

```
<< RISC`HolonomicFunctions`;
<< RISC`Guess`;
<< RISC`LinearSystemSolver`;
SetDirectory[NotebookDirectory[]];
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)  
 written by Christoph Koutschan  
 Copyright Research Institute for Symbolic Computation (RISC),  
 Johannes Kepler University, Linz, Austria

```
--> Type ?HolonomicFunctions for help.
```

Guess Package version 0.52  
 written by Manuel Kauers  
 Copyright Research Institute for Symbolic Computation (RISC),  
 Johannes Kepler University, Linz, Austria

## Section 3: Related Determinants

### Global Definitions

```
(* An auxiliary procedure to simplify certain expressions. *)
MySimp[expr_] := expr /.
  (* Simplify Floors *)
  Floor[a_] => Floor[Expand[a]] //.
  Floor[Optional[a_Integer] * (v : (i | j | k | n)) + b_.] -> a * v + Floor[b] /.
  (* Simplify powers *)
  (2^a_) => 2^Expand[a] /.
  m1^a_ => Simplify[(-1)^a, Element[{i, j, k, n}, Integers]] /.
  (* Expand arguments of Pochhammers. *)
  Pochhammer[a_, b_] => Pochhammer@@Expand[{a, b}] //.
  (* Simplify quotients of conjugate Pochhammers to rational functions. *)
  Pochhammer[a1_, b1_]^c1_ * Pochhammer[a2_, b2_]^c2_ ./;
  IntegerQ[Expand[a1 - a2]] && IntegerQ[Expand[b1 - b2]] && c1 > 0 && c2 < 0 => With[
    {m = Min[c1, -c2]}, FunctionExpand[(Pochhammer[a1, b1] / Pochhammer[a2, b2])^m] *
    Pochhammer[a1, b1]^(c1 - m) * Pochhammer[a2, b2]^(c2 + m)];
```

```

(* Definitions of matrix and determinant D_{s,t}(n) *)
DstMat[s_, t_, n_, mu_] :=
  Table[FunctionExpand[KroneckerDelta[i, j] + Binomial[mu + i + j - 2, j]],
    {i, s, n + s - 1}, {j, t, n + t - 1}];
dst[s_, t_, n_, i_, j_, mu_] := FunctionExpand[
  KroneckerDelta[i + s, j + t] + Binomial[mu + i + s + j + t - 4, j + t - 1]];
dst[s_, t_, n_, i_, j_] := dst[s, t, n, i, j, mu];
DstMat[s_, t_, n_, mu_] := Table[dst[s, t, n, i, j, mu], {i, n}, {j, n}];
DstMat[s_, t_, n_] := DstMat[s, t, n, mu];
Dst[s_, t_, n_, mu_] := Det[DstMat[s, t, n, mu]];
Dst[s_, t_, n_] := Det[DstMat[s, t, n]];

(* Definitions of rational functions in lemmas and propositions of Section 2 *)
R00e[n_] :=
  Pochhammer[mu + 2 n, n] * Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] / Pochhammer[n, n] /
  Pochhammer[mu / 2 + n + 1 / 2, n - 1];
R00o[n_] := Pochhammer[mu + 2 n - 2, n - 1] *
  Pochhammer[mu / 2 + 2 n - 1 / 2, n] / Pochhammer[n, n] / Pochhammer[mu / 2 + n - 1 / 2, n - 1];
R00[n_] := If[EvenQ[n], R00e[n / 2], R00o[(n + 1) / 2]];
R10[n_] := (1 - 2 n) * Pochhammer[mu + 2 n, n]^2 *
  Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1]^2 / (mu + 2 n) / Pochhammer[n, n]^2 /
  Pochhammer[mu / 2 + n + 1 / 2, n - 1]^2;
R10[n_] := -Pochhammer[mu + 2 n, n] * Pochhammer[mu + 2 n + 1, n - 1] *
  Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1]^2 / Pochhammer[n, n] / Pochhammer[n, n - 1] /
  Pochhammer[mu / 2 + n + 1 / 2, n - 1]^2;
R01[n_] := -Pochhammer[mu + 2 n - 2, n + 2] * Pochhammer[mu + 2 n + 1, n - 1] *
  Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1]^2 / Pochhammer[n, n - 1] / Pochhammer[n, n + 2] /
  Pochhammer[mu / 2 + n + 1 / 2, n - 1]^2;
R20[n_] := Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] *
  Pochhammer[mu + 2 n + 1, n - 1] / Pochhammer[n, n - 1] / Pochhammer[mu / 2 + n + 1 / 2, n - 1];
R02[n_] := (2 n - 1) * Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] *
  Pochhammer[mu + 2 n - 2, n + 2] / (mu + 2 n) / Pochhammer[n, n + 2] /
  Pochhammer[mu / 2 + n + 1 / 2, n - 1];

```

(\* Tests \*)

```
Table[Together[{
  Dst[0, 0, 2 n] / Dst[0, 0, 2 n - 1] - R00[2 n - 1],
  Dst[1, 0, 2 n + 1] / Dst[1, 0, 2 n - 1] - R10[n],
  Dst[0, 1, 2 n + 1] / Dst[0, 1, 2 n - 1] - R01[n],
  Dst[2, 0, 2 n] / Dst[2, 0, 2 n - 1] - R20[n],
  Dst[0, 2, 2 n] / Dst[0, 2, 2 n - 1] - R02[n],
  Dst[-1, 1, 2 n + 1] / Dst[-1, 1, 2 n] - R02[n]
}], {n, 5}]
```

```
Table[Together[{
  Dst[0, 0, n] - 2 * Product[R00[i], {i, n - 1}],
  Dst[1, 0, n] - If[EvenQ[n], 0, Product[R10[i], {i, (n - 1) / 2}]],
  Dst[0, 1, n] - If[EvenQ[n], 0, (mu - 1) * Product[R01[i], {i, (n - 1) / 2}]]
}], {n, 10}]
```

```
{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}
```

```
{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0},
{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

(\* Alternative (old) definitions. The above formulas, however, are nicer. \*)

```
R00a[n_] := If[EvenQ[n],
  2^(n/2) * Pochhammer[mu/2 + n/2, Floor[(n+2)/4]] *
  Pochhammer[mu/2 + n + 1/2, (n-2)/2] /
  Pochhammer[n/2, n/2] / Pochhammer[
  mu/2 + Floor[3/4 n] + 1/2, Floor[(n-2)/4]],
  2^((n-1)/2) * Pochhammer[mu/2 + n/2 - 1/2, Floor[(n+1)/4]] *
  Pochhammer[mu/2 + n + 1/2, (n+1)/2] /
  Pochhammer[(n+1)/2, (n+1)/2] /
  Pochhammer[mu/2 + Floor[3/4 (n-1)] + 1/2, Floor[(n+1)/4]];
R10a[n_] := - (mu + 2 n) * Pochhammer[mu/2 + 2 * n + 1/2, n - 1]^2 *
  Pochhammer[mu/2 + n + 1, Floor[(n-1)/2]]^2 / Pochhammer[mu/2 + Floor[3/2 n] + 1/2,
  Floor[(n-1)/2]]^2 / Pochhammer[n, n] / Pochhammer[1/2, n - 1];
R01a[n_] := (-2) * Pochhammer[mu + 2 * n - 2, 3] * Pochhammer[mu/2 + 2 * n + 1/2, n - 1]^2 *
  Pochhammer[mu/2 + n + 1, Floor[(n-1)/2]]^2 /
  Pochhammer[mu/2 + Floor[3/2 n] + 1/2, Floor[(n-1)/2]]^2 / n /
  Pochhammer[n + 2, n + 1] / Pochhammer[1/2, n - 1];
R20a[n_] := Pochhammer[mu/2 + 2 n + 1/2, n - 1] *
  Pochhammer[mu/2 + n + 1, Floor[(n-1)/2]] / 2^(n-1) / Pochhammer[1/2, n - 1] /
  Pochhammer[mu/2 + Floor[3 n/2] + 1/2, Floor[(n-1)/2]];
R02a[n_] := 2^(n-1) * (2 n - 1) * Pochhammer[mu + 2 n - 2, 2] *
  Pochhammer[mu/2 + 2 n + 1/2, n - 1] * Pochhammer[mu/2 + n + 1, Floor[(n-1)/2]] /
  Pochhammer[mu/2 + Floor[3 n/2] + 1/2, Floor[(n-1)/2]] / Pochhammer[n, n + 2];
```

```
(* Test *)
Table[Together[{R00[n] / R00a[n], R10[n] / R10a[n],
  R01[n] / R01a[n], R20[n] / R20a[n], R02[n] / R02a[n]}], {n, 20}]
{{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1},
{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1},
{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1},
{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

## Theorem 2: $D_{1,1}(2n)/D_{1,1}(2n-1)$

```
Table[Together[
  (Dst[1, 1, 2 n] / Dst[1, 1, 2 n - 1]) /
  (Pochhammer[mu + 2 n, n] * Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] / Pochhammer[n, n] /
  Pochhammer[mu / 2 + n + 1 / 2, n - 1]), {n, 5}]
{1, 1, 1, 1, 1}
```

## Lemma 3: $D_{1,0}(2n) = 0$

```
Timing[data = PadRight[Table[ns = LinSolveUniv[DstMat[1, 0, 2 n, mu], mu][[1]];
  Together[ns / ns[[-1]]], {n, 30}]];]
{1801.971000, Null}
```

(\* Directly guess a Groebner basis of annihilating operators for  $c[n, j]$ . \*)

```
Timing[
  ann = NormalizeCoefficients /@ ToOrePolynomial[Join[
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n, j + 2]},
      {n, j}, 7, StartPoint -> {1, 1}, Constraints -> j ≤ 2 n],
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n + 1, j + 1]},
      {n, j}, 8, StartPoint -> {1, 1}, Constraints -> j ≤ 2 n],
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n + 2, j]},
      {n, j}, 17, StartPoint -> {1, 1}, Constraints -> j ≤ 2 n]], c[n, j]];
  GBEqual[ann, OreGroebnerBasis[ann]]
]
{949.971000, True}
```

(\* A faster alternative. Don't guess the Groebner basis directly, but some higher-order recurrences. \*)

(\* Then apply Buchberger. \*)

```
Timing[
  ann1 = OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n, j + 2], c[n + 1, j + 1],
      c[n + 2, j]}, {n, j}, 7, StartPoint -> {1, 1}, Constraints -> j ≤ 2 n], c[n, j]];
  GBEqual[ann, ann1]
]
{83.517000, True}
```

```

(*Put[ann,"ann_c_1_0.m"];*)
ann = Get["ann_c_1_0.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
{1, Sj, Sn}

1106536

{{7, 6}, {6, 5}, {15, 11}}

(* We check whether the recurrence operators in
   ann allow to uniquely define a bivariate sequence. *)
AnnihilatorSingularities[ann, {0, 0}]
$Aborted

(* Indeed, there are only finitely many
   points that have to be given as initial values. *)
AnnihilatorSingularities[ann /. mu -> 67, {0, 0}]
{{{j -> 0, n -> 0}, True}, {{j -> 0, n -> 1}, True}, {{j -> 1, n -> 0}, True},
 {{j -> 2, n -> 0}, True}, {{j -> 3, n -> 0}, True}, {{j -> 4, n -> 0}, True}}

(* Some tests *)
(* 1. Test that the annihilator is valid on the given data. *)
test = ApplyOreOperator[ann, c[n, j]];
Union[Flatten[Together[Table[test, {n, 28}, {j, 28}] /. c[n_, j_] -> data[[n, j]]]]]
(* 2. Verify the identity Sum[d[2n, i, j]*c[n, j], {j, 1, 2n}] = 0 for all 1 ≤ i ≤ 2n. *)
Union[Flatten[Table[Together[
  Sum[dst[1, 0, 2 n, i, j, mu] * data[[n, j], {j, 1, 2 n}], {n, 30}, {i, 2 n}]]]]]
(* 3. Same as 2., but with explicit form of matrix entries. This
   test can be viewed as initial value check. *)
Union[Flatten[Table[
  Together[Sum[FunctionExpand[Binomial[mu + i + j - 3, j - 1]] * data[[n, j], {j, 2 n}] +
    data[[n, i + 1]], {n, 29}, {i, 2 n}]]]]]
{0}
{0}
{0}

(* Hence we want to prove the following: *)
TraditionalForm[
  HoldForm[Sum[Binomial[mu + i + j - 3, j - 1] * c[n, j], {j, 1, 2 n}] == -c[n, i + 1]]]

$$\sum_{j=1}^{2n} \binom{\mu + i + j - 3}{j - 1} c(n, j) = -c(n, i + 1)$$


```

```

(* The sum has natural boundaries. Thus
   creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
  ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
  Annihilator[Binomial[mu + i + j - 3, j - 1], {S[n], S[j], S[i]}], S[j] - 1];]
{2274.377000, Null}

ct = << "ct_1_0.m";

(* Verify that the sum satisfies the same system of recurrences as c[n,i+1]. *)
(* Initial values have been compared before. The proof is complete. *)
GBEqual[DFiniteSubstitute[ann, {j → i + 1}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
True

```

### Lemma 4: $D_{0,1}(2n) = 0$

```

(* Directly guess a Groebner basis of annihilating operators for c[n,j]. *)
Timing[
  data = PadRight[Table[ns = LinSolveUniv[DstMat[0, 1, 2 n, mu], mu][[1]];
    Together[ns/ns[[-1]]], {n, 15}]];
  ann = NormalizeCoefficients /@ ToOrePolynomial[Join[
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j]},
      {n, j}, 9, StartPoint → {1, 1}, Constraints → j ≤ 2 n],
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n, j + 2]}, {n, j},
      6, StartPoint → {1, 1}, Constraints → j ≤ 2 n]
  ], c[n, j]];
  GBEqual[ann, OreGroebnerBasis[ann]]
]
{45.565000, True}

(*Put[ann, "ann_c_0_1.m"];*)
ann = Get["ann_c_0_1.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
{1, Sj}
128 496
{{7, 8}, {2, 6}}

(* We check whether the recurrence operators in
   ann allow to uniquely define a bivariate sequence. *)
AnnihilatorSingularities[ann /. mu → 67, {0, 0}]
{{{n →  $\frac{j}{2}$ },  $\frac{j}{2} \in \text{Integers} \ \&\& \ j \geq 0$ }, {{j → 0, n → 0}, True},
  {{j → 1, n → 0}, True}, {{j → 1, n → 1}, True}}

```

```
(* This recurrence allows to compute
the values for c[n,2n] (they are constant =1). *)
Factor[LeadingCoefficient[
  dfs = DFiniteSubstitute[ann, {j -> 2 n}, Algebra -> OreAlgebra[S[n]]][[1]]]
2 (3 + 2 n) (2 + mu + 2 n) (3 + mu + 4 n) (5 + mu + 4 n)^2 (7 + mu + 4 n)
(-3 mu^2 - 4 mu^3 + 2 mu^4 + 4 mu^5 + mu^6 - 6 n - 12 mu n + 3 mu^2 n + 69 mu^3 n + 83 mu^4 n + 23 mu^5 n -
29 n^2 + 184 mu n^2 + 640 mu^2 n^2 + 708 mu^3 n^2 + 225 mu^4 n^2 + 405 n^3 + 2271 mu n^3 + 3055 mu^2 n^3 +
1189 mu^3 n^3 + 2770 n^4 + 6596 mu n^4 + 3554 mu^2 n^4 + 5676 n^5 + 5676 mu n^5 + 3784 n^6)

OreReduce[dfs, ToOrePolynomial[{S[n] - 1}]]
0

(* Ee want to prove the following: *)
TraditionalForm[
  HoldForm[Sum[Binomial[mu + i + j - 3, j] * c[n, j], {j, 1, 2 n}] == -c[n, i - 1]]]

$$\sum_{j=1}^{2n} \binom{\mu+i+j-3}{j} c(n, j) = -c(n, i-1)$$


(* Test initial values. *)
Union[
  Flatten[Table[Together[Sum[FunctionExpand[Binomial[mu + i + j - 3, j]] * data[[n, j]],
    {j, 2 n}] + If[i == 1, 0, data[[n, i - 1]]]], {n, 14}, {i, 2 n}]]]
{0}

(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
  ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
  Annihilator[Binomial[mu + i + j - 3, j], {S[n], S[j], S[i]}]], S[j] - 1];
]
{46.655000, Null}

ct = << "ct_0_1.m";

GBEqual[DFiniteSubstitute[ann, {j -> i - 1}, Algebra -> OreAlgebra[S[n], S[i]]], ct[[1]]]
True
```

### Lemma 5: $D_{0,0}(2n)/D_{0,0}(2n-1)$

The determinant evaluation of  $D_{0,0}(n)$  was first proven by George E. Andrews. We include our computer proof just for sake of illustration.

```
(* Directly guess a Groebner basis of annihilating operators for c[n,j]. *)
Timing[
  data = PadRight[Table[ns = LinSolveUniv[Most[DstMat[0, 0, 2 n, mu]], mu][[1]];
    Together[ns/ns[[-1]]], {n, 15}]];
  ann = NormalizeCoefficients /@ ToOrePolynomial[Join[
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j]},
      {n, j}, 8, StartPoint -> {1, 1}, Constraints -> j ≤ 2 n],
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n, j + 2]}, {n, j},
      4, StartPoint -> {1, 1}, Constraints -> j ≤ 2 n]
  ], c[n, j]];
  GBEqual[ann, OreGroebnerBasis[ann]]
]
{39.793000, True}
```

```
(*Put[ann, "ann_c_0_0.m"];*)
ann = Get["ann_c_0_0.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
{1, Sj}
91424
{{7, 7}, {2, 4}}
```

```
(* Identity (1): Show that c[n,2n]=1 for all n. *)
diag = DFiniteSubstitute[ann, {j -> 2 n}, Algebra -> OreAlgebra[S[n]]][[1]];
Factor[LeadingCoefficient[diag]]
OreReduce[diag, {ToOrePolynomial[S[n] - 1]}]
2 (3 + 2 n) (mu + 2 n) (1 + mu + 4 n) (3 + mu + 4 n)2 (5 + mu + 4 n)
(6 - 5 mu - 11 mu2 + 10 mu3 + 4 mu4 - 5 mu5 + mu6 - 11 n - 147 mu n + 200 mu2 n + 28 mu3 n - 93 mu4 n +
23 mu5 n - 468 n2 + 1053 mu n2 + 91 mu2 n2 - 741 mu3 n2 + 225 mu4 n2 + 1487 n3 + 431 mu n3 -
3107 mu2 n3 + 1189 mu3 n3 + 878 n4 - 6648 mu n4 + 3554 mu2 n4 - 5676 n5 + 5676 mu n5 + 3784 n6)
0
```

```
(* Identity (2) *)
TraditionalForm[HoldForm[
  Sum[Binomial[μ + i + j - 4, j - 1] * c[n, j], {j, 1, 2 n}] == -c[n, i] " " (1 ≤ i ≤ 2 n - 1)]]

$$\sum_{j=1}^{2n} \binom{\mu + i + j - 4}{j - 1} c(n, j) = -c(n, i) \quad (1 \leq i \leq 2n - 1)$$

```



```
(* Identity (2): numerical check (= initial values) *)
Union[Flatten[Table[Together[Sum[dst[0, 0, 2 n, i, j, mu] * data[[n, j]], {j, 1, 2 n}]],
  {n, 15}, {i, 2 n - 1}]]],
Union[Flatten[Table[Together[Sum[FunctionExpand[Binomial[mu + i + j - 4, j - 1]] *
  data[[n, j]], {j, 1, 2 n}] + data[[n, i]], {n, 15}, {i, 2 n - 1}]]],
{0}
{0}
```

```
(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
  ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
  Annihilator[Binomial[mu + i + j - 4, j - 1], {S[n], S[j], S[i]}], S[j] - 1];
]
{53.147000, Null}

ct = << "ct_0_0_A.m";

GBEqual[DFiniteSubstitute[ann, {j -> i}, Algebra -> OreAlgebra[S[n], S[i]]], ct[[1]]]
True
```

```
(* Identity (3) *)
TraditionalForm[
  HoldForm[Sum[(KroneckerDelta[2 n, j] + Binomial[mu + 2 n + j - 4, j - 1]) * c[n, j],
    {j, 1, 2 n}] == D0,0[2 n] / D0,0[2 n - 1]]]

$$\sum_{j=1}^{2n} \left( \delta_{2n,j} + \binom{\mu + 2n + j - 4}{j - 1} \right) c(n, j) = \frac{D_{0,0}(2n)}{D_{0,0}(2n - 1)}$$

```

```
(* Numerical check of Identity (3) (= initial value check) *)
Table[Together[
  Sum[FunctionExpand[Binomial[mu + 2 n + j - 4, j - 1]] * data[[n, j]], {j, 1, 2 n}] +
  data[[n, 2 n]] - R00[2 n - 1], {n, 10}]
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[
  ct = FindCreativeTelescoping[DFiniteTimes[ann,
    Annihilator[Binomial[mu + 2 n + j - 4, j - 1], {S[n], S[j]}], S[j] - 1];
]
{101.570000, Null}
```

```
ct = << "ct_0_0_B.m";

(* Add the part coming from the KroneckerDelta. *)
ann3 = DFinitePlus[ct[[1]], DFiniteSubstitute[ann, {j -> 2 n}]];
```

(\* The recurrence we get is not too unhandy... \*)

Factor[ann3]

$$\left\{ 2 (3 + 2 n) (\mu + 2 n) (1 + \mu + 4 n) (3 + \mu + 4 n)^2 \right. \\
(5 + \mu + 4 n) \left( 6 - 5 \mu - 11 \mu^2 + 10 \mu^3 + 4 \mu^4 - 5 \mu^5 + \mu^6 - 11 n - 147 \mu n + \right. \\
200 \mu^2 n + 28 \mu^3 n - 93 \mu^4 n + 23 \mu^5 n - 468 n^2 + 1053 \mu n^2 + 91 \mu^2 n^2 - \\
741 \mu^3 n^2 + 225 \mu^4 n^2 + 1487 n^3 + 431 \mu n^3 - 3107 \mu^2 n^3 + 1189 \mu^3 n^3 + \\
878 n^4 - 6648 \mu n^4 + 3554 \mu^2 n^4 - 5676 n^5 + 5676 \mu n^5 + 3784 n^6 \Big) S_n^2 + \\
(-8100 \mu - 5040 \mu^2 + 18531 \mu^3 + 11783 \mu^4 - 12732 \mu^5 - 8564 \mu^6 + 2250 \mu^7 + \\
1938 \mu^8 + 72 \mu^9 - 116 \mu^{10} - 21 \mu^{11} - \mu^{12} - 27540 n - 45144 \mu n + 223443 \mu^2 n + \\
207636 \mu^3 n - 253610 \mu^4 n - 222246 \mu^5 n + 54372 \mu^6 n + 64314 \mu^7 n + 4302 \mu^8 n - \\
4506 \mu^9 n - 967 \mu^{10} n - 54 \mu^{11} n - 102060 n^2 + 940581 \mu n^2 + 1417278 \mu^2 n^2 - \\
1968205 \mu^3 n^2 - 2409495 \mu^4 n^2 + 515897 \mu^5 n^2 + 924553 \mu^6 n^2 + 98657 \mu^7 n^2 - \\
77801 \mu^8 n^2 - 20082 \mu^9 n^2 - 1307 \mu^{10} n^2 + 1303605 n^3 + 4434924 \mu n^3 - \\
7304625 \mu^2 n^3 - 13832520 \mu^3 n^3 + 2324225 \mu^4 n^3 + 7485116 \mu^5 n^3 + 1210077 \mu^6 n^3 - \\
783536 \mu^7 n^3 - 248162 \mu^8 n^3 - 18864 \mu^9 n^3 + 5218875 n^4 - 12699623 \mu n^4 - \\
43976946 \mu^2 n^4 + 4197653 \mu^3 n^4 + 37211456 \mu^4 n^4 + 8973275 \mu^5 n^4 - \\
5081510 \mu^6 n^4 - 2026505 \mu^7 n^4 - 181475 \mu^8 n^4 - 8025685 n^5 - 73041186 \mu n^5 - \\
3061033 \mu^2 n^5 + 116111654 \mu^3 n^5 + 42293153 \mu^4 n^5 - 22112998 \mu^5 n^5 - \\
11477475 \mu^6 n^5 - 1228510 \mu^7 n^5 - 49401675 n^6 - 21869584 \mu n^6 + 221859281 \mu^2 n^6 + \\
127808768 \mu^3 n^6 - 65207089 \mu^4 n^6 - 45992560 \mu^5 n^6 - 6008213 \mu^6 n^6 - 22352020 n^7 + \\
237195716 \mu n^7 + 240300968 \mu^2 n^7 - 128230504 \mu^3 n^7 - 130388308 \mu^4 n^7 - \\
21404572 \mu^5 n^7 + 108556420 n^8 + 256315728 \mu n^8 - 160187384 \mu^2 n^8 - \\
256315728 \mu^3 n^8 - 55152716 \mu^4 n^8 + 118643600 n^9 - 113919680 \mu n^9 - \\
332830160 \mu^2 n^9 - 100266880 \mu^3 n^9 - 34541840 n^{10} - 257023872 \mu n^{10} - \\
122106032 \mu^2 n^{10} - 89453760 n^{11} - 89453760 \mu n^{11} - 29817920 n^{12} \Big) S_n + \\
(-3 + \mu + 3 n) (-2 + \mu + 3 n) (-1 + \mu + 3 n) (-1 + \mu + 6 n) (1 + \mu + 6 n) (3 + \mu + 6 n) \\
(360 \mu + 727 \mu^2 + 486 \mu^3 + 136 \mu^4 + 18 \mu^5 + \mu^6 + 1350 n + 5040 \mu n + 5277 \mu^2 n + \\
2113 \mu^3 n + 357 \mu^4 n + 23 \mu^5 n + 9261 n^2 + 19218 \mu n^2 + 12094 \mu^2 n^2 + \\
2826 \mu^3 n^2 + 225 \mu^4 n^2 + 23919 n^3 + 30599 \mu n^3 + 11109 \mu^2 n^3 + 1189 \mu^3 n^3 + \\
29258 n^4 + 21732 \mu n^4 + 3554 \mu^2 n^4 + 17028 n^5 + 5676 \mu n^5 + 3784 n^6 \Big) \Big\}$$

(\* Plug our expression into the recurrence \*)

test = ApplyOreOperator[ann3[[1]], R00o[n]];

(\* Divide by the (non-zero!) expression itself,

to make simplification easier. \*)

test = If[Head[test] === Plus, #/R00o[n] &/@test, test/R00o[n]];

(\* Simplify \*)

Together[MySimp[test]]

0

## Lemma 6: $D_{2,0}(2n)/D_{2,0}(2n-1)$

```

Timing[
  data = PadRight[Table[ns = LinSolveUniv[Most[DstMat[2, 0, 2 n, mu]], mu][[1]];
    Together[ns/ns[[-1]]], {n, 15}]]];
]
{42.723000, Null}

(* Don't guess the Groebner basis directly,
but some higher-order recurrences. Then apply Buchberger. *)
Timing[
  ann = OreGroebnerBasis[NormalizeCoefficients /@ToOrePolynomial[
    GuessMultRE[data, {c[n, j], c[n, j+1], c[n+1, j], c[n, j+2], c[n+1, j+1]},
      {n, j}, 7, StartPoint -> {1, 1}, Constraints -> j ≤ 2 n], c[n, j]]];
]
{39.076000, Null}

(*Put[ann, "ann_c_2_0.m"];*)
ann = Get["ann_c_2_0.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
{1, Sj, Sn}
1106536
{{7, 6}, {6, 5}, {15, 11}}

(* Identity (1): Show that c[n,2n]=1 for all n. *)
diag = DFiniteSubstitute[ann, {j -> 2 n}, Algebra -> OreAlgebra[S[n]]][[1]];
Select[Factor[LeadingCoefficient[diag]], Exponent[#, n] == 1 || Head[#] != Plus &]
OreReduce[diag, {ToOrePolynomial[S[n] - 1]}]
8 (2 + n) (1 + 2 n) (3 + 2 n) (5 + 2 n) (3 + mu + 2 n) (4 + mu + 2 n)
(3 + mu + 4 n) (5 + mu + 4 n) (6 + mu + 4 n) (7 + mu + 4 n)2 (8 + mu + 4 n) (9 + mu + 4 n)2
0

(* Identity (2) *)
TraditionalForm[HoldForm[Sum[Binomial[μ + i + j - 2, j - 1] * c[n, j], {j, 1, 2 n}] ==
  -c[n, i + 2] " " (1 ≤ i ≤ 2 n - 1)]]]

$$\sum_{j=1}^{2n} \binom{\mu+i+j-2}{j-1} c(n, j) = -c(n, i+2) \quad (1 \leq i \leq 2n-1)$$


```

```

(* Identity (2): numerical check (= initial values) *)
Union[Flatten[Table[Together[Sum[dst[2, 0, 2 n, i, j, mu] * data[[n, j]], {j, 1, 2 n}]],
  {n, 15}, {i, 2 n - 1}]]]]
Union[Flatten[Table[Together[Sum[FunctionExpand[Binomial[mu + i + j - 2, j - 1]] *
  data[[n, j]], {j, 1, 2 n}] + data[[n, i + 2]], {n, 14}, {i, 2 n - 1}]]]]
{0}
{0}

(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
  ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
  Annihilator[Binomial[mu + i + j - 2, j - 1], {S[n], S[j], S[i]}], S[j] - 1];
]
{2211.326000, Null}

ct = << "ct_2_0_A.m";

(* Look at the singularities of the certificate: no pole for 1 ≤ j ≤ 2n. *)
Factor[PolynomialLCM@@
  Denominator[Together[Flatten[OrePolynomialListCoefficients /@ Flatten[ct[[2]]]]]]]
(i + mu) (1 + i + mu) (-1 + i + j + mu) (-4 + j - 2 n) (-3 + j - 2 n) (j + mu + 2 n)
(6 - 19 j + 22 j^2 - 11 j^3 + 2 j^4 - 8 mu + 18 j mu - 13 j^2 mu + 3 j^3 mu + 2 mu^2 -
  3 j mu^2 + j^2 mu^2 + 24 n - 48 j n + 30 j^2 n - 6 j^3 n - 32 mu n + 40 j mu n - 12 j^2 mu n +
  8 mu^2 n - 4 j mu^2 n + 24 n^2 - 36 j n^2 + 12 j^2 n^2 - 32 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2)

GBEqual[DFiniteSubstitute[ann, {j → i + 2}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
True

(* Identity (3) *)
TraditionalForm[
  HoldForm[Sum[(KroneckerDelta[2 n + 1, j - 1] + Binomial[mu + 2 n + j - 2, j - 1]) * c[n, j],
    {j, 1, 2 n}] = D2,0[2 n] / D2,0[2 n - 1]]]

$$\sum_{j=1}^{2n} \left( \delta_{2n+1, j-1} + \binom{\mu + 2n + j - 2}{j-1} \right) c(n, j) = \frac{D_{2,0}(2n)}{D_{2,0}(2n-1)}$$


(* Numerical check of Identity (3) (= initial value check) *)
Table[Together[
  Sum[FunctionExpand[Binomial[mu + 2 n + j - 2, j - 1]] * data[[n, j]], {j, 1, 2 n}] -
  R20[n], {n, 10}]
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```

```

(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[
  ct = FindCreativeTelescoping[DFiniteTimes[ann,
    Annihilator[Binomial[mu + 2 n + j - 4, j - 1], {S[n], S[j]}]], S[j] - 1];
]
{101.570000, Null}

(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[
  ct = CreativeTelescoping[DFiniteTimes[ann,
    Annihilator[Binomial[mu + 2 n + j - 2, j - 1], {S[n], S[j]}]], S[j] - 1];
]
{94.256000, Null}

ct = << "ct_2_0_B.m";

(* Look at the singularities of the certificate: no pole for 1 ≤ j ≤ 2n. *)
Factor[PolynomialLCM@@
  Denominator[Together[OrePolynomialListCoefficients[Flatten[ct][[2]]]]]]
(-4 + j - 2 n) (-3 + j - 2 n) (1 + n) (mu + 2 n) (1 + mu + 2 n) (2 + mu + 2 n)
(3 + mu + 2 n) (j + mu + 2 n) (1 + mu + 4 n) (2 + mu + 4 n) (4 + mu + 4 n) (6 + mu + 4 n)
(6 - 19 j + 22 j^2 - 11 j^3 + 2 j^4 - 8 mu + 18 j mu - 13 j^2 mu + 3 j^3 mu + 2 mu^2 -
  3 j mu^2 + j^2 mu^2 + 24 n - 48 j n + 30 j^2 n - 6 j^3 n - 32 mu n + 40 j mu n - 12 j^2 mu n +
  8 mu^2 n - 4 j mu^2 n + 24 n^2 - 36 j n^2 + 12 j^2 n^2 - 32 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2)

(* We get a nice recurrence *)
Factor[ct[[1]]]
{-8 (-1 + 2 n) (1 + 2 n) (3 + 2 n) (4 + mu + 2 n) (-1 + mu + 4 n) (1 + mu + 4 n) (3 + mu + 4 n)
  (5 + mu + 4 n)^2 (7 + mu + 4 n) (10 mu + 17 mu^2 + 8 mu^3 + mu^4 + 24 n + 104 mu n + 82 mu^2 n +
  16 mu^3 n + 154 n^2 + 270 mu n^2 + 88 mu^2 n^2 + 290 n^3 + 204 mu n^3 + 172 n^4) S_n^2 -
  2 (-1 + 2 n) (-1 + mu + 4 n) (1 + mu + 4 n) (5 + mu + 6 n) (7 + mu + 6 n) (9 + mu + 6 n)
  (240 mu + 1628 mu^2 + 3036 mu^3 + 2469 mu^4 + 1020 mu^5 + 222 mu^6 + 24 mu^7 + mu^8 - 960 n +
  1728 mu n + 13 892 mu^2 n + 19 704 mu^3 n + 11 732 mu^4 n + 3384 mu^5 n + 464 mu^6 n +
  24 mu^7 n - 10 184 n^2 - 2916 mu n^2 + 36 816 mu^2 n^2 + 43 752 mu^3 n^2 + 18 940 mu^4 n^2 +
  3504 mu^5 n^2 + 232 mu^6 n^2 - 50 500 n^3 - 42 576 mu n^3 + 36 696 mu^2 n^3 + 42 864 mu^3 n^3 +
  12 624 mu^4 n^3 + 1152 mu^5 n^3 - 126 892 n^4 - 106 740 mu n^4 + 7244 mu^2 n^4 +
  18 720 mu^3 n^4 + 2976 mu^4 n^4 - 171 820 n^5 - 115 320 mu n^5 - 8448 mu^2 n^5 + 2880 mu^3 n^5 -
  127 516 n^6 - 57 840 mu n^6 - 3568 mu^2 n^6 - 48 880 n^7 - 11 040 mu n^7 - 7568 n^8) S_n +
  (mu + 2 n) (mu + 3 n) (1 + mu + 3 n) (2 + mu + 3 n) (-1 + mu + 6 n) (1 + mu + 6 n)
  (3 + mu + 6 n) (5 + mu + 6 n) (7 + mu + 6 n) (9 + mu + 6 n)
  (640 + 588 mu + 187 mu^2 + 24 mu^3 + mu^4 + 1890 n + 1256 mu n + 258 mu^2 n +
  16 mu^3 n + 2056 n^2 + 882 mu n^2 + 88 mu^2 n^2 + 978 n^3 + 204 mu n^3 + 172 n^4) }

```



```

Together[{rec1, rec2} /. {j -> 1, n -> 1} /. c[n_, j_] := data[[n, j]]]
{
  
$$\frac{1}{16384} (-1209600 - 5120640 \mu - 9591504 \mu^2 - 10576832 \mu^3 - 7681736 \mu^4 - 3884160 \mu^5 -$$


$$1405697 \mu^6 - 367626 \mu^7 - 69063 \mu^8 - 9100 \mu^9 - 799 \mu^{10} - 42 \mu^{11} - \mu^{12}),$$


$$\frac{1}{8} (10080 + 31176 \mu + 39332 \mu^2 + 26670 \mu^3 + 10689 \mu^4 + 2604 \mu^5 + 378 \mu^6 + 30 \mu^7 + \mu^8)}$$

}
ann =
  OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[{rec1, rec2}, c[n, j]]];
Support[
  ann]
{{Sn, Sj, 1}, {Sj2, Sj, 1}}

(* Put[ann, "ann_c_0_2.m"]; *)
ann = Get["ann_c_0_2.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
{1, Sj}
651256
{{10, 11}, {4, 10}}

(* Identity (1): Show that c[n,2n]=1 for all n. *)
diag = DFiniteSubstitute[ann, {j -> 2 n}, Algebra -> OreAlgebra[S[n]]][[1]];
Select[Factor[LeadingCoefficient[diag]], Exponent[#, n] === 1 || Head[#] != Plus &]
OreReduce[diag, {ToOrePolynomial[S[n] - 1]}]
2 (3 + 2 n) (4 + mu + 2 n) (3 + mu + 4 n) (5 + mu + 4 n) (7 + mu + 4 n) (9 + mu + 4 n)
0

(* Identity (2) *)
TraditionalForm[HoldForm[Sum[Binomial[μ + i + j - 2, j + 1] * c[n, j], {j, 1, 2 n}] ==
  -c[n, i - 2] " " (1 ≤ i ≤ 2 n - 1)]]

$$\sum_{j=1}^{2n} \binom{\mu+i+j-2}{j+1} c(n, j) = -c(n, i-2) \quad (1 \leq i \leq 2n-1)$$


(* Identity (2): numerical check (= initial values). Also the special case n=
  1 is covered. *)
Union[Flatten[Table[Together[Sum[dst[0, 2, 2 n, i, j, mu] * data[[n, j]], {j, 1, 2 n}]],
  {n, 15}, {i, 2 n - 1}]]]
Union[Flatten[Table[Together[Sum[FunctionExpand[Binomial[μ + i + j - 2, j + 1]] *
  data[[n, j]], {j, 1, 2 n}] + If[i < 3, 0, data[[n, i - 2]]], {n, 14}, {i, 2 n - 1}]]]
{0}
{0}

```

```

(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
  ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
  Annihilator[Binomial[mu + i + j - 2, j + 1], {S[n], S[j], S[i]}], S[j] - 1];
]
{1643.369000, Null}

ct = << "ct_0_2_A.m";

(* Look at the singularities of the certificate: no pole for 1 ≤ j ≤ 2n. *)
Factor[PolynomialLCM@@
  Denominator[Together[Flatten[OrePolynomialListCoefficients/@Flatten[ct[[2]]]]]]]
(1 + j + mu) (-1 + i + j + mu) (2 + 2 j + mu) (-2 + j - 2 n) (-1 + j - 2 n)
(48 j + 24 j^2 - 4 j mu - 22 j^2 mu - 8 j^3 mu - 2 j^4 mu - 18 j mu^2 - 10 j^2 mu^2 - 4 j^3 mu^2 - 2 j mu^3 -
  2 j^2 mu^3 - 96 n - 144 j n - 88 j^2 n - 16 j^3 n - 4 j^4 n + 8 mu n + 40 j mu n + 56 j^2 mu n + 8 j^3 mu n +
  4 j^4 mu n + 64 mu^2 n + 86 j mu^2 n + 27 j^2 mu^2 n + 8 j^3 mu^2 n + 22 mu^3 n + 17 j mu^3 n + 5 j^2 mu^3 n +
  2 mu^4 n + j mu^4 n + 192 n^2 + 288 j n^2 + 176 j^2 n^2 + 32 j^3 n^2 + 8 j^4 n^2 + 176 mu n^2 + 208 j mu n^2 +
  64 j^2 mu n^2 + 16 j^3 mu n^2 + 48 mu^2 n^2 + 36 j mu^2 n^2 + 10 j^2 mu^2 n^2 + 4 mu^3 n^2 + 2 j mu^3 n^2)

GBEqual[DFiniteSubstitute[ann, {j → i - 2}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
True

(* Identity (3) *)
TraditionalForm[
  HoldForm[Sum[(KroneckerDelta[2 n - 1, j + 1] + Binomial[mu + 2 n + j - 2, j + 1]) * c[n, j],
    {j, 1, 2 n}] = D0,2[2 n] / D0,2[2 n - 1]]]

$$\sum_{j=1}^{2n} \left( \delta_{2n-1, j+1} + \binom{\mu + 2n + j - 2}{j+1} \right) c(n, j) = \frac{D_{0,2}(2n)}{D_{0,2}(2n-1)}$$


(* Numerical check of Identity (3) (= initial value check) *)
Table[Together[
  Sum[FunctionExpand[Binomial[mu + 2 n + j - 2, j + 1]] * data[[n, j]], {j, 1, 2 n}] +
  If[n > 1, data[[n, 2 n - 2], 0] - R02[n], {n, 10}]
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

(* The sum has natural boundaries. Thus
creative telescoping gives its annihilator. *)
Timing[
  ct = FindCreativeTelescoping[DFiniteTimes[ann,
  Annihilator[Binomial[mu + 2 n + j - 2, j + 1], {S[n], S[j]}], S[j] - 1];
]
{1186.696000, Null}

(*Put[ct, "ct_0_2_B.m"];*)
ct = Get["ct_0_2_B.m"];

```



```

(* Look at the singularities of the certificate: no pole for  $1 \leq j \leq 2n$ . *)
Factor[PolynomialLCM@
  Denominator[Together[OrePolynomialListCoefficients[Flatten[ct][[2]]]]]]
2 (1 + j + mu) (2 + 2 j + mu) (-4 + j - 2 n) (-3 + j - 2 n) (-2 + j - 2 n) (-1 + j - 2 n) (1 + n) (3 + 2 n)
(4 + j + mu + 2 n) (-2 + mu + 4 n) (-1 + mu + 4 n) (mu + 4 n) (2 + mu + 4 n) (4 + mu + 4 n) (6 + mu + 4 n)
(48 j + 24 j^2 - 4 j mu - 22 j^2 mu - 8 j^3 mu - 2 j^4 mu - 18 j mu^2 - 10 j^2 mu^2 - 4 j^3 mu^2 - 2 j mu^3 -
  2 j^2 mu^3 - 96 n - 144 j n - 88 j^2 n - 16 j^3 n - 4 j^4 n + 8 mu n + 40 j mu n + 56 j^2 mu n + 8 j^3 mu n +
  4 j^4 mu n + 64 mu^2 n + 86 j mu^2 n + 27 j^2 mu^2 n + 8 j^3 mu^2 n + 22 mu^3 n + 17 j mu^3 n + 5 j^2 mu^3 n +
  2 mu^4 n + j mu^4 n + 192 n^2 + 288 j n^2 + 176 j^2 n^2 + 32 j^3 n^2 + 8 j^4 n^2 + 176 mu n^2 + 208 j mu n^2 +
  64 j^2 mu n^2 + 16 j^3 mu n^2 + 48 mu^2 n^2 + 36 j mu^2 n^2 + 10 j^2 mu^2 n^2 + 4 mu^3 n^2 + 2 j mu^3 n^2)

annSum = DFinitePlus[ct[[1]], DFiniteSubstitute[ann, {j -> 2 n - 2}]];
Support[annSum]
{{S_n^2, S_n, 1}}

(* Plug our expression into the recurrence. *)
test = ApplyOreOperator[annSum[[1]], R02[n]];
(* Divide by the (non-zero!) expression itself,
to make simplification easier. *)
test = If[Head[test] === Plus, #/R02[n] &/@test, test/R02[n]];
(* Simplify *)
Together[MySimp[test]]
0

```

## Section 4: Nice Formula for $D_{1,1}(n)$

Simple formula for  $\prod_{j=1}^{k-1} ((R_{1,0}(j) R_{0,1}(j)) / (R_{0,0}(2j-1) R_{0,0}(2j)))$

```

(* The factor inside the product *)
fac = MySimp[R10[j] * R01[j] / R00o[j] / R00e[j]] /.
  Pochhammer[j, j + 2] -> Pochhammer[j, j - 1] * (2 j - 1) * (2 j) * (2 j + 1)
(((-1 + 2 j + mu) (-3 + 3 j + mu) (-2 + 3 j + mu) (-1 + 3 j + mu)
  Pochhammer[1/2 + 2 j + mu/2, -1 + j]^2 Pochhammer[1 + 2 j + mu, -1 + j]^2) /
(j (1 + 2 j) (-3 + 4 j + mu) (-1 + 4 j + mu) Pochhammer[j, -1 + j]^2
  Pochhammer[1/2 + j + mu/2, -1 + j]^2)

(* Test *)
Table[Together[(R10[j] * R01[j] / R00[2 j - 1] / R00[2 j]) / fac], {j, 10}]
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

```

```
(* Final form for the product (use prod to avoid evaluation) *)
PR0110 = (Pochhammer[mu, 3 k - 3] / Pochhammer[mu / 2 + k - 1 / 2, k - 1] / (2 k - 1)!) *
  prod[Pochhammer[mu / 2 + 2 j + 1 / 2, j - 1] * Pochhammer[mu + 2 j + 1, j - 1] /
    Pochhammer[j, j - 1] / Pochhammer[mu / 2 + j + 1 / 2, j - 1], {j, 1, k - 1}] ^ 2;
TraditionalForm[(HoldForm@@{PR0110}) /. {prod -> Product, mu -> μ}]
```

$$\frac{(\mu)^{-3+3k} \left( \prod_{j=1}^{-1+k} \frac{\binom{\frac{1}{2}+2j+\frac{\mu}{2}}{-1+j} (1+2j+\mu)_{-1+j}}{(j)_{-1+j} \binom{\frac{1}{2}+j+\frac{\mu}{2}}{-1+j}} \right)^2}{(-1+2k)! \binom{-\frac{1}{2}+k+\frac{\mu}{2}}{-1+k}}$$

```
(* Test *)
Table[Together[Product[R10[j] * R01[j] / R00[2 j - 1] / R00[2 j], {j, 1, k - 1}] /
  (PR0110 /. prod -> Product)], {k, 10}]
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

## Simple formula for $\prod_{i=2k}^n R_{0,0}(i)$

```
(* Split the product into even and odd instances of R00 *)
TraditionalForm[Product[R[i], {i, 2 k, n}] ==
  Product[R[2 j], {j, k, Floor[n / 2]}] * Product[R[2 j + 1], {j, k, Floor[(n - 1) / 2]}]]
```

$$\prod_{i=2k}^n R(i) = \left( \prod_{j=k}^{\lfloor \frac{n}{2} \rfloor} R(2j) \right) \left( \prod_{j=k}^{\lfloor \frac{n-1}{2} \rfloor} R(2j+1) \right)$$

```
(* Derivation *)
PR00 = prod[R00e[j], {j, k, Floor[n/2]}] * prod[R00o[j+1], {j, k, Floor[(n-1)/2]}];
PR00 = PR00 /. Pochhammer[a_, b_] -> Pochhammer[Expand[a], Expand[b]];
PR00e = PR00 /. Floor[a_] -> FullSimplify[Floor[a], Element[n/2, Integers]];
PR00o = PR00 /. {Floor[n/2] -> (n-1)/2, Floor[(n-1)/2] -> (n-1)/2} /.
  prod[a_, c_] * prod[b_, c_] -> prod[a * b, c] /.
  Pochhammer[mu/2 + 2j + 1/2, j - 1] ->
  Pochhammer[mu/2 + 2j - 1/2, j] / (mu/2 + 2j - 1/2) /.
  Pochhammer[mu/2 + j + 1/2, j - 1] -> Pochhammer[mu/2 + j + 1/2, j] / (mu/2 + 2j - 1/2);
{PR00e, PR00o} = {PR00e, PR00o} /. prod[a_, b_] -> prod[a, Expand[b]]
{prod[(Pochhammer[1/2 + 2j + mu/2, -1 + j] Pochhammer[2j + mu, j]) /
  (Pochhammer[j, j] Pochhammer[1/2 + j + mu/2, -1 + j]), {j, k, n/2}]
prod[(Pochhammer[3/2 + 2j + mu/2, 1 + j] Pochhammer[2j + mu, j]) /
  (Pochhammer[1 + j, 1 + j] Pochhammer[1/2 + j + mu/2, j]), {j, k, -1 + n/2}], prod[
  (Pochhammer[-1/2 + 2j + mu/2, j] Pochhammer[3/2 + 2j + mu/2, 1 + j] Pochhammer[2j + mu, j]^2) /
  (Pochhammer[j, j] Pochhammer[1 + j, 1 + j] Pochhammer[1/2 + j + mu/2, j]^2)], {j,
  k, -1/2 + n/2}]]}

(* Tests *)
Flatten[Table[Together[(PR00o /. prod -> Product) / Product[R00[i], {i, 2k, n}]],
  {n, 1, 9, 2}, {k, 0, Floor[(n+1)/2]}]]
Flatten[Table[Together[(PR00e /. prod -> Product) / Product[R00[i], {i, 2k, n}]],
  {n, 2, 10, 2}, {k, 0, Floor[(n+1)/2]}]]
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

## Final formula for $D_{1,1}(n)$

```
(* The case distinction (k=0 vs. k>0) *)
Together[PR0110 * (2k - 1)! /. k -> 0 /. prod -> Product]
-----
1
2 (-2 + mu) (-1 + mu)
```

(\* n even \*)

FFe = sum[(mu - 1) / 2 \* If[k == 0, 4 (mu - 2), 1 / (2 k - 1) !] \*  
PR00e \* (PR0110 \* (2 k - 1) !), {k, 0, n / 2}];

(\* n odd \*)

FFo = sum[(mu - 1) / 2 \* If[k == 0, 4 (mu - 2), 1 / (2 k - 1) !] \*  
PR00o \* (PR0110 \* (2 k - 1) !), {k, 0, (n + 1) / 2}];

{FFe, FFo} = {FFe, FFo} /. (mu - 1) \* Pochhammer[mu, 3 k - 3] -> Pochhammer[mu - 1, 3 k - 2];

TraditionalForm[HoldForm[FF] /. FF -> FFe /. {prod -> Product, sum -> Sum, mu -> mu}]

$$\sum_{k=0}^{\frac{n}{2}} \left( \text{If}\left[k=0, 4(\mu-2), \frac{1}{(2k-1)!}\right] (-1+\mu)_{-2+3k} \left( \prod_{j=k}^{\frac{n}{2}} \frac{\left(\frac{1}{2}+2j+\frac{\mu}{2}\right)_{-1+j} (2j+\mu)_j}{(j)_j \left(\frac{1}{2}+j+\frac{\mu}{2}\right)_{-1+j}} \right) \right. \\ \left. \left( \prod_{j=k}^{-1+\frac{n}{2}} \frac{\left(\frac{3}{2}+2j+\frac{\mu}{2}\right)_{1+j} (2j+\mu)_j}{(1+j)_{1+j} \left(\frac{1}{2}+j+\frac{\mu}{2}\right)_j} \right) \left( \prod_{j=1}^{-1+k} \frac{\left(\frac{1}{2}+2j+\frac{\mu}{2}\right)_{-1+j} (1+2j+\mu)_{-1+j}}{(j)_{-1+j} \left(\frac{1}{2}+j+\frac{\mu}{2}\right)_{-1+j}} \right)^2 \right) / \left( 2 \left( -\frac{1}{2} + k + \frac{\mu}{2} \right)_{-1+k} \right)$$

TraditionalForm[HoldForm[FF] /. FF -> FFo /. {prod -> Product, sum -> Sum, mu -> mu}]

$$\sum_{k=0}^{\frac{1+n}{2}} \left( \text{If}\left[k=0, 4(\mu-2), \frac{1}{(2k-1)!}\right] (-1+\mu)_{-2+3k} \left( \prod_{j=k}^{-\frac{1}{2}+\frac{n}{2}} \frac{\left(-\frac{1}{2}+2j+\frac{\mu}{2}\right)_j \left(\frac{3}{2}+2j+\frac{\mu}{2}\right)_{1+j} ((2j+\mu)_j)^2}{(j)_j (1+j)_{1+j} \left(\frac{1}{2}+j+\frac{\mu}{2}\right)_j^2} \right) \right. \\ \left. \left( \prod_{j=1}^{-1+k} \frac{\left(\frac{1}{2}+2j+\frac{\mu}{2}\right)_{-1+j} (1+2j+\mu)_{-1+j}}{(j)_{-1+j} \left(\frac{1}{2}+j+\frac{\mu}{2}\right)_{-1+j}} \right)^2 \right) / \left( 2 \left( -\frac{1}{2} + k + \frac{\mu}{2} \right)_{-1+k} \right)$$

(\* Test \*)

Table[Together[

(If[EvenQ[n], FFe, FFo] /. sum -> Sum /. prod -> Product) / Dst[1, 1, n], {n, 10}]

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

## Section 5: Proof of the Monstrous Conjecture

### The Conjecture

```

Clear[myC, myE, myF, myT, myS1, myS2, myP1, myP2, myG, D34];
myC[n_Integer?Positive] := ((-1)^n + 3) / 2 * Product[Floor[i/2]! / i!, {i, 1, n}];
myE[n_Integer?Positive, mu_] := Pochhammer[mu + 1, n] * Product[
  (mu + 2 i + 6)^(2 * Floor[(i + 2) / 3]), {i, 1, Floor[3 / 2 * Floor[(n - 1) / 2] - 2]}] *
  Product[(mu + 2 i + 2 * Floor[3 / 2 * Floor[n / 2 + 1] - 1])^
  (2 * Floor[Floor[n / 2] / 2 - (i - 1) / 3] - 1), {i, 1, Floor[3 / 2 * Floor[n / 2] - 2]}];
myF[m : (0 | 1), n_Integer?Positive, mu_] :=
  Product[(mu + 2 i + n + m)^(1 - 2 i - m), {i, 1, Floor[(n - 1) / 4]}] *
  Product[(mu - 2 i + 2 n - 2 m + 1)^(1 - 2 i - m), {i, 1, Floor[n / 4 - 1]}];
myF[n_Integer?Positive, mu_] := Which[
  EvenQ[n], myE[n, mu] * myF[0, n, mu],
  OddQ[n], myE[n, mu] * myF[1, n, mu] * Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}]];
myT[k_, mu_] := -12 + 84 * k + 288 * k^2 - 5856 * k^3 + 20352 * k^4 -
  41472 * k^5 + 55296 * k^6 + 10 * mu + 76 * k * mu - 2176 * k^2 * mu + 9888 * k^3 * mu -
  25344 * k^4 * mu + 41472 * k^5 * mu + 10 * mu^2 - 261 * k * mu^2 + 1676 * k^2 * mu^2 -
  5472 * k^3 * mu^2 + 11520 * k^4 * mu^2 - 10 * mu^3 + 115 * k * mu^3 - 488 * k^2 * mu^3 +
  1440 * k^3 * mu^3 + 2 * mu^4 - 15 * k * mu^4 + 76 * k^2 * mu^4 + k * mu^5;
myS1[n_Integer?Positive, mu_] := Sum[myT[k, mu] * 2^(6 k) * (mu + 8 k - 1) *
  Pochhammer[1 / 2, 2 k - 1]^2 * Pochhammer[(mu + 4 k + 2) / 2, 2 n - 2 k - 2] *
  Pochhammer[(mu + 5) / 2, 2 k - 3] * Pochhammer[(mu + 4 k + 2) / 2, k - 2] /
  ((2 k)! * Pochhammer[(mu + 6 k - 3) / 2, 3 k + 4]), {k, 1, n - 1}];
myS2[n_Integer?Positive, mu_] :=
  Sum[myT[k + 1 / 2, mu] * 2^(6 k) * (mu + 8 k + 3) *
  Pochhammer[1 / 2, 2 k]^2 * Pochhammer[(mu + 4 k + 4) / 2, 2 n - 2 k - 2] *
  Pochhammer[(mu + 5) / 2, 2 k - 2] * Pochhammer[(mu + 4 k + 4) / 2, k - 2] /
  ((2 k + 1)! * Pochhammer[(mu + 6 k + 1) / 2, 3 k + 5]), {k, 1, n - 1}];
myP1[n_Integer?Positive, mu_] := Together[
  2^(3 n - 1) * Pochhammer[(mu + 6 n - 3) / 2, 3 n - 2] / Pochhammer[(mu + 5) / 2, 2 n - 3] *
  (2^(-13) * mu * (mu - 1) * myS1[n, mu] +
  1 / (mu + 3)^2 * Pochhammer[(mu + 2) / 2, 2 n - 2]);
myP2[n_Integer?Positive, mu_] := Together[
  2^(3 n - 1) * Pochhammer[(mu + 6 n + 1) / 2, 3 n - 1] / Pochhammer[(mu + 5) / 2, 2 n - 2] *
  (2^(-9) mu * (mu - 1) * myS2[n, mu] +
  (mu + 14) / ((mu + 7) * (mu + 9)) * Pochhammer[(mu + 4) / 2, 2 n - 2]);
myG[n_Integer?Positive, mu_] := If[EvenQ[n], myP2[n / 2, mu], myP1[(n + 1) / 2, mu]];
D34[n_Integer?Positive, mu_] := myC[n] * myF[n, mu] * myG[Floor[(n + 1) / 2], mu];

```

```

(* even n *)
Table[Together[(FFe /. sum → Sum /. prod → Product) /
  (myC[n] * myE[n, mu] * myF[0, n, mu] * myP2[n / 4, mu])], {n, 4, 12, 4}]
Table[Together[(FFe /. sum → Sum /. prod → Product) /
  (myC[n] * myE[n, mu] * myF[0, n, mu] * myP1[(n + 2) / 4, mu])], {n, 2, 10, 4}]
(* odd n *)
Table[
  Together[(FFo /. sum → Sum /. prod → Product) / (myC[n] * myE[n, mu] * myF[1, n, mu] *
    Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}] * myP1[(n + 3) / 4, mu])], {n, 1, 9, 4}]
Table[Together[(FFo /. sum → Sum /. prod → Product) / (myC[n] * myE[n, mu] * myF[1, n, mu] *
  Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}] * myP2[(n + 1) / 4, mu])], {n, 3, 11, 4}]
{1, 1, 1}
{1, 1, 1}
{1, 1, 1}
{1, 1, 1}

(* These are the recurrence operators that we used to define P_1 and P_2. *)
rec1 = NormalizeCoefficients[ToOrePolynomial[
  -myP1fast[n, mu] + (-8 * (-10 + mu + 4 * n) * (-8 + mu + 4 * n) * (-14 + mu + 6 * n) *
    (-12 + mu + 6 * n) * (-10 + mu + 6 * n) * (-9 + mu + 8 * n) * (-31 + mu + 12 * n) *
    (-29 + mu + 12 * n) * (-27 + mu + 12 * n) * (-25 + mu + 12 * n) * (-23 + mu + 12 * n) *
    (-21 + mu + 12 * n) * (81 - 72 * n + 16 * n^2) * (49 - 56 * n + 16 * n^2) *
    (123 168 - 78 946 * mu + 18 939 * mu^2 - 2053 * mu^3 + 93 * mu^4 - mu^5 +
    (-9 + mu) * (70 956 - 30 208 * mu + 3989 * mu^2 - 158 * mu^3 + mu^4) * n +
    4 * (346 032 - 149 656 * mu + 21 803 * mu^2 - 1202 * mu^3 + 19 * mu^4) * n^2 +
    96 * (-9 + mu) * (1861 - 402 * mu + 15 * mu^2) * n^3 +
    384 * (2753 - 606 * mu + 30 * mu^2) * n^4 + 41 472 * (-9 + mu) * n^5 + 55 296 * n^6) *
  myP1fast[-2 + n, mu] + (-13 + mu + 8 * n) * (-6 * (-15 + mu) * (-11 + mu) * (-9 + mu) *
    (77 272 834 343 040 - 90 508 623 095 808 * mu + 46 786 094 223 720 * mu^2 -
    14 041 912 717 156 * mu^3 + 2 707 887 452 266 * mu^4 - 350 541 498 059 * mu^5 +
    30 888 280 625 * mu^6 - 1 838 952 303 * mu^7 + 72 032 193 * mu^8 -
    1 778 033 * mu^9 + 26 555 * mu^10 - 241 * mu^11 + mu^12) +
    (-13 + mu) * (615 591 764 176 296 960 - 787 691 318 438 414 592 * mu +
    453 271 146 257 615 040 * mu^2 - 154 970 921 382 725 880 * mu^3 +
    35 030 740 197 791 460 * mu^4 - 5 511 255 715 119 386 * mu^5 +
    618 465 455 797 003 * mu^6 - 49 890 145 667 170 * mu^7 +
    2 877 469 024 970 * mu^8 - 116 576 723 262 * mu^9 + 3 218 550 024 * mu^10 -
    58 094 110 * mu^11 + 655 730 * mu^12 - 4400 * mu^13 + 13 * mu^14) * n +
    (43 790 163 197 061 415 680 - 55 769 554 581 921 674 496 * mu +
    32 100 807 569 482 408 752 * mu^2 - 11 046 065 343 390 418 896 * mu^3 +
    2 532 539 665 806 086 200 * mu^4 - 408 068 212 472 225 048 * mu^5 +
    47 486 735 062 736 003 * mu^6 - 4 036 853 597 489 641 * mu^7 +
    250 606 824 181 572 * mu^8 - 11 237 476 473 228 * mu^9 +
    356 071 800 098 * mu^10 - 7 704 642 502 * mu^11 + 108 621 484 * mu^12 -
    941 780 * mu^13 + 4611 * mu^14 - 9 * mu^15) * n^2 +
    2 * (-13 + mu) * (5 765 368 315 087 296 000 - 6 423 796 647 403 130 880 * mu +

```

$$\begin{aligned}
& 3\,186\,986\,194\,272\,026\,736 * \mu^2 - 928\,737\,086\,880\,929\,008 * \mu^3 + \\
& 176\,577\,512\,806\,080\,224 * \mu^4 - 23\,002\,876\,518\,214\,396 * \mu^5 + \\
& 2\,097\,912\,117\,891\,133 * \mu^6 - 134\,465\,197\,774\,532 * \mu^7 + \\
& 5\,992\,468\,266\,728 * \mu^8 - 181\,075\,265\,324 * \mu^9 + 3\,560\,096\,842 * \mu^{10} - \\
& 42\,928\,700 * \mu^{11} + 293\,696 * \mu^{12} - 1000 * \mu^{13} + \mu^{14}) * n^3 + \\
8 * & (44\,967\,647\,815\,472\,773\,440 - 49\,875\,119\,477\,893\,931\,904 * \mu + \\
& 24\,771\,543\,294\,236\,452\,512 * \mu^2 - 7\,277\,588\,373\,063\,623\,552 * \mu^3 + \\
& 1\,407\,087\,781\,066\,080\,464 * \mu^4 - 188\,436\,568\,279\,081\,716 * \mu^5 + \\
& 17\,910\,169\,812\,661\,579 * \mu^6 - 1\,217\,322\,600\,443\,922 * \mu^7 + \\
& 58\,827\,888\,448\,174 * \mu^8 - 1\,983\,671\,151\,898 * \mu^9 + 45\,113\,742\,796 * \mu^{10} - \\
& 655\,655\,046 * \mu^{11} + 5\,605\,730 * \mu^{12} - 24\,666 * \mu^{13} + 41 * \mu^{14}) * n^4 + \\
32 * & (-13 + \mu) * (1\,545\,137\,447\,830\,050\,528 - 1\,468\,846\,207\,754\,989\,056 * \mu + \\
& 613\,359\,955\,784\,013\,384 * \mu^2 - 148\,046\,294\,338\,567\,160 * \mu^3 + \\
& 22\,867\,645\,137\,091\,796 * \mu^4 - 2\,363\,768\,523\,778\,396 * \mu^5 + \\
& 166\,104\,951\,524\,749 * \mu^6 - 7\,900\,529\,853\,234 * \mu^7 + 248\,588\,564\,859 * \mu^8 - \\
& 4\,947\,975\,304 * \mu^9 + 57\,722\,923 * \mu^{10} - 345\,266 * \mu^{11} + 785 * \mu^{12}) * \\
n^5 + & 128 * (6\,923\,436\,910\,786\,740\,816 - 6\,551\,979\,917\,272\,781\,760 * \mu + \\
& 2\,741\,775\,205\,145\,125\,620 * \mu^2 - 668\,624\,737\,408\,815\,316 * \mu^3 + \\
& 105\,402\,483\,452\,844\,020 * \mu^4 - 11\,258\,804\,752\,461\,004 * \mu^5 + \\
& 830\,334\,150\,499\,955 * \mu^6 - 42\,256\,983\,681\,030 * \mu^7 + \\
& 1\,457\,399\,275\,653 * \mu^8 - 32\,763\,679\,904 * \mu^9 + 447\,520\,681 * \mu^{10} - \\
& 3\,258\,554 * \mu^{11} + 9319 * \mu^{12}) * n^6 + 1024 * (-13 + \mu) * \\
(72\,414\,477\,952\,775\,604 - & 57\,105\,723\,925\,009\,800 * \mu + 19\,399\,742\,350\,341\,207 * \\
& \mu^2 - 3\,719\,307\,354\,992\,416 * \mu^3 + 442\,850\,412\,559\,382 * \mu^4 - \\
& 33\,955\,375\,237\,500 * \mu^5 + 1\,681\,820\,711\,178 * \mu^6 - 52\,507\,834\,704 * \mu^7 + \\
& 974\,233\,650 * \mu^8 - 9\,518\,828 * \mu^9 + 36\,355 * \mu^{10}) * n^7 + \\
4096 * & (204\,759\,442\,490\,425\,380 - 160\,746\,724\,570\,083\,012 * \mu + \\
& 54\,801\,297\,077\,548\,677 * \mu^2 - 10\,648\,677\,530\,738\,482 * \mu^3 + \\
& 1\,300\,829\,127\,395\,384 * \mu^4 - 103\,865\,351\,431\,818 * \mu^5 + \\
& 5\,455\,145\,057\,379 * \mu^6 - 184\,594\,947\,228 * \mu^7 + 3\,811\,103\,508 * \mu^8 - \\
& 42\,749\,540 * \mu^9 + 194\,248 * \mu^{10}) * n^8 + 49\,152 * (-13 + \mu) * \\
(920\,215\,916\,156\,142 - & 577\,914\,239\,846\,832 * \mu + 151\,701\,784\,373\,213 * \mu^2 - \\
& 21\,614\,250\,577\,806 * \mu^3 + 1\,815\,722\,558\,519 * \mu^4 - 91\,353\,917\,016 * \mu^5 + \\
& 2\,663\,224\,490 * \mu^6 - 40\,669\,644 * \mu^7 + 245\,586 * \mu^8) * n^9 + 196\,608 * \\
(1\,693\,595\,159\,851\,230 - & 1\,058\,822\,980\,698\,432 * \mu + 279\,542\,833\,819\,585 * \mu^2 - \\
& 40\,572\,445\,515\,984 * \mu^3 + 3\,526\,446\,267\,001 * \mu^4 - 187\,021\,320\,840 * \mu^5 + \\
& 5\,872\,755\,784 * \mu^6 - 99\,020\,958 * \mu^7 + 679\,074 * \mu^8) * n^{10} + 21\,233\,664 * \\
(-13 + \mu) * & (550\,446\,775\,412 - 258\,091\,315\,032 * \mu + 47\,985\,773\,125 * \mu^2 - \\
& 4\,496\,668\,860 * \mu^3 + 222\,288\,724 * \mu^4 - 5\,456\,352 * \mu^5 + 51\,547 * \mu^6) * \\
n^{11} + & 28\,311\,552 * (1\,958\,821\,138\,060 - 914\,306\,594\,496 * \mu + \\
& 171\,668\,385\,371 * \mu^2 - 16\,540\,689\,390 * \mu^3 + 859\,090\,262 * \mu^4 - \\
& 22\,689\,546 * \mu^5 + 236\,549 * \mu^6) * n^{12} + 21\,403\,533\,312 * (-13 + \mu) * \\
(57\,395\,792 - & 17\,859\,456 * \mu + 1\,964\,631 * \mu^2 - 89\,610 * \mu^3 + 1425 * \mu^4) * \\
n^{13} + & 12\,230\,590\,464 * (290\,157\,464 - 89\,880\,912 * \mu + \\
& 10\,081\,119 * \mu^2 - 483\,594 * \mu^3 + 8337 * \mu^4) * n^{14} + \\
3\,522\,410\,053\,632 * & (-13 + \mu) * (12\,823 - 1986 * \mu + 69 * \mu^2) * n^{15} + \\
3\,522\,410\,053\,632 * & (19\,340 - 2982 * \mu + 111 * \mu^2) * n^{16} + \\
380\,420\,285\,792\,256 * & (-13 + \mu) * n^{17} +
\end{aligned}$$

```

        169 075 682 574 336 * n^18) * myP1fast[-1 + n, mu] /
(( -1 + n) * ( -3 + 2 * n) * ( -6 + mu + 4 * n) * ( -5 + mu + 4 * n) *
(-4 + mu + 4 * n) *
(-3 + mu + 4 * n) *
(-9 + mu + 6 * n) *
(-7 + mu + 6 * n) * ( -5 + mu + 6 * n) *
(-17 + mu + 8 * n) *
(-2 * ( -2 619 750 + 910 279 * mu - 117 666 * mu^2 + 6856 * mu^3 - 168 * mu^4 + mu^5) +
(-17 + mu) * (862 188 - 199 648 * mu + 14 213 * mu^2 - 302 * mu^3 + mu^4) * n +
4 * (4 278 168 - 996 880 * mu + 77 747 * mu^2 - 2282 * mu^3 + 19 * mu^4) * n^2 +
96 * (-17 + mu) * (6541 - 762 * mu + 15 * mu^2) * n^3 +
384 * (9773 - 1146 * mu + 30 * mu^2) * n^4 +
41 472 * (-17 + mu) * n^5 + 55 296 * n^6)),
myP1fast[n, mu], OreAlgebra[S[n]]];
rec2 = NormalizeCoefficients[ToOrePolynomial[
-myP2fast[n, mu] + (-8 * (-8 + mu + 4 * n) * (-6 + mu + 4 * n) * (-12 + mu + 6 * n) *
(-10 + mu + 6 * n) * (-8 + mu + 6 * n) * (-5 + mu + 8 * n) * (-25 + mu + 12 * n) *
(-23 + mu + 12 * n) * (-21 + mu + 12 * n) * (-19 + mu + 12 * n) * (-17 + mu + 12 * n) *
(-15 + mu + 12 * n) * (49 - 56 * n + 16 * n^2) * (25 - 40 * n + 16 * n^2) *
(-((-3 + mu) * (2788 - 2196 * mu + 577 * mu^2 - 54 * mu^3 + mu^4)) +
2 * (-5 + mu) * (7620 - 5536 * mu + 1253 * mu^2 - 86 * mu^3 + mu^4) * n +
8 * (35 820 - 26 716 * mu + 6791 * mu^2 - 662 * mu^3 + 19 * mu^4) * n^2 +
192 * (-5 + mu) * (601 - 222 * mu + 15 * mu^2) * n^3 +
768 * (863 - 336 * mu + 30 * mu^2) * n^4 + 82 944 * (-5 + mu) * n^5 + 110 592 * n^6) *
myP2fast[-2 + n, mu] + (-9 + mu + 8 * n) * (-((-11 + mu) * (-7 + mu) *
(-5 + mu) * (-3 + mu) * (-941 137 562 880 + 1 369 543 037 568 * mu -
856 059 425 680 * mu^2 + 301 467 356 208 * mu^3 - 65 925 560 840 * mu^4 +
9 300 152 544 * mu^5 - 851 420 265 * mu^6 + 49 707 939 * mu^7 -
1 788 230 * mu^8 + 38 538 * mu^9 - 505 * mu^10 + 3 * mu^11)) + (-9 + mu) *
(2 174 231 624 313 600 - 4 271 307 638 939 136 * mu + 3 746 500 640 981 808 * mu^2 -
1 938 172 937 860 384 * mu^3 + 658 024 132 807 528 * mu^4 - 154 336 161 708 664 *
mu^5 + 25 631 896 940 311 * mu^6 - 3 038 647 883 536 * mu^7 +
255 911 958 856 * mu^8 - 15 059 474 264 * mu^9 + 601 933 862 * mu^10 -
15 728 672 * mu^11 + 258 008 * mu^12 - 2528 * mu^13 + 11 * mu^14) * n -
4 * (-41 108 205 131 322 624 + 79 558 217 840 920 896 * mu -
69 190 984 849 287 408 * mu^2 + 35 769 692 404 688 632 * mu^3 -
12 252 335 726 377 252 * mu^4 + 2 933 722 316 842 738 * mu^5 -
504 752 475 079 572 * mu^6 + 63 145 862 893 203 * mu^7 - 5 745 369 671 196 * mu^
8 + 376 356 342 416 * mu^9 - 17 384 266 580 * mu^10 + 548 066 954 * mu^11 -
11 274 720 * mu^12 + 143 142 * mu^13 - 1032 * mu^14 + 3 * mu^15) * n^2 +
4 * (-9 + mu) * (23 845 345 590 072 960 - 40 269 695 568 954 624 * mu +
30 117 142 128 190 992 * mu^2 - 13 158 415 762 916 400 * mu^3 +
3 730 778 330 679 232 * mu^4 - 721 067 843 021 868 * mu^5 +
97 108 500 711 985 * mu^6 - 9 153 045 269 192 * mu^7 +
597 928 404 668 * mu^8 - 26 432 573 136 * mu^9 + 759 984 806 * mu^10 -
13 416 552 * mu^11 + 134 684 * mu^12 - 676 * mu^13 + mu^14) * n^3 +
16 * (195 243 602 402 676 096 - 325 694 901 477 820 032 * mu +
242 221 596 032 134 128 * mu^2 - 106 098 978 486 724 128 * mu^3 +

```



$$\begin{aligned}
& 30\,459\,989\,915\,673\,992 * \mu^4 - 6\,033\,975\,669\,037\,412 * \mu^5 + \\
& 845\,417\,566\,861\,997 * \mu^6 - 84\,452\,424\,919\,988 * \mu^7 + \\
& 5\,983\,741\,160\,080 * \mu^8 - 295\,319\,349\,276 * \mu^9 + 9\,821\,158\,066 * \mu^{10} - \\
& 208\,695\,676 * \mu^{11} + 2\,610\,160 * \mu^{12} - 16\,816 * \mu^{13} + 41 * \mu^{14}) * n^4 + \\
64 * (-9 + \mu) * (14\,622\,810\,947\,299\,008 - 20\,883\,005\,872\,697\,088 * \mu + \\
13\,042\,640\,269\,010\,160 * \mu^2 - 4\,687\,978\,533\,249\,048 * \mu^3 + \\
1\,073\,821\,472\,622\,084 * \mu^4 - 163\,965\,505\,744\,412 * \mu^5 + 16\,961\,587\,465\,549 * \\
\mu^6 - 1\,184\,203\,363\,074 * \mu^7 + 54\,575\,767\,659 * \mu^8 - 1\,588\,856\,808 * \mu^9 + \\
27\,087\,171 * \mu^{10} - 236\,578 * \mu^{11} + 785 * \mu^{12}) * n^5 + \\
256 * (68\,159\,047\,060\,299\,744 - 96\,275\,531\,839\,385\,520 * \mu + 59\,912\,949\,582\,646\,848 * \\
\mu^2 - 21\,651\,596\,638\,546\,640 * \mu^3 + 5\,041\,402\,403\,618\,604 * \mu^4 - \\
793\,018\,597\,591\,700 * \mu^5 + 85\,896\,040\,596\,299 * \mu^6 - \\
6\,405\,365\,947\,182 * \mu^7 + 323\,083\,532\,589 * \mu^8 - 10\,605\,978\,520 * \mu^9 + \\
211\,271\,829 * \mu^{10} - 2\,240\,614 * \mu^{11} + 9\,319 * \mu^{12}) * n^6 + \\
2048 * (-9 + \mu) * (1\,541\,341\,241\,341\,668 - 1\,813\,373\,921\,002\,968 * \mu + \\
915\,446\,118\,884\,163 * \mu^2 - 259\,814\,716\,685\,092 * \mu^3 + \\
45\,629\,241\,741\,242 * \mu^4 - 5\,143\,009\,129\,752 * \mu^5 + 373\,337\,413\,062 * \mu^6 - \\
17\,038\,328\,436 * \mu^7 + 461\,072\,406 * \mu^8 - 6\,556\,280 * \mu^9 + 36\,355 * \mu^{10}) * \\
n^7 + 8192 * (4\,500\,207\,031\,276\,008 - 5\,239\,264\,901\,634\,576 * \mu + \\
2\,640\,189\,965\,261\,667 * \mu^2 - 755\,987\,780\,804\,488 * \mu^3 + \\
135\,697\,154\,047\,598 * \mu^4 - 15\,878\,627\,119\,200 * \mu^5 + \\
1\,219\,280\,284\,095 * \mu^6 - 60\,190\,646\,760 * \mu^7 + 1\,809\,241\,320 * \mu^8 - \\
29\,487\,896 * \mu^9 + 194\,248 * \mu^{10}) * n^8 + 98\,304 * (-9 + \mu) * \\
(43\,421\,763\,841\,182 - 40\,431\,075\,715\,248 * \mu + 15\,676\,365\,711\,905 * \mu^2 - \\
3\,287\,037\,266\,982 * \mu^3 + 404\,944\,404\,503 * \mu^4 - 29\,779\,385\,976 * \mu^5 + \\
1\,265\,065\,310 * \mu^6 - 28\,070\,508 * \mu^7 + 245\,586 * \mu^8) * n^9 + \\
393\,216 * (81\,960\,492\,523\,446 - 75\,530\,247\,171\,240 * \mu + 29\,304\,166\,747\,543 * \mu^2 - \\
6\,232\,480\,720\,254 * \mu^3 + 791\,707\,261\,321 * \mu^4 - 61\,212\,289\,536 * \mu^5 + \\
2\,795\,677\,186 * \mu^6 - 68\,402\,040 * \mu^7 + 679\,074 * \mu^8) * n^{10} + \\
42\,467\,328 * (-9 + \mu) * (56\,814\,548\,324 - 39\,259\,013\,448 * \mu + \\
10\,716\,187\,369 * \mu^2 - 1\,468\,655\,040 * \mu^3 + \\
105\,783\,400 * \mu^4 - 3\,770\,148 * \mu^5 + 51\,547 * \mu^6) * n^{11} + \\
56\,623\,104 * (206\,001\,269\,260 - 140\,889\,461\,280 * \mu + 38\,650\,036\,817 * \mu^2 - \\
5\,426\,533\,428 * \mu^3 + 409\,650\,908 * \mu^4 - 15\,687\,096 * \mu^5 + 236\,549 * \mu^6) * \\
n^{12} + 42\,807\,066\,624 * (-9 + \mu) * (12\,790\,352 - 5\,830\,560 * \mu + \\
935\,607 * \mu^2 - 61\,962 * \mu^3 + 1425 * \mu^4) * n^{13} + 24\,461\,180\,928 * \\
(65\,459\,144 - 29\,536\,992 * \mu + 4\,812\,879 * \mu^2 - 334\,554 * \mu^3 + 8337 * \mu^4) * \\
n^{14} + 7\,044\,820\,107\,264 * (-9 + \mu) * (6091 - 1374 * \mu + 69 * \mu^2) * n^{15} + \\
7\,044\,820\,107\,264 * (9242 - 2064 * \mu + 111 * \mu^2) * n^{16} + \\
760\,840\,571\,584\,512 * (-9 + \mu) * n^{17} + \\
338\,151\,365\,148\,672 * n^{18}) * \text{myP2fast}[-1 + n, \mu] / \\
((-1 + n) * (-1 + 2 * n) * (-4 + \mu + 4 * n) * (-3 + \mu + 4 * n) * \\
(-2 + \mu + 4 * n) * \\
(-1 + \mu + 4 * n) * \\
(-5 + \mu + 6 * n) * \\
(-3 + \mu + 6 * n) * \\
(-1 + \mu + 6 * n) * \\
(-13 + \mu + 8 * n) *
\end{aligned}$$

```
(2 136 180 - 963 208 * mu + 161 921 * mu^2 - 12 281 * mu^3 + 391 * mu^4 - 3 * mu^5 +
  2 * (-13 + mu) * (298 788 - 89 728 * mu + 8309 * mu^2 - 230 * mu^3 + mu^4) * n +
  8 * (1 475 028 - 447 124 * mu + 45 455 * mu^2 - 1742 * mu^3 + 19 * mu^4) * n^2 +
  192 * (-13 + mu) * (3841 - 582 * mu + 15 * mu^2) * n^3 +
  768 * (5723 - 876 * mu + 30 * mu^2) * n^4 +
  82 944 * (-13 + mu) * n^5 + 110 592 * n^6),
myP2fast[n, mu], OreAlgebra[S[n]]];
```

## Preparing the stage

```
(* Alternative closed form for D_{1,1}(n) *)
{FFeAlt, FFoAlt} =
  {FFe, FFo} /. Pochhammer[j, j - 1] → Pochhammer[1/2, j - 1] * 2^(2 j - 2) /.
    Pochhammer[mu + 2 j + 1, j - 1] / Pochhammer[mu/2 + j + 1/2, j - 1] →
      2^(2 j - 3) * Pochhammer[mu/2 + j + 1, j - 2] / Pochhammer[mu + 3 j, j - 2] /.
    Pochhammer[mu + 2 j, j]^2 / Pochhammer[mu/2 + j + 1/2, j]^2 →
      2^(2 j) * Pochhammer[mu/2 + j, Floor[(j + 1)/2]]^2 /
        Pochhammer[mu/2 + Floor[3 j/2] + 1/2, Floor[(j + 1)/2]]^2 /.
    Pochhammer[mu + 2 j, j] / Pochhammer[mu/2 + j + 1/2, j] →
      2^j * Pochhammer[mu/2 + j, Floor[(j + 1)/2]] /
        Pochhammer[mu/2 + Floor[3 j/2] + 1/2, Floor[(j + 1)/2]] /.
    Pochhammer[mu + 2 j, j] / Pochhammer[mu/2 + j + 1/2, j - 1] →
      2^j * Pochhammer[mu/2 + j, Floor[(j + 1)/2]] /
        Pochhammer[mu/2 + Floor[3 j/2] + 1/2, Floor[(j - 1)/2]] /.
    prod[2^a_ * b_, c_] := Product[2^a, c] * prod[b, c] /.
    prod[a_/2, {j, 1, k - 1}]^2 * If[k == 0, b_, c_] →
      2^(2 - 2 k) * If[k == 0, b/4, c] * prod[a, {j, 1, k - 1}]^2 /.
    2^a_ := 2^Expand[a];

(* Test *)
Table[Together[(FFeAlt/FFe) /. {sum → Sum, prod → Product}], {n, 2, 10, 2}]
Table[Together[(FFoAlt/FFo) /. {sum → Sum, prod → Product}], {n, 1, 9, 2}]
{1, 1, 1, 1, 1}
{1, 1, 1, 1, 1}
```

## even n

```

FFe1 = FFeAlt /. prod[a_, {j, k, b_}] =>
  prod[a, {j, 1, b}]/prod[a, {j, 1, k-1}] * if[k == 0, a /. j -> 0, 1] //.
  prod[a1_, b_] ^ c1_ . * prod[a2_, b_] ^ c2_ . -> prod[a1 ^ c1 * a2 ^ c2, b] //.
  sum[(a_ /; FreeQ[a, k]) * b_, c_] -> a * sum[b, c] //.
  prod[a_*b_, {j, 1, n/2 + c_}] -> prod[a, {j, 1, n/2 + c}] * prod[b, {j, 1, n/2 + c}];
Check0 = FFe1 / (
  (* myC[n] *)
  ((-1) ^ n + 3) / 2 * prod[Floor[i/2]! / i!, {i, 1, n}] *
  (* myE[n,mu] *)
  Pochhammer[mu + 1, n] * prod[
    (mu + 2 i + 6) ^ (2 * Floor[(i + 2) / 3]), {i, 1, Floor[3 / 2 * Floor[(n - 1) / 2] - 2]}] *
  prod[(mu + 2 i + 2 * Floor[3 / 2 * Floor[n / 2 + 1]] - 1) ^
    (2 * Floor[Floor[n / 2] / 2 - (i - 1) / 3] - 1),
    {i, 1, Floor[3 / 2 * Floor[n / 2] - 2]}] *
  (* myF[0,n,mu] *)
  prod[(mu + 2 i + n) ^ (1 - 2 i), {i, 1, Floor[(n - 1) / 4]}] *
  prod[(mu - 2 i + 2 n + 1) ^ (1 - 2 i), {i, 1, Floor[n / 4 - 1]}];
Check0 = Check0 /. {(-1) ^ n -> 1, Floor[n / 2] -> n / 2, Floor[(n - 1) / 2] -> n / 2 - 1} /.
  Floor[a_] -> Floor[Together[a]];

```

 $n = 0 \pmod{4}$ 

```

Check00 =
  Check0 /. {Floor[3 / 4 * (n - 2)] -> 3 / 4 n - 2, Floor[n / 4] -> n / 4, Floor[3 / 4 n] -> 3 / 4 n,
    Floor[(n - 1) / 4] -> n / 4 - 1, Floor[3 / 4 (n + 2)] -> 3 / 4 n + 1,
    Floor[(3 n - 4 i + 4) / 12] -> n / 4 + Floor[(1 - i) / 3]} /.
  prod[Pochhammer[j + mu/2, Floor[(1 + j) / 2]], {j, 1, n/2}] *
  prod[Pochhammer[j + mu/2, Floor[(1 + j) / 2]], {j, 1, -1 + n/2}] /
  (prod[(6 + 2 i + mu) ^ 2 Floor[(2 + i) / 3], {i, 1, -4 + 3 n / 4}] *
  prod[(2 i + mu + n) ^ 1 - 2 i, {i, 1, -1 + n / 4}]) ->
  Pochhammer[mu / 2 + 1, n / 2] * Pochhammer[mu / 2 + 1,
    n / 2 - 1] / 2 ^ (n ^ 2 / 8 - 3 / 4 n + 1) /.
  prod[Pochhammer[1/2 + 2 j + mu/2, -1 + j], {j, 1, n/2}]
  prod[Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1 + n/2}]
  prod[1 / Pochhammer[1/2 + mu/2 + Floor[3 j / 2], Floor[1/2 (-1 + j)]]],
  {j, 1, n/2}] prod[1 / Pochhammer[1/2 + mu/2 + Floor[3 j / 2],

```

$$\begin{aligned} & \text{Floor}\left[\frac{1+j}{2}\right], \left\{j, 1, -1 + \frac{n}{2}\right\} \Big/ \\ & \left( \text{prod}\left[\left(-1 + 2i + \mu + 2\left(1 + \frac{3n}{4}\right)\right)^{-1+2\left(\frac{n}{4} + \text{Floor}\left[\frac{1-i}{3}\right]\right)}, \left\{i, \right. \right. \right. \\ & \quad \left. \left. \left. 1, -2 + \frac{3n}{4}\right\}\right] \text{prod}\left[(1 - 2i + \mu + 2n)^{1-2i}, \left\{i, 1, -1 + \frac{n}{4}\right\}\right] \right) \rightarrow \\ & \text{Pochhammer}\left[\frac{\mu}{2} + \frac{3}{4}n + \frac{1}{2}, \frac{3}{4}n - 1\right] / (\mu + 3) / \\ & 2^{\wedge}(n^{\wedge}2 / 8 - n / 2) /. \\ & \text{prod}\left[\frac{1}{\text{Pochhammer}[j, j]}, \left\{j, 1, \frac{n}{2}\right\}\right] \text{prod}\left[\frac{1}{\text{Pochhammer}[1+j, 1+j]}, \right. \\ & \quad \left. \left\{j, 1, -1 + \frac{n}{2}\right\}\right] / \text{prod}\left[\frac{\text{Floor}\left[\frac{j}{2}\right]!}{i!}, \left\{i, 1, n\right\}\right] \rightarrow 2^{\wedge}(n / 2) /. \\ & (2^{\wedge}a_) \Rightarrow 2^{\wedge}\text{Expand}[a] /. \\ & \text{If} \rightarrow \text{if} //. \text{if}[a_, b1_, c1_] * \text{if}[a_, b2_, c2_] \Rightarrow \\ & \text{if}[a, \text{Together}[b1 * b2], \text{Together}[c1 * c2]] /. \\ & \text{Pochhammer}\left[1 + \frac{\mu}{2}, \frac{n}{2}\right] / \text{Pochhammer}[1 + \mu, n] \rightarrow \\ & 1 / 2^{\wedge}n / \text{Pochhammer}\left[\frac{\mu}{2} + \frac{1}{2}, n / 2\right] /. \\ & (* Now the product expression inside the sum *) \\ & (* We first rewrite this \\ & Pochhammer to separate even and odd factors *) \\ & \text{Pochhammer}[3j + \mu, -2 + j] \rightarrow \text{Pochhammer}\left[\frac{\mu}{2} + \text{Floor}\left[\frac{3}{2}j + \frac{1}{2}\right], \right. \\ & \quad \text{Floor}\left[\frac{j-2}{2}\right] * \text{Pochhammer}\left[\frac{\mu}{2} + \text{Floor}\left[\frac{3}{2}j\right] + \frac{1}{2}, \right. \\ & \quad \left. \left. \text{Floor}\left[\frac{j-1}{2}\right] * 2^{\wedge}(j-2)\right] /. \\ & \text{prod}[a\_Times, b_] \Rightarrow (\text{prod}[\#, b] \& @/a) /. \\ & \text{prod}\left[\frac{1}{\text{Pochhammer}\left[j + \frac{\mu}{2}, \text{Floor}\left[\frac{1+i}{2}\right]\right]^2}, \left\{j, 1, -1+k\right\}\right] \\ & \text{prod}\left[\text{Pochhammer}\left[1+j + \frac{\mu}{2}, -2+j\right]^2, \left\{j, 1, -1+k\right\}\right] \text{prod}\left[1 / \right. \\ & \quad \left. \text{Pochhammer}\left[\frac{\mu}{2} + \text{Floor}\left[\frac{1}{2} + \frac{3j}{2}\right], \text{Floor}\left[\frac{1}{2}(-2+j)\right]\right]^2, \left\{j, 1, -1+k\right\}\right] \rightarrow \\ & \text{if}[k == 0, 4 / \mu^{\wedge}2, 1] / \text{Pochhammer}\left[\frac{\mu}{2} + 1, k-1\right]^{\wedge}2 /. \\ & \text{prod}\left[\text{Pochhammer}\left[\frac{1}{2} + 2j + \frac{\mu}{2}, -1+j\right], \left\{j, 1, -1+k\right\}\right] \\ & \text{prod}\left[\frac{1}{\text{Pochhammer}\left[\frac{3}{2} + 2j + \frac{\mu}{2}, 1+j\right]}, \left\{j, 1, -1+k\right\}\right] \text{prod}\left[ \right. \\ & \quad \left. 1 / \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + \text{Floor}\left[\frac{3j}{2}\right], \text{Floor}\left[\frac{1}{2}(-1+j)\right]\right], \left\{j, 1, -1+k\right\}\right] \\ & \text{prod}\left[\text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + \text{Floor}\left[\frac{3j}{2}\right], \text{Floor}\left[\frac{1+j}{2}\right]\right], \left\{j, 1, -1+k\right\}\right] \rightarrow \\ & \text{if}[k == 0, 1, (\mu + 3) / 2] / \text{Pochhammer}\left[\frac{\mu}{2} + 2k - 1 / 2, k\right] /. \\ & \text{prod}\left[\frac{1}{\text{Pochhammer}\left[\frac{1}{2}, -1+j\right]^2}, \left\{j, 1, -1+k\right\}\right] \text{prod}\left[\text{Pochhammer}[j, j], \left\{j, 1, \right. \right. \\ & \quad \left. \left. -1+k\right\}\right] \text{prod}\left[\text{Pochhammer}[1+j, 1+j], \left\{j, 1, -1+k\right\}\right] \rightarrow \text{if}[k == 0, 1 / 8, 1] * \end{aligned}$$

```

      2 ^ (2 k (k - 1)) * Pochhammer[3 / 2, k - 1] * Pochhammer[1 / 2, k - 1] ^ 2 /.
      prod[2 ^ a_, {j, 1, k - 1}] => With[{cf = Product[2 ^ a, {j, 1, k - 1}]},
      cf * if[k == 0, 1 / (cf /. k -> 0), 1]] // .
      if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] -> if[k == 0, a1 * a2, b1 * b2] /.
      if[k == 0, a_ * b_, a_ * c_] -> a * if[k == 0, b, c] /.
      a_ * sum[b_, c_] -> sum[a * b, c] /.
      Pochhammer[mu / 2 + 1, n / 2 - 1] ->
      Pochhammer[mu / 2 + 1, k - 1] * Pochhammer[mu / 2 + k, n / 2 - k] /.
      Pochhammer[3 / 2, k - 1] * if[k == 0, a_, b_] ->
      if[k == 0, a, b * (2 k - 1)! / (2 k - 2)!!] / 2 ^ (k - 1) /.
      (2 ^ a_) -> 2 ^ FullSimplify[a] /. if[k == 0, a_, b_] -> 4 * if[k == 0, a / 4, b / 4]
      sum[
      (2 ^ (-2 + k + 3 n / 4) if[k == 0, -2 + mu, 1 / (mu^2 * 8 (-2 + 2 k)!!)] Pochhammer[1 / 2, -1 + k]^2 Pochhammer[k + mu / 2,
      -k + n / 2] Pochhammer[-1 + mu, -2 + 3 k] Pochhammer[1 / 2 + mu / 2 + 3 n / 4, -1 + 3 n / 4]) /
      (Pochhammer[1 / 2 + mu / 2, n / 2] Pochhammer[1 + mu / 2, -1 + k] Pochhammer[-1 / 2 + k + mu / 2, -1 + k]
      Pochhammer[-1 / 2 + 2 k + mu / 2, k]), {k, 0, n / 2}]

```

```

Table[Together[(Check00 /. {sum -> Sum, prod -> Product, if -> If}) / myP2[n / 4, mu]],
      {n, 4, 20, 4}]

```

```
{1, 1, 1, 1, 1}
```

```
TraditionalForm[HoldForm@@{Check00} /. {sum -> Sum, prod -> Product, if -> If}]

```

$$\sum_{k=0}^{\frac{n}{2}} \left( 2^{-2+k+\frac{3n}{4}} \text{If}[k=0, \frac{-2+\mu}{\mu^2}, \frac{1}{8(-2+2k)!!}] \left( \left( \frac{1}{2} \right)_{-1+k} \right)^2 \left( k + \frac{\mu}{2} \right)_{-k+\frac{n}{2}} (-1+\mu)_{-2+3k} \left( \frac{1}{2} + \frac{\mu}{2} + \frac{3n}{4} \right)_{-1+\frac{3n}{4}} \right) / \left( \left( \frac{1}{2} + \frac{\mu}{2} \right)_{\frac{n}{2}} \left( 1 + \frac{\mu}{2} \right)_{-1+k} \left( -\frac{1}{2} + k + \frac{\mu}{2} \right)_{-1+k} \left( -\frac{1}{2} + 2k + \frac{\mu}{2} \right)_k \right)$$

```
(* Our expression fits the recurrence (this is the initial value check). *)
```

```
test = ApplyOreOperator[rec2, f[n]];
```

```
Together[Table[test, {n, 0, 4}] /. f[nn_] -> (Check00 /. n -> 4 nn /. sum -> Sum /. if -> If)]
```

```
{0, 0, 0, 0, 0}
```

```
(* the smnd for k=1 to n/2 *)
```

```
smnd = ExpandAll[Check00[[1]] /. n -> 4 n /. if[k == 0, _, a_] -> a]
```

$$\left( 2^{-5+k+3n} \text{Pochhammer}\left[\frac{1}{2}, -1+k\right]^2 \text{Pochhammer}\left[k + \frac{\mu}{2}, -k+2n\right] \text{Pochhammer}[-1+\mu, -2+3k] \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + 3n, -1+3n\right] \right) / \left( (-2+2k)!! \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2}, 2n\right] \text{Pochhammer}\left[1 + \frac{\mu}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2} + k + \frac{\mu}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2} + 2k + \frac{\mu}{2}, k\right] \right)$$

```

Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]
{{(1 + mu + 4 n) (3 + mu + 4 n) (1 + mu + 6 n) (3 + mu + 6 n) (5 + mu + 6 n) S_n -
(mu + 4 n) (2 + mu + 4 n) (-1 + mu + 12 n) (1 + mu + 12 n)
(3 + mu + 12 n) (5 + mu + 12 n) (7 + mu + 12 n) (9 + mu + 12 n)}, {0}}

(* We use the following identity in order
to construct an annihilating operator for the sum *)
TraditionalForm[HoldForm[
0 == Sum[(p1[n] S[n] + p0[n]) * f[n, k], {k, 1, 2 n}] ==
(p1[n] S[n] + p0[n]) * Sum[f[n, k], {k, 1, 2 n}]
- p1[n] * (f[n + 1, 2 n + 1] + f[n + 1, 2 n + 2])
]]
test =
ApplyOreOperator[op, Check00 /. n -> 4 n /. sum[a_, {k, 0, b_}] -> sum[a, {k, 1, b}]] -
LeadingCoefficient[op] *
((smnd /. n -> n + 1 /. k -> 2 n + 1) + (smnd /. n -> n + 1 /. k -> 2 n + 2));
Together[Table[test /. sum -> Sum /. if -> If, {n, 0, 4}]]
0 = Sum_{k=1}^{2n} (p1(n) S(n) + p0(n)) f(n, k) = (p1(n) S(n) + p0(n)) Sum_{k=1}^{2n} f(n, k) - p1(n) (f(n + 1, 2 n + 1) + f(n + 1, 2 n + 2))
{0, 0, 0, 0, 0}

inh = Factor[LeadingCoefficient[op] *
(1 + FullSimplify[(smnd /. n -> n + 1 /. k -> 2 n + 2) / (smnd /. n -> n + 1 /. k -> 2 n + 1)])] *
(smnd /. n -> n + 1 /. k -> 2 n + 1);
rec00 = Annihilator[inh, S[n]][[1]] ** op;

smnd0 = ExpandAll[Together[Check00[[1]] /. n -> 4 n /. if[k == 0, a_, _] -> a /. k -> 0]]
(2^{-2+3 n} Pochhammer[frac{mu}{2}, 2 n] Pochhammer[frac{1}{2} + frac{mu}{2} + 3 n, -1 + 3 n]) /
(mu Pochhammer[frac{1}{2} + frac{mu}{2}, 2 n])

(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec2a. *)
OreReduce[rec00, Annihilator[smnd0, S[n]]]
0

GBEqual[{rec2}, {rec00}]
True

```

## $n = 2 \pmod{4}$

```

Check01 =
Check0 /. Floor[a_] -> (FullSimplify[Floor[a /. n -> 4 l - 2], Element[1, Integers]] /.
Floor[1 + b_] -> Floor[Together[b]] +
1 /. 1 -> (n + 2) / 4) /.

```

```

prod[a_^b_, {i, 1, c_}] := prod[Expand[a]^
  Simplify[b], {i, 1, Expand[c]}] /.
prod[Pochhammer[j + mu/2, Floor[(1+j)/2]], {j, 1, -1 + n/2}] *
  prod[Pochhammer[j + mu/2, Floor[(1+j)/2]], {j, 1, n/2}]/
  (prod[(6 + 2 i + mu)^(2 Floor[(2+i)/3]), {i, 1, -7/2 + 3 n/4}]
  prod[(2 i + mu + n)^(1-2 i), {i, 1, -1/2 + n/4}]) ->
Pochhammer[mu/2 + 1, n/2] * Pochhammer[mu/2 + 1,
  n/2 - 1] / 2^(n^2/8 - 3/4 n + 1) /.
prod[Pochhammer[1/2 + 2 j + mu/2, -1 + j], {j, 1, n/2}]
  prod[Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1 + n/2}]
  prod[1/Pochhammer[1/2 + mu/2 + Floor[3 j/2], Floor[1/2 (-1 + j)]]],
  {j, 1, n/2}] prod[1/Pochhammer[1/2 + mu/2 + Floor[3 j/2],
  Floor[(1+j)/2]], {j, 1, -1 + n/2}]/
  (prod[(2 + 2 i + mu + 3 n/2)^(1/2 (n-4 Ceiling[1/6 + 1/3])), {i, 1, -5/2 + 3 n/4}]
  prod[(1 - 2 i + mu + 2 n)^(1-2 i), {i, 1, -3/2 + n/4}]) ->
Pochhammer[mu/2 + 3/4 n, 3/4 n - 1/2] / (mu + 3) /
  2^(n^2/8 - n/2 - 1/2) /.
prod[1/Pochhammer[j, j], {j, 1, n/2}] prod[1/Pochhammer[1 + j, 1 + j],
  {j, 1, -1 + n/2}]/ prod[Floor[1/2]!, {i, 1, n}] -> 2^(n/2) /.
(2^a_) := 2^Expand[a] /.
If -> if //. if[a_, b1_, c1_] * if[a_, b2_, c2_] :=
  if[a, Together[b1 * b2], Together[c1 * c2]] /.
Pochhammer[1 + mu/2, n/2] / Pochhammer[1 + mu, n] ->
  1/2^n / Pochhammer[mu/2 + 1/2, n/2] /.
(* Now the product expression inside the sum *)
(* We first rewrite this
Pochhammer to separate even and odd factors *)
Pochhammer[3 j + mu, -2 + j] -> Pochhammer[mu/2 + Floor[3/2 j + 1/2],
  Floor[(j - 2)/2]] * Pochhammer[mu/2 + Floor[3/2 j] + 1/2,
  Floor[(j - 1)/2]] * 2^(j - 2) /.
prod[a_Times, b_] := (prod[#, b] & /@ a) /.

```

```

prod[ $\frac{1}{\text{Pochhammer}\left[j + \frac{\mu}{2}, \text{Floor}\left[\frac{1+j}{2}\right]\right]^2}, \{j, 1, -1+k\}$ ]
  prod[ $\text{Pochhammer}\left[1 + j + \frac{\mu}{2}, -2 + j\right]^2, \{j, 1, -1+k\}$ ] prod[1 /
    Pochhammer[ $\frac{\mu}{2} + \text{Floor}\left[\frac{1}{2} + \frac{3j}{2}\right], \text{Floor}\left[\frac{1}{2}(-2 + j)\right]\right]^2, \{j, 1, -1+k\}] \rightarrow$ 
  if[k == 0, 4 / mu^2, 1] / Pochhammer[mu / 2 + 1, k - 1]^2 /.
prod[ $\text{Pochhammer}\left[\frac{1}{2} + 2j + \frac{\mu}{2}, -1 + j\right], \{j, 1, -1+k\}$ ]
  prod[ $\frac{1}{\text{Pochhammer}\left[\frac{3}{2} + 2j + \frac{\mu}{2}, 1 + j\right]}, \{j, 1, -1+k\}$ ] prod[
    1 / Pochhammer[ $\frac{1}{2} + \frac{\mu}{2} + \text{Floor}\left[\frac{3j}{2}\right], \text{Floor}\left[\frac{1}{2}(-1 + j)\right]\right], \{j, 1, -1+k\}]
  prod[ $\text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + \text{Floor}\left[\frac{3j}{2}\right], \text{Floor}\left[\frac{1+j}{2}\right]\right], \{j, 1, -1+k\}] \rightarrow$ 
  if[k == 0, 1, (mu + 3) / 2] / Pochhammer[mu / 2 + 2k - 1 / 2, k] /.
prod[ $\frac{1}{\text{Pochhammer}\left[\frac{1}{2}, -1 + j\right]^2}, \{j, 1, -1+k\}$ ] prod[Pochhammer[j, j], {j, 1,
  -1+k}] prod[Pochhammer[1 + j, 1 + j], {j, 1, -1+k}] \rightarrow if[k == 0, 1 / 8, 1] *
  2^ (2k (k - 1)) * Pochhammer[3 / 2, k - 1] * Pochhammer[1 / 2, k - 1]^2 /.
prod[2^a_, {j, 1, k - 1}] \Rightarrow With[{cf = Product[2^a, {j, 1, k - 1}]},
  cf * if[k == 0, 1 / (cf /. k \to 0), 1]] //.
  if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] \rightarrow if[k == 0, a1 * a2, b1 * b2] /.
  if[k == 0, a_ * b_, a_ * c_] \rightarrow a * if[k == 0, b, c] /.
  a_ * sum[b_, c_] \rightarrow sum[a * b, c] /.
Pochhammer[mu / 2 + 1, n / 2 - 1] \rightarrow
  Pochhammer[mu / 2 + 1, k - 1] * Pochhammer[mu / 2 + k, n / 2 - k] /.
Pochhammer[3 / 2, k - 1] * if[k == 0, a_, b_] \rightarrow
  if[k == 0, a, b * (2k - 1)! / (2k - 2)!] / 2^ (k - 1) /.
(2^a_) \Rightarrow 2^FullSimplify[a] /. if[k == 0, a_, b_] \rightarrow 4 * if[k == 0, a / 4, b / 4]
sum[ $\left(2^{-\frac{3}{2}+k+\frac{3n}{4}} \text{if}[k == 0, \frac{-2 + \mu}{\mu^2}, \frac{1}{8(-2 + 2k)!!}] \text{Pochhammer}\left[\frac{1}{2}, -1 + k\right]^2 \text{Pochhammer}\left[k + \frac{\mu}{2},$ 
   $-k + \frac{n}{2}\right] \text{Pochhammer}[-1 + \mu, -2 + 3k] \text{Pochhammer}\left[\frac{\mu}{2} + \frac{3n}{4}, -\frac{1}{2} + \frac{3n}{4}\right]\right) /$ 
  (Pochhammer[ $\frac{1}{2} + \frac{\mu}{2}, \frac{n}{2}$ ] Pochhammer[ $1 + \frac{\mu}{2}, -1 + k$ ] Pochhammer[ $-\frac{1}{2} + k + \frac{\mu}{2}, -1 + k$ ]
  Pochhammer[ $-\frac{1}{2} + 2k + \frac{\mu}{2}, k$ ]), {k, 0,  $\frac{n}{2}$ }]
Table[Together[(Check01 /. {sum \to Sum, prod \to Product, if \to If}) / myP1[n / 4 + 1 / 2, mu],
  {n, 2, 18, 4}]
{1, 1, 1, 1, 1}$ 
```



TraditionalForm[HoldForm@@{Check01} /. {sum → Sum, prod → Product, if → If}]

$$\sum_{k=0}^{\frac{n}{2}} \left( 2^{-\frac{3}{2}+k+\frac{3n}{4}} \text{If}\left[k=0, \frac{-2+\mu}{\mu^2}, \frac{1}{8(-2+2k)!!}\right] \left(\left(\frac{1}{2}\right)_{-1+k}\right)^2 \left(k+\frac{\mu}{2}\right)_{-k+\frac{n}{2}} (-1+\mu)_{-2+3k} \left(\frac{\mu}{2}+\frac{3n}{4}\right)_{-\frac{1}{2}+\frac{3n}{4}} \right) / \left(\left(\frac{1}{2}+\frac{\mu}{2}\right)_{\frac{n}{2}} \left(1+\frac{\mu}{2}\right)_{-1+k} \left(-\frac{1}{2}+k+\frac{\mu}{2}\right)_{-1+k} \left(-\frac{1}{2}+2k+\frac{\mu}{2}\right)_k\right)$$

(\* Our expression fits the recurrence (this is the initial value check). \*)

test = ApplyOreOperator[rec1, f[n]];

Together[Table[test, {n, 5}] /. f[n\_] => (Check01 /. n → 4 n - 2 /. sum → Sum /. if → If)]

{0, 0, 0, 0, 0}

(\* the smnd for k>=1 \*)

smnd = Check01[[1]] /. n → 4 n - 2 /. if[k == 0, \_, a\_] → a

$$\left( 2^{-\frac{9}{2}+k+\frac{3}{4}(-2+4n)} \text{Pochhammer}\left[\frac{1}{2}, -1+k\right]^2 \text{Pochhammer}\left[k+\frac{\mu}{2}, -k+\frac{1}{2}(-2+4n)\right] \text{Pochhammer}[-1+\mu, -2+3k] \text{Pochhammer}\left[\frac{\mu}{2}+\frac{3}{4}(-2+4n), -\frac{1}{2}+\frac{3}{4}(-2+4n)\right] \right) / \left( (-2+2k)!! \text{Pochhammer}\left[\frac{1}{2}+\frac{\mu}{2}, \frac{1}{2}(-2+4n)\right] \text{Pochhammer}\left[1+\frac{\mu}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2}+k+\frac{\mu}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2}+2k+\frac{\mu}{2}, k\right] \right)$$

Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]

{{(-1+μ+4n)(1+μ+4n)(-3+μ+6n)(-1+μ+6n)(1+μ+6n) S<sub>n</sub> -  
(-2+μ+4n)(μ+4n)(-7+μ+12n)(-5+μ+12n)  
(-3+μ+12n)(-1+μ+12n)(1+μ+12n)(3+μ+12n)}, {0}}

(\* We use the following identity in order

to construct an annihilating operator for the sum \*)

TraditionalForm[HoldForm[

0 == Sum[(p1[n] S[n] + p0[n]) \* f[n, k], {k, 1, 2 n - 1}] ==

(p1[n] S[n] + p0[n]) \* Sum[f[n, k], {k, 1, 2 n - 1}]

- p1[n] \* (f[n+1, 2 n] + f[n+1, 2 n+1])

]]

test =

ApplyOreOperator[op, Check01 /. n → 4 n - 2 /. sum[a\_, {k, 0, b\_}] → sum[a, {k, 1, b}]] -

LeadingCoefficient[op] \*

((smnd /. n → n + 1 /. k → 2 n) + (smnd /. n → n + 1 /. k → 2 n + 1));

Together[Table[test /. sum → Sum /. if → If, {n, 5}]]

$$0 = \sum_{k=1}^{2n-1} (p1(n) S(n) + p0(n)) f(n, k) = (p1(n) S(n) + p0(n)) \sum_{k=1}^{2n-1} f(n, k) - p1(n) (f(n+1, 2n) + f(n+1, 2n+1))$$

{0, 0, 0, 0, 0}

```

inh = Factor[LeadingCoefficient[op] *
  (1 + FullSimplify[(smnd /. n → n + 1 /. k → 2 n + 1) / (smnd /. n → n + 1 /. k → 2 n)])] *
  (smnd /. n → n + 1 /. k → 2 n);
rec01 = Annihilator[inh, S[n]][[1]] ** op;

smnd0 = Together[ExpandAll[Check01[[1]] /. n → 4 n - 2 /. if[k == 0, a_, _] → a /. k → 0]]
  (2-3+3 n Pochhammer[ $\frac{\mu}{2}$ , -1 + 2 n] Pochhammer[ $-\frac{3}{2} + \frac{\mu}{2} + 3 n$ , -2 + 3 n]) /
  (mu Pochhammer[ $\frac{1}{2} + \frac{\mu}{2}$ , -1 + 2 n])

(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec01. *)
OreReduce[rec01, Annihilator[smnd0, S[n]]]
0

GBEqual[{rec1}, rec01]
True

```

## odd n

```

FFo1 = FFoAlt /. prod[a_, {j, k, b_}] ⇒
  prod[a, {j, 1, b}] / prod[a, {j, 1, k - 1}] * if[k == 0, a /. j → 0, 1] //.
  prod[a1_, b_] ^ c1_ * prod[a2_, b_] ^ c2_ → prod[a1 ^ c1 * a2 ^ c2, b] //.
  sum[(a_ /; FreeQ[a, k]) * b_, c_] → a * sum[b, c] //.
  prod[a_ * b_, {j, 1, n / 2 + c_}] → prod[a, {j, 1, n / 2 + c}] * prod[b, {j, 1, n / 2 + c}];
Check1 = FFo1 / (
  (* myC[n] *)
  ((-1) ^ n + 3) / 2 * prod[Floor[i / 2]! / i!, {i, 1, n}] *
  (* myE[n,mu] *)
  Pochhammer[mu + 1, n] * prod[
    (mu + 2 i + 6) ^ (2 * Floor[(i + 2) / 3]), {i, 1, Floor[3 / 2 * Floor[(n - 1) / 2] - 2]}] *
    prod[(mu + 2 i + 2 * Floor[3 / 2 * Floor[n / 2 + 1]] - 1) ^
      (2 * Floor[Floor[n / 2] / 2 - (i - 1) / 3] - 1),
    {i, 1, Floor[3 / 2 * Floor[n / 2] - 2]}] *
  (* myF[1,n,mu] *)
  prod[(mu + 2 i + n + 1) ^ (-2 i), {i, 1, Floor[(n - 1) / 4]}] *
  prod[(mu - 2 i + 2 n - 1) ^ (-2 i), {i, 1, Floor[n / 4 - 1]}] *
  (* remaining factor in myF[n,mu] *)
  Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}]);
Check1 =
  Check1 /. {(-1) ^ n → -1, Floor[n / 2] → (n - 1) / 2, Floor[(n - 1) / 2] → (n - 1) / 2} /.
  Floor[a_] ⇒ Floor[Together[a]];

```

```
(* n=1 and n=3 don't work because of the "remaining factor in myF[n,mu]". *)
Table[Together[(Check1 /. {sum → Sum, prod → Product, if → If})/myP1[(n + 3) / 4, mu]],
  {n, 1, 19, 4}]
Table[Together[(Check1 /. {sum → Sum, prod → Product, if → If})/myP2[(n + 1) / 4, mu]],
  {n, 3, 21, 4}]
{(-1 + mu) (1 + mu), 1, 1, 1, 1}
{5 + mu, 1, 1, 1, 1}
```

## $n = 1 \pmod{4}$

Check10 =

```
Check1 /. Floor[a_] := (FullSimplify[Floor[a /. n → 4 l - 3], Element[1, Integers]] /.
  Floor[1 + b_] := Floor[Together[b]] +
  1 /. 1 → (n + 3) / 4) /.
```

```
prod[a_^b_, {i, 1, c_}] := prod[Expand[a]^
  Simplify[b], {i, 1, Expand[c]}] /.
```

```
prod[Pochhammer[j +  $\frac{\mu}{2}$ , Floor[ $\frac{1+j}{2}$ ]]^2, {j, 1,  $-\frac{1}{2} + \frac{n}{2}$ }] /
```

```
prod[(6 + 2 i + mu)^2 Floor[ $\frac{2+i}{3}$ ], {i, 1,  $-\frac{11}{4} + \frac{3n}{4}$ }] /
```

```
prod[(1 + 2 i + mu + n)^-2 i, {i, 1,  $-\frac{1}{4} + \frac{n}{4}$ }] →
```

```
Pochhammer[mu / 2 + 1, (n - 1) / 2] ^ 2 / 2 ^ ((n ^ 2 - 6 n + 5) / 8) /.
```

```
prod[Pochhammer[- $\frac{1}{2} + 2 j + \frac{\mu}{2}$ , j], {j, 1,  $-\frac{1}{2} + \frac{n}{2}$ }]
```

```
prod[Pochhammer[ $\frac{3}{2} + 2 j + \frac{\mu}{2}$ , 1 + j], {j, 1,  $-\frac{1}{2} + \frac{n}{2}$ }]
```

```
prod[1 / Pochhammer[ $\frac{1}{2} + \frac{\mu}{2} + \text{Floor}[\frac{3j}{2}]$ , Floor[ $\frac{1+j}{2}$ ]]^2, {j,
  1,  $-\frac{1}{2} + \frac{n}{2}$ }] / prod[( $-\frac{1}{2} + 2 i + \mu + \frac{3n}{2}$ )^ $\frac{1}{2}(1+n-4 \text{Ceiling}[\frac{2+i}{3}])$ ,
```

```
{i, 1,  $-\frac{11}{4} + \frac{3n}{4}$ }] / prod[(-1 - 2 i + mu + 2 n)^-2 i,
```

```
{i, 1,  $-\frac{5}{4} + \frac{n}{4}$ }] → Pochhammer[mu / 2 + 3 / 4 (n - 1) + 3 / 2,
```

```
3 / 4 (n - 1) - 2] * Pochhammer[mu / 2 + (n - 1) + 3 / 2,
  (n - 1) / 2 + 1] / (mu + 3) / 2 ^ ((n ^ 2 - 8 n + 15) / 8) /.
```

```
prod[ $\frac{1}{\text{Pochhammer}[j, j]}$ , {j, 1,  $-\frac{1}{2} + \frac{n}{2}$ }] prod[
   $\frac{1}{\text{Pochhammer}[1 + j, 1 + j]}$ , {j, 1,  $-\frac{1}{2} + \frac{n}{2}$ }] /
```

```
prod[ $\frac{\text{Floor}[\frac{i}{2}]!}{i!}$ , {i, 1, n}] → 2 ^ ((n - 1) / 2) /.
```

```
(2 ^ a_) := 2 ^ Expand[a] /.
```

```
If → if /. if[a_, b1_, c1_] * if[a_, b2_, c2_] :=
```

```

if[a, Together[b1 * b2], Together[c1 * c2]] /.
Pochhammer[ $\frac{1}{2} + \frac{\mu}{2} + n, 1 + \frac{1}{2}(-1 + n)$ ] Pochhammer[ $\frac{3}{2} + \frac{\mu}{2} + \frac{3}{4}(-1 + n),$ 
 $-2 + \frac{3}{4}(-1 + n)$ ] / Pochhammer[ $\frac{1}{2}(1 + \mu + 2n), \frac{1}{2}(-5 + n)$ ] →
Pochhammer[ $\frac{3}{2} + \frac{\mu}{2} + \frac{3}{4}(-1 + n), (3n + 1) / 4$ ] /.
Pochhammer[ $1 + \frac{\mu}{2}, \frac{n-1}{2}$ ]^2 / Pochhammer[1 + μ, n] → Pochhammer[
 $\mu / 2 + 1, (n - 1) / 2$ ] / 2^n / Pochhammer[ $\mu / 2 + 1 / 2, (n + 1) / 2$ ] /.
(* Now the product expression inside the sum *)
(* We first rewrite this
Pochhammer to separate even and odd factors *)
Pochhammer[3 j + μ, -2 + j] → Pochhammer[μ / 2 +
Floor[3 / 2 j + 1 / 2], Floor[(j - 2) / 2]] * Pochhammer[
 $\mu / 2 + \text{Floor}[3 / 2 j] + 1 / 2, \text{Floor}[(j - 1) / 2]$ ] * 2^(j - 2) /.
prod[a_Times, b_] := (prod[#, b] & /@ a) /.
prod[ $\frac{1}{\text{Pochhammer}[j + \frac{\mu}{2}, \text{Floor}[\frac{1+j}{2}]]^2}, \{j, 1, -1+k\}$ ] prod[
Pochhammer[ $1 + j + \frac{\mu}{2}, -2 + j$ ], {j, 1, -1+k}] prod[1 /
Pochhammer[ $\frac{\mu}{2} + \text{Floor}[\frac{1}{2} + \frac{3j}{2}], \text{Floor}[\frac{1}{2}(-2 + j)]$ ], {j, 1, -1+k}] →
if[k == 0, 4 / μ^2, 1] / Pochhammer[μ / 2 + 1, k - 1]^2 /.
prod[ $\frac{1}{\text{Pochhammer}[-\frac{1}{2} + 2j + \frac{\mu}{2}, j]}$ , {j, 1, -1+k}]
prod[Pochhammer[ $\frac{1}{2} + 2j + \frac{\mu}{2}, -1 + j$ ], {j, 1, -1+k}]
prod[ $\frac{1}{\text{Pochhammer}[\frac{3}{2} + 2j + \frac{\mu}{2}, 1 + j]}$ , {j, 1, -1+k}] prod[1 /
Pochhammer[ $\frac{1}{2} + \frac{\mu}{2} + \text{Floor}[\frac{3j}{2}], \text{Floor}[\frac{1}{2}(-1 + j)]$ ], {j, 1, -1+k}]
prod[Pochhammer[ $\frac{1}{2} + \frac{\mu}{2} + \text{Floor}[\frac{3j}{2}], \text{Floor}[\frac{1+j}{2}]$ ], {j, 1, -1+k}] →
if[k == 0, 1, (μ + 3) / 2] / Pochhammer[μ / 2 + 2k - 1 / 2, k] /.
prod[ $\frac{1}{\text{Pochhammer}[\frac{1}{2}, -1 + j]^2}, \{j, 1, -1+k\}$ ] prod[Pochhammer[j, j],
{j, 1, -1+k}] prod[Pochhammer[1 + j, 1 + j], {j, 1, -1+k}] →
if[k == 0, 1 / 8, 1] * 2^(2k(k - 1)) * Pochhammer[3 / 2, k - 1] *
Pochhammer[1 / 2, k - 1]^2 /.
prod[2^a_, {j, 1, k - 1}] := With[{cf = Product[2^a, {j, 1, k - 1}]},
cf * if[k == 0, 1 / (cf /. k → 0), 1]] //.
if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] → if[k == 0, a1 * a2, b1 * b2] /.
if[k == 0, a_ * b_, a_ * c_] → a * if[k == 0, b, c] /.
a_ * sum[b_, c_] → sum[a * b, c] /.

```

```

Pochhammer[mu/2 + 1, (n - 1)/2] →
Pochhammer[mu/2 + 1, k - 1] * Pochhammer[mu/2 + k, (n + 1)/2 - k] /.
Pochhammer[3/2, k - 1] * if[k == 0, a_, b_] →
if[k == 0, a, b * (2 k - 1)! / (2 k - 2)!!] / 2^(k - 1) /. (2^a_) := 2^FullSimplify[a] /.
Pochhammer[3/2 + mu/2 + 3/4 (-1 + n), 1/4 (1 + 3 n)] → Pochhammer[3/4 + mu/2 + 3 n/4, 1/4 + 3 n/4] /.
if[k == 0, a_, b_] → 4 * if[k == 0, a/4, b/4]
sum[ (2^(-3/4+k+3n/4) if[k == 0, -2+mu, 1/(8(-2+2k)!!)] Pochhammer[1/2, -1+k]^2 Pochhammer[k + mu/2,
-k + (1+n)/2] Pochhammer[-1+mu, -2+3k] Pochhammer[3/4 + mu/2 + 3n/4, 1/4 + 3n/4]) /
(Pochhammer[1/2 + mu/2, (1+n)/2] Pochhammer[1 + mu/2, -1+k] Pochhammer[-1/2 + k + mu/2, -1+k]
Pochhammer[-1/2 + 2k + mu/2, k]), {k, 0, (1+n)/2} ]

```

```

Table[Factor[(Check10 /. {sum → Sum, prod → Product, if → If}) / myP1[(n + 3)/4, mu]],
{n, 1, 19, 4}]
{1, 1, 1, 1, 1}

```

```

TraditionalForm[HoldForm@@{Check10} /. {sum → Sum, prod → Product, if → If}]

```

$$\sum_{k=0}^{\frac{1+n}{2}} \left( 2^{-\frac{3}{4}+k+\frac{3n}{4}} \text{If}[k=0, \frac{-2+\mu}{\mu^2}, \frac{1}{8(-2+2k)!!}] \left( \left( \frac{1}{2} \right)_{-1+k} \right)^2 \left( k + \frac{\mu}{2} \right)_{-k+\frac{1+n}{2}} (-1+\mu)_{-2+3k} \left( \frac{3}{4} + \frac{\mu}{2} + \frac{3n}{4} \right)_{\frac{1}{4}+\frac{3n}{4}} \right) /$$

$$\left( \left( \frac{1}{2} + \frac{\mu}{2} \right)_{\frac{1+n}{2}} \left( 1 + \frac{\mu}{2} \right)_{-1+k} \left( -\frac{1}{2} + k + \frac{\mu}{2} \right)_{-1+k} \left( -\frac{1}{2} + 2k + \frac{\mu}{2} \right)_k \right)$$

(\* Our expression fits the recurrence (this is the initial value check). \*)

```
test = ApplyOreOperator[rec1, f[n]];
```

```
Together[Table[test, {n, 5}] /. f[n_] := (Check10 /. n → 4 n n - 3 /. sum → Sum /. if → If)]
```

```
{0, 0, 0, 0, 0}
```

(\* the smnd for k>=1 \*)

```
smnd = Check10[[1]] /. n → 4 n - 3 /. if[k == 0, _, a_] → a /.
```

```
Pochhammer[a_, b_] := Pochhammer@@Expand[{a, b}]
```

$$\left( 2^{-\frac{15}{4}+k+\frac{3}{4}(-3+4n)} \text{Pochhammer}\left[\frac{1}{2}, -1+k\right]^2 \text{Pochhammer}\left[k + \frac{\mu}{2}, -1-k+2n\right] \right.$$

$$\left. \text{Pochhammer}[-1+\mu, -2+3k] \text{Pochhammer}\left[-\frac{3}{2} + \frac{\mu}{2} + 3n, -2+3n\right] \right) /$$

$$\left( (-2+2k)!! \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2}, -1+2n\right] \text{Pochhammer}\left[1 + \frac{\mu}{2}, -1+k\right] \right.$$

$$\left. \text{Pochhammer}\left[-\frac{1}{2} + k + \frac{\mu}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2} + 2k + \frac{\mu}{2}, k\right] \right)$$

```

Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]
{{{(-1 + mu + 4 n) (1 + mu + 4 n) (-3 + mu + 6 n) (-1 + mu + 6 n) (1 + mu + 6 n) S_n -
(-2 + mu + 4 n) (mu + 4 n) (-7 + mu + 12 n) (-5 + mu + 12 n)
(-3 + mu + 12 n) (-1 + mu + 12 n) (1 + mu + 12 n) (3 + mu + 12 n)}, {0}}

(* We use the following identity in order
to construct an annihilating operator for the sum *)
TraditionalForm[HoldForm[
0 == Sum[(p1[n] S[n] + p0[n]) * f[n, k], {k, 1, 2 n - 1}] ==
(p1[n] S[n] + p0[n]) * Sum[f[n, k], {k, 1, 2 n - 1}]
- p1[n] * (f[n + 1, 2 n] + f[n + 1, 2 n + 1])
]]
test =
ApplyOreOperator[op, Check10 /. n -> 4 n - 3 /. sum[a_, {k, 0, b_}] -> sum[a, {k, 1, b}]] -
LeadingCoefficient[op] *
((smnd /. n -> n + 1 /. k -> 2 n) + (smnd /. n -> n + 1 /. k -> 2 n + 1));
Together[Table[test /. sum -> Sum /. if -> If, {n, 5}]]
0 =  $\sum_{k=1}^{2n-1} (p1(n) S(n) + p0(n)) f(n, k) = (p1(n) S(n) + p0(n)) \sum_{k=1}^{2n-1} f(n, k) - p1(n) (f(n+1, 2n) + f(n+1, 2n+1))$ 
{0, 0, 0, 0, 0}

inh = Factor[LeadingCoefficient[op] *
(1 + FullSimplify[(smnd /. n -> n + 1 /. k -> 2 n + 1) / (smnd /. n -> n + 1 /. k -> 2 n)])] *
(smnd /. n -> n + 1 /. k -> 2 n);
rec10 = Annihilator[inh, S[n]][[1]] ** op;

smnd0 = Together[ExpandAll[Check10[[1]] /. n -> 4 n - 3 /. if[k == 0, a_, _] -> a /. k -> 0]]
(2-3+3 n Pochhammer[ $\frac{\mu}{2}$ , -1 + 2 n] Pochhammer[ $-\frac{3}{2} + \frac{\mu}{2} + 3 n$ , -2 + 3 n]) /
(mu Pochhammer[ $\frac{1}{2} + \frac{\mu}{2}$ , -1 + 2 n])

(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec01. *)
OreReduce[rec10, Annihilator[smnd0, S[n]]]
0

GBEqual[{rec1}, rec10]
True

```

## $n = 3 \bmod 4$

```

Check11 =
Check1 /. Floor[a_] -> (FullSimplify[Floor[a /. n -> 4 l - 1], Element[1, Integers]] /.
Floor[1 + b_] -> Floor[Together[b]] +
1 /. 1 -> (n + 1) / 4) /.

```

```

prod[a_^b_, {i, 1, c_}] := prod[Expand[a]^
  Simplify[b], {i, 1, Expand[c]}] /.
prod[Pochhammer[j + mu/2, Floor[(1+j)/2]]^2, {j, 1, -1/2 + n/2}]/
  prod[(6 + 2 i + mu)^2 Floor[(2+i)/3], {i, 1, -13/4 + 3 n/4}]/
  prod[(1 + 2 i + mu + n)^-2 i, {i, 1, -3/4 + n/4}] ->
  Pochhammer[mu/2 + 1, (n - 1)/2]^2/2^(n^2 - 6 n + 9)/8) /.
prod[Pochhammer[-1/2 + 2 j + mu/2, j], {j, 1, -1/2 + n/2}]
prod[Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1/2 + n/2}]
prod[1/Pochhammer[1/2 + mu/2 + Floor[3 j/2], Floor[(1+j)/2]]^2, {j,
  1, -1/2 + n/2}]/prod[(1/2 + 2 i + mu + 3 n/2)^(1/2 (-1+n-4 Ceiling[1/6 + i/3])),
  {i, 1, -13/4 + 3 n/4}]/prod[(-1 - 2 i + mu + 2 n)^-2 i,
  {i, 1, -7/4 + n/4}] -> Pochhammer[mu/2 + 3/4 n + 5/4,
  (3 n - 13)/4] Pochhammer[mu/2 + n + 1/2, (n + 1)/2]/
  (mu + 3)/2^(n^2 - 8 n + 15)/8) /.
prod[1/Pochhammer[j, j], {j, 1, -1/2 + n/2}] prod[
  1/Pochhammer[1 + j, 1 + j], {j, 1, -1/2 + n/2}]/
  prod[Floor[i/2]!
  i!, {i, 1, n}] -> 2^(n - 1)/2) /.
(2^a_) := 2^Expand[a] /.
If -> if //. if[a_, b1_, c1_] * if[a_, b2_, c2_] :=
if[a, Together[b1 * b2], Together[c1 * c2]] /.
Pochhammer[1/2 + mu/2 + n, (1+n)/2] Pochhammer[5/4 + mu/2 + 3 n/4,
  1/4 (-13 + 3 n)]/Pochhammer[1/2 (1 + mu + 2 n), 1/2 (-5 + n)] ->
Pochhammer[5/4 + mu/2 + 3 n/4, 3 n/4 - 1/4] /.
Pochhammer[1 + mu/2, (n-1)/2]^2/Pochhammer[1 + mu, n] -> Pochhammer[
  mu/2 + 1, (n - 1)/2]/2^n/Pochhammer[mu/2 + 1/2, (n + 1)/2] /.
(* Now the product expression inside the sum *)
(* We first rewrite this
Pochhammer to separate even and odd factors *)
Pochhammer[3 j + mu, -2 + j] -> Pochhammer[mu/2 + Floor[3/2 j + 1/2],
  Floor[(j - 2)/2]] * Pochhammer[mu/2 + Floor[3/2 j] + 1/2,
  Floor[(j - 1)/2]] * 2^(j - 2) /.

```

```

prod[a_Times, b_] := (prod[#, b] & /@ a) /.
prod[ $\frac{1}{\text{Pochhammer}\left[j + \frac{\mu}{2}, \text{Floor}\left[\frac{1+i}{2}\right]\right]^2}, \{j, 1, -1+k\}$ ]
  prod[Pochhammer[ $1 + j + \frac{\mu}{2}, -2 + j$ ], {j, 1, -1+k}] prod[1/
    Pochhammer[ $\frac{\mu}{2} + \text{Floor}\left[\frac{1}{2} + \frac{3j}{2}\right], \text{Floor}\left[\frac{1}{2}(-2 + j)\right]$ ], {j, 1, -1+k}] →
  if[k == 0, 4 / mu^2, 1] / Pochhammer[mu / 2 + 1, k - 1]^2 /.
prod[ $\frac{1}{\text{Pochhammer}\left[-\frac{1}{2} + 2j + \frac{\mu}{2}, j\right]}$ , {j, 1, -1+k}]
  prod[Pochhammer[ $\frac{1}{2} + 2j + \frac{\mu}{2}, -1 + j$ ], {j, 1, -1+k}]
  prod[ $\frac{1}{\text{Pochhammer}\left[\frac{3}{2} + 2j + \frac{\mu}{2}, 1 + j\right]}$ , {j, 1, -1+k}] prod[
    1 / Pochhammer[ $\frac{1}{2} + \frac{\mu}{2} + \text{Floor}\left[\frac{3j}{2}\right], \text{Floor}\left[\frac{1}{2}(-1 + j)\right]$ ], {j, 1, -1+k}]
  prod[Pochhammer[ $\frac{1}{2} + \frac{\mu}{2} + \text{Floor}\left[\frac{3j}{2}\right], \text{Floor}\left[\frac{1+j}{2}\right]$ ], {j, 1, -1+k}] →
  if[k == 0, 1, (mu + 3) / 2] / Pochhammer[mu / 2 + 2k - 1 / 2, k] /.
prod[ $\frac{1}{\text{Pochhammer}\left[\frac{1}{2}, -1 + j\right]^2}, \{j, 1, -1+k\}$ ] prod[Pochhammer[j, j], {j, 1,
  -1+k}] prod[Pochhammer[1 + j, 1 + j], {j, 1, -1+k}] → if[k == 0, 1 / 8, 1] *
  2^ (2k (k - 1)) * Pochhammer[3 / 2, k - 1] * Pochhammer[1 / 2, k - 1]^2 /.
prod[2^a_, {j, 1, k - 1}] := With[{cf = Product[2^a, {j, 1, k - 1}]},
  cf * if[k == 0, 1 / (cf /. k → 0), 1]] //.
if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] → if[k == 0, a1 * a2, b1 * b2] /.
if[k == 0, a_ * b_, a_ * c_] → a * if[k == 0, b, c] /.
a_ * sum[b_, c_] → sum[a * b, c] /.
Pochhammer[mu / 2 + 1, (n - 1) / 2] →
  Pochhammer[mu / 2 + 1, k - 1] * Pochhammer[mu / 2 + k, (n + 1) / 2 - k] /.
Pochhammer[3 / 2, k - 1] * if[k == 0, a_, b_] →
  if[k == 0, a, b * (2k - 1)! / (2k - 2)!] / 2^(k - 1) /.
(2^a_) := 2^FullSimplify[a] /. if[k == 0, a_, b_] → 4 * if[k == 0, a / 4, b / 4]
sum[ $\left(2^{-\frac{5}{4}k + \frac{3n}{4}} \text{if}[k == 0, \frac{-2 + \mu}{\mu^2}, \frac{1}{8(-2 + 2k)!!}] \text{Pochhammer}\left[\frac{1}{2}, -1 + k\right]^2 \text{Pochhammer}\left[k + \frac{\mu}{2}, -k + \frac{1+n}{2}\right] \text{Pochhammer}[-1 + \mu, -2 + 3k] \text{Pochhammer}\left[\frac{5}{4} + \frac{\mu}{2} + \frac{3n}{4}, -\frac{1}{4} + \frac{3n}{4}\right]\right) /$ 
  (Pochhammer[ $\frac{1}{2} + \frac{\mu}{2}, \frac{1+n}{2}$ ] Pochhammer[ $1 + \frac{\mu}{2}, -1 + k$ ] Pochhammer[ $-\frac{1}{2} + k + \frac{\mu}{2}, -1 + k$ ]
  Pochhammer[ $-\frac{1}{2} + 2k + \frac{\mu}{2}, k$ ]), {k, 0,  $\frac{1+n}{2}$ }]

```



```
Table[Factor[(Check11 /. {sum -> Sum, prod -> Product, if -> If}) / myP2[(n + 1) / 4, mu]],
  {n, 3, 23, 4}]
{1, 1, 1, 1, 1, 1}
```

```
TraditionalForm[HoldForm@@{Check11} /. {sum -> Sum, prod -> Product, if -> If}]
```

$$\sum_{k=0}^{\frac{1+n}{2}} \left( 2^{-\frac{5}{4}k + \frac{3n}{4}} \text{If}[k=0, \frac{-2+\mu}{\mu^2}, \frac{1}{8(-2+2k)!!}] \left( \left( \frac{1}{2} \right)_{-1+k} \right)^2 \left( k + \frac{\mu}{2} \right)_{-k + \frac{1+n}{2}} (-1+\mu)_{-2+3k} \left( \frac{5}{4} + \frac{\mu}{2} + \frac{3n}{4} \right)_{-\frac{1}{4} + \frac{3n}{4}} \right) /$$

$$\left( \left( \frac{1}{2} + \frac{\mu}{2} \right)_{\frac{1+n}{2}} \left( 1 + \frac{\mu}{2} \right)_{-1+k} \left( -\frac{1}{2} + k + \frac{\mu}{2} \right)_{-1+k} \left( -\frac{1}{2} + 2k + \frac{\mu}{2} \right)_k \right)$$

(\* Our expression fits the recurrence (this is the initial value check). \*)

```
test = ApplyOreOperator[rec2, f[n]];
```

```
Together[Table[test, {n, 5}] /. f[nn_] -> (Check11 /. n -> 4 nn - 1 /. sum -> Sum /. if -> If)]
```

```
{0, 0, 0, 0, 0}
```

(\* the smnd for k>=1 \*)

```
smnd = Check11[[1]] /. n -> 4 n - 1 /. if[k == 0, _, a_] -> a /.
```

```
Pochhammer[a_, b_] -> Pochhammer@@Expand[{a, b}]
```

$$\left( 2^{-\frac{17}{4}k + \frac{3}{4}(-1+4n)} \text{Pochhammer}\left[\frac{1}{2}, -1+k\right]^2 \text{Pochhammer}\left[k + \frac{\mu}{2}, -k+2n\right] \right.$$

$$\left. \text{Pochhammer}[-1+\mu, -2+3k] \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + 3n, -1+3n\right] \right) /$$

$$\left( (-2+2k)!! \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2}, 2n\right] \text{Pochhammer}\left[1 + \frac{\mu}{2}, -1+k\right] \right.$$

$$\left. \text{Pochhammer}\left[-\frac{1}{2} + k + \frac{\mu}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2} + 2k + \frac{\mu}{2}, k\right] \right)$$

```
Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]
```

```
{{(1 + mu + 4 n) (3 + mu + 4 n) (1 + mu + 6 n) (3 + mu + 6 n) (5 + mu + 6 n) S_n -
(mu + 4 n) (2 + mu + 4 n) (-1 + mu + 12 n) (1 + mu + 12 n)
(3 + mu + 12 n) (5 + mu + 12 n) (7 + mu + 12 n) (9 + mu + 12 n)}, {0}}
```

```

(* We use the following identity in order
to construct an annihilating operator for the sum *)
TraditionalForm[HoldForm[
  0 == Sum[(p1[n] S[n] + p0[n]) * f[n, k], {k, 1, 2 n}] ==
  (p1[n] S[n] + p0[n]) * Sum[f[n, k], {k, 1, 2 n}]
  - p1[n] * (f[n + 1, 2 n + 1] + f[n + 1, 2 n + 2])
]]
test =
  ApplyOreOperator[op, Check11 /. n -> 4 n - 1 /. sum[a_, {k, 0, b_}] -> sum[a, {k, 1, b}]] -
  LeadingCoefficient[op] *
  ((smnd /. n -> n + 1 /. k -> 2 n + 1) + (smnd /. n -> n + 1 /. k -> 2 n + 2));
Together[Table[test /. sum -> Sum /. if -> If, {n, 5}]]
0 =  $\sum_{k=1}^{2n} (p1(n) S(n) + p0(n)) f(n, k) = (p1(n) S(n) + p0(n)) \sum_{k=1}^{2n} f(n, k) - p1(n) (f(n+1, 2n+1) + f(n+1, 2n+2))$ 
{0, 0, 0, 0, 0}

inh = Factor[LeadingCoefficient[op] *
  (1 + FullSimplify[(smnd /. n -> n + 1 /. k -> 2 n + 2) / (smnd /. n -> n + 1 /. k -> 2 n + 1)])] *
  (smnd /. n -> n + 1 /. k -> 2 n + 1);
rec11 = Annihilator[inh, S[n]][[1]] ** op;

smnd0 = Together[ExpandAll[Check11[[1]] /. n -> 4 n - 1 /. if[k == 0, a_, _] -> a /. k -> 0]]
 $\left(2^{-2+3n} \text{Pochhammer}\left[\frac{\mu}{2}, 2n\right] \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + 3n, -1 + 3n\right]\right) /$ 
 $\left(\mu \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2}, 2n\right]\right)$ 

(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec01. *)
OreReduce[rec11, Annihilator[smnd0, S[n]]]
0

GBEqual[{rec2}, rec11]
True

```

## Summarize

$$\begin{aligned}
& \{\text{Check00, Check01, Check10, Check11}\} /. \\
& (\text{if}[k == 0, (\mu - 2) / \mu^2, 1 / 8 / (2k - 2) !!] * \text{Pochhammer}[1 / 2, k - 1]^2 * \\
& \quad \text{Pochhammer}[\mu - 1, 3k - 2] / \text{Pochhammer}[\mu / 2 + 1, k - 1] / \text{Pochhammer}[ \\
& \quad \mu / 2 + k - 1 / 2, k - 1] / \text{Pochhammer}[\mu / 2 + 2k - 1 / 2, k]) \rightarrow (\text{h}[n, k] / 2^k) \\
& \left\{ \text{sum} \left[ \left( 2^{-2 + \frac{3n}{4}} \text{h}[n, k] \text{Pochhammer} \left[ k + \frac{\mu}{2}, -k + \frac{n}{2} \right] \text{Pochhammer} \left[ \frac{1}{2} + \frac{\mu}{2} + \frac{3n}{4}, -1 + \frac{3n}{4} \right] \right) / \right. \right. \\
& \quad \left. \left. \text{Pochhammer} \left[ \frac{1}{2} + \frac{\mu}{2}, \frac{n}{2} \right], \left\{ k, 0, \frac{n}{2} \right\} \right] \right\}, \\
& \left\{ \text{sum} \left[ \left( 2^{-\frac{3}{2} + \frac{3n}{4}} \text{h}[n, k] \text{Pochhammer} \left[ k + \frac{\mu}{2}, -k + \frac{n}{2} \right] \text{Pochhammer} \left[ \frac{\mu}{2} + \frac{3n}{4}, -\frac{1}{2} + \frac{3n}{4} \right] \right) / \right. \right. \\
& \quad \left. \left. \text{Pochhammer} \left[ \frac{1}{2} + \frac{\mu}{2}, \frac{n}{2} \right], \left\{ k, 0, \frac{n}{2} \right\} \right] \right\}, \\
& \left\{ \text{sum} \left[ \left( 2^{-\frac{3}{4} + \frac{3n}{4}} \text{h}[n, k] \text{Pochhammer} \left[ k + \frac{\mu}{2}, -k + \frac{1+n}{2} \right] \text{Pochhammer} \left[ \frac{3}{4} + \frac{\mu}{2} + \frac{3n}{4}, \frac{1}{4} + \frac{3n}{4} \right] \right) / \right. \right. \\
& \quad \left. \left. \text{Pochhammer} \left[ \frac{1}{2} + \frac{\mu}{2}, \frac{1+n}{2} \right], \left\{ k, 0, \frac{1+n}{2} \right\} \right] \right\}, \\
& \left\{ \text{sum} \left[ \left( 2^{-\frac{5}{4} + \frac{3n}{4}} \text{h}[n, k] \text{Pochhammer} \left[ k + \frac{\mu}{2}, -k + \frac{1+n}{2} \right] \text{Pochhammer} \left[ \frac{5}{4} + \frac{\mu}{2} + \frac{3n}{4}, -\frac{1}{4} + \frac{3n}{4} \right] \right) / \right. \right. \\
& \quad \left. \left. \text{Pochhammer} \left[ \frac{1}{2} + \frac{\mu}{2}, \frac{1+n}{2} \right], \left\{ k, 0, \frac{1+n}{2} \right\} \right] \right\}
\end{aligned}$$

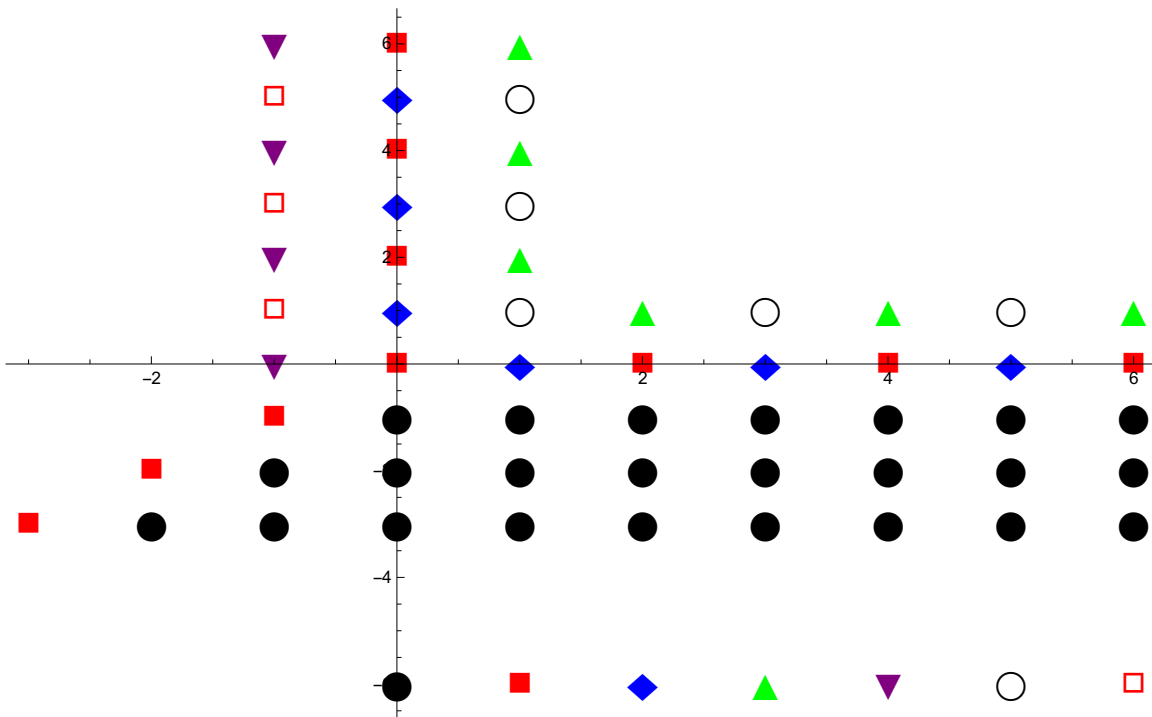
## Section 6: The General Determinant

### Overview

```

Do[Fam[i] = {}, {i, 0, 6}];
NiceQ[x_] :=
  Max[First /@ TimeConstrained[FactorInteger[x], 0.1, {{10^6, 0}}]] < 100;
Do[
  tt = Table[Det[DstMat[s, t, n, 37]], {n, 17, 19}];
  Which[
    MatchQ[tt, {(0) ..}], AppendTo[Fam[0], {s, t}],
    MatchQ[tt, {_?NiceQ, 0, _?NiceQ}], AppendTo[Fam[2], {s, t}],
    MatchQ[tt, {( _?NiceQ) ..}], AppendTo[Fam[1], {s, t}],
    NiceQ[tt[[2]]] && s < 0, AppendTo[Fam[4], {s, t}],
    NiceQ[tt[[2]]], AppendTo[Fam[3], {s, t}],
    tt = Rest[tt] / Most[tt];
    NiceQ[tt[[1]]], AppendTo[Fam[5], {s, t}],
    NiceQ[tt[[2]]], AppendTo[Fam[6], {s, t}]
  ];
  , {s, -3, 6}, {t, -3, 6}];
ListPlot[Table[Append[Fam[i], {i, -6}], {i, 0, 6}],
  PlotStyle -> {Black, Red, Blue, Green, Purple},
  PlotMarkers -> {Automatic, Large}, ImageSize -> 600]

```



Corollary 15:  $D_{-r,-r}(n)$ 

```
TableForm[DstMat[-2, -2, 10, 37]]
```

1	0	1	34	595	7140	66 045	501 942	3 262 623	18 643 560
0	1	1	35	630	7770	73 815	575 757	3 838 380	22 481 940
0	0	2	36	666	8436	82 251	658 008	4 496 388	26 978 328
0	0	1	38	703	9139	91 390	749 398	5 245 786	32 224 114
0	0	1	38	742	9880	101 270	850 668	6 096 454	38 320 568
0	0	1	39	780	10 661	111 930	962 598	7 059 052	45 379 620
0	0	1	40	820	11 480	123 411	1 086 008	8 145 060	53 524 680
0	0	1	41	861	12 341	135 751	1 221 760	9 366 819	62 891 499
0	0	1	42	903	13 244	148 995	1 370 754	10 737 574	73 629 072
0	0	1	43	946	14 190	163 185	1 533 939	12 271 512	85 900 585

```
TableForm[DstMat[-3, -3, 10, 37]]
```

1	0	0	1	33	561	6545	58 905	435 897	2 760 681
0	1	0	1	34	595	7140	66 045	501 942	3 262 623
0	0	1	1	35	630	7770	73 815	575 757	3 838 380
0	0	0	2	36	666	8436	82 251	658 008	4 496 388
0	0	0	1	38	703	9139	91 390	749 398	5 245 786
0	0	0	1	38	742	9880	101 270	850 668	6 096 454
0	0	0	1	39	780	10 661	111 930	962 598	7 059 052
0	0	0	1	40	820	11 480	123 411	1 086 008	8 145 060
0	0	0	1	41	861	12 341	135 751	1 221 760	9 366 819
0	0	0	1	42	903	13 244	148 995	1 370 754	10 737 574

```
Union[Flatten[Table[Factor[Dst[r, r, n + 1] / Dst[r + 1, r + 1, n]], {r, -5, -1}, {n, 10}]]]
{1}
```

```
Table[Together[Dst[-r, -r, n] / If[r < n, Dst[0, 0, n - r], 1]], {r, 0, 5}, {n, 10}]
{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}
```

Proposition 16:  $D_{s,t}(n) = 0$  for  $t \leq -1$  and  $s \geq t + 1$ 

```
TableForm[DstMat[0, -1, 8, 31]]
```

0	2	30	465	4960	40 920	278 256	1 623 160
0	1	32	496	5456	46 376	324 632	1 947 792
0	1	32	529	5984	52 360	376 992	2 324 784
0	1	33	561	6546	58 905	435 897	2 760 681
0	1	34	595	7140	66 046	501 942	3 262 623
0	1	35	630	7770	73 815	575 758	3 838 380
0	1	36	666	8436	82 251	658 008	4 496 389
0	1	37	703	9139	91 390	749 398	5 245 786

```
Union[Flatten[Table[Dst[s, t, n], {t, -1, -5, -1}, {s, t + 1, 6}, {n, 10}]]]
{0}
```



## Theorem 18 (Family A): $D_{2r,0}(n)$ and $D_{0,2r}(n)$

```

quoAe = Pochhammer[mu + 2 n + 4 r, n - r] *
  Pochhammer[mu / 2 + 2 n + r + 1 / 2, n - r - 1] / Pochhammer[n - r, n - r] /
  Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r - 1];
quoAo = Pochhammer[mu + 2 n + 4 r - 2, n - r - 1] * Pochhammer[mu / 2 + 2 n + r - 1 / 2, n - r] /
  Pochhammer[n - r, n - r] / Pochhammer[mu / 2 + n + 2 r - 1 / 2, n - r - 1];
Table[Together[{Dst[2 r, 0, 2 n + 1] / Dst[2 r, 0, 2 n] / quoAe,
  Dst[2 r, 0, 2 n] / Dst[2 r, 0, 2 n - 1] / quoAo}], {n, 5}, {r, 0, n - 1}]
{{{1, 1}}, {{1, 1}, {1, 1}}, {{1, 1}, {1, 1}, {1, 1}},
  {{1, 1}, {1, 1}, {1, 1}, {1, 1}}, {{1, 1}, {1, 1}, {1, 1}, {1, 1}, {1, 1}}}

RA[n_] := If[EvenQ[n],
  Pochhammer[mu + n + 4 r, n / 2 - r] *
  Pochhammer[mu / 2 + n + r + 1 / 2, n / 2 - r - 1] / Pochhammer[n / 2 - r, n / 2 - r] /
  Pochhammer[mu / 2 + n / 2 + 2 r + 1 / 2, n / 2 - r - 1],
  Pochhammer[mu + n + 4 r - 1, (n + 1) / 2 - r - 1] *
  Pochhammer[mu / 2 + n + r + 1 / 2, (n + 1) / 2 - r] / Pochhammer[(n + 1) / 2 - r,
    (n + 1) / 2 - r] / Pochhammer[mu / 2 + (n + 1) / 2 + 2 r - 1 / 2, (n + 1) / 2 - r - 1]];
Table[Together[Dst[2 r, 0, n + 1] / Dst[2 r, 0, n] / RA[n]], {n, 10}, {r, 0, (n - 1) / 2}]
{{1}, {1}, {1, 1}, {1, 1}, {1, 1, 1}, {1, 1, 1},
  {1, 1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}

{quoAei, quoAoi} = {quoAe /. n -> i / 2, quoAo /. n -> (i + 1) / 2};
Table[Together[Dst[2 r, 0, n] / If[n <= 2 r, 1,
  2 * Product[If[Mod[i, 2] == 0, quoAei, quoAoi], {i, 2 r + 1, n - 1}]]], {n, 8}, {r, 8}]
{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
  {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
  {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}

Table[Dst[2 r, 0, n], {r, 0, 4}, {n, 2 r}]
{{}, {1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}

Union[Flatten[Table[Dst[2 r, 0, n, mu] - Dst[0, 0, n - 2 r, mu + 6 r],
  {r, 4}, {n, 2 r + 1, 2 r + 5}, {mu, -2, 6}]]]
{0}

Table[
  Together[Dst[0, 2 r, n] / (Product[Pochhammer[mu + i - 1, 2 r] / Pochhammer[i + 1, 2 r],
    {i, 0, n - 1}] * Dst[2 r, 0, n]), {n, 8}, {r, 8}]
  {{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
    {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
    {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}}

```

(\* Special case: Proposition 8 \*)

(quoAe /. r -> 0) / R00e[n]

1

(\* Special case: Proposition 8 \*)

(quoAo /. r -> 0) / R00o[n]

1

(\* Special case: Lemma 6 \*)

(quoAo /. r -> 1) / R20[n]

$$\left( \text{Pochhammer}[n, -1 + n] \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + n, -1 + n\right] \text{Pochhammer}[2 + \mu + 2n, -2 + n] \right) /$$

$$\left( \text{Pochhammer}[-1 + n, -1 + n] \text{Pochhammer}\left[\frac{3}{2} + \frac{\mu}{2} + n, -2 + n\right] \text{Pochhammer}[1 + \mu + 2n, -1 + n] \right)$$

FullSimplify[%]

1

(\* Special case: Lemma 7 \*)

((quoAo /. r -> 1) \* (mu + 2n - 2) \* (mu + 2n - 1) / (2n) / (2n + 1)) / R02[n]

$$\left( (-2 + \mu + 2n) (-1 + \mu + 2n) (\mu + 2n) \text{Pochhammer}[n, 2 + n] \right.$$

$$\left. \text{Pochhammer}\left[\frac{1}{2} + \frac{\mu}{2} + n, -1 + n\right] \text{Pochhammer}[2 + \mu + 2n, -2 + n] \right) /$$

$$\left( 2n (-1 + 2n) (1 + 2n) \text{Pochhammer}[-1 + n, -1 + n] \right.$$

$$\left. \text{Pochhammer}\left[\frac{3}{2} + \frac{\mu}{2} + n, -2 + n\right] \text{Pochhammer}[-2 + \mu + 2n, 2 + n] \right)$$

FullSimplify[%]

1

## Theorem 19 (Family B): $D_{2r-1,0}(n)$ and $D_{0,2r-1}(n)$

quoB = -Pochhammer[mu + 2n + 4r - 4, n - r + 1] \* Pochhammer[mu + 2n + 4r - 3, n - r] \*  
 Pochhammer[mu / 2 + r + 2n - 1 / 2, n - r]^2 / Pochhammer[n - r + 1, n - r] /  
 Pochhammer[n - r + 1, n - r + 1] / Pochhammer[mu / 2 + n + 2r - 3 / 2, n - r]^2;  
 Table[Factor[(Dst[2r - 1, 0, 2n + 1] / Dst[2r - 1, 0, 2n - 1]) / quoB], {n, 5}, {r, n}] //  
 TableForm

1

1				
1	1			
1	1	1		
1	1	1	1	
1	1	1	1	1

Table[Together[Dst[2r - 1, 0, n] / Product[quoB, {n, r, (n - 1) / 2}]],  
 {n, 1, 9, 2}, {r, 5}]

{{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}



```
TableForm[Table[Dst[2 r - 1, 0, n], {n, 2, 10, 2}, {r, 5}]]
```

```
0 1 1 1 1
0 0 1 1 1
0 0 0 1 1
0 0 0 0 1
0 0 0 0 0
```

```
Table[Together[Dst[2 r - 1, 0, n, mu] - Dst[1, 0, n - 2 r + 2, mu + 6 r - 6]],
{r, 5}, {n, 2 r, 10}]
```

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0}, {0}}
```

```
Table[Dst[0, 2 r - 1, n, 37] /
```

```
(Product[Pochhammer[mu + i - 1, 2 r - 1] / Pochhammer[i + 1, 2 r - 1], {i, 0, n - 1}] *
Product[quoB, {n, r, (n - 1) / 2}] /. mu -> 37), {n, 1, 9, 2}, {r, 1, 10}]
```

```
{{1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1},
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}
```

```
(* Special case: Proposition 9 *)
```

```
(quoB /. r -> 1) / R10[n]
```

```
1
```

```
(* Special case: Proposition 10 *)
```

```
((quoB /. r -> 1) * (mu + 2 n - 1) * (mu + 2 n - 2) / (2 n + 1) / (2 n)) / R01[n]
```

```
((-2 + mu + 2 n) (-1 + mu + 2 n) Pochhammer[n, 2 + n] Pochhammer[mu + 2 n, n]) /
```

```
(2 n (1 + 2 n) Pochhammer[n, n] Pochhammer[-2 + mu + 2 n, 2 + n])
```

```
FullSimplify[%]
```

```
1
```

## Conjecture 20 (Family C): $D_{2r,1}(n)$ and $D_{1,2r}(n)$

```
quoC = - (2 n + 2 r) * (mu + 2 n + 2 r - 1) * (mu + 2 n + 2 r) * Pochhammer[mu + 2 n + 4 r, n - r]^2 *
Pochhammer[mu / 2 + 2 n + r + 3 / 2, n - r + 1]^2 / Pochhammer[n - r + 1, n - r + 1]^2 /
```

```
Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r]^2 / (mu + 2 n + 1) / (2 n + 1) / (2 n + 2);
```

```
Table[Factor[(Dst[2 r, 1, 2 n + 2] / Dst[2 r, 1, 2 n]) / quoC], {n, 6}, {r, n}]
```

```
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1}}
```

```
TableForm[
```

```
Table[Together[Dst[2 r, 1, 2 n] / (Pochhammer[mu + 2 r - 1, 2 n] / (2 n)!), {r, 5}, {n, r}]]
```

```
 $\frac{-1+\mu}{1+\mu}$ 
```

```
1  $\frac{-1+\mu}{3+\mu}$ 
```

```
1 1  $\frac{-1+\mu}{5+\mu}$ 
```

```
1 1 1  $\frac{-1+\mu}{7+\mu}$ 
```

```
1 1 1 1  $\frac{-1+\mu}{9+\mu}$ 
```

```
Table[Together[Dst[2 r, 1, 2 r] / ((mu - 1) * Pochhammer[mu + 2 r, 2 r - 1] / (2 r)!), {r, 5}]
{1, 1, 1, 1, 1}
```

```
Table[Together[Dst[2 r, 1, 2 n] /
  If[n < r, Pochhammer[mu + 2 r - 1, 2 n] / (2 n)!, (mu - 1) *
    Pochhammer[mu + 2 r, 2 r - 1] / (2 r)! * Product[quoC, {n, r, n - 1}]], {n, 5}, {r, 5}]
{{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

```
Table[
  Together[Dst[1, 2 r, n] / (Product[Pochhammer[mu + i, 2 r - 1] / Pochhammer[i + 2, 2 r - 1],
    {i, 0, n - 1}] * Dst[2 r, 1, n]), {n, 8}, {r, 8}]
{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
  {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
  {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}
```

## Conjecture 21 (Family D): $D_{-1,2r}(n)$

```
quoD = -Pochhammer[mu + 2 n - 3, 2 r + 1] *
  Pochhammer[mu + 2 n - 1, 2 r] * Pochhammer[mu + 2 n + 4 r, n - r]^2 *
  Pochhammer[mu / 2 + 2 n + r + 1 / 2, n - r - 1]^2 / Pochhammer[n - r, n - r]^2 /
  Pochhammer[2 n + 2, 2 r + 1] / Pochhammer[2 n + 1, 2 r] /
  Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r - 1]^2;
Table[Together[(Dst[-1, 2 r, 2 n + 2] / Dst[-1, 2 r, 2 n]) / quoD], {n, 5}, {r, 0, n - 1}]
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

```
Table[
  Together[Dst[-1, 2 r, 2 n] / (Product[quoD, {n, r + 1, n - 1}] * Dst[-1, 2 r, 2 r + 2]),
  {n, 5}, {r, 0, n - 1}]
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

```
Table[Together[(Dst[-1, 2 r, 2 n + 2] / Dst[-1, 2 r, 2 n]) /
  (Pochhammer[mu + 2 n - 2, 2 r] * Pochhammer[mu + 2 n - 1, 2 r] / Pochhammer[2 n + 1, 2 r] /
  Pochhammer[2 n + 2, 2 r]), {r, 0, 5}, {n, r - 1}]
{{}, {}, {1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}}
```

```
Table[Together[
  Dst[-1, 2 r, 2 r] / (Product[Pochhammer[mu + 2 i - 2, 2 r] * Pochhammer[mu + 2 i - 1, 2 r] /
  Pochhammer[2 i + 1, 2 r] / Pochhammer[2 i + 2, 2 r], {i, 0, r - 1}]), {r, 4}]
{1, 1, 1, 1}
```

```
Table[Together[Dst[-1, 2 r, 2 n] /
  (Product[quoD, {n, r + 1, n - 1}] * (Dst[-1, 2 r, 2 r + 2] / Dst[-1, 2 r, 2 r]) * Product[
  Pochhammer[mu + 2 i - 2, 2 r] * Pochhammer[mu + 2 i - 1, 2 r] / Pochhammer[2 i + 1, 2 r] /
  Pochhammer[2 i + 2, 2 r], {i, 0, r - 1}]), {n, 5}, {r, 1, n - 1}]
{{}, {1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}}
```

```

Table[Together[(Dst[-1, 2 r, 2 r + 2] / Dst[-1, 2 r, 2 r]) /
  ((3 - mu) * Pochhammer[mu + 2 r - 2, 2 r] * Pochhammer[mu + 2 r - 1, 2 r] /
    Pochhammer[2 r + 1, 2 r] / Pochhammer[2 r + 1, 2 r + 1]), {r, 5}]
{1, 1, 1, 1, 1}

RD[r_, n_] := Which[
  n > r, -Pochhammer[mu + 2 n - 3, 2 r + 1] *
    Pochhammer[mu + 2 n - 1, 2 r] * Pochhammer[mu + 2 n + 4 r, n - r]^2 *
    Pochhammer[mu / 2 + 2 n + r + 1 / 2, n - r - 1]^2 / Pochhammer[n - r, n - r]^2 /
    Pochhammer[2 n + 2, 2 r + 1] / Pochhammer[2 n + 1, 2 r] /
    Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r - 1]^2 (* this is quoD *),
  n == r, (3 - mu) * Pochhammer[mu + 2 r - 2, 2 r] *
    Pochhammer[mu + 2 r - 1, 2 r] / Pochhammer[2 r + 1, 2 r] / Pochhammer[2 r + 1, 2 r + 1],
  n < r, Pochhammer[mu + 2 n - 2, 2 r] *
    Pochhammer[mu + 2 n - 1, 2 r] / Pochhammer[2 n + 1, 2 r] / Pochhammer[2 n + 2, 2 r]];
Table[Together[Dst[-1, 2 r, 2 n] / Product[RD[r, i], {i, 0, n - 1}]], {n, 5}, {r, 0, 10}]
{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

```

## Corollary 22 (Family E): $D_{2r-1,1}(2n)/D_{2r-1,1}(2n-1)$

```

quoE = Pochhammer[mu / 2 + 2 n + r - 1 / 2, n - r] *
  Pochhammer[mu + 2 n + 4 r - 4, n - r + 1] / Pochhammer[n - r + 1, n - r + 1] /
  Pochhammer[mu / 2 + n + 2 r - 3 / 2, n - r];
Table[Together[(Dst[2 r - 1, 1, 2 n] / Dst[2 r - 1, 1, 2 n - 1]) / quoE], {n, 5}, {r, n}]
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}

Table[Together[(Dst[1, 2 r - 1, 2 n] / Dst[1, 2 r - 1, 2 n - 1]) /
  (Pochhammer[mu + 2 n - 1, 2 r - 2] / Pochhammer[2 n + 1, 2 r - 2] * quoE)], {n, 5}, {r, n}]
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}

TableForm[Table[Factor[Dst[2 r - 1, 1, n + 1] / Dst[2 r - 1, 1, n]], {r, 4}, {n, 2 r}]
2 + mu      1/12 (3 + mu) (14 + mu)
3 + mu      (4 + mu) (5 + mu)
2           3 (2 + mu)
8 + mu      (6 + mu) (7 + mu) (52 + 3 mu)
4           40 (4 + mu)
5 + mu      (8 + mu) (9 + mu)
2           5 (4 + mu)
14 + mu     (10 + mu) (11 + mu) (114 + 5 mu)
6           84 (6 + mu)
7 + mu      8 + mu
2           3
9 + mu      10 + mu
4           5
11 + mu     12 + mu
6           7 (6 + mu)

```

## Corollary 23 (Family F): $D_{-1,2r-1}(2n+1)/D_{-1,2r-1}(2n)$

```
quoF = 2 Pochhammer[-2 + 2 n + mu, 2 r] * Pochhammer[-2 + 4 r + 2 n + mu, n - r - 1] *
  Pochhammer[mu / 2 + 2 n + r - 1 / 2, n - r] / Pochhammer[2 n, 2 r] /
  Pochhammer[n - r + 1, n - r] / Pochhammer[mu / 2 + n + 2 r - 1 / 2, n - r - 1];
Table[Factor[(Dst[-1, 2 r - 1, 2 n + 1] / Dst[-1, 2 r - 1, 2 n]) / quoF], {n, 4}, {r, n}]
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}}
```

## Encore: Guessing another closed form for $D_{1,2r-1}(2n)/D_{1,2r-1}(2n-1)$

```
dets = Table[Dst[1, 2 r - 1, n, 100 003], {n, 40}, {r, 20}];
data = Table[dets[[n + 1, r]] / dets[[n, r]], {n, 1, 39, 2}, {r, 20}];
guess = GuessMultRE[data, {f[m, r], f[m + 1, r], f[m, r + 1]},
  {m, r}, 6, StartPoint -> {1, 1}, Constraints -> m > r];
gb = Factor[OreGroebnerBasis[NormalizeCoefficients /@
  ToOrePolynomial[guess, f[m, r]]]]
{2 (m + r) (50 000 + 2 m + r) (50 001 + 2 m + r) (-1 + 2 m + 2 r)
  (99 999 + 2 m + 4 r) (100 001 + 2 m + 4 r) Sr - 3 (1 + 2 m - 2 r) (33 334 + m + r)
  (50 000 + m + r) (100 001 + 2 m + 2 r) (100 000 + 3 m + 3 r) (100 001 + 3 m + 3 r),
  -4 (50 001 + m) (100 003 + 2 m) (3 + 2 m - 2 r) (m + r) (50 000 + 2 m + r)
  (50 001 + 2 m + r)2 (50 002 + 2 m + r) (-1 + 2 m + 2 r) (99 999 + 2 m + 4 r) Sm +
  9 (1 + m) (16 667 + m) (1 + 2 m) (50 002 + 3 m) (50 003 + 3 m) (33 334 + m + r)
  (50 000 + m + r) (100 001 + 2 m + 2 r) (100 000 + 3 m + 3 r) (100 001 + 3 m + 3 r)}

Reconst[x_, v_] := With[{a = Round[v / x]}, (mu + x * a - v) / a];
Factor[gb = NormalizeCoefficients /@
  Expand[gb /. x_Integer /; Abs[x] > 10 000 -> Reconst[x, 100 003]]]
{(m + r) (-1 + 2 m + 2 r) (-3 + 4 m + mu + 2 r) (-1 + 4 m + mu + 2 r)
  (-4 + 2 m + mu + 4 r) (-2 + 2 m + mu + 4 r) Sr - (1 + 2 m - 2 r) (-3 + 2 m + mu + 2 r)
  (-2 + 2 m + mu + 2 r) (-3 + 3 m + mu + 3 r) (-2 + 3 m + mu + 3 r) (-1 + 3 m + mu + 3 r),
  -2 (-1 + 2 m + mu) (2 m + mu) (3 + 2 m - 2 r) (m + r) (-1 + 2 m + 2 r) (-3 + 4 m + mu + 2 r)
  (-1 + 4 m + mu + 2 r)2 (1 + 4 m + mu + 2 r) (-4 + 2 m + mu + 4 r) Sm +
  (1 + m) (1 + 2 m) (-1 + 6 m + mu) (1 + 6 m + mu) (3 + 6 m + mu) (-3 + 2 m + mu + 2 r)
  (-2 + 2 m + mu + 2 r) (-3 + 3 m + mu + 3 r) (-2 + 3 m + mu + 3 r) (-1 + 3 m + mu + 3 r)}

sol = RSolve[ApplyOreOperator[gb[[2]], f[m]] == 0, f[m], m][[1, 1, 2]];
Factor[gb = DFiniteTimes[Annihilator[1/sol, {S[m], S[r]}], gb]]
{r (-1 + mu + 2 r) (-4 + mu + 4 r) (-2 + mu + 4 r) Sr +
  (-2 + mu + 2 r) (-3 + mu + 3 r) (-2 + mu + 3 r) (-1 + mu + 3 r), Sm - 1}
```

```
sol = Simplify[sol * RSolve[ApplyOreOperator[gb[[1]], f[r]] == 0, f[r], r][[1, 1, 2]]]
```

$$\left( (-1)^{1+r} 16^{1-2m-r} 27^{-1+2m+r} C[1]^2 \text{Pochhammer}\left[\frac{1}{2}, m\right] \text{Pochhammer}[1, m] \right. \\ \text{Pochhammer}\left[\frac{1}{6}(-1+mu), m\right] \text{Pochhammer}\left[\frac{mu}{3}, -1+r\right] \text{Pochhammer}\left[\frac{mu}{2}, -1+r\right] \\ \text{Pochhammer}\left[\frac{1+mu}{6}, m\right] \text{Pochhammer}\left[\frac{1+mu}{3}, -1+r\right] \text{Pochhammer}\left[\frac{2+mu}{3}, -1+r\right] \\ \text{Pochhammer}\left[\frac{3+mu}{6}, m\right] \text{Pochhammer}\left[-1+\frac{mu}{3}+r, m\right] \text{Pochhammer}\left[-\frac{2}{3}+\frac{mu}{3}+r, m\right] \\ \left. \text{Pochhammer}\left[-\frac{1}{3}+\frac{mu}{3}+r, m\right] \text{Pochhammer}\left[-\frac{3}{2}+\frac{mu}{2}+r, m\right] \text{Pochhammer}\left[-1+\frac{mu}{2}+r, m\right] \right) / \\ \left( \text{Pochhammer}[1, -1+r] \text{Pochhammer}\left[\frac{1}{2}(-1+mu), m\right] \text{Pochhammer}\left[\frac{mu}{4}, -1+r\right] \right. \\ \text{Pochhammer}\left[\frac{mu}{2}, m\right] \text{Pochhammer}\left[\frac{1+mu}{2}, -1+r\right] \text{Pochhammer}\left[\frac{2+mu}{4}, -1+r\right] \\ \text{Pochhammer}\left[\frac{3}{2}-r, m\right] \text{Pochhammer}\left[-\frac{1}{2}+r, m\right] \text{Pochhammer}[r, m] \\ \text{Pochhammer}\left[-2+\frac{mu}{2}+2r, m\right] \text{Pochhammer}\left[\frac{1}{4}(-3+mu+2r), m\right] \\ \left. \text{Pochhammer}\left[\frac{1}{4}(-1+mu+2r), m\right]^2 \text{Pochhammer}\left[\frac{1}{4}(1+mu+2r), m\right] \right)$$

```
Union[Flatten[Table[(sol /. mu -> 100 003) / C[1]^2 / data[[m, r]], {r, 6}, {m, r, 6}]]]
```

$$\left\{ \frac{1}{2} \right\}$$

```
sol = sol / C[1]^2 / (1 / 2);
```

```
sol = FullSimplify[sol, Element[{r, m}, Integers] && r >= 1 && m >= 1 && m >= r]
```

$$\left( 2^{4-4m-mu-2r} 27^m \pi (2m)! \text{Gamma}\left[-\frac{1}{6}+m+\frac{mu}{6}\right] \text{Gamma}\left[\frac{1}{6}+m+\frac{mu}{6}\right] \text{Gamma}\left[\frac{1}{2}+m+\frac{mu}{6}\right] \right. \\ \left. \text{Gamma}\left[\frac{1}{2}(-1+mu)\right] \text{Gamma}[-3+2m+mu+2r] \text{Gamma}[-3+3m+mu+3r] \right) / \\ \left( (2(-1+m+r))! \text{Gamma}\left[\frac{1}{6}(-1+mu)\right] \text{Gamma}\left[\frac{1+mu}{6}\right] \text{Gamma}\left[\frac{3+mu}{6}\right] \right. \\ \text{Gamma}[-1+2m+mu] \text{Gamma}\left[\frac{3}{2}+m-r\right] \text{Gamma}\left[2m+\frac{1}{2}(-3+mu)+r\right] \\ \left. \text{Gamma}\left[2m+\frac{1}{2}(-1+mu)+r\right] \text{Gamma}\left[-2+m+\frac{mu}{2}+2r\right] \right)$$

```

quo =
sol /. Gamma[a_] / Gamma[b_] /; IntegerQ[Expand[a - b - m]] => Pochhammer[b, Expand[a -
b]] /.
Pochhammer[(mu - 1) / 6, m] Pochhammer[(1 + mu) / 6, m] Pochhammer[
(3 + mu) / 6, m] -> Pochhammer[mu / 2 - 1 / 2, 3 m] / 3^(3 m) /.
Gamma[a_] => Gamma[Expand[a]] /.
Gamma[a_ + Rational[b_, 2]] => With[{z = a + b / 2 - 1 / 2},
2^Expand[1 - 2 z] * Sqrt[Pi] * Gamma[Expand[2 z]] / Gamma[z]] /.
Gamma[m - r + 1] / Gamma[2 m - 2 r + 2] -> 1 / Pochhammer[m - r + 1, m - r + 1] /.
Gamma[mu / 2 - 1 + r + 2 m] / Gamma[mu / 2 - 2 + 2 r + m] ->
Pochhammer[mu / 2 + 2 r + m - 2, m - r + 1] /.
Gamma[mu / 2 + 2 m + r - 2] / Gamma[mu / 2 - 1] ->
Pochhammer[mu / 2 - 1, 2 m + r - 1] /.
Gamma[-3 + 3 r + 3 m + mu] / Gamma[-2 + 2 r + 4 m + mu] ->
1 / Pochhammer[-3 + 3 r + 3 m + mu, 1 - r + m] /.
Gamma[mu - 2] / Gamma[mu + 2 m - 1] -> 1 / Pochhammer[mu - 2, 2 m + 1] /.
Gamma[mu + 2 m + 2 r - 3] / Gamma[mu + 4 m + 2 r - 4] ->
1 / Pochhammer[mu + 2 m + 2 r - 3, 2 m - 1] /.
(2 m)! / (2 (m + r - 1))! -> 1 / Pochhammer[2 m + 1, 2 r - 2] /.
Pochhammer[mu / 2 - 1 / 2, 3 m] -> Pochhammer[mu / 2 - 1 / 2, 2 m + r - 1] *
Pochhammer[mu / 2 - 1 / 2 + 2 m + r - 1, m - r + 1] /.
Pochhammer[mu / 2 - 1, 2 m + r - 1] * Pochhammer[mu / 2 - 1 / 2, 2 m + r - 1] ->
Pochhammer[mu - 2, 4 m + 2 r - 2] / 2^(4 m + 2 r - 2) /.
Pochhammer[mu - 2, 4 m + 2 r - 2] / Pochhammer[mu - 2, 2 m + 1] ->
Pochhammer[mu + 2 m - 1, 4 m + 2 r - 2 m - 3] /.
Pochhammer[mu + 2 m - 1, 2 m + 2 r - 3] -> Pochhammer[mu + 2 m - 1, 2 r - 2] *
Pochhammer[mu + 2 m + 2 r - 3, 2 m - 1]
(2^(2 + 2 m - 2 r) Pochhammer[-1 + 2 m + mu, -2 + 2 r]
Pochhammer[-3/2 + 2 m + mu/2 + r, 1 + m - r] Pochhammer[-2 + m + mu/2 + 2 r, 1 + m - r]) /
(Pochhammer[1 + 2 m, -2 + 2 r] Pochhammer[1 + m - r, 1 + m - r]
Pochhammer[-3 + 3 m + mu + 3 r, 1 + m - r])

```

```

Table[Det[DstMat[1, 2 r - 1, 2 m, 37]] / Det[DstMat[1, 2 r - 1, 2 m - 1, 37]] /
(quo /. {mu -> 37}), {r, 10}, {m, r, 10}]

```

```

{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
{1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1},
{1, 1, 1, 1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1}, {1, 1}, {1}}

```

```

TraditionalForm[quo /. {mu -> mu}]

```

$$\frac{2^{2m-2r+2} \left(2m+r+\frac{\mu}{2}-\frac{3}{2}\right)_{m-r+1} \left(m+2r+\frac{\mu}{2}-2\right)_{m-r+1} (2m+\mu-1)_{2r-2}}{(2m+1)_{2r-2} (m-r+1)_{m-r+1} (3m+3r+\mu-3)_{m-r+1}}$$