, {Sn Si^2, Sn^2, Sn Si, Si^2, Sn, Si, 1},
          {Sn^2 Si, Sn^2, Sn Si, Si^2, Sn, Si, 1}, {Sn^3, Sn^2, Sn Si, Si^2, Sn, Si, 1}}
        degree {n, i}: {{15, 15}, {10, 10}, {15, 16}, {24, 18}}
        Standard Monomials: {1, Si, Sn, Si^2, Sn Si, Sn^2}
        Holonomic Rank: 6

In[•]:= (* We are required to check initial values at the following indices: *)
        AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
         {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 1, n → 3}, True},
         {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
         {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True}, {{i → 3, n → 5}, True}}
```

```
In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

```
In[•]:= (* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]

Out[•]= {0.837697, {151 952, 146 040}}

In[•]:= (* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]

Out[•]= {8.14798, 0}

In[•]:= (* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]

Out[•]= {6.2868, 0}

In[•]:= (* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 67 032
Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$
degree {n}: {{48}}
Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$
Holonomic Rank: 6

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]

Out[•]= {-6}

In[•]:= (* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :→ (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=
$$\frac{\text{Gamma}\left[1+\frac{n}{2}\right]\,\text{Gamma}[4\,n]}{\text{Gamma}[3\,n]\,\text{Gamma}\left[1+\frac{3\,n}{2}\right]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```
Out[•]= {True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[•]= 1.28124 min

{-2,0,-1,-1}

In[•]:=
```
InitializeDeterminantProof[2, 2, {-2, 0, -1, -1},
 -2 prod[(8 (-3 + 2 i) (-1 + 2 i) Gamma[(1+i)/2] Gamma[-5 + 4 i]) / (i Gamma[3/2 (-1 + i)] Gamma[-2 + 3 i]), {i, 2, n}]]
```
We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+2j}{-2+2j}+2^i\binom{-1+i+2j}{-2+2j}\right) = -2\prod_{i=2}^{n}\frac{8(-3+2i)(-1+2i)\,\Gamma\left(\frac{1+i}{2}\right)\Gamma(-5+4i)}{i\,\Gamma\left(\frac{3}{2}(-1+i)\right)\Gamma(-2+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[•]//TableForm=

-2	2	6	10	14	18
1	3	13	31	57	91
0	5	40	140	336	660
0	9	120	560	1680	3960
0	17	337	2016	7392	20 592
0	33	899	6720	29 568	96 096

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
1	1				
-1	-4	1			
$\frac{3}{4}$	10	$-\frac{19}{4}$	1		
$-\frac{13}{25}$	$-\frac{112}{5}$	$\frac{364}{25}$	$-\frac{144}{25}$	1	
$\frac{17}{49}$	$\frac{4440}{91}$	$-\frac{264}{7}$	$\frac{13179}{637}$	$-\frac{4345}{637}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 175 752

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{6, 10}, {8, 5}, {14, 10}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
(* Recurrences are only valid for j>0,
which is ok, because a_{i,0}=0 (for i>1). *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 2, 10}, {j, 1, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {2, 1}]
```

Out[∘]= {{{j → 1, n → 2}, True}, {{j → 1, n → 3}, True}, {{j → 2, n → 2}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-2}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {3.50162, {611 536, 611 560}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]= {42.2379, {0, 0, 0, 0}}

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]= {42.4829, {0, 0, 0, 0}}

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 366 912

Support: $\{\{S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{17, 16}, {13, 11}, {17, 15}, {26, 18}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2\}$

Holonomic Rank: 6

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {2, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{0.891984, {255 064, 263 040}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{10.4404, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]=
```
{10.4558, 0}
```

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 76 088

Support: $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{55}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5\}$

Holonomic Rank: 6

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[∘]= {-6, -5}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[∘]= $$\frac{8\,(-3+2\,n)\,(-1+2\,n)\,\mathrm{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\,\mathrm{Gamma}[-5+4\,n]}{n\,\mathrm{Gamma}\left[-\frac{3}{2}+\frac{3\,n}{2}\right]\,\mathrm{Gamma}[-2+3\,n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```
Out[∘]= {True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[∘]= 1.81127 min

det33: Variations II (Theorem 12)

{0,1,1,-1}

In[∘]:=
```
InitializeDeterminantProof[3, 3, {0, 1, 1, -1},
 prod[(2^(1+i) Gamma[(2+i)/3] Gamma[-2 + 4 i]) / (i Gamma[-1/3 + (4 i)/3] Gamma[-2 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+3\,j}{3\,j}+3^{1+i}\binom{1+i+3\,j}{3\,j}\right)=\prod_{i=1}^{n}\frac{2^{1+i}\,\Gamma\left(\frac{2+i}{3}\right)\Gamma(-2+4\,i)}{i\,\Gamma\left(-\frac{1}{3}+\frac{4\,i}{3}\right)\Gamma(-2+3\,i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

4	12	21	30	39	48
10	90	252	495	819	1224
28	540	2268	5940	12 285	22 032
82	2834	17 010	57 915	147 420	313 956
244	13 604	112 266	486 486	1 503 684	3 767 472
730	61 226	673 596	3 648 645	13 533 156	39 558 456

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-3	1				
$\frac{189}{40}$	$-\frac{133}{40}$	1			
$-\frac{405}{64}$	$\frac{445}{64}$	$-\frac{235}{56}$	1		
$\frac{44307}{5632}$	$-\frac{66987}{5632}$	$\frac{7595}{704}$	$-\frac{2263}{440}$	1	
$-\frac{105705}{11264}$	$\frac{204849}{11264}$	$-\frac{216919}{9856}$	$\frac{2769}{176}$	$-\frac{159}{26}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 66 616

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 4}, {7, 8}, {4, 6}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

```
In[•]:= (* Compute a recurrence for c_{n,n-1} *)
        Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
Out[•]= {S_n^4, S_n^3, S_n^2, S_n, 1}

In[•]:= (* Verify that this recurrence admits a constant sequence as solution. *)
        OreReduce1[cnn1, Annihilator[1, S[n]]]
Out[•]= 0

In[•]:= (* Look at the integer roots of the leading coefficient. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
Out[•]= {-3}

In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]
Out[•]= {3.71909, {823 400, 817 632}}

In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
Out[•]= {171.12, {0, 0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
Out[•]= {201.231, {0, 0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 028 656

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 12}, {8, 8}, {9, 11}, {11, 14}, {14, 16}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{0.948633, {240 232, 182 568}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{544.508, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {472.684, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 182 192

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{77}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-10, -9}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $$\frac{2^{1+n}\,\mathrm{Gamma}\left[\frac{2}{3}+\frac{n}{3}\right]\,\mathrm{Gamma}[-2+4\,n]}{n\,\mathrm{Gamma}\left[-\frac{1}{3}+\frac{4\,n}{3}\right]\,\mathrm{Gamma}[-2+3\,n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 22.3464 min

{0,3,5,-1}

In[•]:=
```
InitializeDeterminantProof[3, 3, {0, 3, 5, -1},
  prod[(2^(1+i) (-2 + 3 i) (-1 + 3 i) Gamma[4 (1 + i)] Gamma[(2 + i)/3]) /
    ((1 + i) (2 + i) (3 + i) (4 + i) Gamma[1 + 3 i] Gamma[1/3 (5 + 4 i)]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+3j}{3j}+3^{3+i}\binom{5+i+3j}{3j}\right)=\prod_{i=1}^{n}\frac{2^{1+i}\,(-2+3\,i)\,(-1+3\,i)\,\Gamma(4\,(1+i))\,\Gamma\left(\frac{2+i}{3}\right)}{(1+i)\,(2+i)\,(3+i)\,(4+i)\,\Gamma(1+3\,i)\,\Gamma\left(\frac{1}{3}\,(5+4\,i)\right)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

28	1512	12 474	54 054	167 076	418 608
82	6804	74 844	405 405	1 503 684	4 395 384
244	29 160	416 988	2 779 920	12 244 284	41 442 192
730	120 284	2 189 187	17 721 990	91 832 130	357 438 906
2188	481 136	10 945 935	106 331 940	642 824 910	2 859 511 248
6562	1 876 436	52 540 488	606 092 058	4 242 644 406	21 446 334 360

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
-54	1				
$\frac{1701}{4}$	$-\frac{129}{8}$	1			
$-\frac{56133}{32}$	$\frac{6545}{64}$	$-\frac{307}{24}$	1		
$\frac{1321677}{256}$	$-\frac{210681}{512}$	$\frac{164377}{2112}$	$-\frac{1075}{88}$	1	
$-\frac{6318243}{512}$	$\frac{9034497}{7168}$	$-\frac{3156491}{9856}$	$\frac{1211823}{16016}$	$-\frac{2259}{182}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 174 400

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{4, 10}, {8, 13}, {4, 11}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```
Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```
Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[•]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-4}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {4.42131, {1 057 224, 1 051 392}}

```
In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {258.606, {0, 0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {321.169, {0, 0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 578 072

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{10, 16}, {9, 14}, {10, 15}, {12, 16}, {15, 19}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

```
In[•]:= (* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}

In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {1.27245, {303 384, 226 216}}

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]= {766.492, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {642.053, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 201 288

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{84}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-14, -13, -12, -11, -9}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $$\frac{2^{1+n} (-2+3n) (-1+3n) \,\mathrm{Gamma}\left[\frac{2}{3}+\frac{n}{3}\right] \mathrm{Gamma}[4+4n]}{(1+n)(2+n)(3+n)(4+n) \,\mathrm{Gamma}\left[\frac{5}{3}+\frac{4n}{3}\right] \mathrm{Gamma}[1+3n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
  {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 31.6188 min

{1,1,2,0}

In[∘]:=
```
InitializeDeterminantProof[3, 3, {1, 1, 2, 0},
  prod[(2^i Gamma[(4+i)/3] Gamma[2 + 4 i]) / (9 i^2 Gamma[7/3 + (4 i)/3] Gamma[-1 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-i+3j}{1+3j}+3^{1+i}\binom{2+i+3j}{1+3j}\right)=\prod_{i=1}^{n}\frac{2^i\,\Gamma\left(\frac{4+i}{3}\right)\Gamma(2+4i)}{9\,i^2\,\Gamma\left(\frac{7}{3}+\frac{4i}{3}\right)\Gamma(-1+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

6	15	24	33	42	51
26	135	324	594	945	1377
106	945	3240	7722	15 120	26 163
402	5670	26 730	81 081	192 780	392 445
1454	30 619	192 456	729 729	2 082 024	4 944 807
5098	153 095	1 250 964	5 837 832	19 779 228	54 392 877

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{5}{2}$	1				
$\frac{27}{7}$	$-\frac{22}{7}$	1			
$-\frac{81}{16}$	$\frac{253}{40}$	$-\frac{65}{16}$	1		
$\frac{5103}{832}$	$-\frac{21763}{2080}$	$\frac{651}{64}$	$-\frac{2877}{572}$	1	
$-\frac{188811}{26624}$	$\frac{206307}{13312}$	$-\frac{41495}{2048}$	$\frac{25107}{1664}$	$-\frac{1347}{224}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 82 312

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {7, 9}, {4, 7}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= $\{\{\{j \to 0, n \to 1\}, \text{True}\}, \{\{j \to 0, n \to 2\}, \text{True}\}, \{\{j \to 1, n \to 1\}, \text{True}\}, \{\{j \to 2, n \to 1\}, \text{True}\}\}$

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]= {-3}

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {2.91868, {928 096, 928 072}}

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]= {206.198, {0, 0, 0, 0, 0}}

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]= {239.393, {0, 0, 0, 0, 0}}

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 269 456

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 14}, {8, 11}, {9, 13}, {11, 16}, {14, 18}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 1, n → 8}, True},
 {{i → 1, n → 9}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}, {{i → 4, n → 6}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{0.972094, {273 304, 209 480}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{662.422, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]=
```
{552.456, 0}
```

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 182 296

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{77}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[•]= {-10, -8}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[•]=
$$\frac{2^{n}\,\text{Gamma}\left[\frac{4}{3}+\frac{n}{3}\right]\,\text{Gamma}[2+4\,n]}{9\,n^{2}\,\text{Gamma}\left[\frac{7}{3}+\frac{4\,n}{3}\right]\,\text{Gamma}[-1+3\,n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```
Out[•]= {True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[•]= 26.452 min

{4,2,4,0}

In[•]:=
```
InitializeDeterminantProof[3, 3, {4, 2, 4, 0},
 prod[(2^(1+i) Gamma[(4+i)/3] Gamma[3 + 4 i])/(3 Gamma[7/3 + 4 i/3] Gamma[2 + 3 i]), {i, 1, n}]]
```
$$\text{prod}\left[\frac{2^{1+i}\,\text{Gamma}\left[\frac{4+i}{3}\right]\,\text{Gamma}[3+4\,i]}{3\,\text{Gamma}\left[\frac{7}{3}+\frac{4\,i}{3}\right]\,\text{Gamma}[2+3\,i]},\{i,1,n\}\right]$$

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=
$$\det_{0\le i,j<n}\left(\binom{-i+3j}{4+3j}+3^{2+i}\binom{4+i+3j}{4+3j}\right)=\prod_{i=1}^{n}\frac{2^{1+i}\,\Gamma\left(\frac{4+i}{3}\right)\Gamma(3+4i)}{3\,\Gamma\left(\frac{7}{3}+\frac{4i}{3}\right)\Gamma(2+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[•]//TableForm=

9	9	9	9	9	9
136	216	297	378	459	540
1220	2916	5346	8505	12 393	17 010
8520	29 160	69 498	136 080	235 467	374 220
51 065	240 569	729 729	1 735 020	3 532 005	6 455 295
275 632	1 732 096	6 567 561	18 738 216	44 503 263	92 956 248

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```
Out[∘]//TableForm=

1					
-1	1				
$\frac{81}{80}$	$-\frac{161}{80}$	1			
$-\frac{729}{704}$	$\frac{2153}{704}$	$-\frac{133}{44}$	1		
$\frac{2187}{2048}$	$-\frac{8507}{2048}$	$\frac{783}{128}$	$-\frac{129}{32}$	1	
$-\frac{1467477}{1323008}$	$\frac{7031205}{1323008}$	$-\frac{856145}{82688}$	$\frac{210591}{20672}$	$-\frac{3255}{646}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```
ByteCount: 96 496

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 6}, {7, 10}, {4, 7}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```
Out[∘]=
```
{0}
```

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```
Out[∘]=
```
{{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}
```

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```
Out[•]=
```
0
```

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]=
```
{-4}
```

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{4.24066, {1102640, 1102616}}
```

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{259.087, {0, 0, 0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]=
```
{319.199, {0, 0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 213 664

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 14}, {8, 10}, {9, 13}, {11, 15}, {14, 17}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{1.14943, {308 304, 286 456}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{958.126, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```
Out[∘]= {861.935, 0}

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 185 152

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{78}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[∘]= {-8}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```
Out[∘]=
$$\frac{2^{1+n}\,\text{Gamma}\left[\frac{4}{3}+\frac{n}{3}\right]\,\text{Gamma}[3+4\,n]}{3\,\text{Gamma}\left[\frac{7}{3}+\frac{4\,n}{3}\right]\,\text{Gamma}[2+3\,n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```
Out[∘]= {True, True, True, True, True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[∘]= 37.8713 min

{4,3,3,-3}

In[∘]:=
```
InitializeDeterminantProof[3, 3, {4, 3, 3, -3},
  5/3 prod[(2^(1+i) Gamma[(4+i)/3] Gamma[3 + 4 i])/(3 Gamma[7/3 + (4 i)/3] Gamma[2 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-3-i+3j}{4+3j}+3^{3+i}\binom{3+i+3j}{4+3j}\right)=\frac{5}{3}\prod_{i=1}^{n}\frac{2^{1+i}\,\Gamma\left(\frac{4+i}{3}\right)\Gamma(3+4i)}{3\,\Gamma\left(\frac{7}{3}+\frac{4i}{3}\right)\Gamma(2+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

15	0	0	0	0	0
116	80	81	81	81	81
1285	1936	2673	3402	4131	4860
11 061	26 208	48 114	76 545	111 537	153 090
76 755	262 320	625 483	1 224 720	2 119 203	3 367 980
459 600	2 164 800	6 567 572	15 615 180	31 788 045	58 097 655

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
0	1				
0	$-\frac{81}{80}$	1			
0	$\frac{729}{704}$	$-\frac{89}{44}$	1		
0	$-\frac{2187}{2048}$	$\frac{395}{128}$	$-\frac{97}{32}$	1	
0	$\frac{1467477}{1323008}$	$-\frac{347733}{82688}$	$\frac{127103}{20672}$	$-\frac{2609}{646}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 82 704

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {7, 9}, {4, 7}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```
Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```
Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[•]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-4}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {3.21614, {973 624, 968 688}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```

Out[•]=
```
{122.05, {0, 0, 0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```

Out[•]=
```
{168.654, {0, 0, 0, 0, 0}}
```

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 496 544

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 9}, {5, 3}, {9, 7}, {10, 9}, {13, 11}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
 {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True},
 {{i → 0, n → 8}, True}, {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True},
 {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True},
 {{i → 1, n → 8}, True}, {{i → 1, n → 9}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True},
 {{i → 2, n → 8}, True}, {{i → 3, n → 5}, True}, {{i → 4, n → 6}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]= {1.04582, {275 568, 249 520}}

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]= {732.785, 0}

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]= {663.431, 0}

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 198 112

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{83}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-8}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```

Out[∘]=
$$\frac{2^{1+n}\,\mathrm{Gamma}\left[\frac{4}{3}+\frac{n}{3}\right]\,\mathrm{Gamma}[3+4n]}{3\,\mathrm{Gamma}\left[\frac{7}{3}+\frac{4n}{3}\right]\,\mathrm{Gamma}[2+3n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
  {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]=
```
{True, True, True, True, True, True, True, True, True, True}
```

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]=
```
26.9473 min
```

{5,2,5,1}

In[•]:=
```
InitializeDeterminantProof[3, 3, {5, 2, 5, 1},
  prod[(2^i Gamma[1 + i/3] Gamma[4 + 4 i]) / (3 Gamma[2 + 4 i/3] Gamma[3 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{1-i+3j}{5+3j}+3^{2+i}\binom{5+i+3j}{5+3j}\right)=\prod_{i=1}^{n}\frac{2^i\,\Gamma\left(1+\frac{i}{3}\right)\Gamma(4+4i)}{3\,\Gamma\left(2+\frac{4i}{3}\right)\Gamma(3+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

9	9	9	9	9	9
162	243	324	405	486	567
1700	3645	6318	9720	13 851	18 711
13 602	40 095	88 452	165 240	277 020	430 353
91 833	360 855	995 085	2 230 740	4 363 065	7 746 354
551 068	2 814 670	9 552 816	25 430 436	57 592 458	116 195 310

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
-1	1				
1	-2	1			
$-\frac{729}{728}$	$\frac{1093}{364}$	$-\frac{2185}{728}$	1		
$\frac{2187}{2176}$	$-\frac{4367}{1088}$	$\frac{13\,083}{2176}$	$-\frac{1089}{272}$	1	
$-\frac{19\,683}{19\,456}$	$\frac{49\,031}{9728}$	$-\frac{195\,571}{19\,456}$	$\frac{24\,393}{2432}$	$-\frac{761}{152}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 113 480

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 7}, {7, 11}, {4, 8}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[•]= {-4}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

```
In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
Out[•]= {5.70828, {1 225 752, 1 222 512}}
```

```
In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
Out[•]= {356.549, {0, 0, 0, 0, 0}}
```

```
In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
Out[•]= {407.016, {0, 0, 0, 0, 0}}
```

```
In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 494 224

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 17}, {8, 12}, {9, 15}, {11, 18}, {14, 19}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

```
In[•]:= (* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}
```

```
In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {1.09035, {342 424, 316 664}}

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]= {1187.06, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {1091.95, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 196 568

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{82}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-7}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=
$$\frac{2^n \text{Gamma}\left[1+\frac{n}{3}\right] \text{Gamma}[4+4n]}{3\, \text{Gamma}\left[2+\frac{4n}{3}\right] \text{Gamma}[3+3n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 47.8075 min

{8,4,8,0}

In[•]:=
```
InitializeDeterminantProof[3, 3, {8, 4, 8, 0},
 prod[(2^(-1+i) Gamma[(1+i)/3] Gamma[7 + 4 i]) / (Gamma[(4 (1+i))/3] Gamma[6 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-i+3j}{8+3j}+3^{4+i}\binom{8+i+3j}{8+3j}\right)=\prod_{i=1}^{n}\frac{2^{-1+i}\,\Gamma\left(\frac{1+i}{3}\right)\Gamma(7+4i)}{\Gamma\left(\frac{4(1+i)}{3}\right)\Gamma(6+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

81	81	81	81	81	81
2188	2916	3645	4374	5103	5832
32 814	56 862	87 480	124 659	168 399	218 700
360 900	796 068	1 487 160	2 493 180	3 873 177	5 686 200
3 247 860	8 955 764	20 076 660	39 267 585	69 717 186	115 145 550
25 332 516	85 975 332	228 873 924	518 332 122	1 045 757 790	1 934 445 240

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-1	1				
$\frac{729}{728}$	$-\frac{1457}{728}$	1			
$-\frac{2187}{2176}$	$\frac{6547}{2176}$	$-\frac{817}{272}$	1		
$\frac{19683}{19456}$	$-\frac{78379}{19456}$	$\frac{771}{128}$	$-\frac{609}{152}$	1	
$-\frac{10058013}{9844736}$	$\frac{49896405}{9844736}$	$-\frac{653053}{64768}$	$\frac{772671}{76912}$	$-\frac{2535}{506}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 101 528

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 6}, {7, 10}, {4, 8}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]=
```
{-5, -3, -2}
```

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{3.9621, {1 102 632, 1 099 544}}
```

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{196.39, {0, 0, 0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]=
```
{254.432, {0, 0, 0, 0, 0}}
```

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 819 944

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 12}, {5, 3}, {9, 10}, {10, 12}, {13, 14}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```
Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True},
 {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}}
```
In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```
Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{1.12026, {305 864, 282 944}}
```
In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```
Out[•]=
```
{946.353, 0}
```
In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```
Out[•]=
```
{881.512, 0}
```
In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```
ByteCount: 200 760

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{83}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-8}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```

Out[•]=
$$\frac{2^{-1+n}\,\text{Gamma}\left[\frac{1}{3}+\frac{n}{3}\right]\,\text{Gamma}[7+4\,n]}{\text{Gamma}\left[\frac{4}{3}+\frac{4\,n}{3}\right]\,\text{Gamma}[6+3\,n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 36.1492 min

{8,5,7,-3}

In[•]:=
```
InitializeDeterminantProof[3, 3, {8, 5, 7, -3},
 5/9 prod[(2^(-1+i) Gamma[(1+i)/3] Gamma[7 + 4 i]) / (Gamma[4 (1+i)/3] Gamma[6 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=
$$\det_{0\le i,j<n}\left(\binom{-3-i+3j}{8+3j}+3^{5+i}\binom{7+i+3j}{8+3j}\right)=\frac{5}{9}\prod_{i=1}^{n}\frac{2^{-1+i}\,\Gamma\left(\frac{1+i}{3}\right)\Gamma(7+4i)}{\Gamma\left(\frac{4(1+i)}{3}\right)\Gamma(6+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

45	0	0	0	0	0
894	728	729	729	729	729
20 178	26 232	32 805	39 366	45 927	52 488
296 532	511 680	787 320	1 121 931	1 515 591	1 968 300
3 250 698	7 164 248	13 384 441	22 438 620	34 858 593	51 175 800
29 235 690	80 600 520	180 689 955	353 408 265	627 454 674	1 036 309 950

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
0	1				
0	$-\frac{729}{728}$	1			
0	$\frac{2187}{2176}$	$-\frac{545}{272}$	1		
0	$-\frac{19683}{19456}$	$\frac{7337}{2432}$	$-\frac{457}{152}$	1	
0	$\frac{10058013}{9844736}$	$-\frac{19683}{4864}$	$\frac{464263}{76912}$	$-\frac{2029}{506}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 70 744

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 4}, {7, 8}, {4, 7}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

```
In[•]:= (* Verify that this recurrence admits a constant sequence as solution. *)
        OreReduce1[cnn1, Annihilator[1, S[n]]]

Out[•]= 0

In[•]:= (* Look at the integer roots of the leading coefficient. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]

Out[•]= {-5, -2, -1}

In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]

Out[•]= {3.45075, {876928, 871656}}

In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {127.32, {0, 0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {170.582, {0, 0, 0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 836 912

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 11}, {8, 8}, {9, 10}, {11, 12}, {14, 14}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
 {{i → 1, n → 7}, True}, {{i → 1, n → 8}, True}, {{i → 1, n → 9}, True},
 {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True},
 {{i → 2, n → 7}, True}, {{i → 2, n → 8}, True}, {{i → 3, n → 5}, True},
 {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True}, {{i → 4, n → 6}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{1.04325, {251 312, 226 392}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{568.145, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {522.777, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 210 352

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{87}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-8}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```

Out[•]= $$\frac{2^{-1+n}\,\text{Gamma}\left[\frac{1}{3}+\frac{n}{3}\right]\,\text{Gamma}[7+4n]}{\text{Gamma}\left[\frac{4}{3}+\frac{4n}{3}\right]\,\text{Gamma}[6+3n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 22.2802 min

{0,0,0,0}

In[•]:=
```
InitializeDeterminantProof[3, 3, {0, 0, 0, 0},
  2 prod[(2^(-1+i) Gamma[(1+i)/3] Gamma[-3 + 4 i])/(Gamma[2/3 (-1 + 2 i)] Gamma[-2 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-i+3j}{3j}+3^i\binom{i+3j}{3j}\right)=2\prod_{i=1}^{n}\frac{2^{-1+i}\,\Gamma\left(\frac{1+i}{3}\right)\Gamma(-3+4i)}{\Gamma\left(\frac{2}{3}(-1+2i)\right)\Gamma(-2+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

2	2	2	2	2	2
4	12	21	30	39	48
10	90	252	495	819	1224
28	540	2268	5940	12 285	22 032
82	2834	17 010	57 915	147 420	313 956
244	13 604	112 266	486 486	1 503 684	3 767 472

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
-1	1				
$\frac{9}{8}$	$-\frac{17}{8}$	1			
$-\frac{81}{64}$	$\frac{217}{64}$	$-\frac{25}{8}$	1		
$\frac{729}{512}$	$-\frac{2465}{512}$	$\frac{417}{64}$	$-\frac{33}{8}$	1	
$-\frac{149\,445}{93\,184}$	$\frac{598\,509}{93\,184}$	$-\frac{132\,077}{11\,648}$	$\frac{15\,501}{1456}$	$-\frac{933}{182}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 78 408

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {7, 9}, {4, 6}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

```
In[•]:= (* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]

Out[•]= {0}

In[•]:= (* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]

Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
  {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}
```

Proof of Identity (H1)

```
In[•]:= (* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

```
In[•]:= (* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]

Out[•]= 0

In[•]:= (* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]

Out[•]= {-3}

In[•]:= (* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];

In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]

Out[•]= {2.73115, {960 448, 960 352}}
```

```
In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {287.709, {0, 0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {283.209, {0, 0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]

ByteCount: 1491072
Support: {{Si^4, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn Si^3, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn^2 Si^2, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn^3 Si, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn^4, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1}}
degree {n, i}: {{9, 16}, {8, 12}, {9, 16}, {11, 18}, {14, 20}}
Standard Monomials: {1, Si, Sn, Si^2, Sn Si, Sn^2, Si^3, Sn Si^2, Sn^2 Si, Sn^3}
Holonomic Rank: 10

In[•]:= (* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
  {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
  {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
  {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
  {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 1, n → 8}, True},
  {{i → 1, n → 9}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
  {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 2, n → 8}, True},
  {{i → 3, n → 5}, True}, {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True},
  {{i → 3, n → 8}, True}, {{i → 3, n → 9}, True}, {{i → 4, n → 6}, True}}

In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]= {0.907112, {273 720, 246 232}}

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]= {813.963, 0}

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]= {759.153, 0}

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 185 928

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{79}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-8}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]= $$\frac{2^{-1+n}\,\mathrm{Gamma}\left[\frac{1}{3}+\frac{n}{3}\right]\,\mathrm{Gamma}[-3+4\,n]}{\mathrm{Gamma}\left[-\frac{2}{3}+\frac{4\,n}{3}\right]\,\mathrm{Gamma}[-2+3\,n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]=
```
{True, True, True, True, True, True, True, True, True, True}
```

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]=
```
34.1447 min
```

{0,1,-1,-3}

In[∘]:=
```
InitializeDeterminantProof[3, 3, {0, 1, -1, -3},
 4 prod[(2^(-1+i) Gamma[(1+i)/3] Gamma[-3 + 4 i]) / (Gamma[2/3 (-1 + 2 i)] Gamma[-2 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-3-i+3j}{3j}+3^{1+i}\binom{-1+i+3j}{3j}\right)=4\prod_{i=1}^{n}\frac{2^{-1+i}\,\Gamma\!\left(\frac{1+i}{3}\right)\Gamma(-3+4i)}{\Gamma\!\left(\frac{2}{3}(-1+2i)\right)\Gamma(-2+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

4	0	0	0	0	0
10	8	9	9	9	9
28	104	189	270	351	432
82	800	2268	4455	7371	11 016
244	4840	20 413	53 460	110 565	198 288
730	25 480	153 097	521 235	1 326 780	2 825 604

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
0	1				
0	$-\frac{9}{8}$	1			
0	$\frac{81}{64}$	$-\frac{17}{8}$	1		
0	$-\frac{729}{512}$	$\frac{217}{64}$	$-\frac{25}{8}$	1	
0	$\frac{149445}{93184}$	$-\frac{8019}{1664}$	$\frac{9493}{1456}$	$-\frac{751}{182}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 76 688

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {7, 9}, {4, 6}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
Out[•]= {-3}
```

```
In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

```
In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]
Out[•]= {3.07574, {946552, 946416}}
```

```
In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
Out[•]= {174.139, {0, 0, 0, 0, 0}}
```

```
In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
Out[•]= {216.026, {0, 0, 0, 0, 0}}
```

```
In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
        id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
        AnnInfo[id2ann]
```

ByteCount: 836080

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 11}, {8, 8}, {9, 10}, {11, 12}, {14, 14}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
 {{i → 1, n → 7}, True}, {{i → 1, n → 8}, True}, {{i → 1, n → 9}, True},
 {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True},
 {{i → 2, n → 7}, True}, {{i → 2, n → 8}, True}, {{i → 3, n → 5}, True},
 {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True}, {{i → 4, n → 6}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{1.05455, {276 040, 250 632}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{702.713, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]=
```
{620.657, 0}
```

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 172 920

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{74}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-8}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $$\frac{2^{-1+n}\,\text{Gamma}\left[\frac{1}{3}+\frac{n}{3}\right]\,\text{Gamma}[-3+4\,n]}{\text{Gamma}\left[-\frac{2}{3}+\frac{4\,n}{3}\right]\,\text{Gamma}[-2+3\,n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 27.3082 min

{0,2,-2,-6}

In[•]:=
```
InitializeDeterminantProof[3, 3, {0, 2, -2, -6},
 10 prod[(2^(-1+i) Gamma[(1+i)/3] Gamma[-3 + 4 i])/(Gamma[2/3 (-1 + 2 i)] Gamma[-2 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-6-i+3j}{3j}+3^{2+i}\binom{-2+i+3j}{3j}\right)=10\prod_{i=1}^{n}\frac{2^{-1+i}\,\Gamma\left(\frac{1+i}{3}\right)\Gamma(-3+4i)}{\Gamma\left(\frac{2}{3}(-1+2i)\right)\Gamma(-2+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

10	-10	0	0	0	0
28	-20	1	0	0	0
82	46	88	81	81	81
244	916	1729	2430	3159	3888
730	7206	20 496	40 094	66 339	99 144
2188	43 620	183 918	481 130	995 085	1 784 592

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
1	1				
$-\frac{1}{8}$	$-\frac{1}{8}$	1			
$\frac{9}{64}$	$\frac{9}{64}$	$-\frac{9}{8}$	1		
$-\frac{81}{512}$	$-\frac{81}{512}$	$\frac{81}{64}$	$-\frac{17}{8}$	1	
$\frac{16605}{93184}$	$\frac{16605}{93184}$	$-\frac{16605}{11648}$	$\frac{4941}{1456}$	$-\frac{569}{182}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 77 496

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {7, 9}, {4, 6}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-3}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {2.87225, {907 056, 903 128}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]= {145.83, {0, 0, 0, 0, 0}}

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]= {160.198, {0, 0, 0, 0, 0}}

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

```
ByteCount: 688528
Support: {{Si^4, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn Si^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn^2 Si^2, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn^3 Si, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1},
  {Sn^4, Sn^3, Sn^2 Si, Sn Si^2, Si^3, Sn^2, Sn Si, Si^2, Sn, Si, 1}}
degree {n, i}: {{9, 11}, {5, 3}, {9, 9}, {10, 11}, {13, 13}}
Standard Monomials: {1, Si, Sn, Si^2, Sn Si, Sn^2, Si^3, Sn Si^2, Sn^2 Si, Sn^3}
Holonomic Rank: 10
```

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0}, n ≥ 2}, {{i → 0, n → 2}, True},
 {{i → 0, n → 3}, True}, {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 0, n → 6}, True}, {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 1, n → 7}, True}, {{i → 1, n → 8}, True},
 {{i → 1, n → 9}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 2, n → 8}, True},
 {{i → 2, n → 9}, True}, {{i → 2, n → 10}, True}, {{i → 3, n → 5}, True},
 {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True}, {{i → 3, n → 8}, True},
 {{i → 3, n → 9}, True}, {{i → 4, n → 6}, True}, {{i → 5, n → 7}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

AnnihilatorSingularities tells us that id2ann cannot be applied for i=0. Therefore we consider this case separately. We have a_{0,0}=10, a_{0,1}=-10, and a_{0,j}=0 for j>1. Hence it suffices to show that c_{n,0}=c_{n,1} for all n.

```
(* Compute a recurrence satisfied by c_{n,0}-c_{n,1}. *)
Factor[
 DFinitePlus[DFiniteSubstitute[annc, {j → 0}], DFiniteSubstitute[annc, {j → 1}]]]
```

Out[•]= $\{27\,(5+n)\,(-2+3n)\,(-1+3n)\,(1+3n)\,(2+3n)\,(4+3n)\,(5+3n)\,(7+3n)\,(8+3n)\,(10+3n)\,(11+3n)\,(13+3n)\,(14+3n)\,S_n^6 + 189\,(7+2n)^3\,(-2+3n)\,(-1+3n)\,(1+3n)\,(2+3n)\,(4+3n)\,(5+3n)\,(7+3n)\,(8+3n)\,(10+3n)\,(11+3n)\,S_n^5 + 36\,(2+n)\,(-2+3n)\,(-1+3n)\,(1+3n)\,(2+3n)\,(4+3n)\,(5+3n)\,(7+3n)\,(8+3n)\,(84315+108912\,n+53288\,n^2+11712\,n^3+976\,n^4)\,S_n^4 + 176\,(1+n)\,(2+n)\,(5+2n)\,(-2+3n)\,(-1+3n)\,(1+3n)\,(2+3n)\,(4+3n)\,(5+3n)\,(52731+80760\,n+46952\,n^2+12320\,n^3+1232\,n^4)\,S_n^3 + 64\,n\,(1+n)\,(2+n)\,(-2+3n)\,(-1+3n)\,(1+3n)\,(2+3n)\,(2816355+8689904\,n+11045692\,n^2+7434880\,n^3+2803280\,n^4+562176\,n^5+46848\,n^6)\,S_n^2 + 256\,(-1+n)\,n\,(1+n)\,(2+n)\,(3+2n)\,(-2+3n)\,(-1+3n)\,(3+4n)\,(9+4n)\,(2165+12612\,n+16300\,n^2+8064\,n^3+1344\,n^4)\,S_n + 1024\,(-2+n)\,(-1+n)\,n\,(1+n)\,(2+n)\,(1+2n)\,(3+2n)\,(-3+4n)\,(-1+4n)\,(3+4n)\,(5+4n)\,(9+4n)\,(11+4n)\}$

In[•]:=
```
(* Check the 6 initial values. *)
Table[myc[n, 0] == myc[n, 1], {n, 2, 7}]
```

Out[•]= {True, True, True, True, True, True}

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {0.969679, {264160, 239760}}

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]= {723.848, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {658.954, 0}

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 186 152
Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$
degree {n}: {{79}}
Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$
Holonomic Rank: 10

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-9}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :→ (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]= $\dfrac{2^{-1+n}\,\text{Gamma}\left[\frac{1}{3}+\frac{n}{3}\right]\,\text{Gamma}[-3+4\,n]}{\text{Gamma}\left[-\frac{2}{3}+\frac{4\,n}{3}\right]\,\text{Gamma}[-2+3\,n]}$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 26.9773 min

{1,0,1,1}

In[•]:=
```
InitializeDeterminantProof[3, 3, {1, 0, 1, 1},
  2 prod[(2^(-2+i) Gamma[i/3] Gamma[-1 + 4 i])/(3 Gamma[4 i/3] Gamma[-1 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{1-i+3j}{1+3j}+3^i\binom{1+i+3j}{1+3j}\right)=2\prod_{i=1}^{n}\frac{2^{-2+i}\,\Gamma\left(\frac{i}{3}\right)\Gamma(-1+4i)}{3\,\Gamma\left(\frac{4i}{3}\right)\Gamma(-1+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

2	2	2	2	2	2
6	15	24	33	42	51
26	135	324	594	945	1377
106	945	3240	7722	15 120	26 163
402	5670	26 730	81 081	192 780	392 445
1454	30 619	192 456	729 729	2 082 024	4 944 807

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
−1	1				
1	−2	1			
$-\frac{81}{80}$	$\frac{121}{40}$	$-\frac{241}{80}$	1		
$\frac{729}{704}$	$-\frac{131}{32}$	$\frac{4281}{704}$	$-\frac{177}{44}$	1	
$-\frac{2187}{2048}$	$\frac{5347}{1024}$	$-\frac{21035}{2048}$	$\frac{1299}{128}$	$-\frac{161}{32}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 94 232

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 6}, {7, 10}, {4, 7}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```
Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```
Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]= {-4}

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {4.88372, {1 067 144, 1 067 072}}

```
In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]

Out[•]= {400.003, {0, 0, 0, 0, 0}}

In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]

Out[•]= {422.584, {0, 0, 0, 0, 0}}

In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
        id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
        AnnInfo[id2ann]
```

ByteCount: 1 754 288

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 18}, {8, 15}, {9, 18}, {11, 20}, {14, 22}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

```
In[•]:= (* We are required to check initial values at the following indices: *)
        AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
         {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
         {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True}, {{i → 1, n → 3}, True},
         {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
         {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
         {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 2, n → 8}, True},
         {{i → 2, n → 9}, True}, {{i → 2, n → 10}, True},
         {{i → 3, n → 5}, True}, {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True},
         {{i → 3, n → 8}, True}, {{i → 3, n → 9}, True}, {{i → 5, n → 7}, True}}

In[•]:= (* Check a few (more than necessary) initial values. *)
        Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

```
In[•]:= (* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
Out[•]= {1.14223, {304 752, 275 200}}
```

```
In[•]:= (* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
Out[•]= {1026.07, 0}
```

```
In[•]:= (* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
Out[•]= {824.494, 0}
```

```
In[•]:= (* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 196 728

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{83}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
Out[•]= {-7}
```

```
In[•]:= (* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```

Out[•]= $$\frac{2^n \,\mathrm{Gamma}\left[1+\frac{n}{3}\right]\,\mathrm{Gamma}[-1+4n]}{3\,\mathrm{Gamma}\left[1+\frac{4n}{3}\right]\,\mathrm{Gamma}[-1+3n]}$$

```
In[•]:= (* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
Out[•]= 0
```

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[•]= {True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[•]= 42.3461 min

{1,3,-2,-8}

In[•]:=
```
InitializeDeterminantProof[3, 3, {1, 3, -2, -8},
 -168 prod[(2^(-2+i) Gamma[i/3] Gamma[-1 + 4 i])/(3 Gamma[4 i/3] Gamma[-1 + 3 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-8-i+3j}{1+3j}+3^{3+i}\binom{-2+i+3j}{1+3j}\right)=-168\prod_{i=1}^{n}\frac{2^{-2+i}\,\Gamma\left(\frac{i}{3}\right)\Gamma(-1+4i)}{3\,\Gamma\left(\frac{4i}{3}\right)\Gamma(-1+3i)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[•]//TableForm=

-62	70	-8	0	0	0
-90	126	-36	0	0	0
-10	210	-120	1	0	0
718	1059	399	740	729	729
4362	11 430	16 704	24 123	30 618	37 179
19 670	99 130	234 480	433 312	688 904	1 003 833

In[•]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[•]= $\left\{\frac{31}{84}, 1, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$\frac{35}{31}$	1				
1	1	1			
$-\frac{1}{80}$	$-\frac{1}{80}$	$-\frac{1}{80}$	1		
$\frac{9}{704}$	$\frac{9}{704}$	$\frac{9}{704}$	$-\frac{45}{44}$	1	
$-\frac{27}{2048}$	$-\frac{27}{2048}$	$-\frac{27}{2048}$	$\frac{135}{128}$	$-\frac{65}{32}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 80 952

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {7, 9}, {4, 7}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 3, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {3, 0}]
```

Out[∘]= {{{j → 0, n → 3}, True}, {{j → 0, n → 4}, True},
 {{j → 1, n → 3}, True}, {{j → 2, n → 3}, True}, {{j → 3, n → 3}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]=
```
{-3}
```

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{3.3215, {906024, 907904}}
```

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{262.748, {0, 0, 0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]=
```
{313.904, {0, 0, 0, 0, 0}}
```

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1428632

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 16}, {8, 11}, {9, 14}, {11, 18}, {14, 19}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {3, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 3}, True}, {{i → 0, n → 4}, True},
 {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
 {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 3, n → 5}, True}, {{i → 6, n → 8}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{1.0333, {269 600, 243 392}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{824.066, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{736.817, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 206 048

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{87}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[∘]= {-9}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[∘]= $\frac{2^n \text{Gamma}\left[1+\frac{n}{3}\right] \text{Gamma}[-1+4 n]}{3 \text{Gamma}\left[1+\frac{4 n}{3}\right] \text{Gamma}[-1+3 n]}$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 3, 2 + LeadingExponent[id3ann][[1]]}]
```
Out[∘]= {True, True, True, True, True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[∘]= 34.0643 min

{2,0,2,2}

In[∘]:=
```
InitializeDeterminantProof[3, 3, {2, 0, 2, 2},
 2 prod[(2^(-3+i) Gamma[(2+i)/3] Gamma[1 + 4 i]) / (Gamma[2/3 + (4 i)/3] Gamma[1 + 3 i]), {i, 1, n}]]
```
We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{2-i+3j}{2+3j}+3^i\binom{2+i+3j}{2+3j}\right)=2\prod_{i=1}^{n}\frac{2^{-3+i}\,\Gamma\left(\frac{2+i}{3}\right)\Gamma(1+4i)}{\Gamma\left(\frac{2}{3}+\frac{4i}{3}\right)\Gamma(1+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[∘]//TableForm=

2	2	2	2	2	2
9	18	27	36	45	54
54	189	405	702	1080	1539
271	1512	4455	9828	18 360	30 780
1218	10 206	40 095	110 565	247 860	484 785
5109	61 236	312 741	1 061 424	2 825 604	6 399 162

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-1	1				
1	-2	1			
-1	3	-3	1		
$\frac{729}{728}$	$-\frac{2915}{728}$	$\frac{4371}{728}$	$-\frac{2913}{728}$	1	
$-\frac{2187}{2176}$	$\frac{10921}{2176}$	$-\frac{21817}{2176}$	$\frac{21795}{2176}$	$-\frac{1361}{272}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 112 296

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 7}, {7, 11}, {4, 8}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]= {-4}

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {3.12449, {1 194 456, 1 194 384}}

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]= {485.173, {0, 0, 0, 0, 0}}

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]= {456.542, {0, 0, 0, 0, 0}}

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 1 979 032

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{9, 20}, {8, 16}, {9, 20}, {11, 22}, {14, 24}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

```
In[•]:= (* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
 {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
 {{i → 1, n → 7}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 2, n → 8}, True},
 {{i → 3, n → 5}, True}, {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True},
 {{i → 3, n → 8}, True}, {{i → 3, n → 9}, True}, {{i → 3, n → 10}, True},
 {{i → 3, n → 11}, True}, {{i → 6, n → 8}, True}}

In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

```
In[•]:= (* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]

Out[•]= {1.06954, {336 896, 305 144}}

In[•]:= (* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]

Out[•]= {1244.63, 0}

In[•]:= (* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]

Out[•]= {1149.84, 0}
```

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 201 456
Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$
degree {n}: {{85}}
Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$
Holonomic Rank: 10

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-6}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :→ (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]= $\frac{2^{-3+n}\,\text{Gamma}\left[\frac{2}{3}+\frac{n}{3}\right]\,\text{Gamma}[1+4\,n]}{\text{Gamma}\left[\frac{2}{3}+\frac{4\,n}{3}\right]\,\text{Gamma}[1+3\,n]}$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 52.7127 min

{-3,0,-1,-1}

In[∘]:=
```
InitializeDeterminantProof[3, 3, {-3, 0, -1, -1},
  2 prod[(2^(1+i) (-1 + 2 i) Gamma[(2+i)/3] Gamma[-5 + 4 i]) / (i (1 + i) Gamma[-1/3 + (4 i)/3] Gamma[-5 + 3 i]), {i, 2, n}]]
```

```
We are going to prove the following determinant evaluation:
```

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-1-i+3j}{-3+3j}+3^i\binom{-1+i+3j}{-3+3j}\right)=2\prod_{i=2}^{n}\frac{2^{1+i}(-1+2i)\,\Gamma\!\left(\frac{2+i}{3}\right)\Gamma(-5+4i)}{i\,(1+i)\,\Gamma\!\left(-\frac{1}{3}+\frac{4i}{3}\right)\Gamma(-5+3i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

2	2	20	56	110	182
-2	4	64	259	670	1378
1	10	316	1891	6436	16381
0	28	1512	12474	54054	167076
0	82	6804	74844	405405	1503684
0	244	29160	416988	2779920	12244284

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
-1	1				
4	-14	1			
$-\frac{32}{5}$	$\frac{1113}{20}$	$-\frac{309}{40}$	1		
$\frac{856}{105}$	$-\frac{24057}{160}$	$\frac{67507}{2240}$	$-\frac{1285}{168}$	1	
$-\frac{720}{77}$	$\frac{85293}{256}$	$-\frac{305903}{3584}$	$\frac{156741}{4928}$	$-\frac{723}{88}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 239 648

Support: $\{\{S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n^2, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{4, 14}, {8, 17}, {4, 15}}

Standard Monomials: $\{1, S_j, S_n, S_j^2\}$

Holonomic Rank: 4

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
(* Recurrences are only valid for j>0,
which is ok, because a_{i,0}=0 (for i>2). *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 1, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 1}]
```

Out[•]=
```
{{{j → 1, n → 1}, True}, {{j → 1, n → 2}, True},
 {{j → 1, n → 3}, True}, {{j → 2, n → 1}, True},
 {{j → 2, n → 2}, True}, {{j → 3, n → 1}, True}, {{j → 3, n → 2}, True}}
```

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^4, S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```

Out[•]= {-3}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

```
In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
Out[•]= {7.94778, {1702920, 1702944}}
```

```
In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
Out[•]= {887.187, {0, 0, 0, 0, 0}}
```

```
In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
Out[•]= {997.897, {0, 0, 0, 0, 0}}
```

```
In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 2634952

Support: $\{\{S_i^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n S_i^3, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^2 S_i^2, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^3 S_i, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_n^4, S_n^3, S_n^2 S_i, S_n S_i^2, S_i^3, S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{10, 25}, {9, 22}, {10, 23}, {12, 26}, {15, 28}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_n^2, S_i^3, S_n S_i^2, S_n^2 S_i, S_n^3\}$

Holonomic Rank: 10

```
In[•]:= (* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
Out[•]= {{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
  {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
  {{i → 0, n → 7}, True}, {{i → 0, n → 8}, True}, {{i → 1, n → 3}, True},
  {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
  {{i → 1, n → 7}, True}, {{i → 1, n → 8}, True}, {{i → 1, n → 9}, True},
  {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True},
  {{i → 2, n → 7}, True}, {{i → 2, n → 8}, True}, {{i → 3, n → 5}, True},
  {{i → 3, n → 6}, True}, {{i → 3, n → 7}, True}, {{i → 4, n → 6}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{1.62934, {472 072, 433 512}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{2608.88, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{2334.13, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 219 840

Support: $\{\{S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{93}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9\}$

Holonomic Rank: 10

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]=
```
{-11, -10, -8}
```

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]=

$$\frac{2^{1+n}\,(-1+2\,n)\,\mathrm{Gamma}\left[\frac{2}{3}+\frac{n}{3}\right]\,\mathrm{Gamma}[-5+4\,n]}{n\,(1+n)\,\mathrm{Gamma}\left[-\frac{1}{3}+\frac{4\,n}{3}\right]\,\mathrm{Gamma}[-5+3\,n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```
Out[∘]= {True, True, True, True, True, True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[∘]= 1.79093 h

det24: Variations III (Theorem 14)

{1,1,-1,-3}

In[∘]:=
```
InitializeDeterminantProof[2, 4, {1, 1, -1, -3},
 3 prod[(3^(-1+i) Gamma[1/2 + i/2] Gamma[-1 + 3 i]) / (Gamma[2 i] Gamma[1/2 (-1 + 3 i)]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-3-i+2j}{1+2j}+4^{1+i}\binom{-1+i+2j}{1+2j}\right)=3\prod_{i=1}^{n}\frac{3^{-1+i}\,\Gamma\left(\frac{1}{2}+\frac{i}{2}\right)\Gamma(-1+3i)}{\Gamma(2i)\,\Gamma\left(\frac{1}{2}(-1+3i)\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[∘]//TableForm=

-7	-1	0	0	0	0
-4	-4	0	0	0	0
59	54	63	64	64	64
506	1004	1530	2048	2560	3072
3065	10 205	21 483	36 863	56 320	79 872
16 376	81 864	229 320	491 512	901 120	1 490 944

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[∘]= $\left\{-\frac{7}{3}, 1, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
$-\frac{1}{7}$	1				
0	0	1			
0	0	$-\frac{64}{63}$	1		
0	0	$\frac{256}{243}$	$-\frac{55}{27}$	1	
0	0	$-\frac{8960}{8019}$	$\frac{256}{81}$	$-\frac{101}{33}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 29 072

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {2, 3}, {5, 5}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 3, 9}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {3, 0}]
```

Out[•]= {{{j → 0, n → 3}, True}, {{j → 0, n → 4}, True}, {{j → 1, n → 3}, True},
 {{j → 2, n → 3}, True}, {{j → 2, n → 4}, True}, {{j → 3, n → 3}, True}}

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
Out[•]= {-2, 0, 1}
```

```
In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

```
In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]
Out[•]= {1.02179, {114 224, 116 480}}
```

```
In[•]:= (* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
Out[•]= {0.560534, {0, 0, 0}}
```

```
In[•]:= (* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
        Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
Out[•]= {1.22934, {0, 0, 0}}
```

```
In[•]:= (* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
        id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
        AnnInfo[id2ann]
```

ByteCount: 47 280

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 4}, {4, 6}, {2, 3}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {3, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0, n → 3}, True}, {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{0.347522, {59 312, 67 568}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]=
```
{0.974772, 0}
```

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]=
```
{1.23519, 0}
```

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 16 440

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{16}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]=
```
{-4, -3, -2, -2}
```

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]=
$$\frac{2^{-2+3n}\times 3^{-1+n}\,\text{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\text{Gamma}\left[\frac{3n}{2}\right]}{\sqrt{\pi}\,\text{Gamma}[2n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 3, 2 + LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 8.12016 s

{1,1,0,-2}

In[∘]:=
$$\text{InitializeDeterminantProof}\left[2, 4, \{1, 1, 0, -2\}, -\text{prod}\left[\frac{3^{-1+i}\,\text{Gamma}\left[\frac{i}{2}\right]\text{Gamma}[3\,i]}{2\,\text{Gamma}\left[\frac{3\,i}{2}\right]\text{Gamma}[2\,i]}, \{i, 1, n\}\right]\right]$$

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=
$$\det_{0\le i,j<n}\left(\binom{-2-i+2j}{1+2j}+4^{1+i}\binom{i+2j}{1+2j}\right) = -\prod_{i=1}^{n}\frac{3^{-1+i}\,\Gamma\left(\frac{i}{2}\right)\Gamma(3\,i)}{2\,\Gamma\left(\frac{3\,i}{2}\right)\Gamma(2\,i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-2	0	0	0	0	0
13	15	16	16	16	16
124	252	384	512	640	768
763	2550	5375	9216	14 080	19 968
4090	20 460	57 338	122 880	225 280	372 736
20 473	143 325	516 075	1 351 679	2 928 640	5 591 040

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
0	1				
0	$-\frac{16}{15}$	1			
0	$\frac{32}{27}$	$-\frac{19}{9}$	1		
0	$-\frac{992}{729}$	$\frac{832}{243}$	$-\frac{85}{27}$	1	
0	$\frac{38656}{24057}$	$-\frac{40448}{8019}$	$\frac{5987}{891}$	$-\frac{46}{11}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 34 168

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 5}, {2, 4}, {5, 6}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```

Out[∘]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
 {{j → 1, n → 1}, True}, {{j → 1, n → 2}, True}, {{j → 2, n → 1}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[∘]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-2}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {0.689036, {143 832, 140 288}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]= {0.82621, {0, 0, 0}}

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]= {1.60231, {0, 0, 0}}

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 99 584

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 6}, {4, 10}, {2, 7}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.3889, {72 240, 81 176}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{1.37936, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{1.44385, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 15 512

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{15}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[•]= {-3}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[•]=
$$\frac{2^{-2+3n}\times 3^{-1+n}\,\text{Gamma}\left[\frac{n}{2}\right]\,\text{Gamma}\left[\frac{1}{2}+\frac{3n}{2}\right]}{\sqrt{\pi}\,\text{Gamma}[2n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```
Out[•]= {True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[•]= 8.87255 s

{1,1,1,-1}

In[•]:=
```
InitializeDeterminantProof[2, 4, {1, 1, 1, -1},
 prod[(3^i Gamma[(1+i)/2] Gamma[-1 + 3 i]) / (Gamma[2 i] Gamma[1/2 (-1 + 3 i)]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=
$$\det_{0\le i,j<n}\left(\binom{-1-i+2j}{1+2j}+4^{1+i}\binom{1+i+2j}{1+2j}\right)=\prod_{i=1}^{n}\frac{3^{i}\,\Gamma\left(\frac{1+i}{2}\right)\Gamma(-1+3i)}{\Gamma(2i)\,\Gamma\left(\frac{1}{2}(-1+3i)\right)}$$

In[•]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```
Out[•]//TableForm=

3	4	4	4	4	4
30	64	96	128	160	192
189	639	1344	2304	3520	4992
1020	5116	14 336	30 720	56 320	93 184
5115	35 830	129 023	337 920	732 160	1 397 760
24 570	229 356	1 032 186	3 244 032	8 200 192	17 891 328

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```
Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```
Out[∘]//TableForm=

1					
$-\frac{4}{3}$	1				
$\frac{16}{9}$	$-\frac{7}{3}$	1			
$-\frac{1360}{567}$	$\frac{112}{27}$	$-\frac{211}{63}$	1		
$\frac{7168}{2187}$	$-\frac{4864}{729}$	$\frac{1843}{243}$	$-\frac{118}{27}$	1	
$-\frac{326\,912}{72\,171}$	$\frac{246\,272}{24\,057}$	$-\frac{116\,480}{8019}$	$\frac{10\,763}{891}$	$-\frac{178}{33}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```
ByteCount: 31 088

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 4}, {2, 3}, {5, 6}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 9}, {j, 0, n - 1}]]]]
```
Out[∘]=
```
{0}
```

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {1, 0}]
```
Out[∘]=
```
{{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True}, {{j → 1, n → 1}, True}}
```

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[∘]=
```
0
```

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]= {-2}

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {0.945368, {131 072, 126 944}}

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]= {0.840515, {0, 0, 0}}

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]= {1.18006, {0, 0, 0}}

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 73 776

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 6}, {4, 8}, {2, 4}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

In[•]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1]
```

Out[•]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}}
```

In[•]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[•]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]=
```
{0.336661, {63 368, 74 912}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]=
```
{1.31498, 0}
```

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]=
```
{1.78047, 0}
```

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 14 600

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{14}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-4}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]=
$$\frac{2^{-2+3n} \times 3^{n} \,\mathrm{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right] \mathrm{Gamma}\left[\frac{3n}{2}\right]}{\sqrt{\pi}\,\mathrm{Gamma}[2n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 8.89929 s

{1,2,-3,-7}

In[∘]:=
```
InitializeDeterminantProof[2, 4, {1, 2, -3, -7},
 42 prod[(3^(-1+i) Gamma[(1+i)/2] Gamma[-1 + 3 i]) / (Gamma[2 i] Gamma[1/2 (-1 + 3 i)]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=
$$\det_{0\le i,j<n}\left(\binom{-7-i+2j}{1+2j}+4^{2+i}\binom{-3+i+2j}{1+2j}\right)=42\prod_{i=1}^{n}\frac{3^{-1+i}\,\Gamma\left(\frac{1+i}{2}\right)\Gamma(-1+3i)}{\Gamma(2i)\,\Gamma\left(\frac{1}{2}(-1+3i)\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-55	-51	-21	-1	0	0
-136	-56	-56	-8	0	0
-265	-84	-126	-36	-1	0
-10	-120	-252	-120	-10	0
4085	3931	3634	3766	4041	4095
32 756	65 316	97 512	130 280	163 620	196 596

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{-\frac{55}{42}, -\frac{241}{21}, \frac{59}{21}, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{51}{55}$	1				
$-\frac{105}{241}$	$\frac{14}{241}$	1			
$\frac{76}{177}$	$\frac{5}{531}$	$-\frac{4442}{3717}$	1		
$\frac{1}{81}$	$-\frac{1}{243}$	$-\frac{5}{243}$	$-\frac{1}{27}$	1	
$-\frac{35}{2673}$	$\frac{35}{8019}$	$\frac{175}{8019}$	$\frac{35}{891}$	$-\frac{35}{33}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 50 152

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 6}, {2, 5}, {5, 8}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 5, 15}, {j, 0, n - 1}]]]]
```

Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {5, 0}]
```

Out[∘]= {{{j → 0, n → 5}, True}, {{j → 0, n → 6}, True},
 {{j → 1, n → 5}, True}, {{j → 5, n → 5}, True}}

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]= {-2, 4}

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]= {1.00335, {192 144, 193 600}}

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]= {2.31475, {0, 0, 0}}

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]= {4.4185, {0, 0, 0}}

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 204 464

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 10}, {4, 18}, {2, 14}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

```
In[•]:= (* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {5, 0}, Assumptions → i < n - 1]

Out[•]= {{{i → 1}, n ≥ 5}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True},
  {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True}, {{i → 2, n → 5}, True},
  {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}}

In[•]:= (* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]

Out[•]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

AnnihilatorSingularities tells us that id2ann cannot be applied for i=1. Therefore we consider the cases i=0 and i=1 separately. We have a_{0,0}=-55, a_{0,1}=-51, a_{0,2}=-21, a_{0,3}=-1, and a_{0,j}=0 for j>3. Moreover, we have a_{1,0}=-136, a_{1,1}=-56, a_{1,2}=-56, a_{1,3}=-8, and a_{1,j}=0 for j>3. Hence it suffices to show that -55*c_{n,0}-51*c_{n,1}-21*c_{n,2}-c{n,3}=0 and -136*c_{n,0}-56*c_{n,1}-56*c_{n,2}-8*c{n,3}=0 for all n.

```
(* Compute a recurrence satisfied by C_0*c_{n,0}+...+C_3*c_{n,3}. *)
Factor[
 DFinitePlus[DFiniteSubstitute[annc, {j → 0}], DFiniteSubstitute[annc, {j → 1}],
  DFiniteSubstitute[annc, {j → 2}], DFiniteSubstitute[annc, {j → 3}]]]
```

Out[•]=

$$\{96 (4+n) (5+n) (1+2n) (3+2n) (5+2n) (7+2n) (9+2n) (11+2n) S_n^6 + 16 (4+n) (1+2n) (3+2n) (5+2n) (7+2n) (9+2n) (14+3n) (25+21n) S_n^5 + 240 n (1+2n) (3+2n) (5+2n) (7+2n) (388+489n+171n^2+18n^3) S_n^4 + 360 (-1+n) n (1+2n) (3+2n) (5+2n) (512+652n+243n^2+27n^3) S_n^3 + 270 (-2+n) (-1+n) n (1+2n) (3+2n) (676+913n+378n^2+45n^3) S_n^2 + 81 (-3+n) (-2+n) (-1+n) n (1+2n) (1000+1566n+765n^2+99n^3) S_n + 243 (-4+n) (-3+n) (-2+n) (-1+n) n (5+n) (2+3n) (4+3n)\}$$

```
(* Check the necessary 6 initial values (related to i=0). *)
Table[-55 * myc[n, 0] - 51 * myc[n, 1] - 21 * myc[n, 2] - myc[n, 3], {n, 4, 9}]

Out[•]= {0, 0, 0, 0, 0, 0}

(* Check the necessary 6 initial values (related to i=1). *)
Table[-136 * myc[n, 0] - 56 * myc[n, 1] - 56 * myc[n, 2] - 8 * myc[n, 3], {n, 4, 9}]

Out[•]= {0, 0, 0, 0, 0, 0}
```

Proof of Identity (H3)

In[•]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[•]= {0.546844, {99 648, 111 592}}

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[•]= {2.40897, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[•]= {2.47539, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 23 680

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{22}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[•]= {-4}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $$\frac{2^{-2+3n} \times 3^{-1+n}\,\text{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\,\text{Gamma}\left[\frac{3n}{2}\right]}{\sqrt{\pi}\,\text{Gamma}[2n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[•]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 5, 4 + LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 15.9443 s

{1,2,-2,-6}

In[∘]:=
```
InitializeDeterminantProof[2, 4, {1, 2, -2, -6},
 7 prod[(3^(-1+i) Gamma[i/2] Gamma[3 i]) / (2 Gamma[3 i/2] Gamma[2 i]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-6-i+2\,j}{1+2\,j}+4^{2+i}\binom{-2+i+2\,j}{1+2\,j}\right)=7\prod_{i=1}^{n}\frac{3^{-1+i}\,\Gamma\left(\frac{i}{2}\right)\Gamma(3\,i)}{2\,\Gamma\left(\frac{3\,i}{2}\right)\Gamma(2\,i)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-38	-20	-6	0	0	0
-71	-35	-21	-1	0	0
-8	-56	-56	-8	0	0
1015	940	898	988	1023	1024
8182	16 264	24 324	32 648	40 950	49 152
49 141	163 675	343 602	589 494	901 065	1 277 951

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{-\frac{19}{7}, -\frac{3}{7}, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1					
$-\frac{10}{19}$	1				
$-\frac{7}{3}$	$\frac{62}{15}$	1			
$\frac{1}{27}$	$-\frac{1}{27}$	$-\frac{1}{9}$	1		
$-\frac{31}{729}$	$\frac{31}{729}$	$\frac{31}{243}$	$-\frac{31}{27}$	1	
$\frac{1208}{24057}$	$-\frac{1208}{24057}$	$-\frac{1208}{8019}$	$\frac{1208}{891}$	$-\frac{24}{11}$	1

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 40 288
Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$
degree {n, j}: {{3, 5}, {2, 4}, {5, 7}}
Standard Monomials: $\{1, S_j, S_n\}$
Holonomic Rank: 3

In[•]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 4, 14}, {j, 0, n - 1}]]]]
```

Out[•]= {0}

In[•]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {4, 0}]
```

Out[•]= $\{\{\{j \to 0, n \to 4\}, \text{True}\}, \{\{j \to 0, n \to 5\}, \text{True}\}, \{\{j \to 1, n \to 4\}, \text{True}\}, \{\{j \to 4, n \to 4\}, \text{True}\}\}$

Proof of Identity (H1)

In[•]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

In[•]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[•]=
```
{-2, 3}
```

In[•]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[•]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[•]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[•]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[•]=
```
{0.929313, {157456, 162984}}
```

In[•]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[•]=
```
{1.52562, {0, 0, 0}}
```

In[•]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[•]=
```
{2.27922, {0, 0, 0}}
```

In[•]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 148280

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\},$
$\{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 8}, {4, 14}, {2, 10}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {4, 0}, Assumptions → i < n - 1]
```

Out[∘]=
```
{{{i → 0}, n ≥ 4}, {{i → 0, n → 4}, True},
 {{i → 0, n → 5}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

AnnihilatorSingularities tells us that id2ann cannot be applied for i=0. Therefore we consider this case separately. We have a_{0,0}=-38, a_{0,1}=-20, a_{0,2}=-6, and a_{0,j}=0 for j>2. Hence it suffices to show that -38*c_{n,0}-20*c_{n,1}-6*c_{n,2}=0 for all n.

```
(* Compute a recurrence satisfied by -38*c_{n,0}-20*c_{n,1}-6*c_{n,2}. *)
Factor[DFinitePlus[DFiniteSubstitute[annc, {j → 0}],
  DFiniteSubstitute[annc, {j → 1}], DFiniteSubstitute[annc, {j → 2}]]]
```

Out[∘]=

$$\{48 (3+n) (4+n) (1+2n) (3+2n) (5+2n) (7+2n) (9+2n) S_n^5 + 16 (3+n) (1+2n) (3+2n) (5+2n) (7+2n) (124+135n+27n^2) S_n^4 + 24 n (1+2n) (3+2n) (5+2n) (812+1151n+486n^2+63n^3) S_n^3 + 216 (-1+n) n (1+2n) (3+2n) (124+189n+87n^2+12n^3) S_n^2 + 27 (-2+n) (-1+n) n (1+2n) (540+1007n+540n^2+81n^3) S_n + 81 (-3+n) (-2+n) (-1+n) n (4+n) (1+3n) (5+3n)\}$$

In[∘]:=
```
(* Check the necessary 5 initial values. *)
Table[-38 * myc[n, 0] - 20 * myc[n, 1] - 6 * myc[n, 2], {n, 3, 7}]
```

Out[∘]=
```
{0, 0, 0, 0, 0}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]=
```
{0.723458, {83 632, 94 808}}
```

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```
Out[•]= {2.15186, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```
Out[•]= {2.33414, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```
ByteCount: 20 968

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{20}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[•]= {-4}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[•]= $\dfrac{2^{-2+3n} \times 3^{-1+n}\, \mathrm{Gamma}\left[\frac{n}{2}\right] \mathrm{Gamma}\left[\frac{1}{2} + \frac{3n}{2}\right]}{\sqrt{\pi}\, \mathrm{Gamma}[2n]}$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 4, 3 + LeadingExponent[id3ann][[1]]}]
```
Out[•]= {True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[•]= 12.4926 s

{1,2,-1,-5}

In[∘]:=
```
InitializeDeterminantProof[2, 4, {1, 2, -1, -5},
  5 prod[(3^i Gamma[(1+i)/2] Gamma[-1 + 3 i])/(Gamma[2 i] Gamma[1/2 (-1 + 3 i)]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-5-i+2j}{1+2j}+4^{2+i}\binom{-1+i+2j}{1+2j}\right)=5\prod_{i=1}^{n}\frac{3^i\,\Gamma\left(\frac{1+i}{2}\right)\Gamma(-1+3i)}{\Gamma(2i)\,\Gamma\left(\frac{1}{2}(-1+3i)\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-21	-10	-1	0	0	0
-6	-20	-6	0	0	0
249	221	235	255	256	256
2040	4040	6088	8184	10 240	12 288
12 279	40 876	85 890	147 420	225 279	319 488
65 526	327 560	917 252	1 965 960	3 604 470	5 963 776

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{-\frac{7}{5}, 1, 1, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{10}{21}$	1				
$\frac{1}{9}$	$-\frac{1}{3}$	1			
$-\frac{85}{567}$	$\frac{85}{189}$	$-\frac{85}{63}$	1		
$\frac{448}{2187}$	$-\frac{448}{729}$	$\frac{448}{243}$	$-\frac{64}{27}$	1	
$-\frac{20432}{72171}$	$\frac{20432}{24057}$	$-\frac{20432}{8019}$	$\frac{3824}{891}$	$-\frac{112}{33}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 30 392

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 4}, {2, 3}, {5, 6}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 3, 12}, {j, 0, n - 1}]]]]
```
Out[∘]=
```
{0}
```

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {3, 0}]
```
Out[∘]=
```
{{{j → 0, n → 3}, True}, {{j → 0, n → 4}, True},
 {{j → 1, n → 3}, True}, {{j → 3, n → 3}, True}}
```

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[∘]=
```
0
```

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]=
```
{-2, 2}
```

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]=
```
{0.698035, {127 968, 132 144}}
```

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]=
```
{0.790013, {0, 0, 0}}
```

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]=
```
{1.79715, {0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 96 488

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 6}, {4, 10}, {2, 6}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {3, 0}, Assumptions → i < n - 1]
```
Out[∘]=
```
{{{i → 0, n → 3}, True}, {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True},
 {{i → 1, n → 3}, True}, {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True},
 {{i → 1, n → 6}, True}, {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True},
 {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```
Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]=
```
{0.494255, {67 744, 78 712}}
```

In[∘]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```

Out[∘]= {1.83392, 0}

In[∘]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```

Out[∘]= {1.78558, 0}

In[∘]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 18 264

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{18}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```

Out[∘]= {-4}

In[∘]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i -> n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod -> Product]
```

Out[∘]=
$$\frac{2^{-2+3n} \times 3^n \, \text{Gamma}\left[\frac{1}{2} + \frac{n}{2}\right] \text{Gamma}\left[\frac{3n}{2}\right]}{\sqrt{\pi} \, \text{Gamma}[2n]}$$

In[∘]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```

Out[∘]= 0

In[∘]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 3, 2 + LeadingExponent[id3ann][[1]]}]
```

Out[∘]= {True, True, True, True, True}

In[∘]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```

Out[∘]= 10.0473 s

{1,3,-3,-9}

In[∘]:=
```
InitializeDeterminantProof[2, 4, {1, 3, -3, -9},
  198 prod[(3^i Gamma[(1+i)/2] Gamma[-1 + 3 i])/(Gamma[2 i] Gamma[1/2 (-1 + 3 i)]), {i, 1, n}]]
```

We are going to prove the following determinant evaluation:

Out[∘]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{-9-i+2j}{1+2j}+4^{3+i}\binom{-3+i+2j}{1+2j}\right)=198\prod_{i=1}^{n}\frac{3^{i}\,\Gamma\left(\frac{1+i}{2}\right)\Gamma(-1+3i)}{\Gamma(2i)\,\Gamma\left(\frac{1}{2}(-1+3i)\right)}$$

In[∘]:=
```
(* Display the matrix A_6. *)
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[∘]//TableForm=

-201	-148	-126	-36	-1	0
-522	-120	-252	-120	-10	0
-1035	-165	-462	-330	-55	-1
-12	-220	-792	-792	-220	-12
16371	16098	15097	14668	15669	16306
131058	261780	391214	520856	653358	786068

In[∘]:=
```
(* Test the conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n], {n, 10}]
```

Out[∘]= $\left\{-\frac{67}{198}, -\frac{41}{11}, -\frac{1}{3}, 1, 1, 1, 1, 1, 1, 1\right\}$

Holonomic description of $c_{n,j}$

In[∘]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 6}, {j, 0, n - 1}]]
```

Out[∘]//TableForm=

1					
$-\frac{148}{201}$	1				
$-\frac{154}{369}$	$-\frac{35}{123}$	1			
$-\frac{190}{81}$	$-\frac{394}{297}$	$\frac{3475}{693}$	1		
$\frac{55}{2187}$	$-\frac{28}{729}$	$\frac{25}{243}$	$-\frac{10}{27}$	1	
$-\frac{202}{6561}$	$\frac{1277}{24057}$	$-\frac{1172}{8019}$	$\frac{449}{891}$	$-\frac{46}{33}$	1

In[∘]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 31008

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 4}, {2, 3}, {5, 6}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

In[∘]:=
```
(* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
With[{test = ApplyOreOperator[annc, myc[n, j]]},
 Union[Flatten[Table[test, {n, 5, 15}, {j, 0, n - 1}]]]]
```
Out[∘]= {0}

In[∘]:=
```
(* The values at these indices have to be given as initial conditions,
in order to uniquely define c_{n,j} via the recurrences in annc. *)
AnnihilatorSingularities[annc, {5, 0}]
```
Out[∘]=
```
{{{j → 0, n → 5}, True}, {{j → 0, n → 6}, True},
 {{j → 1, n → 5}, True}, {{j → 6, n → 5}, True}}
```

Proof of Identity (H1)

In[∘]:=
```
(* Compute a recurrence for c_{n,n-1} *)
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```
Out[∘]= $\{S_n^3, S_n^2, S_n, 1\}$

In[∘]:=
```
(* Verify that this recurrence admits a constant sequence as solution. *)
OreReduce[cnn1, Annihilator[1, S[n]]]
```
Out[∘]= 0

In[∘]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
```
Out[∘]= {-2, 6}

In[∘]:=
```
(* Check the first few initial values. *)
Table[myc[n, n - 1], {n, 9}]
```
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}

Proof of Identity (H2)

In[∘]:=
```
(* Include the variable i into annc. *)
annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

In[∘]:=
```
(* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
 annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]= {1.02216, {134 432, 135 664}}

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```

Out[∘]= {1.03246, {0, 0, 0}}

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```

Out[∘]= {1.5714, {0, 0, 0}}

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 96 120

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 6}, {4, 10}, {2, 6}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
AnnihilatorSingularities[id2ann, {5, 0}, Assumptions → i < n - 1]
```

Out[∘]= {{{i → 1}, n ≥ 5}, {{i → 0, n → 5}, True}, {{i → 0, n → 6}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True}, {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True}, {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}}

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[mya[i, j] * myc[n, j], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```

Out[∘]= {{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

AnnihilatorSingularities tells us that id2ann cannot be applied for i=1. Therefore we consider the cases i=0 and i=1 separately. We have a_{0,0}=-201, a_{0,1}=-148, a_{0,2}=-126, a_{0,3}=-36, a_{0,4}=-1, and a_{0,j}=0 for j>4. Moreover, we have a_{1,0}=-522, a_{1,1}=-120, a_{1,2}=-252, a_{1,3}=-120, a_{1,4}=-10, and a_{1,j}=0 for j>4. Hence it suffices to show that -201*c_{n,0}-148*c_{n,1}-126*c_{n,2}-36*c{n,3}-c_{n,4}=0 and -522*c_{n,0}-120*c_{n,1}-252*c_{n,2}-120*c{n,3}-10*c_{n,4}=0 for all n.

```
(* Compute a recurrence satisfied by C_0*c_{n,0}+...+C_4*c_{n,4}. *)
Factor[DFinitePlus[DFiniteSubstitute[annc, {j → 0}],
  DFiniteSubstitute[annc, {j → 1}], DFiniteSubstitute[annc, {j → 2}],
  DFiniteSubstitute[annc, {j → 3}], DFiniteSubstitute[annc, {j → 4}]]]
```

Out[∘]= $\{576\,(4+n)\,(5+n)\,(6+n)\,(7+n)\,(1+2n)\,(3+2n)\,(5+2n)\,(7+2n)\,(9+2n)\,(11+2n)\,(13+2n)\,(15+2n)\,S_n^8 + 192\,(4+n)\,(5+n)\,(6+n)\,(1+2n)\,(3+2n)\,(5+2n)\,(7+2n)\,(9+2n)\,(11+2n)\,(13+2n)\,(962+441n+45n^2)\,S_n^7 + 16\,(4+n)\,(5+n)\,(1+2n)\,(3+2n)\,(5+2n)\,(7+2n)\,(9+2n)\,(11+2n)\,(861\,688+958\,986\,n+372\,681\,n^2+60\,426\,n^3+3483\,n^4)\,S_n^6 + 48\,(1+n)\,(4+n)\,(1+2n)\,(3+2n)\,(5+2n)\,(7+2n)\,(9+2n)\,(4\,660\,520+6\,165\,130\,n+3\,132\,117\,n^2+768\,222\,n^3+91\,287\,n^4+4212\,n^5)\,S_n^5 + 180\,n\,(1+n)\,(1+2n)\,(3+2n)\,(5+2n)\,(7+2n)\,(9\,143\,408+14\,733\,756\,n+9\,681\,892\,n^2+3\,325\,275\,n^3+629\,685\,n^4+62\,289\,n^5+2511\,n^6)\,S_n^4 + 108\,(-1+n)\,n\,(1+n)\,(1+2n)\,(3+2n)\,(5+2n)\,(16\,054\,400+27\,132\,144\,n+18\,774\,292\,n^2+6\,797\,700\,n^3+1\,355\,175\,n^4+140\,616\,n^5+5913\,n^6)\,S_n^3 + 81\,(-2+n)\,(-1+n)\,n\,(1+n)\,(1+2n)\,(3+2n)\,(12\,201\,280+22\,431\,768\,n+16\,867\,958\,n^2+6\,604\,389\,n^3+1\,412\,757\,n^4+155\,763\,n^5+6885\,n^6)\,S_n^2 + 243\,(-3+n)\,(-2+n)\,(-1+n)\,n\,(1+n)\,(1+2n)\,(1\,050\,560+2\,263\,176\,n+1\,958\,986\,n^2+863\,865\,n^3+203\,580\,n^4+24\,219\,n^5+1134\,n^6)\,S_n + 729\,(-4+n)\,(-3+n)\,(-2+n)\,(-1+n)\,n\,(1+n)\,(5+n)\,(7+n)\,(2+3n)\,(4+3n)\,(8+3n)\,(10+3n)\}$

In[∘]:=
```
(* Check the necessary 6 initial values (related to i=0). *)
Table[-201 * myc[n, 0] - 148 * myc[n, 1] -
  126 * myc[n, 2] - 36 * myc[n, 3] - myc[n, 4], {n, 5, 12}]
```

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0}

In[∘]:=
```
(* Check the necessary 6 initial values (related to i=1). *)
Table[-522 * myc[n, 0] - 120 * myc[n, 1] -
  252 * myc[n, 2] - 120 * myc[n, 3] - 10 * myc[n, 4], {n, 5, 12}]
```

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0}

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```

Out[∘]= {0.363041, {69 968, 79 864}}

In[•]:=
```
(* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]
```
Out[•]= {1.78108, 0}

In[•]:=
```
(* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]
```
Out[•]= {2.15678, 0}

In[•]:=
```
(* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```
ByteCount: 21 264

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{20}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

In[•]:=
```
(* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]
```
Out[•]= {-4}

In[•]:=
```
(* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```
Out[•]= $$\frac{2^{-2+3n} \times 3^n \, \mathrm{Gamma}\left[\frac{1}{2} + \frac{n}{2}\right] \mathrm{Gamma}\left[\frac{3n}{2}\right]}{\sqrt{\pi}\ \mathrm{Gamma}[2n]}$$

In[•]:=
```
(* Verify that b_n/b_{n-1} satisfies
 the recurrence derived for the LHS of (H3). *)
OreReduce[id3ann, Annihilator[quot, S[n]]]
```
Out[•]= 0

In[•]:=
```
(* Compare initial values. *)
Table[Sum[mya[n - 1, j] * myc[n, j], {j, 0, n - 1}] == myb[n] / myb[n - 1],
 {n, 5, 4 + LeadingExponent[id3ann][[1]]}]
```
Out[•]= {True, True, True, True, True}

In[•]:=
```
(* How long the calculations in this section took. *)
CurrentDate[] - start
```
Out[•]= 10.4526 s

detx41: Variations III (Theorem 15)

In[•]:=
```
InitializeDeterminantProof[2, 4, {1, 0, x + 1, x + 1},
  2 prod[(2^(2 i-1) * 3^(i-1) * Gamma[i] * Gamma[(3 i + x) / 2]) / (Gamma[2 i] * Gamma[(i + x) / 2]), {i, 1, n}], "detx41/"]
```

We are going to prove the following determinant evaluation:

Out[•]//TraditionalForm=

$$\det_{0\le i,j<n}\left(\binom{1-i+2j+x}{1+2j}+4^i\binom{1+i+2j+x}{1+2j}\right)=2\prod_{i=1}^{n}\frac{2^{-1+2i}\times 3^{-1+i}\,\Gamma(i)\,\Gamma\left(\frac{1}{2}(3i+x)\right)}{\Gamma(2i)\,\Gamma\left(\frac{i+x}{2}\right)}$$

In[•]:=
```
(* Display the matrix A_4. *)
TableForm[Table[mya[i, j], {i, 0, 3}, {j, 0, 3}]]
```

Out[•]//TableForm=

$2+2x$	$\frac{1}{3}(1+x)(2+x)(3+x)$	$\frac{1}{60}(1+x)(2+x)(3+x)$ (
$x+4(2+x)$	$\frac{1}{6}x(1+x)(2+x)+\frac{2}{3}(2+x)(3+x)(4+x)$	$\frac{1}{120}x(1+x)(2+x)(3+x$
$-1+x+16(3+x)$	$\frac{1}{6}(-1+x)x(1+x)+\frac{8}{3}(3+x)(4+x)(5+x)$	$\frac{1}{120}(-1+x)x(1+x)(2+$
$-2+x+64(4+x)$	$\frac{1}{6}(-2+x)(-1+x)x+\frac{32}{3}(4+x)(5+x)(6+x)$	$\frac{1}{120}(-2+x)(-1+x)x(1$

In[•]:=
```
(* Test the conjectured identity. *)
Table[Together[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] / myb[n]], {n, 8}]
```

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1}

Holonomic description of $c_{n,j}$

In[•]:=
```
(* The first few values of c_{n,j}. *)
TableForm[Table[myc[n, j], {n, 4}, {j, 0, n - 1}]]
```

Out[•]//TableForm=

1		
$-\frac{1}{6}(2+x)(3+x)$	1	
$\frac{1}{360}(360+486x+233x^2+46x^3+3x^4)$	$\frac{1}{30}(-60-29x-3x^2)$	1
$\frac{-46080-80136x-54826x^2-18665x^3-3285x^4-279x^5-9x^6}{45360}$	$\frac{22920+19214x+5613x^2+666x^3+27x^4}{7560}$	$\frac{1}{126}(-380-129x-9$

In[•]:=
```
(* This is the guessed annihilator for c_{n,j}. *)
AnnInfo[annc]
```

ByteCount: 269 608

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree {n, j}: {{3, 8}, {2, 6}, {5, 9}}

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

```
In[•]:= (* Check whether the first values of c_{n,j} satisfy the guessed recurrences. *)
        With[{test = ApplyOreOperator[annc, myc[n, j]]},
         Union[Flatten[Table[Together[test], {n, 9}, {j, 0, n - 1}]]]]
Out[•]= {0}
```

```
In[•]:= (* The values at these indices have to be given as initial conditions,
        in order to uniquely define c_{n,j} via the recurrences in annc. *)
        Select[AnnihilatorSingularities[annc, {1, 0}], FreeQ[#, x] &]
Out[•]= {{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True}, {{j → 1, n → 1}, True}}
```

Proof of Identity (H1)

```
In[•]:= (* Compute a recurrence for c_{n,n-1} *)
        Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]
```

Out[•]= $\{S_n^3, S_n^2, S_n, 1\}$

```
In[•]:= (* Verify that this recurrence admits a constant sequence as solution. *)
        OreReduce1[cnn1, Annihilator[1, S[n]]]
Out[•]= 0
```

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
        Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
Out[•]= {-2}
```

```
In[•]:= (* Check the first few initial values. *)
        Table[myc[n, n - 1], {n, 9}]
Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of Identity (H2)

```
In[•]:= (* Include the variable i into annc. *)
        annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

```
In[•]:= (* Annihilator for a_{i,j}*c_{n,j}, split into two parts. *)
        Timing[
         annSmnd1 = DFiniteTimesHyper[annci, mya1[i, j]];
         annSmnd2 = DFiniteTimesHyper[annci, mya2[i, j]];
         ByteCount /@ {annSmnd1, annSmnd2}
        ]
Out[•]= {5.20495, {1123208, 1123184}}
```

In[∘]:=
```
(* Verify that id2ct1 constitutes a set of telescopers for annSmnd1. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct1], annSmnd1]]
```
Out[∘]=
```
{39.8309, {0, 0, 0}}
```

In[∘]:=
```
(* Verify that id2ct2 constitutes a set of telescopers for annSmnd2. *)
Timing[OreReduce1[MapThread[#1 + (S[j] - 1) ** #2[[1]] &, id2ct2], annSmnd2]]
```
Out[∘]=
```
{249.481, {0, 0, 0}}
```

In[∘]:=
```
(* Combine the telescopers to an annihilator of the sum on the LHS of (H2). *)
id2ann = DFinitePlus[id2ct1[[1]], id2ct2[[1]]];
AnnInfo[id2ann]
```

ByteCount: 2 183 016

Support: $\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_i^4, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^3, S_n S_i^2, S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

degree {n, i}: {{4, 8}, {4, 20}, {2, 16}}

Standard Monomials: $\{1, S_i, S_n, S_i^2, S_n S_i, S_i^3, S_n S_i^2\}$

Holonomic Rank: 7

In[∘]:=
```
(* We are required to check initial values at the following indices: *)
Select[AnnihilatorSingularities[id2ann, {1, 0}, Assumptions → i < n - 1], FreeQ[#, x] &]
```
Out[∘]=
```
{{{i → 0, n → 2}, True}, {{i → 0, n → 3}, True},
 {{i → 0, n → 4}, True}, {{i → 0, n → 5}, True}, {{i → 1, n → 3}, True},
 {{i → 1, n → 4}, True}, {{i → 1, n → 5}, True}, {{i → 1, n → 6}, True},
 {{i → 2, n → 4}, True}, {{i → 2, n → 5}, True}, {{i → 2, n → 6}, True},
 {{i → 2, n → 7}, True}, {{i → 3, n → 5}, True}, {{i → 3, n → 6}, True},
 {{i → 3, n → 7}, True}, {{i → 3, n → 8}, True}, {{i → 4, n → 6}, True}}
```

In[∘]:=
```
(* Check a few (more than necessary) initial values. *)
Table[Sum[Expand[mya[i, j] * myc[n, j]], {j, 0, n - 1}], {n, 12}, {i, 0, n - 2}]
```
Out[∘]=
```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Proof of Identity (H3)

In[∘]:=
```
(* Annihilator for a_{n-1,j}*c_{n,j}, split into two parts. *)
Timing[
 annSmnd1 = DFiniteTimesHyper[annc, mya1[n - 1, j]];
 annSmnd2 = DFiniteTimesHyper[annc, mya2[n - 1, j]];
 ByteCount /@ {annSmnd1, annSmnd2}
]
```
Out[∘]=
```
{2.79343, {606 088, 697 016}}
```

```
In[•]:= (* Verify that id3ct1 constitutes a telescopic relation for annSmnd1. *)
Timing[OreReduce1[id3ct1[[1, 1]] + (S[j] - 1) ** id3ct1[[2, 1, 1]], annSmnd1]]

Out[•]= {121.335, 0}

In[•]:= (* Verify that id3ct2 constitutes a telescopic relation for annSmnd2. *)
Timing[OreReduce1[id3ct2[[1, 1]] + (S[j] - 1) ** id3ct2[[2, 1, 1]], annSmnd2]]

Out[•]= {182.916, 0}

In[•]:= (* Combine the telescopers to a recurrence for the sum on the LHS of (H3). *)
id3ann = DFinitePlus[id3ct1[[1]], id3ct2[[1]]][[1]];
AnnInfo[{id3ann}]
```

ByteCount: 401 664

Support: $\{\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{24}}

Standard Monomials: $\{1, S_n, S_n^2, S_n^3, S_n^4\}$

Holonomic Rank: 5

```
In[•]:= (* Look at the integer roots of the leading coefficient. *)
Select[n /. Solve[LeadingCoefficient[id3ann] == 0, n], IntegerQ]

Out[•]= {-3, -2}

In[•]:= (* Simplify the quotient b_n/b_{n-1}. *)
quot = myb[n] / myb[n - 1] /. prod[f_, {i, a_, n}] :> (f /. i → n) * prod[f, {i, a, n - 1}];
quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[•]= $$\frac{3^{-1+n}\sqrt{\pi}\,\text{Gamma}\left[\frac{3n}{2}+\frac{x}{2}\right]}{\text{Gamma}\left[\frac{1}{2}+n\right]\text{Gamma}\left[\frac{n}{2}+\frac{x}{2}\right]}$$

```
In[•]:= (* Verify that b_n/b_{n-1} satisfies
         the recurrence derived for the LHS of (H3). *)
OreReduce1[id3ann, Annihilator[quot, S[n]]]

Out[•]= 0

In[•]:= (* Compare initial values. *)
Table[Sum[Expand[mya[n - 1, j] * myc[n, j]], {j, 0, n - 1}] == Expand[myb[n] / myb[n - 1]],
  {n, LeadingExponent[id3ann][[1]]}]

Out[•]= {True, True, True, True, True}

In[•]:= (* How long the calculations in this section took. *)
CurrentDate[] - start

Out[•]= 10.0323 min
```