This Mathematica notebook accompanies the paper by Christoph Koutschan and Thotsaporn Thanatipanonda with the title

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# A curious family of binomial determinants that count rhombus tilings of a holey hexagon

The following Mathematica packages are required. They are part of the RISCErgoSum bundle, which can be downloaded from

http://www.risc.jku.at/research/combinat/software/ergosum/installation.html

For the download a password is required. It can be obtained by sending an e-mail to Peter Paule (ppaule@risc.uni-linz.ac.at).

In[40]:=
```
<< RISC`HolonomicFunctions`;
<< RISC`Guess`;
<< RISC`LinearSystemSolver`;
SetDirectory[NotebookDirectory[]];
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan


```
--> Type  ?HolonomicFunctions  for help.
```

Guess Package version 0.52
written by Manuel Kauers

# Section 3: Related Determinants

## Global Definitions

In[5]:=
```
(* An auxiliary procedure to simplify certain expressions. *)
MySimp[expr_] := expr /.
          (* Simplify Floors *)
          Floor[a_] :> Floor[Expand[a]] //.
         Floor[Optional[a_Integer] * (v : (i | j | k | n)) + b_.] → a * v + Floor[b] /.
        (* Simplify powers *)
        (2 ^ a_) :> 2 ^ Expand[a] /.
      m1 ^ a_ :> Simplify[(-1) ^ a, Element[{i, j, k, n}, Integers]] /.
     (* Expand arguments of Pochhammers. *)
     Pochhammer[a_, b_] :> Pochhammer @@ Expand[{a, b}] //.
    (* Simplify quotients of conjugate Pochhammers to rational functions. *)
    Pochhammer[a1_, b1_] ^ c1_. * Pochhammer[a2_, b2_] ^ c2_. /;
      IntegerQ[Expand[a1 - a2]] && IntegerQ[Expand[b1 - b2]] && c1 > 0 && c2 < 0 :> With[
      {m = Min[c1, -c2]}, FunctionExpand[(Pochhammer[a1, b1] / Pochhammer[a2, b2]) ^ m] *
       Pochhammer[a1, b1] ^ (c1 - m) * Pochhammer[a2, b2] ^ (c2 + m)];
```

In[6]:=
```
(* Definitions of matrix and determinant D_{s,t}(n) *)
DstMat[s_, t_, n_, mu_] :=
  Table[FunctionExpand[KroneckerDelta[i, j] + Binomial[mu + i + j - 2, j]],
   {i, s, n + s - 1}, {j, t, n + t - 1}];
dst[s_, t_, n_, i_, j_, mu_] := FunctionExpand[
   KroneckerDelta[i + s, j + t] + Binomial[mu + i + s + j + t - 4, j + t - 1]];
dst[s_, t_, n_, i_, j_] := dst[s, t, n, i, j, mu];
DstMat[s_, t_, n_, mu_] := Table[dst[s, t, n, i, j, mu], {i, n}, {j, n}];
DstMat[s_, t_, n_] := DstMat[s, t, n, mu];
Dst[s_, t_, n_, mu_] := Det[DstMat[s, t, n, mu]];
Dst[s_, t_, n_] := Det[DstMat[s, t, n]];
```

In[13]:=
```
(* Definitions of rational functions in lemmas and propositions of Section 2 *)
R00e[n_] :=
  Pochhammer[mu + 2 n, n] * Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] / Pochhammer[n, n] /
   Pochhammer[mu / 2 + n + 1 / 2, n - 1];
R00o[n_] := Pochhammer[mu + 2 n - 2, n - 1] *
   Pochhammer[mu / 2 + 2 n - 1 / 2, n] / Pochhammer[n, n] / Pochhammer[mu / 2 + n - 1 / 2, n - 1];
R00[n_] := If[EvenQ[n], R00e[n / 2], R00o[(n + 1) / 2]];
R10[n_] := (1 - 2 n) * Pochhammer[mu + 2 n, n] ^ 2 *
   Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] ^ 2 / (mu + 2 n) / Pochhammer[n, n] ^ 2 /
    Pochhammer[mu / 2 + n + 1 / 2, n - 1] ^ 2;
R10[n_] := - Pochhammer[mu + 2 n, n] * Pochhammer[mu + 2 n + 1, n - 1] *
   Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] ^ 2 / Pochhammer[n, n] / Pochhammer[n, n - 1] /
    Pochhammer[mu / 2 + n + 1 / 2, n - 1] ^ 2;
R01[n_] := - Pochhammer[mu + 2 n - 2, n + 2] * Pochhammer[mu + 2 n + 1, n - 1] *
   Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] ^ 2 / Pochhammer[n, n - 1] / Pochhammer[n, n + 2] /
    Pochhammer[mu / 2 + n + 1 / 2, n - 1] ^ 2;
R20[n_] := Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] *
   Pochhammer[mu + 2 n + 1, n - 1] / Pochhammer[n, n - 1] / Pochhammer[mu / 2 + n + 1 / 2, n - 1];
R02[n_] := (2 n - 1) * Pochhammer[mu / 2 + 2 n + 1 / 2, n - 1] *
   Pochhammer[mu + 2 n - 2, n + 2] / (mu + 2 n) / Pochhammer[n, n + 2] /
    Pochhammer[mu / 2 + n + 1 / 2, n - 1];

(* Tests *)
Table[Together[{
   Dst[0, 0, 2 n] / Dst[0, 0, 2 n - 1] - R00[2 n - 1],
   Dst[1, 0, 2 n + 1] / Dst[1, 0, 2 n - 1] - R10[n],
   Dst[0, 1, 2 n + 1] / Dst[0, 1, 2 n - 1] - R01[n],
   Dst[2, 0, 2 n] / Dst[2, 0, 2 n - 1] - R20[n],
   Dst[0, 2, 2 n] / Dst[0, 2, 2 n - 1] - R02[n],
   Dst[-1, 1, 2 n + 1] / Dst[-1, 1, 2 n] - R02[n]
  }], {n, 5}]
Table[Together[{
   Dst[0, 0, n] - 2 * Product[R00[i], {i, n - 1}],
   Dst[1, 0, n] - If[EvenQ[n], 0, Product[R10[i], {i, (n - 1) / 2}]],
   Dst[0, 1, n] - If[EvenQ[n], 0, (mu - 1) * Product[R01[i], {i, (n - 1) / 2}]]
  }], {n, 10}]
```

```
{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0},
 {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
In[21]:= (* Alternative (old) definitions. The above formulas, however, are nicer. *)
R00a[n_] := If[EvenQ[n],
   2^(n/2) * Pochhammer[mu/2 + n/2, Floor[(n + 2)/4]] *
    Pochhammer[mu/2 + n + 1/2, (n - 2)/2] /
      Pochhammer[n/2, n/2] / Pochhammer[
      mu/2 + Floor[3/4 n] + 1/2, Floor[(n - 2)/4]],
   2^((n - 1)/2) * Pochhammer[mu/2 + n/2 - 1/2, Floor[(n + 1)/4]] *
    Pochhammer[mu/2 + n + 1/2, (n + 1)/2] /
      Pochhammer[(n + 1)/2, (n + 1)/2] /
     Pochhammer[mu/2 + Floor[3/4 (n - 1)] + 1/2, Floor[(n + 1)/4]]];
R10a[n_] := -(mu + 2 n) * Pochhammer[mu/2 + 2 * n + 1/2, n - 1]^2 *
   Pochhammer[mu/2 + n + 1, Floor[(n - 1)/2]]^2 / Pochhammer[mu/2 + Floor[3/2 n] + 1/2,
       Floor[(n - 1)/2]]^2 / Pochhammer[n, n] / Pochhammer[1/2, n - 1];
R01a[n_] := (-2) * Pochhammer[mu + 2 * n - 2, 3] * Pochhammer[mu/2 + 2 * n + 1/2, n - 1]^2 *
   Pochhammer[mu/2 + n + 1, Floor[(n - 1)/2]]^2 /
       Pochhammer[mu/2 + Floor[3/2 n] + 1/2, Floor[(n - 1)/2]]^2 / n /
     Pochhammer[n + 2, n + 1] / Pochhammer[1/2, n - 1];
R20a[n_] := Pochhammer[mu/2 + 2 n + 1/2, n - 1] *
   Pochhammer[mu/2 + n + 1, Floor[(n - 1)/2]] / 2^(n - 1) / Pochhammer[1/2, n - 1] /
    Pochhammer[mu/2 + Floor[3 n/2] + 1/2, Floor[(n - 1)/2]];
R02a[n_] := 2^(n - 1) * (2 n - 1) * Pochhammer[mu + 2 n - 2, 2] *
   Pochhammer[mu/2 + 2 n + 1/2, n - 1] * Pochhammer[mu/2 + n + 1, Floor[(n - 1)/2]] /
     Pochhammer[mu/2 + Floor[3 n/2] + 1/2, Floor[(n - 1)/2]] / Pochhammer[n, n + 2];

(* Tests *)
Table[Together[{R00[n] / R00a[n], R10[n] / R10a[n],
   R01[n] / R01a[n], R20[n] / R20a[n], R02[n] / R02a[n]}], {n, 20}]

{{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

## Theorem 2: $D_{1,1}(2n)/D_{1,1}(2n-1)$

The proof of this theorem is not part of the present paper. Here we verify it only on a few instances.

```
Table[Together[
  (Dst[1, 1, 2 n] / Dst[1, 1, 2 n - 1]) /
   (Pochhammer[mu + 2 n, n] * Pochhammer[mu/2 + 2 n + 1/2, n - 1] / Pochhammer[n, n] /
      Pochhammer[mu/2 + n + 1/2, n - 1])], {n, 5}]

{1, 1, 1, 1, 1}
```

## Lemma 3: $D_{1,0}(2n) = 0$

We generate some data for the bivariate sequence $c_{n,j}$. For $1 \leq n \leq 30$ we compute the nullspace of the matrix of $D_{1,0}(2n)$ using the command LinSolveUniv from the LinearSystemSolver package. For each $n$ this nullspace is spanned by a single vector. We divide all entries of this vector by its last entry, therefore normalizing it such that the last entry is 1.We fill the obtained triangular array with zeros, since the guessing procedure requires a rectangular array as input.

```
Timing[data = PadRight[Table[ns = LinSolveUniv[DstMat[1, 0, 2 n, mu], mu][[1]];
     Together[ns / ns[[-1]]], {n, 30}]];]
```

```
{1801.971000, Null}
```

We now use this data to guess recurrences. The settings (structure set and degree) have been found by trial and error. The Constraints option ensures that only the relevant data is used for the guessing, and not the padded zeros. The recurrences are converted into Ore polynomials (a data structure provided by the HolonomicFunctions package that represents recurrences as operators). The command NormalizeCoefficients removes any denominators and common factors among the coefficients of the recurrences. We verify that the guessed operators form a left Groebner basis, by feeding them into OreGroebnerBasis command and check that the output essentially equals the input (up to sign, this is achieved by GBEqual).

```
(* Directly guess a Groebner basis of annihilating operators for c[n,j]. *)
Timing[
 ann = NormalizeCoefficients /@ ToOrePolynomial[Join[
     GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n, j + 2]},
      {n, j}, 7, StartPoint → {1, 1}, Constraints → j ≤ 2 n],
     GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n + 1, j + 1]},
      {n, j}, 8, StartPoint → {1, 1}, Constraints → j ≤ 2 n],
     GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n + 2, j]},
      {n, j}, 17, StartPoint → {1, 1}, Constraints → j ≤ 2 n]], c[n, j]];
 GBEqual[ann, OreGroebnerBasis[ann]]
]
```

```
{949.971000, True}
```

Alternatively, we can guess recurrences of higher orders (and lower degree), which requires fewer data points and is therefore faster. The corresponding operators do not form a Groebner basis, so we have to call Buchberger’s algorithm (provided by the OreGroebnerBasis command) to obtain it. We verify that we get exactly the same result as before.

```
(* A faster alternative. Don't guess the Groebner basis directly,
but some higher-order recurrences. *)
(* Then apply Buchberger. *)
Timing[
 ann1 = OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[
     GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n, j + 2], c[n + 1, j + 1],
        c[n + 2, j]}, {n, j}, 7, StartPoint → {1, 1}, Constraints → j ≤ 2 n], c[n, j]]];
 GBEqual[ann, ann1]
]
```

```
{83.517000, True}
```

So far, we have constructed a set ann of operators that generate an annihilating ideal in the Ore algebra defined by the shift operators $S_n$ and $S_j$. Together with a few initial values, these operators uniquely define a bivariate sequence $c_{n,j}$. We now show that this sequence has the desired property, namely that the vector $c_{n,j}$ with $1 \le j \le 2n$ is in the nullspace of the matrix, exhibiting that $D_{1,0}(2n) = 0$. Note that for the proof of Lemma 3, it is not relevant how the set ann of operators was obtained. So in some sense, the proof of Lemma 3 starts here. We can load the precomputed set ann.

```
(*Put[ann,"ann_c_1_0.m"];*)
ann = Get["ann_c_1_0.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
```

```
{1, Sj, Sn}
```

```
1 106 536
```

```
{{7, 6}, {6, 5}, {15, 11}}
```

We first check whether the recurrence operators in ann allow to uniquely define a bivariate sequence. The command AnnihilatorSingularities gives all points (n,j) that cannot be computed using the recurrences (either because they lie under the staircase or because all applicable recurrence have a vanishing leading coefficient at the point). The command does not finish in reasonable time for symbolic $\mu$, but for concrete $\mu$ it does. This is sufficient to conclude that there cannot be infinitely many singularities for symbolic $\mu$.

```
AnnihilatorSingularities[ann, {0, 0}]
```

```
$Aborted
```

```
AnnihilatorSingularities[ann /. mu → 67, {0, 0}]
```

```
{{{j → 0, n → 0}, True}, {{j → 0, n → 1}, True}, {{j → 1, n → 0}, True},
 {{j → 2, n → 0}, True}, {{j → 3, n → 0}, True}, {{j → 4, n → 0}, True}}
```

```
(* Some tests *)
(* 1. Test that the annihilator is valid on the given data. *)
test = ApplyOreOperator[ann, c[n, j]];
Union[Flatten[Together[Table[test, {n, 28}, {j, 28}] /. c[n_, j_] :→ data[[n, j]]]]]
(* 2. Verify the identity Sum[d[2n,i,j]*c[n,j],{j,1,2n}]==0 for all 1≤i≤2n. *)
Union[Flatten[Table[Together[
    Sum[dst[1, 0, 2 n, i, j, mu] * data[[n, j]], {j, 1, 2 n}]], {n, 30}, {i, 2 n}]]]
(* 3. Same as 2., but with explicit form of matrix entries. This
 test can be viewed as initial value check. *)
Union[Flatten[Table[
   Together[Sum[FunctionExpand[Binomial[mu + i + j - 3, j - 1]] * data[[n, j]], {j, 2 n}] +
     data[[n, i + 1]]], {n, 29}, {i, 2 n}]]]
```

```
{0}
```

```
{0}
```

```
{0}
```

Hence we want to prove the following identity, which shows that the vector $c_{n,j}$, $1 \le j \le 2\,n$, is in the nullspace of the matrix of the determinant $D_{1,0}(2\,n)$.

```
TraditionalForm[
 HoldForm[Sum[Binomial[mu + i + j - 3, j - 1] * c[n, j], {j, 1, 2 n}] == -c[n, i + 1]]]
```

$$\sum_{j=1}^{2n} \binom{\text{mu} + i + j - 3}{j - 1} c(n, j) = -c(n, i + 1)$$

The sum has natural boundaries, since for $j \leq 0$ the binomial coefficient vanishes, and for $j > 2n$ we have $c_{n,j} = 0$. Thus creative telescoping yields an annihilator for the sum on the left hand side. Before calling the creative telescoping command, we have to compute the annihilator for the product of the binomial coefficient and $c_{n,j}$, which is achieved by the command DFiniteTimes. It requires that both annihilators are given with respect to the same Ore algebra. Therefore, the Annihilator command for the binomial also includes the shift operator S[n], and to the annihilator ann, we have to add the shift operator S[i].

```
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
     ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
     Annihilator[Binomial[mu + i + j - 3, j - 1], {S[n], S[j], S[i]}]], S[j] - 1];]
```

```
{2274.377000, Null}
```

Since this computation takes some time, we can load the precomputed result.

```
ct = << "ct_1_0.m";
```

It remains to verify that the sum satisfies the same system of recurrences as the $c_{n,i+1}$ on the right hand side. Indeed, we find that the two annihilators agree. In addition, we have to compare a few initial values, which has already been done before. The proof is complete.

```
GBEqual[DFiniteSubstitute[ann, {j → i + 1}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
```

```
True
```

## Lemma 4: $D_{0,1}(2n) = 0$

The proof of Lemma 4 is very similar to the proof of Lemma 3. Therefore we give only few comments here and refer to the proof of Lemma 3 for more details.

```
(* Directly guess a Groebner basis of annihilating operators for c[n,j]. *)
Timing[
 data = PadRight[Table[ns = LinSolveUniv[DstMat[0, 1, 2 n, mu], mu][[1]];
    Together[ns / ns[[-1]]], {n, 15}]];
 ann = NormalizeCoefficients /@ ToOrePolynomial[Join[
      GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j]},
       {n, j}, 9, StartPoint → {1, 1}, Constraints → j ≤ 2 n],
      GuessMultRE[data, {c[n, j], c[n, j + 1], c[n, j + 2]}, {n, j},
       6, StartPoint → {1, 1}, Constraints → j ≤ 2 n]
    ], c[n, j]];
 GBEqual[ann, OreGroebnerBasis[ann]]
]
```

```
{45.565000, True}
```

```
(*Put[ann,"ann_c_0_1.m"];*)
ann = Get["ann_c_0_1.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
```

```
{1, Sj}
```

```
128 496
```

```
{{7, 8}, {2, 6}}
```

We check whether the recurrence operators in ann allow to uniquely define a bivariate sequence. We find that there are infinitely many singularities: the recurrences in ann are not applicable to compute the values $c_{n,2\,n}$.

```
AnnihilatorSingularities[ann /. mu → 67, {0, 0}]
```

```
{{{n → j/2}, j/2 ∈ Integers && j ≥ 0}, {{j → 0, n → 0}, True},
 {{j → 1, n → 0}, True}, {{j → 1, n → 1}, True}}
```

However, we can find another recurrence that is a consequence of the previous ones. This recurrence allows us to compute the values for $c_{n,2\,n}$ (they are constant =1).

```
Factor[LeadingCoefficient[
  dfs = DFiniteSubstitute[ann, {j → 2 n}, Algebra → OreAlgebra[S[n]]][[1]]]]
```

```
2 (3 + 2 n) (2 + mu + 2 n) (3 + mu + 4 n) (5 + mu + 4 n)^2 (7 + mu + 4 n)
 (-3 mu^2 - 4 mu^3 + 2 mu^4 + 4 mu^5 + mu^6 - 6 n - 12 mu n + 3 mu^2 n + 69 mu^3 n + 83 mu^4 n + 23 mu^5 n -
   29 n^2 + 184 mu n^2 + 640 mu^2 n^2 + 708 mu^3 n^2 + 225 mu^4 n^2 + 405 n^3 + 2271 mu n^3 + 3055 mu^2 n^3 +
   1189 mu^3 n^3 + 2770 n^4 + 6596 mu n^4 + 3554 mu^2 n^4 + 5676 n^5 + 5676 mu n^5 + 3784 n^6)
```

To see that the constant sequence is a solution of the above recurrence, we reduce its operator with the operator $S_n - 1$ and obtain 0. Hence $S_n - 1$ is a right factor of the recurrence dfs.

```
OreReduce[dfs, ToOrePolynomial[{S[n] - 1}]]
```

```
0
```

```
(* Ee want to prove the following: *)
TraditionalForm[
 HoldForm[Sum[Binomial[mu + i + j - 3, j] * c[n, j], {j, 1, 2 n}] == -c[n, i - 1]]]
```

$$\sum_{j=1}^{2\,n} \binom{\text{mu} + i + j - 3}{j} c(n,\, j) = -c(n,\; i - 1)$$

```
(* Test initial values. *)
Union[
 Flatten[Table[Together[Sum[FunctionExpand[Binomial[mu + i + j - 3, j]] * data[[n, j]],
      {j, 2 n}] + If[i === 1, 0, data[[n, i - 1]]]], {n, 14}, {i, 2 n}]]]
```

```
{0}
```

```
(* The sum has natural boundaries. Thus
 creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
      ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
      Annihilator[Binomial[mu + i + j - 3, j], {S[n], S[j], S[i]}]], S[j] - 1];
]
```

```
{46.655000, Null}
```

```
ct = << "ct_0_1.m";
```

```
GBEqual[DFiniteSubstitute[ann, {j → i - 1}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
```

```
True
```

## Lemma 5: $D_{0,0}(2\,n)\,/\,D_{0,0}(2\,n-1)$

The determinant evaluation of $D_{0,0}(n)$ was first proven by George E. Andrews. We include our computer proof just for sake of illustration. It follows the very same strategy as the proof of Lemma 6 (where also more detailed explanations can be found, see below).

```
(* Directly guess a Groebner basis of annihilating operators for c[n,j]. *)
Timing[
 data = PadRight[Table[ns = LinSolveUniv[Most[DstMat[0, 0, 2 n, mu]], mu][[1]];
    Together[ns / ns[[-1]]], {n, 15}]];
 ann = NormalizeCoefficients /@ ToOrePolynomial[Join[
      GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j]},
       {n, j}, 8, StartPoint → {1, 1}, Constraints → j ≤ 2 n],
      GuessMultRE[data, {c[n, j], c[n, j + 1], c[n, j + 2]}, {n, j},
       4, StartPoint → {1, 1}, Constraints → j ≤ 2 n]
    ], c[n, j]];
 GBEqual[ann, OreGroebnerBasis[ann]]
]
```

```
{39.793000, True}
```

```
(*Put[ann,"ann_c_0_0.m"];*)
ann = Get["ann_c_0_0.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
```

```
{1, Sj}
```

```
91 424
```

```
{{7, 7}, {2, 4}}
```

```
(* Identity (1): Show that c[n,2n]=1 for all n. *)
diag = DFiniteSubstitute[ann, {j → 2 n}, Algebra → OreAlgebra[S[n]]][[1]];
Factor[LeadingCoefficient[diag]]
OreReduce[diag, {ToOrePolynomial[S[n] - 1]}]
```

$2\,(3+2\,n)\,(\text{mu}+2\,n)\,(1+\text{mu}+4\,n)\,(3+\text{mu}+4\,n)^2\,(5+\text{mu}+4\,n)$
$(6-5\,\text{mu}-11\,\text{mu}^2+10\,\text{mu}^3+4\,\text{mu}^4-5\,\text{mu}^5+\text{mu}^6-11\,n-147\,\text{mu}\,n+200\,\text{mu}^2\,n+28\,\text{mu}^3\,n-93\,\text{mu}^4\,n+23\,\text{mu}^5\,n-468\,n^2+1053\,\text{mu}\,n^2+91\,\text{mu}^2\,n^2-741\,\text{mu}^3\,n^2+225\,\text{mu}^4\,n^2+1487\,n^3+431\,\text{mu}\,n^3-3107\,\text{mu}^2\,n^3+1189\,\text{mu}^3\,n^3+878\,n^4-6648\,\text{mu}\,n^4+3554\,\text{mu}^2\,n^4-5676\,n^5+5676\,\text{mu}\,n^5+3784\,n^6)$

0

```
(* Identity (2) *)
TraditionalForm[HoldForm[
  Sum[Binomial[μ + i + j - 4, j - 1] * c[n, j], {j, 1, 2 n}] == -c[n, i] "      " (1 ≤ i ≤ 2 n - 1)]]
```

$$\sum_{j=1}^{2n}\binom{\mu+i+j-4}{j-1}c(n,j)=-c(n,i)\quad (1\le i\le 2\,n-1)$$

```
(* Identity (2): numerical check (= initial values) *)
Union[Flatten[Table[Together[Sum[dst[0, 0, 2 n, i, j, mu] * data[[n, j]], {j, 1, 2 n}]],
   {n, 15}, {i, 2 n - 1}]]]
Union[Flatten[Table[Together[Sum[FunctionExpand[Binomial[mu + i + j - 4, j - 1]] *
      data[[n, j]], {j, 1, 2 n}] + data[[n, i]]], {n, 15}, {i, 2 n - 1}]]]
```

{0}

{0}

```
(* The sum has natural boundaries. Thus
 creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
     ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
     Annihilator[Binomial[mu + i + j - 4, j - 1], {S[n], S[j], S[i]}]], S[j] - 1];
]
```

{53.147000, Null}

```
ct = << "ct_0_0_A.m";

GBEqual[DFiniteSubstitute[ann, {j → i}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
```

True

```
(* Identity (3) *)
TraditionalForm[
 HoldForm[Sum[(KroneckerDelta[2 n, j] + Binomial[μ + 2 n + j - 4, j - 1]) * c[n, j],
    {j, 1, 2 n}] == D0,0[2 n] / D0,0[2 n - 1]]]
```

$$\sum_{j=1}^{2n}\left(\delta_{2n,j}+\binom{\mu+2\,n+j-4}{j-1}\right)c(n,j)=\frac{D_{0,0}(2\,n)}{D_{0,0}(2\,n-1)}$$

```
(* Numerical check of Identity (3) (= initial value check) *)
Table[Together[
  Sum[FunctionExpand[Binomial[mu + 2 n + j - 4, j - 1]] * data[[n, j]], {j, 1, 2 n}] +
   data[[n, 2 n]] - R00[2 n - 1]], {n, 10}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
(* The sum has natural boundaries. Thus
 creative telescoping gives its annihilator. *)
Timing[
 ct = FindCreativeTelescoping[DFiniteTimes[ann,
     Annihilator[Binomial[mu + 2 n + j - 4, j - 1], {S[n], S[j]}]], S[j] - 1];
]
```

```
{101.570000, Null}
```

```
ct = << "ct_0_0_B.m";
```

```
(* Add the part coming from the KroneckerDelta. *)
ann3 = DFinitePlus[ct[[1]], DFiniteSubstitute[ann, {j → 2 n}]];
```

```
(* The recurrence we get is not too unhandy... *)
Factor[ann3]
```

```
{2 (3 + 2 n) (mu + 2 n) (1 + mu + 4 n) (3 + mu + 4 n)^2
    (5 + mu + 4 n) (6 - 5 mu - 11 mu^2 + 10 mu^3 + 4 mu^4 - 5 mu^5 + mu^6 - 11 n - 147 mu n +
      200 mu^2 n + 28 mu^3 n - 93 mu^4 n + 23 mu^5 n - 468 n^2 + 1053 mu n^2 + 91 mu^2 n^2 -
      741 mu^3 n^2 + 225 mu^4 n^2 + 1487 n^3 + 431 mu n^3 - 3107 mu^2 n^3 + 1189 mu^3 n^3 +
      878 n^4 - 6648 mu n^4 + 3554 mu^2 n^4 - 5676 n^5 + 5676 mu n^5 + 3784 n^6) S_n^2 +
  (-8100 mu - 5040 mu^2 + 18 531 mu^3 + 11 783 mu^4 - 12 732 mu^5 - 8564 mu^6 + 2250 mu^7 +
    1938 mu^8 + 72 mu^9 - 116 mu^10 - 21 mu^11 - mu^12 - 27 540 n - 45 144 mu n + 223 443 mu^2 n +
    207 636 mu^3 n - 253 610 mu^4 n - 222 246 mu^5 n + 54 372 mu^6 n + 64 314 mu^7 n + 4302 mu^8 n -
    4506 mu^9 n - 967 mu^10 n - 54 mu^11 n - 102 060 n^2 + 940 581 mu n^2 + 1 417 278 mu^2 n^2 -
    1 968 205 mu^3 n^2 - 2 409 495 mu^4 n^2 + 515 897 mu^5 n^2 + 924 553 mu^6 n^2 + 98 657 mu^7 n^2 -
    77 801 mu^8 n^2 - 20 082 mu^9 n^2 - 1307 mu^10 n^2 + 1 303 605 n^3 + 4 434 924 mu n^3 -
    7 304 625 mu^2 n^3 - 13 832 520 mu^3 n^3 + 2 324 225 mu^4 n^3 + 7 485 116 mu^5 n^3 + 1 210 077 mu^6 n^3 -
    783 536 mu^7 n^3 - 248 162 mu^8 n^3 - 18 864 mu^9 n^3 + 5 218 875 n^4 - 12 699 623 mu n^4 -
    43 976 946 mu^2 n^4 + 4 197 653 mu^3 n^4 + 37 211 456 mu^4 n^4 + 8 973 275 mu^5 n^4 -
    5 081 510 mu^6 n^4 - 2 026 505 mu^7 n^4 - 181 475 mu^8 n^4 - 8 025 685 n^5 - 73 041 186 mu n^5 -
    3 061 033 mu^2 n^5 + 116 111 654 mu^3 n^5 + 42 293 153 mu^4 n^5 - 22 112 998 mu^5 n^5 -
    11 477 475 mu^6 n^5 - 1 228 510 mu^7 n^5 - 49 401 675 n^6 - 21 869 584 mu n^6 + 221 859 281 mu^2 n^6 +
    127 808 768 mu^3 n^6 - 65 207 089 mu^4 n^6 - 45 992 560 mu^5 n^6 - 6 008 213 mu^6 n^6 - 22 352 020 n^7 +
    237 195 716 mu n^7 + 240 300 968 mu^2 n^7 - 128 230 504 mu^3 n^7 - 130 388 308 mu^4 n^7 -
    21 404 572 mu^5 n^7 + 108 556 420 n^8 + 256 315 728 mu n^8 - 160 187 384 mu^2 n^8 -
    256 315 728 mu^3 n^8 - 55 152 716 mu^4 n^8 + 118 643 600 n^9 - 113 919 680 mu n^9 -
    332 830 160 mu^2 n^9 - 100 266 880 mu^3 n^9 - 34 541 840 n^10 - 257 023 872 mu n^10 -
    122 106 032 mu^2 n^10 - 89 453 760 n^11 - 89 453 760 mu n^11 - 29 817 920 n^12) S_n +
  (-3 + mu + 3 n) (-2 + mu + 3 n) (-1 + mu + 3 n) (-1 + mu + 6 n) (1 + mu + 6 n) (3 + mu + 6 n)
   (360 mu + 727 mu^2 + 486 mu^3 + 136 mu^4 + 18 mu^5 + mu^6 + 1350 n + 5040 mu n + 5277 mu^2 n +
     2113 mu^3 n + 357 mu^4 n + 23 mu^5 n + 9261 n^2 + 19 218 mu n^2 + 12 094 mu^2 n^2 +
     2826 mu^3 n^2 + 225 mu^4 n^2 + 23 919 n^3 + 30 599 mu n^3 + 11 109 mu^2 n^3 + 1189 mu^3 n^3 +
     29 258 n^4 + 21 732 mu n^4 + 3554 mu^2 n^4 + 17 028 n^5 + 5676 mu n^5 + 3784 n^6)}
```

```
(* Plug our expression into the recurrence *)
test = ApplyOreOperator[ann3[[1]], R00o[n]];
(* Divide by the (non-zero!) expression itself,
to make simplification easier. *)
test = If[Head[test] === Plus, # / R00o[n] & /@ test, test / R00o[n]];
(* Simplify *)
Together[MySimp[test]]
```

```
0
```

## Lemma 6: $D_{2,0}(2n)/D_{2,0}(2n-1)$

For $1 \le n \le 15$ we compute the nullspace of the $(2n-1)\times(2n)$ matrix that is obtained by removing the last row from the matrix of $D_{2,0}(2n)$. In all cases we encounter a one-dimensional nullspace, hence it is spanned by a single vector. We normalize this vector by dividing it by its last entry. For each $n$ we obtain a vector of length $2n$. The triangular array formed by these vectors is filled with 0's such that we get a rectangular array.

```
Timing[
 data = PadRight[Table[ns = LinSolveUniv[Most[DstMat[2, 0, 2 n, mu]], mu][[1]];
     Together[ns / ns[[-1]]], {n, 15}]];
]
```

```
{42.723000, Null}
```

As described in the proof of Lemma 3, we use the GuessMultRE command to construct recurrence equations that are satisfied by the computed (finite amount of) data. The recurrences are converted into operators and a left Groebner basis of the left ideal generated by these operators is computed.

```
(* Don't guess the Groebner basis directly,
but some higher-order recurrences. Then apply Buchberger. *)
Timing[
 ann = OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[
      GuessMultRE[data, {c[n, j], c[n, j + 1], c[n + 1, j], c[n, j + 2], c[n + 1, j + 1]},
       {n, j}, 7, StartPoint → {1, 1}, Constraints → j ≤ 2 n], c[n, j]]];
]
```

```
{39.076000, Null}
```

In the following we prove certain properties and identities concerning the infinite bivariate sequence $c_{n,j}$ defined by the recurrences (ann) and initial values (data).

```
In[51]:= (*Put[ann,"ann_c_2_0.m"];*)
ann = Get["ann_c_2_0.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
```

```
Out[52]= {1, Sj, Sn}

Out[53]= 1 106 536

Out[54]= {{7, 6}, {6, 5}, {15, 11}}
```

The first identity is $c_{n,2n} = 1$. From the annihilator of $c_{n,j}$ we compute a recurrence operator for the diagonal sequence $c_{n,2n}$. In order to study the solutions of this univariate recurrence, we have to look at its singularities (integer roots of its leading coefficient). By listing all linear factors of degree 1 in $n$, we see that there are no positive integer roots ($\mu$ is considered a symbolic parameter, and hence we need not care about special values of $\mu$). When we reduce the diag operator with the operator $S_n - 1$, we obtain 0, which tells us that $S_n - 1$ is a right factor, and therefore that any constant sequence is a solution of diag. Since the initial values for $c_{n,2n}$ are 1 (by construction), we conclude that $c_{n,2n} = 1$ for all $n$.

```
(* Identity (1): Show that c[n,2n]=1 for all n. *)
diag = DFiniteSubstitute[ann, {j → 2 n}, Algebra → OreAlgebra[S[n]]][[1]];
Select[Factor[LeadingCoefficient[diag]], Exponent[#, n] === 1 || Head[#] =!= Plus &]
OreReduce[diag, {ToOrePolynomial[S[n] - 1]}]
```

$$8\,(2+n)\,(1+2n)\,(3+2n)\,(5+2n)\,(3+mu+2n)\,(4+mu+2n)\,(3+mu+4n)\,(5+mu+4n)\,(6+mu+4n)\,(7+mu+4n)^2\,(8+mu+4n)\,(9+mu+4n)^2$$

```
0
```

The second identity that we want to prove is displayed below. We first perform a numerical check with the available data. This check also serves as comparison of initial values that we will have to do at the end.

```
(* Identity (2) *)
TraditionalForm[HoldForm[Sum[Binomial[μ + i + j - 2, j - 1] * c[n, j], {j, 1, 2 n}] ==
    -c[n, i + 2] "      " (1 ≤ i ≤ 2 n - 1)]]
```

$$\sum_{j=1}^{2n} \binom{\mu+i+j-2}{j-1} c(n, j) = -c(n, i+2) \quad (1 \le i \le 2n-1)$$

```
(* Identity (2): numerical check (= initial values) *)
Union[Flatten[Table[Together[Sum[dst[2, 0, 2 n, i, j, mu] * data[[n, j]], {j, 1, 2 n}]],
    {n, 15}, {i, 2 n - 1}]]]
Union[Flatten[Table[Together[Sum[FunctionExpand[Binomial[mu + i + j - 2, j - 1]] *
        data[[n, j]], {j, 1, 2 n}] + data[[n, i + 2]]], {n, 14}, {i, 2 n - 1}]]]
{0}
{0}
```

The sum has natural boundaries (meaning that the summand is zero outside the given summation range), since for $j < 1$ the binomial coefficient is 0 and for $j > 2n$ we have $c_{n,j} = 0$. Thus creative telescoping gives the annihilator of the sum (ct[[1]]), provided that the certificate (ct[[2]]) does not have poles inside the summation range.

```
(* The sum has natural boundaries. Thus
 creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
      ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
      Annihilator[Binomial[mu + i + j - 2, j - 1], {S[n], S[j], S[i]}]], S[j] - 1];
]
{2211.326000, Null}

(* Load the precomputed expression *)
ct = << "ct_2_0_A.m";

(* Look at the singularities of the certificate: no pole for 1 ≤ j ≤ 2n. *)
Factor[PolynomialLCM @@
  Denominator[Together[Flatten[OrePolynomialListCoefficients /@ Flatten[ct[[2]]]]]]]
(i + mu) (1 + i + mu) (-1 + i + j + mu) (-4 + j - 2 n) (-3 + j - 2 n) (j + mu + 2 n)
 (6 - 19 j + 22 j^2 - 11 j^3 + 2 j^4 - 8 mu + 18 j mu - 13 j^2 mu + 3 j^3 mu + 2 mu^2 -
   3 j mu^2 + j^2 mu^2 + 24 n - 48 j n + 30 j^2 n - 6 j^3 n - 32 mu n + 40 j mu n - 12 j^2 mu n +
   8 mu^2 n - 4 j mu^2 n + 24 n^2 - 36 j n^2 + 12 j^2 n^2 - 32 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2)
```

We see that the recurrences satisfied by the sum and constructed by creative telescoping are identical with the recurrences satisfied by $c_{n,i+2}$. Hence by comparing initial values (already done), we conclude that both expressions agree, thereby establishing identity (2).

```
GBEqual[DFiniteSubstitute[ann, {j → i + 2}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
True
```

Identitites (1) and (2) together imply that $c_{n,j}$ is the cofactor of the Laplace expansion of the matrix of

$D_{2,0}(2\,n)$ with respect to the last row, divided by $D_{2,0}(2\,n-1)$. Hence, by proving the following identity (3), we establish the statement of Lemma 6.

In[49]:=
```
(* Identity (3) *)
TraditionalForm[
 HoldForm[Sum[(KroneckerDelta[2 n + 1, j - 1] + Binomial[μ + 2 n + j - 2, j - 1]) * c[n, j],
     {j, 1, 2 n}] == D2,0[2 n] / D2,0[2 n - 1] == r20] /. r20 → R20[n] /. mu → μ]
```

Out[49]//TraditionalForm=

$$\sum_{j=1}^{2\,n}\left(\delta_{2\,n+1,j-1}+\binom{\mu+2\,n+j-2}{j-1}\right)c(n,\,j)=\frac{D_{2,0}(2\,n)}{D_{2,0}(2\,n-1)}=\frac{\left(\frac{1}{2}+\frac{\mu}{2}+2\,n\right)_{-1+n}(1+\mu+2\,n)_{-1+n}}{(n)_{-1+n}\left(\frac{1}{2}+\frac{\mu}{2}+n\right)_{-1+n}}$$

```
(* Numerical check of Identity (3) (= initial value check) *)
Table[Together[
  Sum[FunctionExpand[Binomial[mu + 2 n + j - 2, j - 1]] * data[[n, j]], {j, 1, 2 n}] -
   R20[n]], {n, 10}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

As before, we find that the sum in identity (3) has natural boundaries. Note that the Kronecker delta can be omitted since it does not contribute to the sum. Using the DFiniteTimes command, we construct an annihilator for the product of the binomial times $c_{n,j}$.

```
(* The sum has natural boundaries. Thus
 creative telescoping gives its annihilator. *)
Timing[
 ct = CreativeTelescoping[DFiniteTimes[ann,
     Annihilator[Binomial[mu + 2 n + j - 2, j - 1], {S[n], S[j]}]], S[j] - 1];
]
```

Out[59]=
```
{106.212, Null}
```

In[56]:=
```
ct = << "ct_2_0_B.m";
```

```
(* Look at the singularities of the certificate: no pole for 1 ≤ j ≤ 2n. *)
Factor[PolynomialLCM @@
  Denominator[Together[OrePolynomialListCoefficients[Flatten[ct][[2]]]]]]
```

```
(-4 + j - 2 n) (-3 + j - 2 n) (1 + n) (mu + 2 n) (1 + mu + 2 n) (2 + mu + 2 n)
 (3 + mu + 2 n) (j + mu + 2 n) (1 + mu + 4 n) (2 + mu + 4 n) (4 + mu + 4 n) (6 + mu + 4 n)
 (6 - 19 j + 22 j^2 - 11 j^3 + 2 j^4 - 8 mu + 18 j mu - 13 j^2 mu + 3 j^3 mu + 2 mu^2 -
   3 j mu^2 + j^2 mu^2 + 24 n - 48 j n + 30 j^2 n - 6 j^3 n - 32 mu n + 40 j mu n - 12 j^2 mu n +
   8 mu^2 n - 4 j mu^2 n + 24 n^2 - 36 j n^2 + 12 j^2 n^2 - 32 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2)
```

As a result, we get a nice recurrence (in $n$) for the sum on the left-hand side of identity (3):

```
Factor[ct[[1]]]
```

$$\{-8\,(-1+2n)\,(1+2n)\,(3+2n)\,(4+mu+2n)\,(-1+mu+4n)\,(1+mu+4n)\,(3+mu+4n)\,(5+mu+4n)^2\,(7+mu+4n)\,\big(10\,mu+17\,mu^2+8\,mu^3+mu^4+24\,n+104\,mu\,n+82\,mu^2\,n+16\,mu^3\,n+154\,n^2+270\,mu\,n^2+88\,mu^2\,n^2+290\,n^3+204\,mu\,n^3+172\,n^4\big)\,S_n^2-2\,(-1+2n)\,(-1+mu+4n)\,(1+mu+4n)\,(5+mu+6n)\,(7+mu+6n)\,(9+mu+6n)\,\big(240\,mu+1628\,mu^2+3036\,mu^3+2469\,mu^4+1020\,mu^5+222\,mu^6+24\,mu^7+mu^8-960\,n+1728\,mu\,n+13\,892\,mu^2\,n+19\,704\,mu^3\,n+11\,732\,mu^4\,n+3384\,mu^5\,n+464\,mu^6\,n+24\,mu^7\,n-10\,184\,n^2-2916\,mu\,n^2+36\,816\,mu^2\,n^2+43\,752\,mu^3\,n^2+18\,940\,mu^4\,n^2+3504\,mu^5\,n^2+232\,mu^6\,n^2-50\,500\,n^3-42\,576\,mu\,n^3+36\,696\,mu^2\,n^3+42\,864\,mu^3\,n^3+12\,624\,mu^4\,n^3+1152\,mu^5\,n^3-126\,892\,n^4-106\,740\,mu\,n^4+7244\,mu^2\,n^4+18\,720\,mu^3\,n^4+2976\,mu^4\,n^4-171\,820\,n^5-115\,320\,mu\,n^5-8448\,mu^2\,n^5+2880\,mu^3\,n^5-127\,516\,n^6-57\,840\,mu\,n^6-3568\,mu^2\,n^6-48\,880\,n^7-11\,040\,mu\,n^7-7568\,n^8\big)\,S_n+(mu+2n)\,(mu+3n)\,(1+mu+3n)\,(2+mu+3n)\,(-1+mu+6n)\,(1+mu+6n)\,(3+mu+6n)\,(5+mu+6n)\,(7+mu+6n)\,(9+mu+6n)\,\big(640+588\,mu+187\,mu^2+24\,mu^3+mu^4+1890\,n+1256\,mu\,n+258\,mu^2\,n+16\,mu^3\,n+2056\,n^2+882\,mu\,n^2+88\,mu^2\,n^2+978\,n^3+204\,mu\,n^3+172\,n^4\big)\}$$

It remains to show that the expression $R_{2,0}(n)$ also satisfies this recurrence (initial values have already been checked). This can, for example, be done by plugging $R_{2,0}(n)$ into the recurrence and by simplifying:

```
(* Plug our expression R_{2,0} into the recurrence *)
test = ApplyOreOperator[ct[[1, 1]], R20[n]];
(* Divide by the (non-zero!) expression itself,
to make simplification easier. *)
test = If[Head[test] === Plus, # / R20[n] & /@ test, test / R20[n]];
(* Simplify *)
Together[MySimp[test]]
```

0

Alternatively, we can compute an annihilator for $R_{2,0}(n)$ and show that it is a right factor of the recurrence operator ct[[1,1]]; the latter is done via the OreReduce command that gives a zero remainder.

```
In[62]:= OreReduce[ct[[1, 1]], Annihilator[R20[n], S[n]]]
```

Out[62]= 0

## Lemma 7: $D_{0,2}(2\,n)\,/\,D_{0,2}(2\,n-1)$

The proof of Lemma 7 follows the very same strategy as the proof of Lemma 6. There, detailed explanations have been given that apply equally to the following calculations. Hence, we list only the Mathematica commands for the computations, and refer to Lemma 6 for the explanations.

```
Timing[
 data = PadRight[Table[ns = LinSolveUniv[Most[DstMat[0, 2, 2 n, mu]], mu][[1]];
     Together[ns / ns[[-1]]], {n, 23}]];
]
```

```
{219.130000, Null}
```

```
Timing[
 rec1 = First[GuessMultRE[data,
      {c[n, j], c[n, j + 1], c[n + 1, j]}, {n, j}, {10, 11}, StartPoint → {1, 1},
      Constraints → j ≤ 2 n && n ≥ 1, AdditionalEquations → Infinity]];
]
```

{175.605000, Null}

```
Timing[
 rec2 = First[
    GuessMultRE[data, {c[n, j], c[n, j + 1], c[n, j + 2]}, {n, j}, 12, StartPoint → {1, 1},
      Constraints → j ≤ 2 n && n ≥ 1, AdditionalEquations → Infinity]];
]
```

{91.951000, Null}

```
(* CAVEAT: these recurrences do not hold for n=j=1. *)
TableForm[Map[If[# =!= {0, 0}, "*", 0] &,
  Together[Table[{rec1, rec2}, {n, 10}, {j, 2 n}] /. c[n_, j_] :→ data[[n, j]]], {2}]]
```

```
*  0
0  0  0  0
0  0  0  0  0  0
0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
```

```
Together[{rec1, rec2} /. {j → 1, n → 1} /. c[n_, j_] :→ data[[n, j]]]
```

$$\left\{\frac{1}{16384}\left(-1209600-5120640\,\mathrm{mu}-9591504\,\mathrm{mu}^2-10576832\,\mathrm{mu}^3-7681736\,\mathrm{mu}^4-3884160\,\mathrm{mu}^5-1405697\,\mathrm{mu}^6-367626\,\mathrm{mu}^7-69063\,\mathrm{mu}^8-9100\,\mathrm{mu}^9-799\,\mathrm{mu}^{10}-42\,\mathrm{mu}^{11}-\mathrm{mu}^{12}\right),\ \frac{1}{8}\left(10080+31176\,\mathrm{mu}+39332\,\mathrm{mu}^2+26670\,\mathrm{mu}^3+10689\,\mathrm{mu}^4+2604\,\mathrm{mu}^5+378\,\mathrm{mu}^6+30\,\mathrm{mu}^7+\mathrm{mu}^8\right)\right\}$$

```
ann =
  OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[{rec1, rec2}, c[n, j]]];
Support[
 ann]
```

$\{\{S_n, S_j, 1\}, \{S_j^2, S_j, 1\}\}$

In[66]:=
```
(* Put[ann,"ann_c_0_2.m"]; *)
ann = Get["ann_c_0_2.m"];
UnderTheStaircase[ann]
ByteCount[ann]
Exponent[#, {n, j}] & /@ ann
```

Out[67]= {1, $S_j$}

Out[68]= 651 256

Out[69]= {{10, 11}, {4, 10}}

```
(* Identity (1): Show that c[n,2n]=1 for all n. *)
diag = DFiniteSubstitute[ann, {j → 2 n}, Algebra → OreAlgebra[S[n]]][[1]];
Select[Factor[LeadingCoefficient[diag]], Exponent[#, n] === 1 || Head[#] =!= Plus &]
OreReduce[diag, {ToOrePolynomial[S[n] - 1]}]
```

2 (3 + 2 n) (4 + mu + 2 n) (3 + mu + 4 n) (5 + mu + 4 n) (7 + mu + 4 n) (9 + mu + 4 n)

0

```
(* Identity (2) *)
TraditionalForm[HoldForm[Sum[Binomial[μ + i + j - 2, j + 1] * c[n, j], {j, 1, 2 n}] ==
   -c[n, i - 2] "     " (1 ≤ i ≤ 2 n - 1)]]
```

$$\sum_{j=1}^{2n}\binom{\mu+i+j-2}{j+1}c(n,j)=-c(n,i-2)\quad(1\le i\le 2n-1)$$

```
(* Identity (2): numerical check (= initial values). Also the special case n=
 1 is covered. *)
Union[Flatten[Table[Together[Sum[dst[0, 2, 2 n, i, j, mu] * data[[n, j]], {j, 1, 2 n}]],
   {n, 15}, {i, 2 n - 1}]]]
Union[Flatten[Table[Together[Sum[FunctionExpand[Binomial[mu + i + j - 2, j + 1]] *
       data[[n, j]], {j, 1, 2 n}] + If[i < 3, 0, data[[n, i - 2]]]], {n, 14}, {i, 2 n - 1}]]]
```

{0}

{0}

```
(* The sum has natural boundaries. Thus
 creative telescoping gives its annihilator. *)
Timing[ct = FindCreativeTelescoping[DFiniteTimes[
     ToOrePolynomial[Append[ann, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]],
     Annihilator[Binomial[mu + i + j - 2, j + 1], {S[n], S[j], S[i]}]], S[j] - 1];
]
```

{1643.369000, Null}

```
ct = << "ct_0_2_A.m";
```

```
(* Look at the singularities of the certificate: no pole for 1 ≤ j ≤ 2n. *)
Factor[PolynomialLCM @@
  Denominator[Together[Flatten[OrePolynomialListCoefficients /@ Flatten[ct[[2]]]]]]]
```

$(1 + j + mu)\ (-1 + i + j + mu)\ (2 + 2\,j + mu)\ (-2 + j - 2\,n)\ (-1 + j - 2\,n)$
$(48\,j + 24\,j^2 - 4\,j\,mu - 22\,j^2\,mu - 8\,j^3\,mu - 2\,j^4\,mu - 18\,j\,mu^2 - 10\,j^2\,mu^2 - 4\,j^3\,mu^2 - 2\,j\,mu^3 - 2\,j^2\,mu^3 - 96\,n - 144\,j\,n - 88\,j^2\,n - 16\,j^3\,n - 4\,j^4\,n + 8\,mu\,n + 40\,j\,mu\,n + 56\,j^2\,mu\,n + 8\,j^3\,mu\,n + 4\,j^4\,mu\,n + 64\,mu^2\,n + 86\,j\,mu^2\,n + 27\,j^2\,mu^2\,n + 8\,j^3\,mu^2\,n + 22\,mu^3\,n + 17\,j\,mu^3\,n + 5\,j^2\,mu^3\,n + 2\,mu^4\,n + j\,mu^4\,n + 192\,n^2 + 288\,j\,n^2 + 176\,j^2\,n^2 + 32\,j^3\,n^2 + 8\,j^4\,n^2 + 176\,mu\,n^2 + 208\,j\,mu\,n^2 + 64\,j^2\,mu\,n^2 + 16\,j^3\,mu\,n^2 + 48\,mu^2\,n^2 + 36\,j\,mu^2\,n^2 + 10\,j^2\,mu^2\,n^2 + 4\,mu^3\,n^2 + 2\,j\,mu^3\,n^2)$

```
GBEqual[DFiniteSubstitute[ann, {j → i - 2}, Algebra → OreAlgebra[S[n], S[i]]], ct[[1]]]
```

True

In[64]:=
```
(* Identity (3) *)
TraditionalForm[
 HoldForm[Sum[(KroneckerDelta[2 n - 1, j + 1] + Binomial[μ + 2 n + j - 2, j + 1]) * c[n, j],
     {j, 1, 2 n}] == D0,2[2 n] / D0,2[2 n - 1] == r02] /. r02 → R02[n] /. mu → μ]
```

Out[64]//TraditionalForm=

$$\sum_{j=1}^{2n}\left(\delta_{2n-1,j+1} + \binom{\mu+2n+j-2}{j+1}\right) c(n,j) = \frac{D_{0,2}(2n)}{D_{0,2}(2n-1)} = \frac{(-1+2n)\left(\frac{1}{2}+\frac{\mu}{2}+2n\right)_{-1+n}(-2+\mu+2n)_{2+n}}{(\mu+2n)\,(n)_{2+n}\left(\frac{1}{2}+\frac{\mu}{2}+n\right)_{-1+n}}$$

```
(* Numerical check of Identity (3) (= initial value check) *)
Table[Together[
  Sum[FunctionExpand[Binomial[mu + 2 n + j - 2, j + 1]] * data[[n, j]], {j, 1, 2 n}] +
   If[n > 1, data[[n, 2 n - 2]], 0] - R02[n]], {n, 10}]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
(* The sum has natural boundaries. Thus
 creative telescoping gives its annihilator. *)
Timing[
 ct = FindCreativeTelescoping[DFiniteTimes[ann,
     Annihilator[Binomial[mu + 2 n + j - 2, j + 1], {S[n], S[j]}]], S[j] - 1];
]
```

{1186.696000, Null}

In[65]:=
```
(*Put[ct,"ct_0_2_B.m"];*)
ct = Get["ct_0_2_B.m"];
```

```
(* Look at the singularities of the certificate: no pole for 1 ≤ j ≤ 2n. *)
Factor[PolynomialLCM @@
  Denominator[Together[OrePolynomialListCoefficients[Flatten[ct][[2]]]]]]
```

$$2\,(1+j+mu)\,(2+2j+mu)\,(-4+j-2n)\,(-3+j-2n)\,(-2+j-2n)\,(-1+j-2n)\,(1+n)\,(3+2n)\,(4+j+mu+2n)\,(-2+mu+4n)\,(-1+mu+4n)\,(mu+4n)\,(2+mu+4n)\,(4+mu+4n)\,(6+mu+4n)\,(48j+24j^2-4j\,mu-22j^2\,mu-8j^3\,mu-2j^4\,mu-18j\,mu^2-10j^2\,mu^2-4j^3\,mu^2-2j\,mu^3-2j^2\,mu^3-96n-144j\,n-88j^2\,n-16j^3\,n-4j^4\,n+8\,mu\,n+40j\,mu\,n+56j^2\,mu\,n+8j^3\,mu\,n+4j^4\,mu\,n+64\,mu^2\,n+86j\,mu^2\,n+27j^2\,mu^2\,n+8j^3\,mu^2\,n+22\,mu^3\,n+17j\,mu^3\,n+5j^2\,mu^3\,n+2\,mu^4\,n+j\,mu^4\,n+192n^2+288j\,n^2+176j^2\,n^2+32j^3\,n^2+8j^4\,n^2+176\,mu\,n^2+208j\,mu\,n^2+64j^2\,mu\,n^2+16j^3\,mu\,n^2+48\,mu^2\,n^2+36j\,mu^2\,n^2+10j^2\,mu^2\,n^2+4\,mu^3\,n^2+2j\,mu^3\,n^2)$$

In contrast to Lemma 6, here the Kronecker delta in identity (3) plays a role (it gives 1 for $j = 2n-2$). Hence we have to add the corresponding term $c_{n,2n-2}$. The command DFinitePlus yields a recurrence operator (of order 2) for the sum (without Kronecker delta) plus $c_{n,2n-2}$.

```
In[70]:= annSum = DFinitePlus[ct[[1]], DFiniteSubstitute[ann, {j → 2 n - 2}]];
         Support[annSum]
```

Out[71]= $\{\{S_n^2, S_n, 1\}\}$

```
(* Plug our expression into the recurrence. *)
test = ApplyOreOperator[annSum[[1]], R02[n]];
(* Divide by the (non-zero!) expression itself,
to make simplification easier. *)
test = If[Head[test] === Plus, # / R02[n] & /@ test, test / R02[n]];
(* Simplify *)
Together[MySimp[test]]
```

0

```
(* Alternative: reduce annSum with the annihilator of R02 *)
OreReduce[annSum[[1]], Annihilator[R02[n], S[n]]]
```

Out[72]= 0

# Section 4: Nice Formula for $D_{1,1}(n)$

## Simple formula for $\prod_{j=1}^{k-1}((R_{1,0}(j)\,R_{0,1}(j))\,/\,(R_{0,0}(2\,j-1)\,R_{0,0}(2\,j)))$

```
(* The factor inside the product *)
fac = MySimp[R10[j] * R01[j] / R00o[j] / R00e[j]] /.
  Pochhammer[j, j + 2] → Pochhammer[j, j - 1] * (2 j - 1) * (2 j) * (2 j + 1)
```

$$\left((-1+2\,j+\text{mu})\,(-3+3\,j+\text{mu})\,(-2+3\,j+\text{mu})\,(-1+3\,j+\text{mu})\,\text{Pochhammer}\left[\frac{1}{2}+2\,j+\frac{\text{mu}}{2},\,-1+j\right]^2\text{Pochhammer}[1+2\,j+\text{mu},\,-1+j]^2\right)\Big/\left(j\,(1+2\,j)\,(-3+4\,j+\text{mu})\,(-1+4\,j+\text{mu})\,\text{Pochhammer}[j,\,-1+j]^2\,\text{Pochhammer}\left[\frac{1}{2}+j+\frac{\text{mu}}{2},\,-1+j\right]^2\right)$$

```
(* Test *)
Table[Together[(R10[j] * R01[j] / R00[2 j - 1] / R00[2 j]) / fac], {j, 10}]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

```
(* Final form for the product (use prod to avoid evaluation) *)
PR0110 = (Pochhammer[mu, 3 k - 3] / Pochhammer[mu / 2 + k - 1 / 2, k - 1] / (2 k - 1) !) *
   prod[Pochhammer[mu / 2 + 2 j + 1 / 2, j - 1] * Pochhammer[mu + 2 j + 1, j - 1] /
       Pochhammer[j, j - 1] / Pochhammer[mu / 2 + j + 1 / 2, j - 1], {j, 1, k - 1}] ^2;
TraditionalForm[(HoldForm @@ {PR0110}) /. {prod → Product, mu → μ}]
```

$$\frac{(\mu)_{-3+3\,k}\left(\prod_{j=1}^{-1+k}\frac{\left(\frac{1}{2}+2\,j+\frac{\mu}{2}\right)_{-1+j}(1+2\,j+\mu)_{-1+j}}{(j)_{-1+j}\left(\frac{1}{2}+j+\frac{\mu}{2}\right)_{-1+j}}\right)^2}{(-1+2\,k)!\left(-\frac{1}{2}+k+\frac{\mu}{2}\right)_{-1+k}}$$

```
(* Test *)
Table[Together[Product[R10[j] * R01[j] / R00[2 j - 1] / R00[2 j], {j, 1, k - 1}] /
   (PR0110 /. prod → Product)], {k, 10}]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

## Simple formula for $\prod_{i=2\,k}^{n} R_{0,0}(i)$

```
(* Split the product into even and odd instances of R00 *)
TraditionalForm[Product[R[i], {i, 2 k, n}] ==
  Product[R[2 j], {j, k, Floor[n / 2]}] * Product[R[2 j + 1], {j, k, Floor[(n - 1) / 2]}]]
```

$$\prod_{i=2\,k}^{n}R(i)=\left(\prod_{j=k}^{\left\lfloor\frac{n}{2}\right\rfloor}R(2\,j)\right)\prod_{j=k}^{\left\lfloor\frac{n-1}{2}\right\rfloor}R(2\,j+1)$$

```
(* Derivation *)
PR00 = prod[R00e[j], {j, k, Floor[n / 2]}] * prod[R00o[j + 1], {j, k, Floor[(n - 1) / 2]}];
PR00 = PR00 /. Pochhammer[a_, b_] :> Pochhammer[Expand[a], Expand[b]];
PR00e = PR00 /. Floor[a_] :> FullSimplify[Floor[a], Element[n / 2, Integers]];
PR00o = PR00 /. {Floor[n / 2] → (n - 1) / 2, Floor[(n - 1) / 2] → (n - 1) / 2} /.
      prod[a_, c_] * prod[b_, c_] → prod[a * b, c] /.
     Pochhammer[mu / 2 + 2 j + 1 / 2, j - 1] →
      Pochhammer[mu / 2 + 2 j - 1 / 2, j] / (mu / 2 + 2 j - 1 / 2) /.
    Pochhammer[mu / 2 + j + 1 / 2, j - 1] → Pochhammer[mu / 2 + j + 1 / 2, j] / (mu / 2 + 2 j - 1 / 2);
{PR00e, PR00o} = {PR00e, PR00o} /. prod[a_, b_] :> prod[a, Expand[b]]
```

$$\left\{\text{prod}\left[\left(\text{Pochhammer}\left[\frac{1}{2}+2j+\frac{mu}{2},-1+j\right]\text{Pochhammer}[2j+mu,j]\right)\Big/\left(\text{Pochhammer}[j,j]\,\text{Pochhammer}\left[\frac{1}{2}+j+\frac{mu}{2},-1+j\right]\right),\left\{j,k,\frac{n}{2}\right\}\right]\right.$$

$$\text{prod}\left[\left(\text{Pochhammer}\left[\frac{3}{2}+2j+\frac{mu}{2},1+j\right]\text{Pochhammer}[2j+mu,j]\right)\Big/\left(\text{Pochhammer}[1+j,1+j]\,\text{Pochhammer}\left[\frac{1}{2}+j+\frac{mu}{2},j\right]\right),\left\{j,k,-1+\frac{n}{2}\right\}\right],\ \text{prod}\left[\right.$$

$$\left(\text{Pochhammer}\left[-\frac{1}{2}+2j+\frac{mu}{2},j\right]\text{Pochhammer}\left[\frac{3}{2}+2j+\frac{mu}{2},1+j\right]\text{Pochhammer}[2j+mu,j]^2\right)\Big/\left(\text{Pochhammer}[j,j]\,\text{Pochhammer}[1+j,1+j]\,\text{Pochhammer}\left[\frac{1}{2}+j+\frac{mu}{2},j\right]^2\right),\left\{j,\right.$$

$$\left.\left.k,-\frac{1}{2}+\frac{n}{2}\right\}\right]\right\}$$

```
(* Tests *)
Flatten[Table[Together[(PR00o /. prod → Product) / Product[R00[i], {i, 2 k, n}]],
   {n, 1, 9, 2}, {k, 0, Floor[(n + 1) / 2]}]]
Flatten[Table[Together[(PR00e /. prod → Product) / Product[R00[i], {i, 2 k, n}]],
   {n, 2, 10, 2}, {k, 0, Floor[(n + 1) / 2]}]]
```

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

## Final formula for $D_{1,1}(n)$

```
(* The case distinction (k=0 vs. k>0) *)
Together[PR0110 * (2 k - 1) ! /. k → 0 /. prod → Product]
```

$$\frac{1}{2\ (-2+mu)\ (-1+mu)}$$

```
(* n even *)
FFe = sum[(mu - 1) / 2 * If[k == 0, 4 (mu - 2), 1 / (2 k - 1) !] *
     PR00e * (PR0110 * (2 k - 1) !), {k, 0, n / 2}];
(* n odd *)
FFo = sum[(mu - 1) / 2 * If[k == 0, 4 (mu - 2), 1 / (2 k - 1) !] *
     PR00o * (PR0110 * (2 k - 1) !), {k, 0, (n + 1) / 2}];
{FFe, FFo} = {FFe, FFo} /. (mu - 1) * Pochhammer[mu, 3 k - 3] → Pochhammer[mu - 1, 3 k - 2];
```

```
TraditionalForm[HoldForm[FF] /. FF → FFe /. {prod → Product, sum → Sum, mu → μ}]
```

$$\sum_{k=0}^{\frac{n}{2}}\left(\mathrm{If}\left[k=0,\,4\,(\mu-2),\,\frac{1}{(2\,k-1)!}\right](-1+\mu)_{-2+3\,k}\left(\prod_{j=k}^{\frac{n}{2}}\frac{\left(\frac{1}{2}+2\,j+\frac{\mu}{2}\right)_{-1+j}(2\,j+\mu)_j}{(j)_j\left(\frac{1}{2}+j+\frac{\mu}{2}\right)_{-1+j}}\right)\left(\prod_{j=k}^{-1+\frac{n}{2}}\frac{\left(\frac{3}{2}+2\,j+\frac{\mu}{2}\right)_{1+j}(2\,j+\mu)_j}{(1+j)_{1+j}\left(\frac{1}{2}+j+\frac{\mu}{2}\right)_j}\right)\left(\prod_{j=1}^{-1+k}\frac{\left(\frac{1}{2}+2\,j+\frac{\mu}{2}\right)_{-1+j}(1+2\,j+\mu)_{-1+j}}{(j)_{-1+j}\left(\frac{1}{2}+j+\frac{\mu}{2}\right)_{-1+j}}\right)^2\right)\Bigg/\left(2\left(-\frac{1}{2}+k+\frac{\mu}{2}\right)_{-1+k}\right)$$

```
TraditionalForm[HoldForm[FF] /. FF → FFo /. {prod → Product, sum → Sum, mu → μ}]
```

$$\sum_{k=0}^{\frac{1+n}{2}}\left(\mathrm{If}\left[k=0,\,4\,(\mu-2),\,\frac{1}{(2\,k-1)!}\right](-1+\mu)_{-2+3\,k}\left(\prod_{j=k}^{-\frac{1}{2}+\frac{n}{2}}\frac{\left(-\frac{1}{2}+2\,j+\frac{\mu}{2}\right)_j\left(\frac{3}{2}+2\,j+\frac{\mu}{2}\right)_{1+j}((2\,j+\mu)_j)^2}{(j)_j\,(1+j)_{1+j}\left(\left(\frac{1}{2}+j+\frac{\mu}{2}\right)_j\right)^2}\right)\left(\prod_{j=1}^{-1+k}\frac{\left(\frac{1}{2}+2\,j+\frac{\mu}{2}\right)_{-1+j}(1+2\,j+\mu)_{-1+j}}{(j)_{-1+j}\left(\frac{1}{2}+j+\frac{\mu}{2}\right)_{-1+j}}\right)^2\right)\Bigg/\left(2\left(-\frac{1}{2}+k+\frac{\mu}{2}\right)_{-1+k}\right)$$

```
(* Test *)
Table[Together[
   (If[EvenQ[n], FFe, FFo] /. sum → Sum /. prod → Product) / Dst[1, 1, n]], {n, 10}]
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

# Section 5: Proof of the Monstrous Conjecture

## The Conjecture

In[26]:=
```
Clear[myC, myE, myF, myT, myS1, myS2, myP1, myP2, myG, D34];
myC[n_Integer?Positive] := ((-1)^n + 3) / 2 * Product[Floor[i / 2]! / i!, {i, 1, n}];
myE[n_Integer?Positive, mu_] := Pochhammer[mu + 1, n] * Product[
    (mu + 2 i + 6)^(2 * Floor[(i + 2) / 3]), {i, 1, Floor[3 / 2 * Floor[(n - 1) / 2] - 2]}] *
   Product[(mu + 2 i + 2 * Floor[3 / 2 * Floor[n / 2 + 1]] - 1)^
     (2 * Floor[Floor[n / 2] / 2 - (i - 1) / 3] - 1), {i, 1, Floor[3 / 2 * Floor[n / 2] - 2]}];
myF[m : (0 | 1), n_Integer?Positive, mu_] :=
  Product[(mu + 2 i + n + m)^(1 - 2 i - m), {i, 1, Floor[(n - 1) / 4]}] *
   Product[(mu - 2 i + 2 n - 2 m + 1)^(1 - 2 i - m), {i, 1, Floor[n / 4 - 1]}];
myF[n_Integer?Positive, mu_] := Which[
   EvenQ[n], myE[n, mu] * myF[0, n, mu],
   OddQ[n], myE[n, mu] * myF[1, n, mu] * Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}]];
myT[k_, mu_] := -12 + 84 * k + 288 * k^2 - 5856 * k^3 + 20352 * k^4 -
   41472 * k^5 + 55296 * k^6 + 10 * mu + 76 * k * mu - 2176 * k^2 * mu + 9888 * k^3 * mu -
   25344 * k^4 * mu + 41472 * k^5 * mu + 10 * mu^2 - 261 * k * mu^2 + 1676 * k^2 * mu^2 -
   5472 * k^3 * mu^2 + 11520 * k^4 * mu^2 - 10 * mu^3 + 115 * k * mu^3 - 488 * k^2 * mu^3 +
   1440 * k^3 * mu^3 + 2 * mu^4 - 15 * k * mu^4 + 76 * k^2 * mu^4 + k * mu^5;
myS1[n_Integer?Positive, mu_] := Sum[myT[k, mu] * 2^(6 k) * (mu + 8 k - 1) *
     Pochhammer[1 / 2, 2 k - 1]^2 * Pochhammer[(mu + 4 k + 2) / 2, 2 n - 2 k - 2] *
     Pochhammer[(mu + 5) / 2, 2 k - 3] * Pochhammer[(mu + 4 k + 2) / 2, k - 2] /
      ((2 k)! * Pochhammer[(mu + 6 k - 3) / 2, 3 k + 4]), {k, 1, n - 1}];
myS2[n_Integer?Positive, mu_] :=
  Sum[myT[k + 1 / 2, mu] * 2^(6 k) * (mu + 8 k + 3) *
     Pochhammer[1 / 2, 2 k]^2 * Pochhammer[(mu + 4 k + 4) / 2, 2 n - 2 k - 2] *
     Pochhammer[(mu + 5) / 2, 2 k - 2] * Pochhammer[(mu + 4 k + 4) / 2, k - 2] /
      ((2 k + 1)! * Pochhammer[(mu + 6 k + 1) / 2, 3 k + 5]), {k, 1, n - 1}];
myP1[n_Integer?Positive, mu_] := Together[
   2^(3 n - 1) * Pochhammer[(mu + 6 n - 3) / 2, 3 n - 2] / Pochhammer[(mu + 5) / 2, 2 n - 3] *
    (2^(-13) * mu * (mu - 1) * myS1[n, mu] +
      1 / (mu + 3)^2 * Pochhammer[(mu + 2) / 2, 2 n - 2])];
myP2[n_Integer?Positive, mu_] := Together[
   2^(3 n - 1) * Pochhammer[(mu + 6 n + 1) / 2, 3 n - 1] / Pochhammer[(mu + 5) / 2, 2 n - 2] *
    (2^(-9) mu * (mu - 1) * myS2[n, mu] +
      (mu + 14) / ((mu + 7) * (mu + 9)) * Pochhammer[(mu + 4) / 2, 2 n - 2])];
myG[n_Integer?Positive, mu_] := If[EvenQ[n], myP2[n / 2, mu], myP1[(n + 1) / 2, mu]];
D34[n_Integer?Positive, mu_] := myC[n] * myF[n, mu] * myG[Floor[(n + 1) / 2], mu];
```

```
(* even n *)
Table[Together[(FFe /. sum → Sum /. prod → Product) /
    (myC[n] * myE[n, mu] * myF[0, n, mu] * myP2[n / 4, mu])], {n, 4, 12, 4}]
Table[Together[(FFe /. sum → Sum /. prod → Product) /
    (myC[n] * myE[n, mu] * myF[0, n, mu] * myP1[(n + 2) / 4, mu])], {n, 2, 10, 4}]
(* odd n *)
Table[
 Together[(FFo /. sum → Sum /. prod → Product) / (myC[n] * myE[n, mu] * myF[1, n, mu] *
      Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}] * myP1[(n + 3) / 4, mu])], {n, 1, 9, 4}]
Table[Together[(FFo /. sum → Sum /. prod → Product) / (myC[n] * myE[n, mu] * myF[1, n, mu] *
      Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}] * myP2[(n + 1) / 4, mu])], {n, 3, 11, 4}]
```

```
{1, 1, 1}

{1, 1, 1}

{1, 1, 1}

{1, 1, 1}
```

In[38]:=
```
(* These are the recurrence operators that we used to define P_1 and P_2. *)
rec1 = NormalizeCoefficients[ToOrePolynomial[
    -myP1fast[n, mu] + (-8 * (-10 + mu + 4 * n) * (-8 + mu + 4 * n) * (-14 + mu + 6 * n) *
          (-12 + mu + 6 * n) * (-10 + mu + 6 * n) * (-9 + mu + 8 * n) * (-31 + mu + 12 * n) *
          (-29 + mu + 12 * n) * (-27 + mu + 12 * n) * (-25 + mu + 12 * n) * (-23 + mu + 12 * n) *
          (-21 + mu + 12 * n) * (81 - 72 * n + 16 * n^2) * (49 - 56 * n + 16 * n^2) *
          (123 168 - 78 946 * mu + 18 939 * mu^2 - 2053 * mu^3 + 93 * mu^4 - mu^5 +
             (-9 + mu) * (70 956 - 30 208 * mu + 3989 * mu^2 - 158 * mu^3 + mu^4) * n +
             4 * (346 032 - 149 656 * mu + 21 803 * mu^2 - 1202 * mu^3 + 19 * mu^4) * n^2 +
             96 * (-9 + mu) * (1861 - 402 * mu + 15 * mu^2) * n^3 +
             384 * (2753 - 606 * mu + 30 * mu^2) * n^4 + 41 472 * (-9 + mu) * n^5 + 55 296 * n^6) *
          myP1fast[-2 + n, mu] + (-13 + mu + 8 * n) * (-6 * (-15 + mu) * (-11 + mu) * (-9 + mu) *
               (77 272 834 343 040 - 90 508 623 095 808 * mu + 46 786 094 223 720 * mu^2 -
                  14 041 912 717 156 * mu^3 + 2 707 887 452 266 * mu^4 - 350 541 498 059 * mu^5 +
                  30 888 280 625 * mu^6 - 1 838 952 303 * mu^7 + 72 032 193 * mu^8 -
                  1 778 033 * mu^9 + 26 555 * mu^10 - 241 * mu^11 + mu^12) +
             (-13 + mu) * (615 591 764 176 296 960 - 787 691 318 438 414 592 * mu +
                  453 271 146 257 615 040 * mu^2 - 154 970 921 382 725 880 * mu^3 +
                  35 030 740 197 791 460 * mu^4 - 5 511 255 715 119 386 * mu^5 +
                  618 465 455 797 003 * mu^6 - 49 890 145 667 170 * mu^7 +
                  2 877 469 024 970 * mu^8 - 116 576 723 262 * mu^9 + 3 218 550 024 * mu^10 -
                  58 094 110 * mu^11 + 655 730 * mu^12 - 4400 * mu^13 + 13 * mu^14) * n +
             (43 790 163 197 061 415 680 - 55 769 554 581 921 674 496 * mu +
                  32 100 807 569 482 408 752 * mu^2 - 11 046 065 343 390 418 896 * mu^3 +
                  2 532 539 665 806 086 200 * mu^4 - 408 068 212 472 225 048 * mu^5 +
                  47 486 735 062 736 003 * mu^6 - 4 036 853 597 489 641 * mu^7 +
                  250 606 824 181 572 * mu^8 - 11 237 476 473 228 * mu^9 +
                  356 071 800 098 * mu^10 - 7 704 642 502 * mu^11 + 108 621 484 * mu^12 -
                  941 780 * mu^13 + 4611 * mu^14 - 9 * mu^15) * n^2 +
             2 * (-13 + mu) * (5 765 368 315 087 296 000 - 6 423 796 647 403 130 880 * mu +
```

```
    3 186 986 194 272 026 736 * mu ^ 2 - 928 737 086 880 929 008 * mu ^ 3 +
    176 577 512 806 080 224 * mu ^ 4 - 23 002 876 518 214 396 * mu ^ 5 +
    2 097 912 117 891 133 * mu ^ 6 - 134 465 197 774 532 * mu ^ 7 +
    5 992 468 266 728 * mu ^ 8 - 181 075 265 324 * mu ^ 9 + 3 560 096 842 * mu ^ 10 -
    42 928 700 * mu ^ 11 + 293 696 * mu ^ 12 - 1000 * mu ^ 13 + mu ^ 14) * n ^ 3 +
8 * (44 967 647 815 472 773 440 - 49 875 119 477 893 931 904 * mu +
    24 771 543 294 236 452 512 * mu ^ 2 - 7 277 588 373 063 623 552 * mu ^ 3 +
    1 407 087 781 066 080 464 * mu ^ 4 - 188 436 568 279 081 716 * mu ^ 5 +
    17 910 169 812 661 579 * mu ^ 6 - 1 217 322 600 443 922 * mu ^ 7 +
    58 827 888 448 174 * mu ^ 8 - 1 983 671 151 898 * mu ^ 9 + 45 113 742 796 * mu ^ 10 -
    655 655 046 * mu ^ 11 + 5 605 730 * mu ^ 12 - 24 666 * mu ^ 13 + 41 * mu ^ 14) * n ^ 4 +
32 * (-13 + mu) * (1 545 137 447 830 050 528 - 1 468 846 207 754 989 056 * mu +
    613 359 955 784 013 384 * mu ^ 2 - 148 046 294 338 567 160 * mu ^ 3 +
    22 867 645 137 091 796 * mu ^ 4 - 2 363 768 523 778 396 * mu ^ 5 +
    166 104 951 524 749 * mu ^ 6 - 7 900 529 853 234 * mu ^ 7 + 248 588 564 859 * mu ^ 8 -
    4 947 975 304 * mu ^ 9 + 57 722 923 * mu ^ 10 - 345 266 * mu ^ 11 + 785 * mu ^ 12) *
  n ^ 5 + 128 * (6 923 436 910 786 740 816 - 6 551 979 917 272 781 760 * mu +
    2 741 775 205 145 125 620 * mu ^ 2 - 668 624 737 408 815 316 * mu ^ 3 +
    105 402 483 452 844 020 * mu ^ 4 - 11 258 804 752 461 004 * mu ^ 5 +
    830 334 150 499 955 * mu ^ 6 - 42 256 983 681 030 * mu ^ 7 +
    1 457 399 275 653 * mu ^ 8 - 32 763 679 904 * mu ^ 9 + 447 520 681 * mu ^ 10 -
    3 258 554 * mu ^ 11 + 9319 * mu ^ 12) * n ^ 6 + 1024 * (-13 + mu) *
  (72 414 477 952 775 604 - 57 105 723 925 009 800 * mu + 19 399 742 350 341 207 *
     mu ^ 2 - 3 719 307 354 992 416 * mu ^ 3 + 442 850 412 559 382 * mu ^ 4 -
    33 955 375 237 500 * mu ^ 5 + 1 681 820 711 178 * mu ^ 6 - 52 507 834 704 * mu ^ 7 +
    974 233 650 * mu ^ 8 - 9 518 828 * mu ^ 9 + 36 355 * mu ^ 10) * n ^ 7 +
4096 * (204 759 442 490 425 380 - 160 746 724 570 083 012 * mu +
    54 801 297 077 548 677 * mu ^ 2 - 10 648 677 530 738 482 * mu ^ 3 +
    1 300 829 127 395 384 * mu ^ 4 - 103 865 351 431 818 * mu ^ 5 +
    5 455 145 057 379 * mu ^ 6 - 184 594 947 228 * mu ^ 7 + 3 811 103 508 * mu ^ 8 -
    42 749 540 * mu ^ 9 + 194 248 * mu ^ 10) * n ^ 8 + 49 152 * (-13 + mu) *
  (920 215 916 156 142 - 577 914 239 846 832 * mu + 151 701 784 373 213 * mu ^ 2 -
    21 614 250 577 806 * mu ^ 3 + 1 815 722 558 519 * mu ^ 4 - 91 353 917 016 * mu ^ 5 +
    2 663 224 490 * mu ^ 6 - 40 669 644 * mu ^ 7 + 245 586 * mu ^ 8) * n ^ 9 + 196 608 *
  (1 693 595 159 851 230 - 1 058 822 980 698 432 * mu + 279 542 833 819 585 * mu ^ 2 -
    40 572 445 515 984 * mu ^ 3 + 3 526 446 267 001 * mu ^ 4 - 187 021 320 840 * mu ^ 5 +
    5 872 755 784 * mu ^ 6 - 99 020 958 * mu ^ 7 + 679 074 * mu ^ 8) * n ^ 10 + 21 233 664 *
  (-13 + mu) * (550 446 775 412 - 258 091 315 032 * mu + 47 985 773 125 * mu ^ 2 -
    4 496 668 860 * mu ^ 3 + 222 288 724 * mu ^ 4 - 5 456 352 * mu ^ 5 + 51 547 * mu ^ 6) *
  n ^ 11 + 28 311 552 * (1 958 821 138 060 - 914 306 594 496 * mu +
    171 668 385 371 * mu ^ 2 - 16 540 689 390 * mu ^ 3 + 859 090 262 * mu ^ 4 -
    22 689 546 * mu ^ 5 + 236 549 * mu ^ 6) * n ^ 12 + 21 403 533 312 * (-13 + mu) *
  (57 395 792 - 17 859 456 * mu + 1 964 631 * mu ^ 2 - 89 610 * mu ^ 3 + 1425 * mu ^ 4) *
  n ^ 13 + 12 230 590 464 * (290 157 464 - 89 880 912 * mu +
    10 081 119 * mu ^ 2 - 483 594 * mu ^ 3 + 8337 * mu ^ 4) * n ^ 14 +
3 522 410 053 632 * (-13 + mu) * (12 823 - 1986 * mu + 69 * mu ^ 2) * n ^ 15 +
3 522 410 053 632 * (19 340 - 2982 * mu + 111 * mu ^ 2) * n ^ 16 +
380 420 285 792 256 * (-13 + mu) * n ^ 17 +
```

```
            169 075 682 574 336 * n^18) * myP1fast[-1 + n, mu]) /
        ((-1 + n) * (-3 + 2 * n) * (-6 + mu + 4 * n) * (-5 + mu + 4 * n) *
          (-4 + mu + 4 * n) *
          (-3 + mu + 4 * n) *
          (-9 + mu + 6 * n) *
          (-7 + mu + 6 * n) * (-5 + mu + 6 * n) *
          (-17 + mu + 8 * n) *
          (-2 * (-2 619 750 + 910 279 * mu - 117 666 * mu^2 + 6856 * mu^3 - 168 * mu^4 + mu^5) +
            (-17 + mu) * (862 188 - 199 648 * mu + 14 213 * mu^2 - 302 * mu^3 + mu^4) * n +
            4 * (4 278 168 - 996 880 * mu + 77 747 * mu^2 - 2282 * mu^3 + 19 * mu^4) * n^2 +
            96 * (-17 + mu) * (6541 - 762 * mu + 15 * mu^2) * n^3 +
            384 * (9773 - 1146 * mu + 30 * mu^2) * n^4 +
            41 472 * (-17 + mu) * n^5 + 55 296 * n^6)),
      myP1fast[n, mu], OreAlgebra[S[n]]]];
rec2 = NormalizeCoefficients[ToOrePolynomial[
      -myP2fast[n, mu] + (-8 * (-8 + mu + 4 * n) * (-6 + mu + 4 * n) * (-12 + mu + 6 * n) *
           (-10 + mu + 6 * n) * (-8 + mu + 6 * n) * (-5 + mu + 8 * n) * (-25 + mu + 12 * n) *
           (-23 + mu + 12 * n) * (-21 + mu + 12 * n) * (-19 + mu + 12 * n) * (-17 + mu + 12 * n) *
           (-15 + mu + 12 * n) * (49 - 56 * n + 16 * n^2) * (25 - 40 * n + 16 * n^2) *
           (-((-3 + mu) * (2788 - 2196 * mu + 577 * mu^2 - 54 * mu^3 + mu^4)) +
             2 * (-5 + mu) * (7620 - 5536 * mu + 1253 * mu^2 - 86 * mu^3 + mu^4) * n +
             8 * (35 820 - 26 716 * mu + 6791 * mu^2 - 662 * mu^3 + 19 * mu^4) * n^2 +
             192 * (-5 + mu) * (601 - 222 * mu + 15 * mu^2) * n^3 +
             768 * (863 - 336 * mu + 30 * mu^2) * n^4 + 82 944 * (-5 + mu) * n^5 + 110 592 * n^6) *
           myP2fast[-2 + n, mu] + (-9 + mu + 8 * n) * (-((-11 + mu) * (-7 + mu) *
                 (-5 + mu) * (-3 + mu) * (-941 137 562 880 + 1 369 543 037 568 * mu -
                   856 059 425 680 * mu^2 + 301 467 356 208 * mu^3 - 65 925 560 840 * mu^4 +
                   9 300 152 544 * mu^5 - 851 420 265 * mu^6 + 49 707 939 * mu^7 -
                   1 788 230 * mu^8 + 38 538 * mu^9 - 505 * mu^10 + 3 * mu^11)) + (-9 + mu) *
               (2 174 231 624 313 600 - 4 271 307 638 939 136 * mu + 3 746 500 640 981 808 * mu^2 -
                 1 938 172 937 860 384 * mu^3 + 658 024 132 807 528 * mu^4 - 154 336 161 708 664 *
                  mu^5 + 25 631 896 940 311 * mu^6 - 3 038 647 883 536 * mu^7 +
                 255 911 958 856 * mu^8 - 15 059 474 264 * mu^9 + 601 933 862 * mu^10 -
                 15 728 672 * mu^11 + 258 008 * mu^12 - 2528 * mu^13 + 11 * mu^14) * n -
             4 * (-41 108 205 131 322 624 + 79 558 217 840 920 896 * mu -
                 69 190 984 849 287 408 * mu^2 + 35 769 692 404 688 632 * mu^3 -
                 12 252 335 726 377 252 * mu^4 + 2 933 722 316 842 738 * mu^5 -
                 504 752 475 079 572 * mu^6 + 63 145 862 893 203 * mu^7 - 5 745 369 671 196 * mu^
                   8 + 376 356 342 416 * mu^9 - 17 384 266 580 * mu^10 + 548 066 954 * mu^11 -
                 11 274 720 * mu^12 + 143 142 * mu^13 - 1032 * mu^14 + 3 * mu^15) * n^2 +
             4 * (-9 + mu) * (23 845 345 590 072 960 - 40 269 695 568 954 624 * mu +
                 30 117 142 128 190 992 * mu^2 - 13 158 415 762 916 400 * mu^3 +
                 3 730 778 330 679 232 * mu^4 - 721 067 843 021 868 * mu^5 +
                 97 108 500 711 985 * mu^6 - 9 153 045 269 192 * mu^7 +
                 597 928 404 668 * mu^8 - 26 432 573 136 * mu^9 + 759 984 806 * mu^10 -
                 13 416 552 * mu^11 + 134 684 * mu^12 - 676 * mu^13 + mu^14) * n^3 +
             16 * (195 243 602 402 676 096 - 325 694 901 477 820 032 * mu +
                 242 221 596 032 134 128 * mu^2 - 106 098 978 486 724 128 * mu^3 +
```

```
30 459 989 915 673 992 * mu ^ 4 - 6 033 975 669 037 412 * mu ^ 5 +
    845 417 566 861 997 * mu ^ 6 - 84 452 424 919 988 * mu ^ 7 +
    5 983 741 160 080 * mu ^ 8 - 295 319 349 276 * mu ^ 9 + 9 821 158 066 * mu ^ 10 -
    208 695 676 * mu ^ 11 + 2 610 160 * mu ^ 12 - 16 816 * mu ^ 13 + 41 * mu ^ 14) * n ^ 4 +
 64 * (-9 + mu) * (14 622 810 947 299 008 - 20 883 005 872 697 088 * mu +
    13 042 640 269 010 160 * mu ^ 2 - 4 687 978 533 249 048 * mu ^ 3 +
    1 073 821 472 622 084 * mu ^ 4 - 163 965 505 744 412 * mu ^ 5 + 16 961 587 465 549 *
     mu ^ 6 - 1 184 203 363 074 * mu ^ 7 + 54 575 767 659 * mu ^ 8 - 1 588 856 808 * mu ^ 9 +
    27 087 171 * mu ^ 10 - 236 578 * mu ^ 11 + 785 * mu ^ 12) * n ^ 5 +
 256 * (68 159 047 060 299 744 - 96 275 531 839 385 520 * mu + 59 912 949 582 646 848 *
     mu ^ 2 - 21 651 596 638 546 640 * mu ^ 3 + 5 041 402 403 618 604 * mu ^ 4 -
    793 018 597 591 700 * mu ^ 5 + 85 896 040 596 299 * mu ^ 6 -
    6 405 365 947 182 * mu ^ 7 + 323 083 532 589 * mu ^ 8 - 10 605 978 520 * mu ^ 9 +
    211 271 829 * mu ^ 10 - 2 240 614 * mu ^ 11 + 9319 * mu ^ 12) * n ^ 6 +
 2048 * (-9 + mu) * (1 541 341 241 341 668 - 1 813 373 921 002 968 * mu +
    915 446 118 884 163 * mu ^ 2 - 259 814 716 685 092 * mu ^ 3 +
    45 629 241 741 242 * mu ^ 4 - 5 143 009 129 752 * mu ^ 5 + 373 337 413 062 * mu ^ 6 -
    17 038 328 436 * mu ^ 7 + 461 072 406 * mu ^ 8 - 6 556 280 * mu ^ 9 + 36 355 * mu ^ 10) *
  n ^ 7 + 8192 * (4 500 207 031 276 008 - 5 239 264 901 634 576 * mu +
    2 640 189 965 261 667 * mu ^ 2 - 755 987 780 804 488 * mu ^ 3 +
    135 697 154 047 598 * mu ^ 4 - 15 878 627 119 200 * mu ^ 5 +
    1 219 280 284 095 * mu ^ 6 - 60 190 646 760 * mu ^ 7 + 1 809 241 320 * mu ^ 8 -
    29 487 896 * mu ^ 9 + 194 248 * mu ^ 10) * n ^ 8 + 98 304 * (-9 + mu) *
   (43 421 763 841 182 - 40 431 075 715 248 * mu + 15 676 365 711 905 * mu ^ 2 -
    3 287 037 266 982 * mu ^ 3 + 404 944 404 503 * mu ^ 4 - 29 779 385 976 * mu ^ 5 +
    1 265 065 310 * mu ^ 6 - 28 070 508 * mu ^ 7 + 245 586 * mu ^ 8) * n ^ 9 +
 393 216 * (81 960 492 523 446 - 75 530 247 171 240 * mu + 29 304 166 747 543 * mu ^ 2 -
    6 232 480 720 254 * mu ^ 3 + 791 707 261 321 * mu ^ 4 - 61 212 289 536 * mu ^ 5 +
    2 795 677 186 * mu ^ 6 - 68 402 040 * mu ^ 7 + 679 074 * mu ^ 8) * n ^ 10 +
 42 467 328 * (-9 + mu) * (56 814 548 324 - 39 259 013 448 * mu +
    10 716 187 369 * mu ^ 2 - 1 468 655 040 * mu ^ 3 +
    105 783 400 * mu ^ 4 - 3 770 148 * mu ^ 5 + 51 547 * mu ^ 6) * n ^ 11 +
 56 623 104 * (206 001 269 260 - 140 889 461 280 * mu + 38 650 036 817 * mu ^ 2 -
    5 426 533 428 * mu ^ 3 + 409 650 908 * mu ^ 4 - 15 687 096 * mu ^ 5 + 236 549 * mu ^ 6) *
  n ^ 12 + 42 807 066 624 * (-9 + mu) * (12 790 352 - 5 830 560 * mu +
    935 607 * mu ^ 2 - 61 962 * mu ^ 3 + 1425 * mu ^ 4) * n ^ 13 + 24 461 180 928 *
   (65 459 144 - 29 536 992 * mu + 4 812 879 * mu ^ 2 - 334 554 * mu ^ 3 + 8337 * mu ^ 4) *
  n ^ 14 + 7 044 820 107 264 * (-9 + mu) * (6091 - 1374 * mu + 69 * mu ^ 2) * n ^ 15 +
 7 044 820 107 264 * (9242 - 2064 * mu + 111 * mu ^ 2) * n ^ 16 +
 760 840 571 584 512 * (-9 + mu) * n ^ 17 +
 338 151 365 148 672 * n ^ 18) * myP2fast[-1 + n, mu]) /
((-1 + n) * (-1 + 2 * n) * (-4 + mu + 4 * n) * (-3 + mu + 4 * n) *
  (-2 + mu + 4 * n) *
  (-1 + mu + 4 * n) *
  (-5 + mu + 6 * n) *
  (-3 + mu + 6 * n) *
  (-1 + mu + 6 * n) *
  (-13 + mu + 8 * n) *
```

```
        (2 136 180 - 963 208 * mu + 161 921 * mu^2 - 12 281 * mu^3 + 391 * mu^4 - 3 * mu^5 +
          2 * (-13 + mu) * (298 788 - 89 728 * mu + 8309 * mu^2 - 230 * mu^3 + mu^4) * n +
          8 * (1 475 028 - 447 124 * mu + 45 455 * mu^2 - 1742 * mu^3 + 19 * mu^4) * n^2 +
          192 * (-13 + mu) * (3841 - 582 * mu + 15 * mu^2) * n^3 +
          768 * (5723 - 876 * mu + 30 * mu^2) * n^4 +
          82 944 * (-13 + mu) * n^5 + 110 592 * n^6)),
    myP2fast[n, mu], OreAlgebra[S[n]]]];
```

## Preparing the stage

```
(* Alternative closed form for D_{1,1}(n) *)
{FFeAlt, FFoAlt} =
  {FFe, FFo} /. Pochhammer[j, j - 1] → Pochhammer[1 / 2, j - 1] * 2^(2 j - 2) /.
          Pochhammer[mu + 2 j + 1, j - 1] / Pochhammer[mu / 2 + j + 1 / 2, j - 1] →
           2^(2 j - 3) * Pochhammer[mu / 2 + j + 1, j - 2] / Pochhammer[mu + 3 j, j - 2] /.
         Pochhammer[mu + 2 j, j]^2 / Pochhammer[mu / 2 + j + 1 / 2, j]^2 →
          2^(2 j) * Pochhammer[mu / 2 + j, Floor[(j + 1) / 2]]^2 /
            Pochhammer[mu / 2 + Floor[3 j / 2] + 1 / 2, Floor[(j + 1) / 2]]^2 /.
        Pochhammer[mu + 2 j, j] / Pochhammer[mu / 2 + j + 1 / 2, j] →
         2^j * Pochhammer[mu / 2 + j, Floor[(j + 1) / 2]] /
           Pochhammer[mu / 2 + Floor[3 j / 2] + 1 / 2, Floor[(j + 1) / 2]] /.
       Pochhammer[mu + 2 j, j] / Pochhammer[mu / 2 + j + 1 / 2, j - 1] →
        2^j * Pochhammer[mu / 2 + j, Floor[(j + 1) / 2]] /
          Pochhammer[mu / 2 + Floor[3 j / 2] + 1 / 2, Floor[(j - 1) / 2]] /.
     prod[2^a_ * b_, c_] :→ Product[2^a, c] * prod[b, c] /.
    prod[a_ / 2, {j, 1, k - 1}]^2 * If[k == 0, b_, c_] →
     2^(2 - 2 k) * If[k == 0, b / 4, c] * prod[a, {j, 1, k - 1}]^2 /.
   2^a_ :→ 2^Expand[a];

(* Test *)
Table[Together[(FFeAlt / FFe) /. {sum → Sum, prod → Product}], {n, 2, 10, 2}]
Table[Together[(FFoAlt / FFo) /. {sum → Sum, prod → Product}], {n, 1, 9, 2}]
```

```
{1, 1, 1, 1, 1}

{1, 1, 1, 1, 1}
```

## even n

```
FFe1 = FFeAlt /. prod[a_, {j, k, b_}] :>
        prod[a, {j, 1, b}] / prod[a, {j, 1, k - 1}] * if[k == 0, a /. j -> 0, 1] //.
      prod[a1_, b_] ^c1_. * prod[a2_, b_] ^c2_. -> prod[a1^c1 * a2^c2, b] //.
     sum[(a_ /; FreeQ[a, k]) * b_, c_] -> a * sum[b, c] //.
    prod[a_ * b_, {j, 1, n / 2 + c_.}] -> prod[a, {j, 1, n / 2 + c}] * prod[b, {j, 1, n / 2 + c}];
Check0 = FFe1 / (
     (* myC[n] *)
     ((-1) ^n + 3) / 2 * prod[Floor[i / 2] ! / i !, {i, 1, n}] *
      (* myE[n,mu] *)
      Pochhammer[mu + 1, n] * prod[
        (mu + 2 i + 6) ^ (2 * Floor[(i + 2) / 3]), {i, 1, Floor[3 / 2 * Floor[(n - 1) / 2] - 2]}] *
      prod[(mu + 2 i + 2 * Floor[3 / 2 * Floor[n / 2 + 1]] - 1) ^
         (2 * Floor[Floor[n / 2] / 2 - (i - 1) / 3] - 1),
        {i, 1, Floor[3 / 2 * Floor[n / 2] - 2]}] *
      (* myF[0,n,mu] *)
      prod[(mu + 2 i + n) ^ (1 - 2 i), {i, 1, Floor[(n - 1) / 4]}] *
      prod[(mu - 2 i + 2 n + 1) ^ (1 - 2 i), {i, 1, Floor[n / 4 - 1]}]);
Check0 = Check0 /. {(-1) ^n -> 1, Floor[n / 2] -> n / 2, Floor[(n - 1) / 2] -> n / 2 - 1} /.
    Floor[a_] :> Floor[Together[a]];
```

## $n$ = 0 mod 4

```
Check00 =
 Check0 /. {Floor[3 / 4 * (n - 2)] -> 3 / 4 n - 2, Floor[n / 4] -> n / 4, Floor[3 / 4 n] -> 3 / 4 n,
                           Floor[(n - 1) / 4] -> n / 4 - 1, Floor[3 / 4 (n + 2)] -> 3 / 4 n + 1,
                           Floor[(3 n - 4 i + 4) / 12] -> n / 4 + Floor[(1 - i) / 3]} /.
```

$$\text{prod}\left[\text{Pochhammer}\left[j+\frac{mu}{2},\ \text{Floor}\left[\frac{1+j}{2}\right]\right],\ \left\{j,\ 1,\ \frac{n}{2}\right\}\right]*$$

$$\text{prod}\left[\text{Pochhammer}\left[j+\frac{mu}{2},\ \text{Floor}\left[\frac{1+j}{2}\right]\right],\ \left\{j,\ 1,\ -1+\frac{n}{2}\right\}\right]\Big/$$

$$\left(\text{prod}\left[(6+2\,i+mu)^{2\,\text{Floor}\left[\frac{2+i}{3}\right]},\ \left\{i,\ 1,\ -4+\frac{3\,n}{4}\right\}\right]*\right.$$

$$\left.\text{prod}\left[(2\,i+mu+n)^{1-2\,i},\ \left\{i,\ 1,\ -1+\frac{n}{4}\right\}\right]\right)\to$$

```
Pochhammer[mu / 2 + 1, n / 2] * Pochhammer[mu / 2 + 1,
   n / 2 - 1] / 2^ (n^2 / 8 - 3 / 4 n + 1) /.
```

$$\text{prod}\left[\text{Pochhammer}\left[\frac{1}{2}+2\,j+\frac{mu}{2},\ -1+j\right],\ \left\{j,\ 1,\ \frac{n}{2}\right\}\right]$$

$$\text{prod}\left[\text{Pochhammer}\left[\frac{3}{2}+2\,j+\frac{mu}{2},\ 1+j\right],\ \left\{j,\ 1,\ -1+\frac{n}{2}\right\}\right]$$

$$\text{prod}\left[1\Big/\ \text{Pochhammer}\left[\frac{1}{2}+\frac{mu}{2}+\text{Floor}\left[\frac{3\,j}{2}\right],\ \text{Floor}\left[\frac{1}{2}\ (-1+j)\right]\right],\right.$$

$$\left.\left\{j,\ 1,\ \frac{n}{2}\right\}\right]\ \text{prod}\left[1\Big/\ \text{Pochhammer}\left[\frac{1}{2}+\frac{mu}{2}+\text{Floor}\left[\frac{3\,j}{2}\right],\right.\right.$$

```
Floor[(1 + j)/2]], {j, 1, -1 + n/2}]/
  (prod[(-1 + 2 i + mu + 2 (1 + (3 n)/4))^(-1+2 (n/4+Floor[(1-i)/3])), {i,
      1, -2 + (3 n)/4}] prod[(1 - 2 i + mu + 2 n)^(1-2 i), {i, 1, -1 + n/4}]) →
   Pochhammer[mu / 2 + 3 / 4 n + 1 / 2, 3 / 4 n - 1] / (mu + 3) /
    2 ^ (n ^ 2 / 8 - n / 2) /.
  prod[1/Pochhammer[j, j], {j, 1, n/2}] prod[1/Pochhammer[1 + j, 1 + j],
      {j, 1, -1 + n/2}] / prod[Floor[i/2] !/i !, {i, 1, n}] → 2 ^ (n / 2) /.
    (2 ^ a_) :→ 2 ^ Expand[a] /.
   If → if //. if[a_, b1_, c1_] * if[a_, b2_, c2_] :→
   if[a, Together[b1 * b2], Together[c1 * c2]] /.
  Pochhammer[1 + mu/2, n/2]/ Pochhammer[1 + mu, n] →
   1 / 2 ^ n / Pochhammer[mu / 2 + 1 / 2, n / 2] /.
 (* Now the product expression inside the sum *)
 (* We first rewrite this
  Pochhammer to separate even and odd factors *)
 Pochhammer[3 j + mu, -2 + j] → Pochhammer[mu / 2 + Floor[3 / 2 j + 1 / 2],
    Floor[(j - 2) / 2]] * Pochhammer[mu / 2 + Floor[3 / 2 j] + 1 / 2,
    Floor[(j - 1) / 2]] * 2 ^ (j - 2) /.
 prod[a_Times, b_] :→ (prod[#, b] & /@ a) /.
prod[1/Pochhammer[j + mu/2, Floor[(1+j)/2]]^2, {j, 1, -1 + k}]
  prod[Pochhammer[1 + j + mu/2, -2 + j]^2, {j, 1, -1 + k}] prod[1/
    Pochhammer[mu/2 + Floor[1/2 + (3 j)/2], Floor[1/2 (-2 + j)]]^2, {j, 1, -1 + k}] →
  if[k == 0, 4 / mu ^ 2, 1] / Pochhammer[mu / 2 + 1, k - 1] ^ 2 /.
prod[Pochhammer[1/2 + 2 j + mu/2, -1 + j], {j, 1, -1 + k}]
  prod[1/Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1 + k}] prod[
   1/ Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[1/2 (-1 + j)]], {j, 1, -1 + k}]
  prod[Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[(1 + j)/2]], {j, 1, -1 + k}] →
 if[k == 0, 1, (mu + 3) / 2] / Pochhammer[mu / 2 + 2 k - 1 / 2, k] /.
prod[1/Pochhammer[1/2, -1 + j]^2, {j, 1, -1 + k}] prod[Pochhammer[j, j], {j, 1,
   -1 + k}] prod[Pochhammer[1 + j, 1 + j], {j, 1, -1 + k}] → if[k == 0, 1 / 8, 1] *
```

```
            2^(2 k (k-1)) * Pochhammer[3/2, k-1] * Pochhammer[1/2, k-1]^2 /.
          prod[2^a_, {j, 1, k-1}] :> With[{cf = Product[2^a, {j, 1, k-1}]},
            cf * if[k == 0, 1/(cf /. k -> 0), 1]] //.
         if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] -> if[k == 0, a1 * a2, b1 * b2] /.
        if[k == 0, a_ * b_, a_ * c_] -> a * if[k == 0, b, c] /.
       a_ * sum[b_, c_] -> sum[a * b, c] /.
      Pochhammer[mu/2 + 1, n/2 - 1] ->
       Pochhammer[mu/2 + 1, k - 1] * Pochhammer[mu/2 + k, n/2 - k] /.
     Pochhammer[3/2, k - 1] * if[k == 0, a_, b_] ->
      if[k == 0, a, b * (2 k - 1)!/(2 k - 2)!!]/2^(k - 1) /.
    (2^a_) :> 2^FullSimplify[a] /. if[k == 0, a_, b_] -> 4 * if[k == 0, a/4, b/4]
```

$$\text{sum}\Big[\Big(2^{-2+k+\frac{3n}{4}}\,\text{if}\Big[k==0,\ \frac{-2+\text{mu}}{\text{mu}^2},\ \frac{1}{8\,(-2+2k)!!}\Big]\,\text{Pochhammer}\Big[\frac{1}{2},-1+k\Big]^2\,\text{Pochhammer}\Big[k+\frac{\text{mu}}{2},-k+\frac{n}{2}\Big]\,\text{Pochhammer}[-1+\text{mu},-2+3k]\,\text{Pochhammer}\Big[\frac{1}{2}+\frac{\text{mu}}{2}+\frac{3n}{4},-1+\frac{3n}{4}\Big]\Big)\Big/\Big(\text{Pochhammer}\Big[\frac{1}{2}+\frac{\text{mu}}{2},\frac{n}{2}\Big]\,\text{Pochhammer}\Big[1+\frac{\text{mu}}{2},-1+k\Big]\,\text{Pochhammer}\Big[-\frac{1}{2}+k+\frac{\text{mu}}{2},-1+k\Big]\,\text{Pochhammer}\Big[-\frac{1}{2}+2k+\frac{\text{mu}}{2},k\Big]\Big),\ \Big\{k,0,\frac{n}{2}\Big\}\Big]$$

```
Table[Together[(Check00 /. {sum -> Sum, prod -> Product, if -> If}) / myP2[n/4, mu]],
  {n, 4, 20, 4}]
```

{1, 1, 1, 1, 1}

```
TraditionalForm[HoldForm @@ {Check00} /. {sum -> Sum, prod -> Product, if -> If}]
```

$$\sum_{k=0}^{\frac{n}{2}}\left(2^{-2+k+\frac{3n}{4}}\,\text{If}\left[k=0,\ \frac{-2+\text{mu}}{\text{mu}^2},\ \frac{1}{8\,(-2+2k)!!}\right]\left(\left(\frac{1}{2}\right)_{-1+k}\right)^2\left(k+\frac{\text{mu}}{2}\right)_{-k+\frac{n}{2}}(-1+\text{mu})_{-2+3k}\left(\frac{1}{2}+\frac{\text{mu}}{2}+\frac{3n}{4}\right)_{-1+\frac{3n}{4}}\right)\Big/\left(\left(\frac{1}{2}+\frac{\text{mu}}{2}\right)_{\frac{n}{2}}\left(1+\frac{\text{mu}}{2}\right)_{-1+k}\left(-\frac{1}{2}+k+\frac{\text{mu}}{2}\right)_{-1+k}\left(-\frac{1}{2}+2k+\frac{\text{mu}}{2}\right)_{k}\right)$$

```
(* Our expression fits the recurrence (this is the initial value check). *)
test = ApplyOreOperator[rec2, f[n]];
Together[Table[test, {n, 0, 4}] /. f[nn_] :> (Check00 /. n -> 4 nn /. sum -> Sum /. if -> If)]
```

{0, 0, 0, 0, 0}

```
(* the smnd for k=1 to n/2 *)
smnd = ExpandAll[Check00[[1]] /. n -> 4 n /. if[k == 0, _, a_] -> a]
```

$$\Big(2^{-5+k+3n}\,\text{Pochhammer}\Big[\frac{1}{2},-1+k\Big]^2\,\text{Pochhammer}\Big[k+\frac{\text{mu}}{2},-k+2n\Big]\,\text{Pochhammer}[-1+\text{mu},-2+3k]\,\text{Pochhammer}\Big[\frac{1}{2}+\frac{\text{mu}}{2}+3n,-1+3n\Big]\Big)\Big/\Big((-2+2k)!!\,\text{Pochhammer}\Big[\frac{1}{2}+\frac{\text{mu}}{2},2n\Big]\,\text{Pochhammer}\Big[1+\frac{\text{mu}}{2},-1+k\Big]\,\text{Pochhammer}\Big[-\frac{1}{2}+k+\frac{\text{mu}}{2},-1+k\Big]\,\text{Pochhammer}\Big[-\frac{1}{2}+2k+\frac{\text{mu}}{2},k\Big]\Big)$$

```
Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]
```

```
{{(1 + mu + 4 n) (3 + mu + 4 n) (1 + mu + 6 n) (3 + mu + 6 n) (5 + mu + 6 n) Sn -
   (mu + 4 n) (2 + mu + 4 n) (-1 + mu + 12 n) (1 + mu + 12 n)
    (3 + mu + 12 n) (5 + mu + 12 n) (7 + mu + 12 n) (9 + mu + 12 n)}, {0}}
```

```
(* We use the following identity in order
 to construct an annihilating operator for the sum *)
TraditionalForm[HoldForm[
  0 == Sum[(p1[n] S[n] + p0[n]) * f[n, k], {k, 1, 2 n}] ==
   (p1[n] S[n] + p0[n]) * Sum[f[n, k], {k, 1, 2 n}]
    - p1[n] * (f[n + 1, 2 n + 1] + f[n + 1, 2 n + 2])
 ]]
test =
  ApplyOreOperator[op, Check00 /. n → 4 n /. sum[a_, {k, 0, b_}] → sum[a, {k, 1, b}]] -
   LeadingCoefficient[op] *
    ((smnd /. n → n + 1 /. k → 2 n + 1) + (smnd /. n → n + 1 /. k → 2 n + 2));
Together[Table[test /. sum → Sum /. if → If, {n, 0, 4}]]
```

$$0 = \sum_{k=1}^{2n} (\mathrm{p1}(n)\, S(n) + \mathrm{p0}(n))\, f(n,\, k) = (\mathrm{p1}(n)\, S(n) + \mathrm{p0}(n)) \sum_{k=1}^{2n} f(n,\, k) - \mathrm{p1}(n)\, (f(n+1,\, 2\, n+1) + f(n+1,\, 2\, n+2))$$

```
{0, 0, 0, 0, 0}
```

```
inh = Factor[LeadingCoefficient[op] *
      (1 + FullSimplify[(smnd /. n → n + 1 /. k → 2 n + 2) / (smnd /. n → n + 1 /. k → 2 n + 1)])] *
   (smnd /. n → n + 1 /. k → 2 n + 1);
rec00 = Annihilator[inh, S[n]][[1]] ** op;
```

```
smnd0 = ExpandAll[Together[Check00[[1]] /. n → 4 n /. if[k == 0, a_, _] → a /. k → 0]]
```

$$\left(2^{-2+3\,n}\, \mathrm{Pochhammer}\left[\frac{\mathrm{mu}}{2},\, 2\,n\right] \mathrm{Pochhammer}\left[\frac{1}{2} + \frac{\mathrm{mu}}{2} + 3\,n,\, -1 + 3\,n\right]\right) \Big/ \left(\mathrm{mu}\, \mathrm{Pochhammer}\left[\frac{1}{2} + \frac{\mathrm{mu}}{2},\, 2\,n\right]\right)$$

```
(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec2a. *)
OreReduce[rec00, Annihilator[smnd0, S[n]]]
```

```
0
```

```
GBEqual[{rec2}, {rec00}]
```

```
True
```

## $n = 2 \bmod 4$

```
Check01 =
 Check0 /. Floor[a_] :→ (FullSimplify[Floor[a /. n → 4 l - 2], Element[l, Integers]] /.
                          Floor[l + b_] :→ Floor[Together[b]] +
                            l /. l → (n + 2) / 4) /.
```

```
prod[a_^b_, {i, 1, c_}] :> prod[Expand[a]^
    Simplify[b], {i, 1, Expand[c]}] /.
   prod[Pochhammer[j + mu/2, Floor[(1 + j)/2]], {j, 1, -1 + n/2}] *
     prod[Pochhammer[j + mu/2, Floor[(1 + j)/2]], {j, 1, n/2}] /
     (prod[(6 + 2 i + mu)^(2 Floor[(2+i)/3]), {i, 1, -7/2 + (3 n)/4}]
       prod[(2 i + mu + n)^(1-2 i), {i, 1, -1/2 + n/4}]) ->
    Pochhammer[mu / 2 + 1, n / 2] * Pochhammer[mu / 2 + 1,
      n / 2 - 1] / 2^(n^2 / 8 - 3 / 4 n + 1) /.
  prod[Pochhammer[1/2 + 2 j + mu/2, -1 + j], {j, 1, n/2}]
    prod[Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1 + n/2}]
    prod[1 / Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[1/2 (-1 + j)]],
     {j, 1, n/2}] prod[1 / Pochhammer[1/2 + mu/2 + Floor[(3 j)/2],
       Floor[(1 + j)/2]], {j, 1, -1 + n/2}] /
     (prod[(2 + 2 i + mu + (3 n)/2)^(1/2 (n-4 Ceiling[1/6+i/3])), {i, 1, -5/2 + (3 n)/4}]
       prod[(1 - 2 i + mu + 2 n)^(1-2 i), {i, 1, -3/2 + n/4}]) ->
   Pochhammer[mu / 2 + 3 / 4 n, 3 / 4 n - 1 / 2] / (mu + 3) /
    2^(n^2 / 8 - n / 2 - 1 / 2) /.
 prod[1/Pochhammer[j, j], {j, 1, n/2}] prod[1/Pochhammer[1 + j, 1 + j],
    {j, 1, -1 + n/2}] / prod[Floor[i/2]!/i!, {i, 1, n}] -> 2^(n / 2) /.
  (2^a_) :> 2^Expand[a] /.
  If -> if //. if[a_, b1_, c1_] * if[a_, b2_, c2_] :>
   if[a, Together[b1 * b2], Together[c1 * c2]] /.
 Pochhammer[1 + mu/2, n/2] / Pochhammer[1 + mu, n] ->
  1 / 2^n / Pochhammer[mu / 2 + 1 / 2, n / 2] /.
(* Now the product expression inside the sum *)
(* We first rewrite this
 Pochhammer to separate even and odd factors *)
Pochhammer[3 j + mu, -2 + j] -> Pochhammer[mu / 2 + Floor[3 / 2 j + 1 / 2],
   Floor[(j - 2) / 2]] * Pochhammer[mu / 2 + Floor[3 / 2 j] + 1 / 2,
   Floor[(j - 1) / 2]] * 2^(j - 2) /.
prod[a_Times, b_] :> (prod[#, b] & /@ a) /.
```

```
prod[1 / Pochhammer[j + mu/2, Floor[(1+j)/2]]^2, {j, 1, -1 + k}]
   prod[Pochhammer[1 + j + mu/2, -2 + j]^2, {j, 1, -1 + k}] prod[1 /
      Pochhammer[mu/2 + Floor[1/2 + (3 j)/2], Floor[1/2 (-2 + j)]]^2, {j, 1, -1 + k}] →
  if[k == 0, 4 / mu^2, 1] / Pochhammer[mu / 2 + 1, k - 1]^2 /.
 prod[Pochhammer[1/2 + 2 j + mu/2, -1 + j], {j, 1, -1 + k}]
   prod[1 / Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1 + k}] prod[
    1 / Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[1/2 (-1 + j)]], {j, 1, -1 + k}]
   prod[Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[(1 + j)/2]], {j, 1, -1 + k}] →
  if[k == 0, 1, (mu + 3) / 2] / Pochhammer[mu / 2 + 2 k - 1 / 2, k] /.
 prod[1 / Pochhammer[1/2, -1 + j]^2, {j, 1, -1 + k}] prod[Pochhammer[j, j], {j, 1,
     -1 + k}] prod[Pochhammer[1 + j, 1 + j], {j, 1, -1 + k}] → if[k == 0, 1 / 8, 1] *
   2^(2 k (k - 1)) * Pochhammer[3 / 2, k - 1] * Pochhammer[1 / 2, k - 1]^2 /.
  prod[2^a_, {j, 1, k - 1}] :→ With[{cf = Product[2^a, {j, 1, k - 1}]},
    cf * if[k == 0, 1 / (cf /. k → 0), 1]] //.
   if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] → if[k == 0, a1 * a2, b1 * b2] /.
  if[k == 0, a_ * b_, a_ * c_] → a * if[k == 0, b, c] /.
 a_ * sum[b_, c_] → sum[a * b, c] /.
 Pochhammer[mu / 2 + 1, n / 2 - 1] →
  Pochhammer[mu / 2 + 1, k - 1] * Pochhammer[mu / 2 + k, n / 2 - k] /.
 Pochhammer[3 / 2, k - 1] * if[k == 0, a_, b_] →
  if[k == 0, a, b * (2 k - 1) ! / (2 k - 2) !!] / 2^(k - 1) /.
(2^a_) :→ 2^FullSimplify[a] /. if[k == 0, a_, b_] → 4 * if[k == 0, a / 4, b / 4]
```

```
sum[(2^(-3/2+k+(3 n)/4) if[k == 0, (-2 + mu)/mu^2, 1/(8 (-2 + 2 k) !!)] Pochhammer[1/2, -1 + k]^2 Pochhammer[k + mu/2,
    -k + n/2] Pochhammer[-1 + mu, -2 + 3 k] Pochhammer[mu/2 + (3 n)/4, -1/2 + (3 n)/4]) /
  (Pochhammer[1/2 + mu/2, n/2] Pochhammer[1 + mu/2, -1 + k] Pochhammer[-1/2 + k + mu/2, -1 + k]
    Pochhammer[-1/2 + 2 k + mu/2, k]), {k, 0, n/2}]
```

```
Table[Together[(Check01 /. {sum → Sum, prod → Product, if → If}) / myP1[n / 4 + 1 / 2, mu]],
 {n, 2, 18, 4}]
```

```
{1, 1, 1, 1, 1}
```

```
TraditionalForm[HoldForm @@ {Check01} /. {sum → Sum, prod → Product, if → If}]
```

$$\sum_{k=0}^{\frac{n}{2}}\left(2^{-\frac{3}{2}+k+\frac{3n}{4}}\,\mathrm{If}\!\left[k=0,\ \frac{-2+\mathrm{mu}}{\mathrm{mu}^2},\ \frac{1}{8\,(-2+2\,k)!!}\right]\left(\left(\frac{1}{2}\right)_{-1+k}\right)^2\left(k+\frac{\mathrm{mu}}{2}\right)_{-k+\frac{n}{2}}(-1+\mathrm{mu})_{-2+3\,k}\left(\frac{\mathrm{mu}}{2}+\frac{3\,n}{4}\right)_{-\frac{1}{2}+\frac{3n}{4}}\right)\Big/\left(\left(\frac{1}{2}+\frac{\mathrm{mu}}{2}\right)_{\frac{n}{2}}\left(1+\frac{\mathrm{mu}}{2}\right)_{-1+k}\left(-\frac{1}{2}+k+\frac{\mathrm{mu}}{2}\right)_{-1+k}\left(-\frac{1}{2}+2\,k+\frac{\mathrm{mu}}{2}\right)_{k}\right)$$

```
(* Our expression fits the recurrence (this is the initial value check). *)
test = ApplyOreOperator[rec1, f[n]];
Together[Table[test, {n, 5}] /. f[nn_] :→ (Check01 /. n → 4 nn - 2 /. sum → Sum /. if → If)]
```

{0, 0, 0, 0, 0}

```
(* the smnd for k>=1 *)
smnd = Check01[[1]] /. n → 4 n - 2 /. if[k == 0, _, a_] → a
```

$$\left(2^{-\frac{9}{2}+k+\frac{3}{4}(-2+4\,n)}\,\mathrm{Pochhammer}\left[\frac{1}{2},\,-1+k\right]^2\mathrm{Pochhammer}\left[k+\frac{\mathrm{mu}}{2},\,-k+\frac{1}{2}(-2+4\,n)\right]\mathrm{Pochhammer}[-1+\mathrm{mu},\,-2+3\,k]\,\mathrm{Pochhammer}\left[\frac{\mathrm{mu}}{2}+\frac{3}{4}(-2+4\,n),\,-\frac{1}{2}+\frac{3}{4}(-2+4\,n)\right]\right)\Big/\left((-2+2\,k)!!\,\mathrm{Pochhammer}\left[\frac{1}{2}+\frac{\mathrm{mu}}{2},\,\frac{1}{2}(-2+4\,n)\right]\mathrm{Pochhammer}\left[1+\frac{\mathrm{mu}}{2},\,-1+k\right]\mathrm{Pochhammer}\left[-\frac{1}{2}+k+\frac{\mathrm{mu}}{2},\,-1+k\right]\mathrm{Pochhammer}\left[-\frac{1}{2}+2\,k+\frac{\mathrm{mu}}{2},\,k\right]\right)$$

```
Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]
```

$\{\{(-1+\mathrm{mu}+4\,n)\,(1+\mathrm{mu}+4\,n)\,(-3+\mathrm{mu}+6\,n)\,(-1+\mathrm{mu}+6\,n)\,(1+\mathrm{mu}+6\,n)\,S_n-(-2+\mathrm{mu}+4\,n)\,(\mathrm{mu}+4\,n)\,(-7+\mathrm{mu}+12\,n)\,(-5+\mathrm{mu}+12\,n)\,(-3+\mathrm{mu}+12\,n)\,(-1+\mathrm{mu}+12\,n)\,(1+\mathrm{mu}+12\,n)\,(3+\mathrm{mu}+12\,n)\},\,\{0\}\}$

```
(* We use the following identity in order
 to construct an annihilating operator for the sum *)
TraditionalForm[HoldForm[
  0 == Sum[(p1[n] S[n] + p0[n]) * f[n, k], {k, 1, 2 n - 1}] ==
   (p1[n] S[n] + p0[n]) * Sum[f[n, k], {k, 1, 2 n - 1}]
    - p1[n] * (f[n + 1, 2 n] + f[n + 1, 2 n + 1])
 ]]
test =
  ApplyOreOperator[op, Check01 /. n → 4 n - 2 /. sum[a_, {k, 0, b_}] → sum[a, {k, 1, b}]] -
   LeadingCoefficient[op] *
    ((smnd /. n → n + 1 /. k → 2 n) + (smnd /. n → n + 1 /. k → 2 n + 1));
Together[Table[test /. sum → Sum /. if → If, {n, 5}]]
```

$$0=\sum_{k=1}^{2\,n-1}(\mathrm{p1}(n)\,S(n)+\mathrm{p0}(n))\,f(n,\,k)=(\mathrm{p1}(n)\,S(n)+\mathrm{p0}(n))\sum_{k=1}^{2\,n-1}f(n,\,k)-\mathrm{p1}(n)\,(f(n+1,\,2\,n)+f(n+1,\,2\,n+1))$$

{0, 0, 0, 0, 0}

```
inh = Factor[LeadingCoefficient[op] *
      (1 + FullSimplify[(smnd /. n → n + 1 /. k → 2 n + 1) / (smnd /. n → n + 1 /. k → 2 n)])] *
    (smnd /. n → n + 1 /. k → 2 n);
rec01 = Annihilator[inh, S[n]][[1]] ** op;

smnd0 = Together[ExpandAll[Check01[[1]] /. n → 4 n - 2 /. if[k == 0, a_, _] → a /. k → 0]]
```

$$\left(2^{-3+3\,n}\,\text{Pochhammer}\left[\frac{\text{mu}}{2},\,-1+2\,n\right]\,\text{Pochhammer}\left[-\frac{3}{2}+\frac{\text{mu}}{2}+3\,n,\,-2+3\,n\right]\right)\Big/\left(\text{mu Pochhammer}\left[\frac{1}{2}+\frac{\text{mu}}{2},\,-1+2\,n\right]\right)$$

```
(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec01. *)
OreReduce[rec01, Annihilator[smnd0, S[n]]]
```

0

```
GBEqual[{rec1}, rec01]
```

True

## odd n

```
FFo1 = FFoAlt /. prod[a_, {j, k, b_}] :>
       prod[a, {j, 1, b}] / prod[a, {j, 1, k - 1}] * if[k == 0, a /. j → 0, 1] //.
     prod[a1_, b_] ^ c1_. * prod[a2_, b_] ^ c2_. → prod[a1 ^ c1 * a2 ^ c2, b] //.
    sum[(a_ /; FreeQ[a, k]) * b_, c_] → a * sum[b, c] //.
   prod[a_ * b_, {j, 1, n / 2 + c_.}] → prod[a, {j, 1, n / 2 + c}] * prod[b, {j, 1, n / 2 + c}];
Check1 = FFo1 / (
    (* myC[n] *)
    ((-1) ^ n + 3) / 2 * prod[Floor[i / 2] ! / i !, {i, 1, n}] *
     (* myE[n,mu] *)
     Pochhammer[mu + 1, n] * prod[
       (mu + 2 i + 6) ^ (2 * Floor[(i + 2) / 3]), {i, 1, Floor[3 / 2 * Floor[(n - 1) / 2] - 2]}] *
     prod[(mu + 2 i + 2 * Floor[3 / 2 * Floor[n / 2 + 1]] - 1) ^
        (2 * Floor[Floor[n / 2] / 2 - (i - 1) / 3] - 1),
       {i, 1, Floor[3 / 2 * Floor[n / 2] - 2]}] *
     (* myF[1,n,mu] *)
     prod[(mu + 2 i + n + 1) ^ (-2 i), {i, 1, Floor[(n - 1) / 4]}] *
     prod[(mu - 2 i + 2 n - 1) ^ (-2 i), {i, 1, Floor[n / 4 - 1]}] *
     (* remaining factor in myF[n,mu] *)
     Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}]);
Check1 =
  Check1 /. {(-1) ^ n → -1, Floor[n / 2] → (n - 1) / 2, Floor[(n - 1) / 2] → (n - 1) / 2} /.
   Floor[a_] :> Floor[Together[a]];
```

```
(* n=1 and n=3 don't work because of the "remaining factor in myF[n,mu]". *)
Table[Together[(Check1 /. {sum → Sum, prod → Product, if → If}) / myP1[(n + 3) / 4, mu]],
  {n, 1, 19, 4}]
Table[Together[(Check1 /. {sum → Sum, prod → Product, if → If}) / myP2[(n + 1) / 4, mu]],
  {n, 3, 21, 4}]
```

```
{(-1 + mu) (1 + mu), 1, 1, 1, 1}
```

```
{5 + mu, 1, 1, 1, 1}
```

## $n = 1 \bmod 4$

```
Check10 =
  Check1 /. Floor[a_] :> (FullSimplify[Floor[a /. n → 4 l - 3], Element[l, Integers]] /.
                            Floor[l + b_] :> Floor[Together[b]] +
                              l /. l → (n + 3) / 4) /.
                      prod[a_^b_, {i, 1, c_}] :> prod[Expand[a]^
                          Simplify[b], {i, 1, Expand[c]}] /.
                    prod[Pochhammer[j + mu/2, Floor[(1 + j)/2]]^2, {j, 1, -1/2 + n/2}] /
                       prod[(6 + 2 i + mu)^(2 Floor[(2+i)/3]), {i, 1, -11/4 + 3 n/4}] /
                      prod[(1 + 2 i + mu + n)^(-2 i), {i, 1, -1/4 + n/4}] →
                    Pochhammer[mu / 2 + 1, (n - 1) / 2]^2 / 2^((n^2 - 6 n + 5) / 8) /.
                  prod[Pochhammer[-1/2 + 2 j + mu/2, j], {j, 1, -1/2 + n/2}]
                    prod[Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1/2 + n/2}]
                    prod[1 / Pochhammer[1/2 + mu/2 + Floor[3 j/2], Floor[(1 + j)/2]]^2, {j,
                        1, -1/2 + n/2}] / prod[(-1/2 + 2 i + mu + 3 n/2)^(1/2 (1+n-4 Ceiling[(2+i)/3])),
                      {i, 1, -11/4 + 3 n/4}] / prod[(-1 - 2 i + mu + 2 n)^(-2 i),
                     {i, 1, -5/4 + n/4}] → Pochhammer[mu / 2 + 3 / 4 (n - 1) + 3 / 2,
                    3 / 4 (n - 1) - 2] * Pochhammer[mu / 2 + (n - 1) + 3 / 2,
                       (n - 1) / 2 + 1] / (mu + 3) / 2^((n^2 - 8 n + 15) / 8) /.
                prod[1/Pochhammer[j, j], {j, 1, -1/2 + n/2}] prod[
                   1/Pochhammer[1 + j, 1 + j], {j, 1, -1/2 + n/2}] /
                    prod[Floor[i/2]!/i!, {i, 1, n}] → 2^((n - 1) / 2) /.
              (2^a_) :> 2^Expand[a] /.
            If → if //. if[a_, b1_, c1_] * if[a_, b2_, c2_] :>
```

```
if[a, Together[b1 * b2], Together[c1 * c2]] /.
 Pochhammer[1/2 + mu/2 + n, 1 + 1/2 (-1 + n)] Pochhammer[3/2 + mu/2 + 3/4 (-1 + n),
   -2 + 3/4 (-1 + n)] / Pochhammer[1/2 (1 + mu + 2 n), 1/2 (-5 + n)] →
  Pochhammer[3/2 + mu/2 + 3/4 (-1 + n), (3 n + 1) / 4] /.
 Pochhammer[1 + mu/2, (n - 1)/2] ^2 / Pochhammer[1 + mu, n] → Pochhammer[
   mu / 2 + 1, (n - 1) / 2] / 2 ^ n / Pochhammer[mu / 2 + 1 / 2, (n + 1) / 2] /.
 (* Now the product expression inside the sum *)
 (* We first rewrite this
  Pochhammer to separate even and odd factors *)
 Pochhammer[3 j + mu, -2 + j] → Pochhammer[mu / 2 +
     Floor[3 / 2 j + 1 / 2], Floor[(j - 2) / 2]] * Pochhammer[
    mu / 2 + Floor[3 / 2 j] + 1 / 2, Floor[(j - 1) / 2]] * 2 ^ (j - 2) /.
prod[a_Times, b_] :> (prod[#, b] & /@ a) /.
prod[1 / Pochhammer[j + mu/2, Floor[(1+j)/2]]^2, {j, 1, -1 + k}] prod[
   Pochhammer[1 + j + mu/2, -2 + j]^2, {j, 1, -1 + k}] prod[1 /
    Pochhammer[mu/2 + Floor[1/2 + 3 j/2], Floor[1/2 (-2 + j)]]^2, {j, 1, -1 + k}] →
 if[k == 0, 4 / mu ^ 2, 1] / Pochhammer[mu / 2 + 1, k - 1] ^ 2 /.
prod[1 / Pochhammer[-1/2 + 2 j + mu/2, j], {j, 1, -1 + k}]
 prod[Pochhammer[1/2 + 2 j + mu/2, -1 + j]^2, {j, 1, -1 + k}]
 prod[1 / Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1 + k}] prod[1 /
   Pochhammer[1/2 + mu/2 + Floor[3 j/2], Floor[1/2 (-1 + j)]]^2, {j, 1, -1 + k}]
 prod[Pochhammer[1/2 + mu/2 + Floor[3 j/2], Floor[(1 + j)/2]]^2, {j, 1, -1 + k}] →
 if[k == 0, 1, (mu + 3) / 2] / Pochhammer[mu / 2 + 2 k - 1 / 2, k] /.
prod[1 / Pochhammer[1/2, -1 + j]^2, {j, 1, -1 + k}] prod[Pochhammer[j, j],
   {j, 1, -1 + k}] prod[Pochhammer[1 + j, 1 + j], {j, 1, -1 + k}] →
  if[k == 0, 1 / 8, 1] * 2 ^ (2 k (k - 1)) * Pochhammer[3 / 2, k - 1] *
   Pochhammer[1 / 2, k - 1] ^ 2 /.
 prod[2 ^ a_, {j, 1, k - 1}] :> With[{cf = Product[2 ^ a, {j, 1, k - 1}]},
   cf * if[k == 0, 1 / (cf /. k → 0), 1]] //.
 if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] → if[k == 0, a1 * a2, b1 * b2] /.
if[k == 0, a_ * b_, a_ * c_] → a * if[k == 0, b, c] /.
a_ * sum[b_, c_] → sum[a * b, c] /.
```

```
      Pochhammer[mu / 2 + 1, (n - 1) / 2] →
        Pochhammer[mu / 2 + 1, k - 1] * Pochhammer[mu / 2 + k, (n + 1) / 2 - k] /.
      Pochhammer[3 / 2, k - 1] * if[k == 0, a_, b_] →
        if[k == 0, a, b * (2 k - 1) ! / (2 k - 2) !!] / 2^(k - 1) /. (2^a_) :> 2^FullSimplify[a] /.
    Pochhammer[3/2 + mu/2 + 3/4 (-1 + n), 1/4 (1 + 3 n)] → Pochhammer[3/4 + mu/2 + 3 n/4, 1 / 4 + 3 n / 4] /.
   if[k == 0, a_, b_] → 4 * if[k == 0, a / 4, b / 4]
```

```
sum[(2^(-3/4+k+3 n/4) if[k == 0, (-2 + mu)/mu^2, 1/(8 (-2 + 2 k) !!)] Pochhammer[1/2, -1 + k]^2 Pochhammer[k + mu/2,
    -k + (1 + n)/2] Pochhammer[-1 + mu, -2 + 3 k] Pochhammer[3/4 + mu/2 + 3 n/4, 1/4 + 3 n/4]) /
  (Pochhammer[1/2 + mu/2, (1 + n)/2] Pochhammer[1 + mu/2, -1 + k] Pochhammer[-1/2 + k + mu/2, -1 + k]
    Pochhammer[-1/2 + 2 k + mu/2, k]), {k, 0, (1 + n)/2}]
```

```
Table[Factor[(Check10 /. {sum → Sum, prod → Product, if → If}) / myP1[(n + 3) / 4, mu]],
  {n, 1, 19, 4}]
```

```
{1, 1, 1, 1, 1}
```

```
TraditionalForm[HoldForm @@ {Check10} /. {sum → Sum, prod → Product, if → If}]
```

$$\sum_{k=0}^{\frac{1+n}{2}} \left(2^{-\frac{3}{4}+k+\frac{3n}{4}} \operatorname{If}\left[k=0, \frac{-2+\mathrm{mu}}{\mathrm{mu}^2}, \frac{1}{8\,(-2+2\,k)!!}\right] \left(\left(\frac{1}{2}\right)_{-1+k}\right)^2 \left(k+\frac{\mathrm{mu}}{2}\right)_{-k+\frac{1+n}{2}} (-1+\mathrm{mu})_{-2+3\,k} \left(\frac{3}{4}+\frac{\mathrm{mu}}{2}+\frac{3\,n}{4}\right)_{\frac{1}{4}+\frac{3\,n}{4}}\right) \Big/ \left(\left(\frac{1}{2}+\frac{\mathrm{mu}}{2}\right)_{\frac{1+n}{2}} \left(1+\frac{\mathrm{mu}}{2}\right)_{-1+k} \left(-\frac{1}{2}+k+\frac{\mathrm{mu}}{2}\right)_{-1+k} \left(-\frac{1}{2}+2\,k+\frac{\mathrm{mu}}{2}\right)_{k}\right)$$

```
(* Our expression fits the recurrence (this is the initial value check). *)
test = ApplyOreOperator[rec1, f[n]];
Together[Table[test, {n, 5}] /. f[nn_] :> (Check10 /. n → 4 nn - 3 /. sum → Sum /. if → If)]
```

```
{0, 0, 0, 0, 0}
```

```
(* the smnd for k>=1 *)
smnd = Check10[[1]] /. n → 4 n - 3 /. if[k == 0, _, a_] → a /.
   Pochhammer[a_, b_] :> Pochhammer @@ Expand[{a, b}]
```

```
(2^(-15/4+k+3/4 (-3+4 n)) Pochhammer[1/2, -1 + k]^2 Pochhammer[k + mu/2, -1 - k + 2 n]
   Pochhammer[-1 + mu, -2 + 3 k] Pochhammer[-3/2 + mu/2 + 3 n, -2 + 3 n]) /
 ((-2 + 2 k) !! Pochhammer[1/2 + mu/2, -1 + 2 n] Pochhammer[1 + mu/2, -1 + k]
   Pochhammer[-1/2 + k + mu/2, -1 + k] Pochhammer[-1/2 + 2 k + mu/2, k])
```

```
Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]
```

```
{{(-1 + mu + 4 n) (1 + mu + 4 n) (-3 + mu + 6 n) (-1 + mu + 6 n) (1 + mu + 6 n) S_n -
   (-2 + mu + 4 n) (mu + 4 n) (-7 + mu + 12 n) (-5 + mu + 12 n)
    (-3 + mu + 12 n) (-1 + mu + 12 n) (1 + mu + 12 n) (3 + mu + 12 n)}, {0}}
```

```
(* We use the following identity in order
 to construct an annihilating operator for the sum *)
TraditionalForm[HoldForm[
  0 == Sum[(p1[n] S[n] + p0[n]) * f[n, k], {k, 1, 2 n - 1}] ==
   (p1[n] S[n] + p0[n]) * Sum[f[n, k], {k, 1, 2 n - 1}]
    - p1[n] * (f[n + 1, 2 n] + f[n + 1, 2 n + 1])
 ]]
test =
  ApplyOreOperator[op, Check10 /. n → 4 n - 3 /. sum[a_, {k, 0, b_}] → sum[a, {k, 1, b}]] -
   LeadingCoefficient[op] *
    ((smnd /. n → n + 1 /. k → 2 n) + (smnd /. n → n + 1 /. k → 2 n + 1));
Together[Table[test /. sum → Sum /. if → If, {n, 5}]]
```

$$0=\sum_{k=1}^{2n-1}(\mathrm{p1}(n)\,S(n)+\mathrm{p0}(n))\,f(n,\,k)=(\mathrm{p1}(n)\,S(n)+\mathrm{p0}(n))\sum_{k=1}^{2n-1}f(n,\,k)-\mathrm{p1}(n)\,(f(n+1,\,2\,n)+f(n+1,\,2\,n+1))$$

```
{0, 0, 0, 0, 0}
```

```
inh = Factor[LeadingCoefficient[op] *
      (1 + FullSimplify[(smnd /. n → n + 1 /. k → 2 n + 1) / (smnd /. n → n + 1 /. k → 2 n)])] *
   (smnd /. n → n + 1 /. k → 2 n);
rec10 = Annihilator[inh, S[n]][[1]] ** op;
```

```
smnd0 = Together[ExpandAll[Check10[[1]] /. n → 4 n - 3 /. if[k == 0, a_, _] → a /. k → 0]]
```

$$\left(2^{-3+3n}\,\mathrm{Pochhammer}\left[\frac{mu}{2},\,-1+2n\right]\,\mathrm{Pochhammer}\left[-\frac{3}{2}+\frac{mu}{2}+3n,\,-2+3n\right]\right)\Big/\left(mu\,\mathrm{Pochhammer}\left[\frac{1}{2}+\frac{mu}{2},\,-1+2n\right]\right)$$

```
(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec01. *)
OreReduce[rec10, Annihilator[smnd0, S[n]]]
```

```
0
```

```
GBEqual[{rec1}, rec10]
```

```
True
```

## $n = 3 \bmod 4$

```
Check11 =
 Check1 /. Floor[a_] :→ (FullSimplify[Floor[a /. n → 4 l - 1], Element[l, Integers]] /.
          Floor[l + b_] :→ Floor[Together[b]] +
           l /. l → (n + 1) / 4) /.
```

```
prod[a_^b_, {i, 1, c_}] :> prod[Expand[a]^
    Simplify[b], {i, 1, Expand[c]}] /.
  prod[Pochhammer[j + mu/2, Floor[(1 + j)/2]]^2, {j, 1, -1/2 + n/2}] /
    prod[(6 + 2 i + mu)^(2 Floor[(2+i)/3]), {i, 1, -13/4 + (3 n)/4}] /
   prod[(1 + 2 i + mu + n)^(-2 i), {i, 1, -3/4 + n/4}] ->
  Pochhammer[mu / 2 + 1, (n - 1) / 2] ^ 2 / 2 ^ ((n^2 - 6 n + 9) / 8) /.
 prod[Pochhammer[-1/2 + 2 j + mu/2, j], {j, 1, -1/2 + n/2}]
  prod[Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1/2 + n/2}]
  prod[1 / Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[(1 + j)/2]]^2, {j,
     1, -1/2 + n/2}] / prod[(1/2 + 2 i + mu + (3 n)/2)^(1/2 (-1+n-4 Ceiling[1/6+i/3])),
    {i, 1, -13/4 + (3 n)/4}] / prod[(-1 - 2 i + mu + 2 n)^(-2 i),
   {i, 1, -7/4 + n/4}] -> Pochhammer[mu / 2 + 3 / 4 n + 5 / 4,
    (3 n - 13) / 4] Pochhammer[mu / 2 + n + 1 / 2, (n + 1) / 2] /
    (mu + 3) / 2 ^ ((n^2 - 8 n + 15) / 8) /.
prod[1/Pochhammer[j, j], {j, 1, -1/2 + n/2}] prod[
   1/Pochhammer[1 + j, 1 + j], {j, 1, -1/2 + n/2}] /
  prod[Floor[i/2]!/i!, {i, 1, n}] -> 2 ^ ((n - 1) / 2) /.
 (2 ^ a_) :> 2 ^ Expand[a] /.
 If -> if //. if[a_, b1_, c1_] * if[a_, b2_, c2_] :>
  if[a, Together[b1 * b2], Together[c1 * c2]] /.
Pochhammer[1/2 + mu/2 + n, (1 + n)/2] Pochhammer[5/4 + mu/2 + (3 n)/4,
   1/4 (-13 + 3 n)] / Pochhammer[1/2 (1 + mu + 2 n), 1/2 (-5 + n)] ->
  Pochhammer[5/4 + mu/2 + (3 n)/4, 3 n / 4 - 1 / 4] /.
Pochhammer[1 + mu/2, (n - 1)/2]^2 / Pochhammer[1 + mu, n] -> Pochhammer[
   mu / 2 + 1, (n - 1) / 2] / 2^n / Pochhammer[mu / 2 + 1 / 2, (n + 1) / 2] /.
(* Now the product expression inside the sum *)
(* We first rewrite this
 Pochhammer to separate even and odd factors *)
Pochhammer[3 j + mu, -2 + j] -> Pochhammer[mu / 2 + Floor[3 / 2 j + 1 / 2],
   Floor[(j - 2) / 2]] * Pochhammer[mu / 2 + Floor[3 / 2 j] + 1 / 2,
   Floor[(j - 1) / 2]] * 2 ^ (j - 2) /.
```

```
prod[a_Times, b_] :> (prod[#, b] & /@ a) /.
  prod[1/Pochhammer[j + mu/2, Floor[(1+j)/2]]^2, {j, 1, -1 + k}]
    prod[Pochhammer[1 + j + mu/2, -2 + j]^2, {j, 1, -1 + k}] prod[1/
      Pochhammer[mu/2 + Floor[1/2 + (3 j)/2], Floor[1/2 (-2 + j)]]^2, {j, 1, -1 + k}] ->
   if[k == 0, 4 / mu^2, 1] / Pochhammer[mu / 2 + 1, k - 1]^2 /.
  prod[1/Pochhammer[-1/2 + 2 j + mu/2, j], {j, 1, -1 + k}]
    prod[Pochhammer[1/2 + 2 j + mu/2, -1 + j]^2, {j, 1, -1 + k}]
    prod[1/Pochhammer[3/2 + 2 j + mu/2, 1 + j], {j, 1, -1 + k}] prod[
     1/ Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[1/2 (-1 + j)]]^2, {j, 1, -1 + k}]
    prod[Pochhammer[1/2 + mu/2 + Floor[(3 j)/2], Floor[(1 + j)/2]]^2, {j, 1, -1 + k}] ->
   if[k == 0, 1, (mu + 3) / 2] / Pochhammer[mu / 2 + 2 k - 1 / 2, k] /.
  prod[1/Pochhammer[1/2, -1 + j]^2, {j, 1, -1 + k}] prod[Pochhammer[j, j], {j, 1,
     -1 + k}] prod[Pochhammer[1 + j, 1 + j], {j, 1, -1 + k}] -> if[k == 0, 1 / 8, 1] *
   2^(2 k (k - 1)) * Pochhammer[3 / 2, k - 1] * Pochhammer[1 / 2, k - 1]^2 /.
  prod[2^a_, {j, 1, k - 1}] :> With[{cf = Product[2^a, {j, 1, k - 1}]},
    cf * if[k == 0, 1/(cf /. k -> 0), 1]] //.
 if[k == 0, a1_, b1_] * if[k == 0, a2_, b2_] -> if[k == 0, a1 * a2, b1 * b2] /.
 if[k == 0, a_ * b_, a_ * c_] -> a * if[k == 0, b, c] /.
 a_ * sum[b_, c_] -> sum[a * b, c] /.
 Pochhammer[mu / 2 + 1, (n - 1) / 2] ->
  Pochhammer[mu / 2 + 1, k - 1] * Pochhammer[mu / 2 + k, (n + 1) / 2 - k] /.
 Pochhammer[3 / 2, k - 1] * if[k == 0, a_, b_] ->
  if[k == 0, a, b * (2 k - 1) ! / (2 k - 2) !!] / 2^(k - 1) /.
(2^a_) :> 2^FullSimplify[a] /. if[k == 0, a_, b_] -> 4 * if[k == 0, a / 4, b/4]
```

```
sum[(2^(-5/4 + k + (3 n)/4) if[k == 0, (-2 + mu)/mu^2, 1/(8 (-2 + 2 k) !!)] Pochhammer[1/2, -1 + k]^2 Pochhammer[k + mu/2,
   -k + (1 + n)/2] Pochhammer[-1 + mu, -2 + 3 k] Pochhammer[5/4 + mu/2 + (3 n)/4, -1/4 + (3 n)/4])/
  (Pochhammer[1/2 + mu/2, (1 + n)/2] Pochhammer[1 + mu/2, -1 + k] Pochhammer[-1/2 + k + mu/2, -1 + k]
   Pochhammer[-1/2 + 2 k + mu/2, k]), {k, 0, (1 + n)/2}]
```

```
Table[Factor[(Check11 /. {sum → Sum, prod → Product, if → If}) / myP2[(n + 1) / 4, mu]],
  {n, 3, 23, 4}]
```

{1, 1, 1, 1, 1, 1}

```
TraditionalForm[HoldForm @@ {Check11} /. {sum → Sum, prod → Product, if → If}]
```

$$\sum_{k=0}^{\frac{1+n}{2}} \left(2^{-\frac{5}{4}+k+\frac{3n}{4}} \operatorname{If}\left[k=0, \frac{-2+\text{mu}}{\text{mu}^2}, \frac{1}{8(-2+2k)!!}\right] \left(\left(\frac{1}{2}\right)_{-1+k}\right)^2 \left(k+\frac{\text{mu}}{2}\right)_{-k+\frac{1+n}{2}} (-1+\text{mu})_{-2+3k} \left(\frac{5}{4}+\frac{\text{mu}}{2}+\frac{3n}{4}\right)_{-\frac{1}{4}+\frac{3n}{4}}\right) \Big/ \left(\left(\frac{1}{2}+\frac{\text{mu}}{2}\right)_{\frac{1+n}{2}} \left(1+\frac{\text{mu}}{2}\right)_{-1+k} \left(-\frac{1}{2}+k+\frac{\text{mu}}{2}\right)_{-1+k} \left(-\frac{1}{2}+2k+\frac{\text{mu}}{2}\right)_k\right)$$

```
(* Our expression fits the recurrence (this is the initial value check). *)
test = ApplyOreOperator[rec2, f[n]];
Together[Table[test, {n, 5}] /. f[nn_] :> (Check11 /. n → 4 nn - 1 /. sum → Sum /. if → If)]
```

{0, 0, 0, 0, 0}

```
(* the smnd for k>=1 *)
smnd = Check11[[1]] /. n → 4 n - 1 /. if[k == 0, _, a_] → a /.
   Pochhammer[a_, b_] :> Pochhammer @@ Expand[{a, b}]
```

$$\left(2^{-\frac{17}{4}+k+\frac{3}{4}(-1+4n)} \text{Pochhammer}\left[\frac{1}{2}, -1+k\right]^2 \text{Pochhammer}\left[k+\frac{\text{mu}}{2}, -k+2n\right] \text{Pochhammer}[-1+\text{mu}, -2+3k] \text{Pochhammer}\left[\frac{1}{2}+\frac{\text{mu}}{2}+3n, -1+3n\right]\right) \Big/ \left((-2+2k)!! \, \text{Pochhammer}\left[\frac{1}{2}+\frac{\text{mu}}{2}, 2n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2}+k+\frac{\text{mu}}{2}, -1+k\right] \text{Pochhammer}\left[-\frac{1}{2}+2k+\frac{\text{mu}}{2}, k\right]\right)$$

```
Factor[{{op}, {cert}} = CreativeTelescoping[smnd, S[k] - 1, S[n]]]
```

$\{\{(1+\text{mu}+4n)(3+\text{mu}+4n)(1+\text{mu}+6n)(3+\text{mu}+6n)(5+\text{mu}+6n) S_n - (\text{mu}+4n)(2+\text{mu}+4n)(-1+\text{mu}+12n)(1+\text{mu}+12n)(3+\text{mu}+12n)(5+\text{mu}+12n)(7+\text{mu}+12n)(9+\text{mu}+12n)\}, \{0\}\}$

```
(* We use the following identity in order
 to construct an annihilating operator for the sum *)
TraditionalForm[HoldForm[
  0 == Sum[(p1[n] S[n] + p0[n]) * f[n, k], {k, 1, 2 n}] ==
   (p1[n] S[n] + p0[n]) * Sum[f[n, k], {k, 1, 2 n}]
    - p1[n] * (f[n + 1, 2 n + 1] + f[n + 1, 2 n + 2])
 ]]
test =
  ApplyOreOperator[op, Check11 /. n → 4 n - 1 /. sum[a_, {k, 0, b_}] → sum[a, {k, 1, b}]] -
   LeadingCoefficient[op] *
    ((smnd /. n → n + 1 /. k → 2 n + 1) + (smnd /. n → n + 1 /. k → 2 n + 2));
Together[Table[test /. sum → Sum /. if → If, {n, 5}]]
```

$$0 = \sum_{k=1}^{2n} (\mathrm{p1}(n)\, S(n) + \mathrm{p0}(n))\, f(n,\ k) = (\mathrm{p1}(n)\, S(n) + \mathrm{p0}(n)) \sum_{k=1}^{2n} f(n,\ k) - \mathrm{p1}(n)\, (f(n+1,\ 2\, n+1) + f(n+1,\ 2\, n+2))$$

```
{0, 0, 0, 0, 0}
```

```
inh = Factor[LeadingCoefficient[op] *
      (1 + FullSimplify[(smnd /. n → n + 1 /. k → 2 n + 2) / (smnd /. n → n + 1 /. k → 2 n + 1)])] *
   (smnd /. n → n + 1 /. k → 2 n + 1);
rec11 = Annihilator[inh, S[n]][[1]] ** op;
```

```
smnd0 = Together[ExpandAll[Check11[[1]] /. n → 4 n - 1 /. if[k == 0, a_, _] → a /. k → 0]]
```

$$\left(2^{-2+3\,n}\,\mathrm{Pochhammer}\left[\frac{\mathrm{mu}}{2},\ 2\,n\right]\,\mathrm{Pochhammer}\left[\frac{1}{2}+\frac{\mathrm{mu}}{2}+3\,n,\ -1+3\,n\right]\right)\Big/\left(\mathrm{mu}\,\mathrm{Pochhammer}\left[\frac{1}{2}+\frac{\mathrm{mu}}{2},\ 2\,n\right]\right)$$

```
(* smnd0 is also in the inhomogeneous part,
but is already annihilated by rec01. *)
OreReduce[rec11, Annihilator[smnd0, S[n]]]
```

```
0
```

```
GBEqual[{rec2}, rec11]
```

```
True
```

## Summarize

```
{Check00, Check01, Check10, Check11} /.
 (if[k == 0, (mu - 2) / mu^2, 1 / 8 / (2 k - 2) !!] * Pochhammer[1 / 2, k - 1] ^2 *
    Pochhammer[mu - 1, 3 k - 2] / Pochhammer[mu / 2 + 1, k - 1] / Pochhammer[
      mu / 2 + k - 1 / 2, k - 1] / Pochhammer[mu / 2 + 2 k - 1 / 2, k]) → (h[n, k] / 2^k)
```

$$\left\{\text{sum}\left[\left(2^{-2+\frac{3n}{4}} h[n, k]\ \text{Pochhammer}\left[k + \frac{mu}{2}, -k + \frac{n}{2}\right] \text{Pochhammer}\left[\frac{1}{2} + \frac{mu}{2} + \frac{3n}{4}, -1 + \frac{3n}{4}\right]\right) \Big/ \text{Pochhammer}\left[\frac{1}{2} + \frac{mu}{2}, \frac{n}{2}\right], \left\{k, 0, \frac{n}{2}\right\}\right],\right.$$

$$\text{sum}\left[\left(2^{-\frac{3}{2}+\frac{3n}{4}} h[n, k]\ \text{Pochhammer}\left[k + \frac{mu}{2}, -k + \frac{n}{2}\right] \text{Pochhammer}\left[\frac{mu}{2} + \frac{3n}{4}, -\frac{1}{2} + \frac{3n}{4}\right]\right) \Big/ \text{Pochhammer}\left[\frac{1}{2} + \frac{mu}{2}, \frac{n}{2}\right], \left\{k, 0, \frac{n}{2}\right\}\right],$$

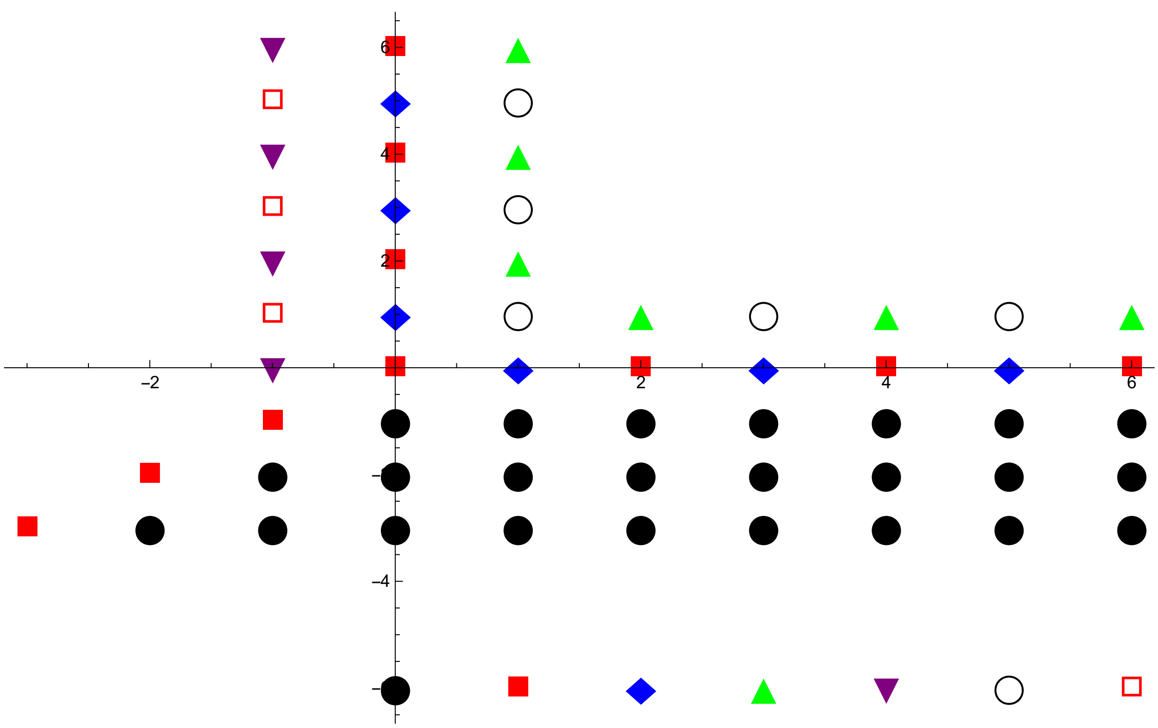
$$\text{sum}\left[\left(2^{-\frac{3}{4}+\frac{3n}{4}} h[n, k]\ \text{Pochhammer}\left[k + \frac{mu}{2}, -k + \frac{1+n}{2}\right] \text{Pochhammer}\left[\frac{3}{4} + \frac{mu}{2} + \frac{3n}{4}, \frac{1}{4} + \frac{3n}{4}\right]\right) \Big/ \text{Pochhammer}\left[\frac{1}{2} + \frac{mu}{2}, \frac{1+n}{2}\right], \left\{k, 0, \frac{1+n}{2}\right\}\right],$$

$$\left.\text{sum}\left[\left(2^{-\frac{5}{4}+\frac{3n}{4}} h[n, k]\ \text{Pochhammer}\left[k + \frac{mu}{2}, -k + \frac{1+n}{2}\right] \text{Pochhammer}\left[\frac{5}{4} + \frac{mu}{2} + \frac{3n}{4}, -\frac{1}{4} + \frac{3n}{4}\right]\right) \Big/ \text{Pochhammer}\left[\frac{1}{2} + \frac{mu}{2}, \frac{1+n}{2}\right], \left\{k, 0, \frac{1+n}{2}\right\}\right]\right\}$$

# Section 6: The General Determinant

## Overview

```
Do[Fam[i] = {}, {i, 0, 6}];
NiceQ[x_] :=
  Max[First /@ TimeConstrained[FactorInteger[x], 0.1, {{10^6, 0}}]] < 100;
Do[
  tt = Table[Det[DstMat[s, t, n, 37]], {n, 17, 19}];
  Which[
   MatchQ[tt, {(0) ..}], AppendTo[Fam[0], {s, t}],
   MatchQ[tt, {_?NiceQ, 0, _?NiceQ}], AppendTo[Fam[2], {s, t}],
   MatchQ[tt, {(_?NiceQ) ..}], AppendTo[Fam[1], {s, t}],
   NiceQ[tt[[2]]] && s < 0, AppendTo[Fam[4], {s, t}],
   NiceQ[tt[[2]]], AppendTo[Fam[3], {s, t}],
   tt = Rest[tt] / Most[tt];
   NiceQ[tt[[1]]], AppendTo[Fam[5], {s, t}],
   NiceQ[tt[[2]]], AppendTo[Fam[6], {s, t}]
  ];
  , {s, -3, 6}, {t, -3, 6}];
ListPlot[Table[Append[Fam[i], {i, -6}], {i, 0, 6}],
 PlotStyle → {Black, Red, Blue, Green, Purple},
 PlotMarkers → {Automatic, Large}, ImageSize → 600]
```



# Corollary 15: $D_{-r,-r}(n)$

```
TableForm[DstMat[-2, -2, 10, 37]]
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 34 | 595 | 7140 | 66 045 | 501 942 | 3 262 623 | 18 643 560 |
| 0 | 1 | 1 | 35 | 630 | 7770 | 73 815 | 575 757 | 3 838 380 | 22 481 940 |
| 0 | 0 | 2 | 36 | 666 | 8436 | 82 251 | 658 008 | 4 496 388 | 26 978 328 |
| 0 | 0 | 1 | 38 | 703 | 9139 | 91 390 | 749 398 | 5 245 786 | 32 224 114 |
| 0 | 0 | 1 | 38 | 742 | 9880 | 101 270 | 850 668 | 6 096 454 | 38 320 568 |
| 0 | 0 | 1 | 39 | 780 | 10 661 | 111 930 | 962 598 | 7 059 052 | 45 379 620 |
| 0 | 0 | 1 | 40 | 820 | 11 480 | 123 411 | 1 086 008 | 8 145 060 | 53 524 680 |
| 0 | 0 | 1 | 41 | 861 | 12 341 | 135 751 | 1 221 760 | 9 366 819 | 62 891 499 |
| 0 | 0 | 1 | 42 | 903 | 13 244 | 148 995 | 1 370 754 | 10 737 574 | 73 629 072 |
| 0 | 0 | 1 | 43 | 946 | 14 190 | 163 185 | 1 533 939 | 12 271 512 | 85 900 585 |

```
TableForm[DstMat[-3, -3, 10, 37]]
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 33 | 561 | 6545 | 58 905 | 435 897 | 2 760 681 |
| 0 | 1 | 0 | 1 | 34 | 595 | 7140 | 66 045 | 501 942 | 3 262 623 |
| 0 | 0 | 1 | 1 | 35 | 630 | 7770 | 73 815 | 575 757 | 3 838 380 |
| 0 | 0 | 0 | 2 | 36 | 666 | 8436 | 82 251 | 658 008 | 4 496 388 |
| 0 | 0 | 0 | 1 | 38 | 703 | 9139 | 91 390 | 749 398 | 5 245 786 |
| 0 | 0 | 0 | 1 | 38 | 742 | 9880 | 101 270 | 850 668 | 6 096 454 |
| 0 | 0 | 0 | 1 | 39 | 780 | 10 661 | 111 930 | 962 598 | 7 059 052 |
| 0 | 0 | 0 | 1 | 40 | 820 | 11 480 | 123 411 | 1 086 008 | 8 145 060 |
| 0 | 0 | 0 | 1 | 41 | 861 | 12 341 | 135 751 | 1 221 760 | 9 366 819 |
| 0 | 0 | 0 | 1 | 42 | 903 | 13 244 | 148 995 | 1 370 754 | 10 737 574 |

```
Union[Flatten[Table[Factor[Dst[r, r, n + 1] / Dst[r + 1, r + 1, n]], {r, -5, -1}, {n, 10}]]]
```

```
{1}
```

```
Table[Together[Dst[-r, -r, n] / If[r < n, Dst[0, 0, n - r], 1]], {r, 0, 5}, {n, 10}]
```

```
{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}
```

# Proposition 16: $D_{s,t}(n) = 0$ for $t \leq -1$ and $s \geq t + 1$

```
TableForm[DstMat[0, -1, 8, 31]]
```

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 2 | 30 | 465 | 4960 | 40 920 | 278 256 | 1 623 160 |
| 0 | 1 | 32 | 496 | 5456 | 46 376 | 324 632 | 1 947 792 |
| 0 | 1 | 32 | 529 | 5984 | 52 360 | 376 992 | 2 324 784 |
| 0 | 1 | 33 | 561 | 6546 | 58 905 | 435 897 | 2 760 681 |
| 0 | 1 | 34 | 595 | 7140 | 66 046 | 501 942 | 3 262 623 |
| 0 | 1 | 35 | 630 | 7770 | 73 815 | 575 758 | 3 838 380 |
| 0 | 1 | 36 | 666 | 8436 | 82 251 | 658 008 | 4 496 389 |
| 0 | 1 | 37 | 703 | 9139 | 91 390 | 749 398 | 5 245 786 |

```
Union[Flatten[Table[Dst[s, t, n], {t, -1, -5, -1}, {s, t + 1, 6}, {n, 10}]]]
```

```
{0}
```

## Theorem 17: switching *s* and *t*

```
Table[
 Together[Dst[s, t, n] - Product[Pochhammer[mu + i + s - 1, n] / Pochhammer[i + s + 1, n],
     {i, 0, t - s - 1}] * Dst[t, s, n]], {s, 0, 5}, {t, s, 5}, {n, 8}]
```

```
{{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0, 0, 0, 0}}}
```

```
(* Base case n=
 1 (we may assume t>s, hence Kronecker delta is not present at all). *)
(Dst[s, t, 1] / Dst[t, s, 1] /. KroneckerDelta[__] → 0) /
 Product[Pochhammer[mu + i + s - 1, 1] / Pochhammer[i + s + 1, 1], {i, 0, t - s - 1}]
```

```
((1 + s) Gamma[1 + s] Gamma[-1 + mu + t] Pochhammer[2 + s, -1 - s + t]) /
 ((-1 + mu + s) Gamma[-1 + mu + s] Gamma[1 + t] Pochhammer[mu + s, -1 - s + t])
```

```
FullSimplify[%]
```

```
1
```

```
(* Base case n=2 (we may assume t>s). We start by assuming t=s+1. *)
Simplify[Dst[s, t, 2] / Dst[t, s, 2] /. t → s + 1] /
 Product[Pochhammer[mu + i + s - 1, 2] / Pochhammer[i + s + 1, 2], {i, 0, s + 1 - s - 1}]
```

$$\frac{(1+s)\ (2+s)\ \text{Gamma}[1+s]\ \text{Gamma}[1+mu+s]}{(-1+mu+s)\ (mu+s)\ \text{Gamma}[3+s]\ \text{Gamma}[-1+mu+s]}$$

```
FullSimplify[%]
```

```
1
```

```
(* Base case n=2; now assume t≥s+2,
hence Kronecker delta is not present at all. *)
Simplify[Dst[s, t, 2] / Dst[t, s, 2] /. KroneckerDelta[__] → 0] /
 Product[Pochhammer[mu + i + s - 1, 2] / Pochhammer[i + s + 1, 2], {i, 0, t - s - 1}]
```

```
((1 + s) (2 + s) Gamma[1 + s] Gamma[2 + s] Gamma[-1 + mu + t]
   Gamma[mu + t] Pochhammer[2 + s, -1 - s + t] Pochhammer[3 + s, -1 - s + t]) /
 ((-1 + mu + s) (mu + s) Gamma[-1 + mu + s] Gamma[mu + s] Gamma[1 + t] Gamma[2 + t]
   Pochhammer[mu + s, -1 - s + t] Pochhammer[1 + mu + s, -1 - s + t])
```

```
FullSimplify[%]
```

```
1
```

## Theorem 18 (Family A): $D_{2r,0}(n)$ and $D_{0,2r}(n)$

```
quoAe = Pochhammer[mu + 2 n + 4 r, n - r] *
   Pochhammer[mu / 2 + 2 n + r + 1 / 2, n - r - 1] / Pochhammer[n - r, n - r] /
    Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r - 1];
quoAo = Pochhammer[mu + 2 n + 4 r - 2, n - r - 1] * Pochhammer[mu / 2 + 2 n + r - 1 / 2, n - r] /
     Pochhammer[n - r, n - r] / Pochhammer[mu / 2 + n + 2 r - 1 / 2, n - r - 1];
Table[Together[{Dst[2 r, 0, 2 n + 1] / Dst[2 r, 0, 2 n] / quoAe,
   Dst[2 r, 0, 2 n] / Dst[2 r, 0, 2 n - 1] / quoAo}], {n, 5}, {r, 0, n - 1}]
```

```
{{{1, 1}}, {{1, 1}, {1, 1}}, {{1, 1}, {1, 1}, {1, 1}},
 {{1, 1}, {1, 1}, {1, 1}, {1, 1}}, {{1, 1}, {1, 1}, {1, 1}, {1, 1}, {1, 1}}}
```

```
RA[n_] := If[EvenQ[n],
   Pochhammer[mu + n + 4 r, n / 2 - r] *
    Pochhammer[mu / 2 + n + r + 1 / 2, n / 2 - r - 1] / Pochhammer[n / 2 - r, n / 2 - r] /
     Pochhammer[mu / 2 + n / 2 + 2 r + 1 / 2, n / 2 - r - 1],
   Pochhammer[mu + n + 4 r - 1, (n + 1) / 2 - r - 1] *
    Pochhammer[mu / 2 + n + r + 1 / 2, (n + 1) / 2 - r] / Pochhammer[(n + 1) / 2 - r,
       (n + 1) / 2 - r] / Pochhammer[mu / 2 + (n + 1) / 2 + 2 r - 1 / 2, (n + 1) / 2 - r - 1]];
Table[Together[Dst[2 r, 0, n + 1] / Dst[2 r, 0, n] / RA[n]], {n, 10}, {r, 0, (n - 1) / 2}]
```

```
{{1}, {1}, {1, 1}, {1, 1}, {1, 1, 1}, {1, 1, 1},
 {1, 1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

```
{quoAei, quoAoi} = {quoAe /. n → i / 2, quoAo /. n → (i + 1) / 2};
Table[Together[Dst[2 r, 0, n] / If[n ≤ 2 r, 1,
    2 * Product[If[Mod[i, 2] == 0, quoAei, quoAoi], {i, 2 r + 1, n - 1}]]], {n, 8}, {r, 8}]
```

```
{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}
```

```
Table[Dst[2 r, 0, n], {r, 0, 4}, {n, 2 r}]
```

```
{{}, {1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}
```

```
Union[Flatten[Table[Dst[2 r, 0, n, mu] - Dst[0, 0, n - 2 r, mu + 6 r],
   {r, 4}, {n, 2 r + 1, 2 r + 5}, {mu, -2, 6}]]]
```

```
{0}
```

```
Table[
 Together[Dst[0, 2 r, n] / (Product[Pochhammer[mu + i - 1, 2 r] / Pochhammer[i + 1, 2 r],
      {i, 0, n - 1}] * Dst[2 r, 0, n])], {n, 8}, {r, 8}]
```

```
{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}
```

```
(* Special case: Proposition 8 *)
(quoAe /. r → 0) / R00e[n]
```

1

```
(* Special case: Proposition 8 *)
(quoAo /. r → 0) / R00o[n]
```

1

```
(* Special case: Lemma 6 *)
(quoAo /. r → 1) / R20[n]
```

$$\left(\text{Pochhammer}[n, -1+n]\ \text{Pochhammer}\left[\frac{1}{2}+\frac{mu}{2}+n, -1+n\right]\ \text{Pochhammer}[2+mu+2n, -2+n]\right) \Big/ \left(\text{Pochhammer}[-1+n, -1+n]\ \text{Pochhammer}\left[\frac{3}{2}+\frac{mu}{2}+n, -2+n\right]\ \text{Pochhammer}[1+mu+2n, -1+n]\right)$$

```
FullSimplify[%]
```

1

```
(* Special case: Lemma 7 *)
((quoAo /. r → 1) * (mu + 2 n - 2) * (mu + 2 n - 1) / (2 n) / (2 n + 1)) / R02[n]
```

$$\left((-2+mu+2n)\ (-1+mu+2n)\ (mu+2n)\ \text{Pochhammer}[n, 2+n]\ \text{Pochhammer}\left[\frac{1}{2}+\frac{mu}{2}+n, -1+n\right]\ \text{Pochhammer}[2+mu+2n, -2+n]\right) \Big/ \left(2n\ (-1+2n)\ (1+2n)\ \text{Pochhammer}[-1+n, -1+n]\ \text{Pochhammer}\left[\frac{3}{2}+\frac{mu}{2}+n, -2+n\right]\ \text{Pochhammer}[-2+mu+2n, 2+n]\right)$$

```
FullSimplify[%]
```

1

## Theorem 19 (Family B): $D_{2r-1,0}(n)$ and $D_{0,2r-1}(n)$

```
quoB = -Pochhammer[mu + 2 n + 4 r - 4, n - r + 1] * Pochhammer[mu + 2 n + 4 r - 3, n - r] *
    Pochhammer[mu / 2 + r + 2 n - 1 / 2, n - r] ^ 2 / Pochhammer[n - r + 1, n - r] /
      Pochhammer[n - r + 1, n - r + 1] / Pochhammer[mu / 2 + n + 2 r - 3 / 2, n - r] ^ 2;
Table[Factor[(Dst[2 r - 1, 0, 2 n + 1] / Dst[2 r - 1, 0, 2 n - 1]) / quoB], {n, 5}, {r, n}] //
 TableForm
```

```
1
1    1
1    1    1
1    1    1    1
1    1    1    1    1
```

```
Table[Together[Dst[2 r - 1, 0, n] / Product[quoB, {n, r, (n - 1) / 2}]],
 {n, 1, 9, 2}, {r, 5}]
```

```
{{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

```
TableForm[Table[Dst[2 r - 1, 0, n], {n, 2, 10, 2}, {r, 5}]]
```

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 |

```
Table[Together[Dst[2 r - 1, 0, n, mu] - Dst[1, 0, n - 2 r + 2, mu + 6 r - 6]],
 {r, 5}, {n, 2 r, 10}]

{{0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0}, {0}}

Table[Dst[0, 2 r - 1, n, 37] /
  (Product[Pochhammer[mu + i - 1, 2 r - 1] / Pochhammer[i + 1, 2 r - 1], {i, 0, n - 1}] *
    Product[quoB, {n, r, (n - 1) / 2}] /. mu → 37), {n, 1, 9, 2}, {r, 1, 10}]

{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

(* Special case: Proposition 9 *)
(quoB /. r → 1) / R10[n]

1

(* Special case: Proposition 10 *)
((quoB /. r → 1) * (mu + 2 n - 1) * (mu + 2 n - 2) / (2 n + 1) / (2 n)) / R01[n]

((-2 + mu + 2 n) (-1 + mu + 2 n) Pochhammer[n, 2 + n] Pochhammer[mu + 2 n, n]) /
 (2 n (1 + 2 n) Pochhammer[n, n] Pochhammer[-2 + mu + 2 n, 2 + n])

FullSimplify[%]

1
```

## Conjecture 20 (Family C): $D_{2r,1}(n)$ and $D_{1,2r}(n)$

```
quoC = - (2 n + 2 r) * (mu + 2 n + 2 r - 1) * (mu + 2 n + 2 r) * Pochhammer[mu + 2 n + 4 r, n - r] ^ 2 *
   Pochhammer[mu / 2 + 2 n + r + 3 / 2, n - r + 1] ^ 2 / Pochhammer[n - r + 1, n - r + 1] ^ 2 /
     Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r] ^ 2 / (mu + 2 n + 1) / (2 n + 1) / (2 n + 2);
Table[Factor[(Dst[2 r, 1, 2 n + 2] / Dst[2 r, 1, 2 n]) / quoC], {n, 6}, {r, n}]

{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1}}

TableForm[
 Table[Together[Dst[2 r, 1, 2 n] / (Pochhammer[mu + 2 r - 1, 2 n] / (2 n) !)], {r, 5}, {n, r}]]
```

| | | | | |
|---|---|---|---|---|
| $\frac{-1+mu}{1+mu}$ | | | | |
| 1 | $\frac{-1+mu}{3+mu}$ | | | |
| 1 | 1 | $\frac{-1+mu}{5+mu}$ | | |
| 1 | 1 | 1 | $\frac{-1+mu}{7+mu}$ | |
| 1 | 1 | 1 | 1 | $\frac{-1+mu}{9+mu}$ |

```
Table[Together[Dst[2 r, 1, 2 r] / ((mu - 1) * Pochhammer[mu + 2 r, 2 r - 1] / (2 r) !)], {r, 5}]
{1, 1, 1, 1, 1}

Table[Together[Dst[2 r, 1, 2 n] /
   If[n < r, Pochhammer[mu + 2 r - 1, 2 n] / (2 n) !, (mu - 1) *
     Pochhammer[mu + 2 r, 2 r - 1] / (2 r) ! * Product[quoC, {n, r, n - 1}]]], {n, 5}, {r, 5}]
{{1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1}}

Table[
 Together[Dst[1, 2 r, n] / (Product[Pochhammer[mu + i, 2 r - 1] / Pochhammer[i + 2, 2 r - 1],
      {i, 0, n - 1}] * Dst[2 r, 1, n])], {n, 8}, {r, 8}]
{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}}
```

## Conjecture 21 (Family D): $D_{-1,2\,r}(n)$

```
quoD = -Pochhammer[mu + 2 n - 3, 2 r + 1] *
   Pochhammer[mu + 2 n - 1, 2 r] * Pochhammer[mu + 2 n + 4 r, n - r] ^ 2 *
   Pochhammer[mu / 2 + 2 n + r + 1 / 2, n - r - 1] ^ 2 / Pochhammer[n - r, n - r] ^ 2 /
      Pochhammer[2 n + 2, 2 r + 1] / Pochhammer[2 n + 1, 2 r] /
    Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r - 1] ^ 2;
Table[Together[(Dst[-1, 2 r, 2 n + 2] / Dst[-1, 2 r, 2 n]) / quoD], {n, 5}, {r, 0, n - 1}]
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}

Table[
 Together[Dst[-1, 2 r, 2 n] / (Product[quoD, {n, r + 1, n - 1}] * Dst[-1, 2 r, 2 r + 2])],
 {n, 5}, {r, 0, n - 1}]
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}

Table[Together[(Dst[-1, 2 r, 2 n + 2] / Dst[-1, 2 r, 2 n]) /
   (Pochhammer[mu + 2 n - 2, 2 r] * Pochhammer[mu + 2 n - 1, 2 r] / Pochhammer[2 n + 1, 2 r] /
      Pochhammer[2 n + 2, 2 r])], {r, 0, 5}, {n, r - 1}]
{{}, {}, {1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}}

Table[Together[
  Dst[-1, 2 r, 2 r] / (Product[Pochhammer[mu + 2 i - 2, 2 r] * Pochhammer[mu + 2 i - 1, 2 r] /
        Pochhammer[2 i + 1, 2 r] / Pochhammer[2 i + 2, 2 r], {i, 0, r - 1}])], {r, 4}]
{1, 1, 1, 1}

Table[Together[Dst[-1, 2 r, 2 n] /
   (Product[quoD, {n, r + 1, n - 1}] * (Dst[-1, 2 r, 2 r + 2] / Dst[-1, 2 r, 2 r]) * Product[
      Pochhammer[mu + 2 i - 2, 2 r] * Pochhammer[mu + 2 i - 1, 2 r] / Pochhammer[2 i + 1, 2 r] /
        Pochhammer[2 i + 2, 2 r], {i, 0, r - 1}])], {n, 5}, {r, 1, n - 1}]
{{}, {1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}}
```

```
Table[Together[(Dst[-1, 2 r, 2 r + 2] / Dst[-1, 2 r, 2 r]) /
    ((3 - mu) * Pochhammer[mu + 2 r - 2, 2 r] * Pochhammer[mu + 2 r - 1, 2 r] /
        Pochhammer[2 r + 1, 2 r] / Pochhammer[2 r + 1, 2 r + 1])], {r, 5}]
```

```
{1, 1, 1, 1, 1}
```

```
RD[r_, n_] := Which[
   n > r, -Pochhammer[mu + 2 n - 3, 2 r + 1] *
    Pochhammer[mu + 2 n - 1, 2 r] * Pochhammer[mu + 2 n + 4 r, n - r] ^ 2 *
    Pochhammer[mu / 2 + 2 n + r + 1 / 2, n - r - 1] ^ 2 / Pochhammer[n - r, n - r] ^ 2 /
       Pochhammer[2 n + 2, 2 r + 1] / Pochhammer[2 n + 1, 2 r] /
     Pochhammer[mu / 2 + n + 2 r + 1 / 2, n - r - 1] ^ 2 (* this is quoD *),
   n == r, (3 - mu) * Pochhammer[mu + 2 r - 2, 2 r] *
    Pochhammer[mu + 2 r - 1, 2 r] / Pochhammer[2 r + 1, 2 r] / Pochhammer[2 r + 1, 2 r + 1],
   n < r, Pochhammer[mu + 2 n - 2, 2 r] *
    Pochhammer[mu + 2 n - 1, 2 r] / Pochhammer[2 n + 1, 2 r] / Pochhammer[2 n + 2, 2 r]];
Table[Together[Dst[-1, 2 r, 2 n] / Product[RD[r, i], {i, 0, n - 1}]], {n, 5}, {r, 0, 10}]
```

```
{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}
```

## Corollary 22 (Family E): $D_{2r-1,1}(2n)/D_{2r-1,1}(2n-1)$

```
quoE = Pochhammer[mu / 2 + 2 n + r - 1 / 2, n - r] *
    Pochhammer[mu + 2 n + 4 r - 4, n - r + 1] / Pochhammer[n - r + 1, n - r + 1] /
     Pochhammer[mu / 2 + n + 2 r - 3 / 2, n - r];
Table[Together[(Dst[2 r - 1, 1, 2 n] / Dst[2 r - 1, 1, 2 n - 1]) / quoE], {n, 5}, {r, n}]
```

```
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

```
Table[Together[(Dst[1, 2 r - 1, 2 n] / Dst[1, 2 r - 1, 2 n - 1]) /
    (Pochhammer[mu + 2 n - 1, 2 r - 2] / Pochhammer[2 n + 1, 2 r - 2] * quoE)], {n, 5}, {r, n}]
```

```
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

```
TableForm[Table[Factor[Dst[2 r - 1, 1, n + 1] / Dst[2 r - 1, 1, n]], {r, 4}, {n, 2 r}]]
```

| | | | | | |
|---|---|---|---|---|---|
| $2+mu$ | $\frac{1}{12}(3+mu)(14+mu)$ | | | | |
| $\frac{3+mu}{2}$ | $\frac{(4+mu)(5+mu)}{3(2+mu)}$ | $8+mu$ | $\frac{(6+mu)(7+mu)(52+3\,mu)}{40(4+mu)}$ | | |
| $\frac{5+mu}{2}$ | $\frac{6+mu}{3}$ | $\frac{7+mu}{4}$ | $\frac{(8+mu)(9+mu)}{5(4+mu)}$ | $14+mu$ | $\frac{(10+mu)(11+mu)(114+5\,mu}{84(6+mu)}$ |
| $\frac{7+mu}{2}$ | $\frac{8+mu}{3}$ | $\frac{9+mu}{4}$ | $\frac{10+mu}{5}$ | $\frac{11+mu}{6}$ | $\frac{(12+mu)(13+mu)}{7(6+mu)}$ |

## Corollary 23 (Family F): $D_{-1,2r-1}(2n+1)/D_{-1,2r-1}(2n)$

```
quoF = 2 Pochhammer[-2 + 2 n + mu, 2 r] * Pochhammer[-2 + 4 r + 2 n + mu, n - r - 1] *
   Pochhammer[mu / 2 + 2 n + r - 1 / 2, n - r] / Pochhammer[2 n, 2 r] /
     Pochhammer[n - r + 1, n - r] / Pochhammer[mu / 2 + n + 2 r - 1 / 2, n - r - 1];
Table[Factor[(Dst[-1, 2 r - 1, 2 n + 1] / Dst[-1, 2 r - 1, 2 n]) / quoF], {n, 4}, {r, n}]
```

```
{{1}, {1, 1}, {1, 1, 1}, {1, 1, 1, 1}}
```

## Encore: Guessing another closed form for $D_{1,2r-1}(2n)/D_{1,2r-1}(2n-1)$

```
dets = Table[Dst[1, 2 r - 1, n, 100 003], {n, 40}, {r, 20}];
data = Table[dets[[n + 1, r]] / dets[[n, r]], {n, 1, 39, 2}, {r, 20}];
guess = GuessMultRE[data, {f[m, r], f[m + 1, r], f[m, r + 1]},
   {m, r}, 6, StartPoint → {1, 1}, Constraints → m > r];
gb = Factor[OreGroebnerBasis[NormalizeCoefficients /@
    ToOrePolynomial[guess, f[m, r]]]]
```

```
{2 (m + r) (50 000 + 2 m + r) (50 001 + 2 m + r) (-1 + 2 m + 2 r)
    (99 999 + 2 m + 4 r) (100 001 + 2 m + 4 r) S_r - 3 (1 + 2 m - 2 r) (33 334 + m + r)
   (50 000 + m + r) (100 001 + 2 m + 2 r) (100 000 + 3 m + 3 r) (100 001 + 3 m + 3 r),
 -4 (50 001 + m) (100 003 + 2 m) (3 + 2 m - 2 r) (m + r) (50 000 + 2 m + r)
    (50 001 + 2 m + r)^2 (50 002 + 2 m + r) (-1 + 2 m + 2 r) (99 999 + 2 m + 4 r) S_m +
  9 (1 + m) (16 667 + m) (1 + 2 m) (50 002 + 3 m) (50 003 + 3 m) (33 334 + m + r)
   (50 000 + m + r) (100 001 + 2 m + 2 r) (100 000 + 3 m + 3 r) (100 001 + 3 m + 3 r)}
```

```
Reconst[x_, v_] := With[{a = Round[v / x]}, (mu + x * a - v) / a];
Factor[gb = NormalizeCoefficients /@
   Expand[gb /. x_Integer /; Abs[x] > 10 000 :> Reconst[x, 100 003]]]
```

```
{(m + r) (-1 + 2 m + 2 r) (-3 + 4 m + mu + 2 r) (-1 + 4 m + mu + 2 r)
    (-4 + 2 m + mu + 4 r) (-2 + 2 m + mu + 4 r) S_r - (1 + 2 m - 2 r) (-3 + 2 m + mu + 2 r)
   (-2 + 2 m + mu + 2 r) (-3 + 3 m + mu + 3 r) (-2 + 3 m + mu + 3 r) (-1 + 3 m + mu + 3 r),
 -2 (-1 + 2 m + mu) (2 m + mu) (3 + 2 m - 2 r) (m + r) (-1 + 2 m + 2 r) (-3 + 4 m + mu + 2 r)
    (-1 + 4 m + mu + 2 r)^2 (1 + 4 m + mu + 2 r) (-4 + 2 m + mu + 4 r) S_m +
  (1 + m) (1 + 2 m) (-1 + 6 m + mu) (1 + 6 m + mu) (3 + 6 m + mu) (-3 + 2 m + mu + 2 r)
   (-2 + 2 m + mu + 2 r) (-3 + 3 m + mu + 3 r) (-2 + 3 m + mu + 3 r) (-1 + 3 m + mu + 3 r)}
```

```
sol = RSolve[ApplyOreOperator[gb[[2]], f[m]] == 0, f[m], m][[1, 1, 2]];
Factor[gb = DFiniteTimes[Annihilator[1 / sol, {S[m], S[r]}], gb]]
```

```
{r (-1 + mu + 2 r) (-4 + mu + 4 r) (-2 + mu + 4 r) S_r +
  (-2 + mu + 2 r) (-3 + mu + 3 r) (-2 + mu + 3 r) (-1 + mu + 3 r), S_m - 1}
```

```
sol = Simplify[sol * RSolve[ApplyOreOperator[gb[[1]], f[r]] == 0, f[r], r][[1, 1, 2]]]
```

$$\Big((-1)^{1+r}\,16^{1-2m-r}\,27^{-1+2m+r}\,C[1]^2\,\text{Pochhammer}\Big[\frac{1}{2}, m\Big]\,\text{Pochhammer}[1, m]\,\text{Pochhammer}\Big[\frac{1}{6}(-1+mu), m\Big]\,\text{Pochhammer}\Big[\frac{mu}{3}, -1+r\Big]\,\text{Pochhammer}\Big[\frac{mu}{2}, -1+r\Big]\,\text{Pochhammer}\Big[\frac{1+mu}{6}, m\Big]\,\text{Pochhammer}\Big[\frac{1+mu}{3}, -1+r\Big]\,\text{Pochhammer}\Big[\frac{2+mu}{3}, -1+r\Big]\,\text{Pochhammer}\Big[\frac{3+mu}{6}, m\Big]\,\text{Pochhammer}\Big[-1+\frac{mu}{3}+r, m\Big]\,\text{Pochhammer}\Big[-\frac{2}{3}+\frac{mu}{3}+r, m\Big]\,\text{Pochhammer}\Big[-\frac{1}{3}+\frac{mu}{3}+r, m\Big]\,\text{Pochhammer}\Big[-\frac{3}{2}+\frac{mu}{2}+r, m\Big]\,\text{Pochhammer}\Big[-1+\frac{mu}{2}+r, m\Big]\Big)\Big/\Big(\text{Pochhammer}[1, -1+r]\,\text{Pochhammer}\Big[\frac{1}{2}(-1+mu), m\Big]\,\text{Pochhammer}\Big[\frac{mu}{4}, -1+r\Big]\,\text{Pochhammer}\Big[\frac{mu}{2}, m\Big]\,\text{Pochhammer}\Big[\frac{1+mu}{2}, -1+r\Big]\,\text{Pochhammer}\Big[\frac{2+mu}{4}, -1+r\Big]\,\text{Pochhammer}\Big[\frac{3}{2}-r, m\Big]\,\text{Pochhammer}\Big[-\frac{1}{2}+r, m\Big]\,\text{Pochhammer}[r, m]\,\text{Pochhammer}\Big[-2+\frac{mu}{2}+2r, m\Big]\,\text{Pochhammer}\Big[\frac{1}{4}(-3+mu+2r), m\Big]\,\text{Pochhammer}\Big[\frac{1}{4}(-1+mu+2r), m\Big]^2\,\text{Pochhammer}\Big[\frac{1}{4}(1+mu+2r), m\Big]\Big)$$

```
Union[Flatten[Table[(sol /. mu → 100 003) / C[1] ^ 2 / data[[m, r]], {r, 6}, {m, r, 6}]]]
```

$$\Big\{\frac{1}{2}\Big\}$$

```
sol = sol / C[1] ^ 2 / (1 / 2);
sol = FullSimplify[sol, Element[{r, m}, Integers] && r ≥ 1 && m ≥ 1 && m ≥ r]
```

$$\Big(2^{4-4m-mu-2r}\,27^m\,\pi\,(2m)!\,\text{Gamma}\Big[-\frac{1}{6}+m+\frac{mu}{6}\Big]\,\text{Gamma}\Big[\frac{1}{6}+m+\frac{mu}{6}\Big]\,\text{Gamma}\Big[\frac{1}{2}+m+\frac{mu}{6}\Big]\,\text{Gamma}\Big[\frac{1}{2}(-1+mu)\Big]\,\text{Gamma}[-3+2m+mu+2r]\,\text{Gamma}[-3+3m+mu+3r]\Big)\Big/\Big((2(-1+m+r))!\,\text{Gamma}\Big[\frac{1}{6}(-1+mu)\Big]\,\text{Gamma}\Big[\frac{1+mu}{6}\Big]\,\text{Gamma}\Big[\frac{3+mu}{6}\Big]\,\text{Gamma}[-1+2m+mu]\,\text{Gamma}\Big[\frac{3}{2}+m-r\Big]\,\text{Gamma}\Big[2m+\frac{1}{2}(-3+mu)+r\Big]\,\text{Gamma}\Big[2m+\frac{1}{2}(-1+mu)+r\Big]\,\text{Gamma}\Big[-2+m+\frac{mu}{2}+2r\Big]\Big)$$

```
quo =
  sol //. Gamma[a_] / Gamma[b_] /; IntegerQ[Expand[a - b - m]] :> Pochhammer[b, Expand[a -
                      b]] /.
                  Pochhammer[(mu - 1) / 6, m] Pochhammer[(1 + mu) / 6, m] Pochhammer[
                    (3 + mu) / 6, m] → Pochhammer[mu / 2 - 1 / 2, 3 m] / 3 ^ (3 m) /.
                Gamma[a_] :> Gamma[Expand[a]] /.
               Gamma[a_ + Rational[b_, 2]] :> With[{z = a + b / 2 - 1 / 2},
                  2 ^ Expand[1 - 2 z] * Sqrt[Pi] * Gamma[Expand[2 z]] / Gamma[z]] /.
              Gamma[m - r + 1] / Gamma[2 m - 2 r + 2] → 1 / Pochhammer[m - r + 1, m - r + 1] /.
             Gamma[mu / 2 - 1 + r + 2 m] / Gamma[mu / 2 - 2 + 2 r + m] →
              Pochhammer[mu / 2 + 2 r + m - 2, m - r + 1] /.
            Gamma[mu / 2 + 2 m + r - 2] / Gamma[mu / 2 - 1] →
             Pochhammer[mu / 2 - 1, 2 m + r - 1] /.
           Gamma[-3 + 3 r + 3 m + mu] / Gamma[-2 + 2 r + 4 m + mu] →
            1 / Pochhammer[-3 + 3 r + 3 m + mu, 1 - r + m] /.
          Gamma[mu - 2] / Gamma[mu + 2 m - 1] → 1 / Pochhammer[mu - 2, 2 m + 1] /.
         Gamma[mu + 2 m + 2 r - 3] / Gamma[mu + 4 m + 2 r - 4] →
          1 / Pochhammer[mu + 2 m + 2 r - 3, 2 m - 1] /.
        (2 m) ! / (2 (m + r - 1)) ! → 1 / Pochhammer[2 m + 1, 2 r - 2] /.
       Pochhammer[mu / 2 - 1 / 2, 3 m] → Pochhammer[mu / 2 - 1 / 2, 2 m + r - 1] *
         Pochhammer[mu / 2 - 1 / 2 + 2 m + r - 1, m - r + 1] /.
      Pochhammer[mu / 2 - 1, 2 m + r - 1] * Pochhammer[mu / 2 - 1 / 2, 2 m + r - 1] →
       Pochhammer[mu - 2, 4 m + 2 r - 2] / 2 ^ (4 m + 2 r - 2) /.
     Pochhammer[mu - 2, 4 m + 2 r - 2] / Pochhammer[mu - 2, 2 m + 1] →
      Pochhammer[mu + 2 m - 1, 4 m + 2 r - 2 m - 3] /.
    Pochhammer[mu + 2 m - 1, 2 m + 2 r - 3] → Pochhammer[mu + 2 m - 1, 2 r - 2] *
      Pochhammer[mu + 2 m + 2 r - 3, 2 m - 1]
```

$$\left(2^{2+2\,m-2\,r}\,\text{Pochhammer}[-1+2\,m+\text{mu},\,-2+2\,r]\;\text{Pochhammer}\left[-\frac{3}{2}+2\,m+\frac{\text{mu}}{2}+r,\,1+m-r\right]\text{Pochhammer}\left[-2+m+\frac{\text{mu}}{2}+2\,r,\,1+m-r\right]\right)\Big/\\(\text{Pochhammer}[1+2\,m,\,-2+2\,r]\;\text{Pochhammer}[1+m-r,\,1+m-r]\;\text{Pochhammer}[-3+3\,m+\text{mu}+3\,r,\,1+m-r])$$

```
Table[Det[DstMat[1, 2 r - 1, 2 m, 37]] / Det[DstMat[1, 2 r - 1, 2 m - 1, 37]] /
   (quo /. {mu → 37}), {r, 10}, {m, r, 10}]
```

```
{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1}, {1, 1, 1, 1}, {1, 1, 1}, {1, 1}, {1}}
```

```
TraditionalForm[quo /. {mu → μ}]
```

$$\left(2^{2\,m-2\,r+2}\left(2\,m+r+\frac{\mu}{2}-\frac{3}{2}\right)_{m-r+1}\left(m+2\,r+\frac{\mu}{2}-2\right)_{m-r+1}(2\,m+\mu-1)_{2\,r-2}\right)\Big/\\((2\,m+1)_{2\,r-2}\,(m-r+1)_{m-r+1}\,(3\,m+3\,r+\mu-3)_{m-r+1})$$